The Music of the Primes

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Acknowledgements.

This work started off when the author was nearing the end of his first year as a Graduate Student at Stanford under Brian Conrad. It took nearly 12 years to assemble. Brian Conrad and Ravi Vakil were in the process of making arrangements for me to return to Stanford for a year to finish the polishing of the paper.

In light of the current climate I am being advised to release the current version as it is. The math is complete, and only filling in some writing remains on several parts across this long paper. The plan was to take a year of doing just this, with Brian Conrad helping me completing the task of polishing and improving the exposition.

The introduction itself is unfinished and will be soon replaced by a proper one, and the plan will continue as stated above.

I have included excerpts from a letter to Peter Sarnak that was sent to him mid October 2026, containing an overview of all the methods. Brian Conrad and I went visiting him at IAS last Sunday, for a four and a half hour long blackboard discussion with him.

Brian Conrad and I are aware of several glitches here and there that I was in the process of addressing on the way to completing the typewriting. The ideas are however all here, and many of the chapters are already fairly stable. The excerpts from the letter to Sarnak explain the overall strategy (generality: all number fields, all unitary Hecke characters).

If any message at all this work is meant to send to the mathematician who feels discouraged by the pace and tone of the current climate, it is this: we are still here. We are very much alive. We are not going anywhere.

Alessandro

Notation and terminology.

We denote by Π the set of all prime numbers. We denote by $\mathbf{Q}$ the field of rational numbers.

We reserve the letter F for finite field extensions of $\mathbf{Q}$, and denote by $[F:\mathbf{Q}]$ the degree of the extension. We refer to finite field extensions of $\mathbf{Q}$ as *number fields*. We denote by $\mathcal{O}_F$ the ring of integers of F, i.e. the normalization of $\mathbf{Z}$ in F, a finitely generated free (or "finite free", for brevity) $\mathbf{Z}$-module of rank $[F:\mathbf{Q}]$.

We adopt the lexicon customarily used within Number Theory and refer to equivalence classes of absolute values (in the sense of [CF, Ch.II,§2]) on F as "places". We denote by Σ_F the set of all non-trivial places F.

To a prime ideal $\mathfrak{p}$ of the ring of integers of F, we will associate the $\mathfrak{p}$-adic absolute value on F, denoted $|\cdot|_{\mathfrak{p}}$. In particular, to a prime number p we will associate the p-adic absolute value.

For a non-trivial place of F, written "v", we denote by $|\cdot|_v$ the corresponding normalized absolute value on F (in the sense of [CF, Ch.II,p.51] if v is finite, and induced by the Euclidean absolute value on $\mathbf{R}$, respectively the square of the Euclidean absolute value on $\mathbf{C}$, if v is infinite).

We refer to the absolute values on F corresponding to the prime ideals of $\mathcal{O}_F$ as the *finite places* of F, and denote by $\Sigma_{F,f}$ the set of all finite places of F.

We refer to the Archimedean absolute values as the *infinite places* of F, and denote by S_∞ the (finite) set of all infinite places of F.

We reserve the letter v to denote the arbitrary place of F, and write F_v for the v-adic completion of F, by which we mean the completion of the field F with respect to the topology induced by any absolute value corresponding to v: if v corresponds to the prime ideal $\mathfrak{p}$, this is, up to isomorphism of topological fields, the p-adic field $F_{\mathfrak{p}}$, a finite extension of the field of p-adic rationals $\mathbf{Q}_p$, where p is the residue characteristic of $F_{\mathfrak{p}}$. We write $v \mid p$ to mean that F_v is an extension of $\mathbf{Q}_p$.

If v is Archimedean, then F_v is, up to isomorphism of topological fields, either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. We call v "real" or "complex", accordingly.

For v a finite place of F, we denote by $\mathcal{O}_{F_v}$ the subset defined by the condition $\{x \in F_v : |x|_v \leq 1\}$, i.e. the unit ball within the metric space F_v. We denote by $\mathcal{O}_{F_v}^\times$ the "unit circle" within F_v, i.e. the subset defined by the condition $\{x \in F_v : |x|_v = 1\}$. This is the unit group of $\mathcal{O}_{F_v}$.

We denote by $\mathbf{A}_F$ the adele ring of the number field F. We recall that $\mathbf{A}_F$ is the topological ring obtained as the restricted product

$$\mathbf{A}_F = \widehat{\prod_v}(F_v, R_v)$$

topologized with the restricted product topology, where v varies over all the places of F, including the Archimedean places, and with $R_v = \mathcal{O}_{F_v}$ for every finite place v, and $R_v = \{0\}$ for every infinite place v.

We denote by $\mathrm{GL}_1(\mathbf{A}_F)$ the group of units of $\mathbf{A}_F$ topologized with the idelic topology, i.e. the topology induced by the anti-diagonal map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{A}_F \times \mathbf{A}_F$ given by $x \mapsto (x, x^{-1})$. We refer to $\mathrm{GL}_1(\mathbf{A}_F)$ as the *idele group* of the number field F.

We denote by

$$|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$$

the *idelic norm*, defined on $\mathrm{GL}_1(\mathbf{A}_F)$ as $\prod_v |\cdot|_v$, where the product is indexed by all the places of F.

For a Hausdorff locally compact abelian group, we denote by $\widehat{G}$ the Pontryagin dual group, the Hausdorff locally compact abelian group of all continuous characters $\chi : G \to \mathrm{U}(1)$ endowed with the compact-open topology. We refer to $\widehat{G}$ as the "dual group".

Index of Symbols

$\mathcal{O}_{F,S}$ S-integer ring of the number field F for S a finite set of places of F containing all the Archimedean places 6
Φ_F global Adelic θ-function operator 6
$\Phi_{F,S}$ S-local Adelic θ-function operator 6
C_F idele class group of the number field F.
The quotient of the idele group $\mathrm{GL}_1(\mathbf{A}_F)$ by the discrete subgroup $F^\times$. 6
$O_{F,S}^\times$ group of S-units of the number field F 6
Z global Idelic Mellin transform 6
Z_S S-local Idelic Mellin transform 6

[1]

[1]Include comment that $\mathbf{N}$ includes 0.

CHAPTER 1

Introduction

1.1. The paradigm of families

1.1.1. A "wonderful exercise": Tate's Thesis.

1.1.2. Spaces of Schwartz-Bruhat functions.

- $\mathcal{S}(G)$ the space of Schwartz-Bruhat functions on a Hausdorff locally compact abelian group G. We will be usually assuming G is second-countable. The examples of main interest are $G = F_v, F_v^\times, \mathbf{A}_F, \mathbf{A}_{F,S} = \prod_{v\in S} F_v$, $C_F = \mathrm{GL}_1(\mathbf{A}_F)/F^\times$,$J_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$.
- Assume G is a second-countable $\mathbf{R}_{>0}$-modulated group. For $I \subset \mathbf{R}$ a non-empty interval, $\mathcal{S}_I(G)$ is the space of I-*exponentially decaying* Schwartz-Bruhat functions, i.e. $\phi \in C^\infty(G)$ such that $|\cdot|_G^a\phi \in \mathcal{S}(G)$ for all $a \in I$. We topologize $\mathcal{S}_I(G)$ with the locally convex inductive limit topology of the nuclear Fréchet topologies on $\mathcal{S}_I(k, G)$ for $k \subset G$ varying over the compact subgroups of G whose orthogonal $k^\perp \subset \widehat{G}$ is compactly generated. The topology on $\mathcal{S}_I(k, G)$ is the subspace topology induced by the linear injection

$$T : \mathcal{S}_I(k, G) \to \prod_{a\in I} \mathcal{S}(k, G) \tag{1.1.2.1}$$

  sending ϕ to $(|\cdot|_G^a\phi)_{a\in I}$. This is also $\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi \mathcal{S}(G^1/k)$.
- For $I = \mathbf{R}$, we denote $\mathcal{S}_I(G)$ by $\mathcal{S}_{\mathrm{E}}(G)$, and call this the space of *exponentially decaying Schwartz-Bruhat functions on* G.
- Under the above assumptions on G, $\mathcal{S}_I(G)$ is a strict LF-space. Nuclear (this is used to get good properties of topological tensor products and to get a handle on continuous duals of quotient linear maps, such that they are linear topological embeddings; nuclearity often ensures that quotient maps lift bounded sets, which is the crux of the matter behind the good behavior of the continuous duals of quotient maps), complete (used for vector valued integration), Montel (used for the Heine-Borel property: to get constants in some estimates that are uniform on bounded sets of functions).

1.1.3. Hardy-Sobolev spaces. Let $I \subset \mathbf{R}$ be a compact interval, i.e. $I = [a, b]$ for $a, b \in \mathbf{R}$, with $a < b$.

For G a second-countable Hausdorff locally compact abelian group and an $\mathbf{R}_{>0}$-modulated group, we denote by

$$W_I^{s,2}(G)$$

the completion of $\mathcal{S}_{\mathrm{E}}(G)$ (since I is compact, it is also the completion of $\mathcal{S}_I(G)$) with respect to the linear topology generated by the norm:

$$\|\phi\|_{s,I} := \left(\int_G |\phi(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\}(1 + |\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu(g)\right)^{1/2}$$

with μ a fixed choice of Haar measure on G.

Even with $s = 0$, these Hilbert spaces were never considered in the literature. In particular, they were never considered by Connes nor Meyer, in any instance.

We refer to $W_I^{s,2}(G)$ as the *Hardy-Sobolev space* of G with base I.
If $I = \{0\}$, then $W_I^{s,2}(G)$ agrees with the Sobolev space $W^{s,2}(G)$ introduced in the paper.
The topology on $W_I^{s,2}(G)$ is also generated by the finitely many semi-norms:

$$\|\phi\|_{\alpha,k} := \|\partial_z^k M(\phi)(z)\|_{L^2(\alpha+i\mathbf{R},\mathrm{d}x)},\ \alpha = a, b$$

for all $0 \leq k \leq s/2$ (assume that s is an even integer), i.e. by integration of the Mellin transform $M(\phi)$ and some of its holomorphic derivatives along the edge of the vertical strip $\mathbf{C}_I$ of bounded width.

1.1.4. Adelic θ-functions: Sarnak's paradigm of families. Recall that we denote by:

- $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ the quotient of $\mathcal{S}(\mathbf{A}_{F,S})$ by the kernel of the θ-function operator $\Phi_{F,S}$

$$\Phi_{F,S}(f)(g) := \sum_{\xi \in \mathcal{O}_{F,S}^\times} f(\xi g),\ g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$$

  for $f \in \mathcal{S}(\mathbf{A}_{F,S})$. We endow $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ with the quotient topology. As a consequence of the Structure Theorems, this is a strict LF-space. We refer to $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ as the *space of S-local θ-functions*.
- $\chi \in \widehat{C_F}$ the arbitrary continuous character of C_F, and by χ_S, for S a finite set of places of F containing all the Archimedean places, and such that χ is unramified outside S, the unique continuous character of $J_{F,S}$ such that the composition of χ_S by the quotient

$$(\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}_{\Sigma_F - S}^\times)/\mathcal{O}_{F,S}^\times \to J_{F,S}$$

  is the restriction of χ. If $\chi|_{C_F^1} = 1$, we will be using $\chi = 1$ as unitary base point for the connected component $\Omega(C_F)(\chi)$ of the Riemann surface $\Omega(C_F)$, rather than $\chi = |\cdot|_F^{ix_0}$ for some $x_0 \in \mathbf{R}$.
- $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ the quotient of $\mathcal{S}(\mathbf{A}_F)$ by $\ker(\Phi_F)$, with Φ_F the global θ-function operator defined by

$$\Phi_F(f)(g) := \sum_{\xi \in F^\times} f(\xi g),\ g \in \mathrm{GL}_1(\mathbf{A}_F).$$

  As a consequence of the Structure Theorems, $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ is a strict LF-space. We refer to $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ as the *space of global θ-functions*.
- If $f(0) = \widehat{f}(0) = 0$, then $\Phi_F(f)$ is an exponentially decaying Schwartz-Bruhat function on C_F, i.e. for any integer $N \geq 1$, $|\Phi_F(f)(g)| \ll e^{-N|\log|g|_F|}$ as $|\log|g|_F| \to +\infty$.
- We denote by $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J_{F,S}^1}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F))$ the isotypes of the spaces of θ-functions. These are nuclear Fréchet spaces. All equipped with a continuous left-translation action of $J_{F,S}$, respectively C_F, and hence a convolution action by the elements of $\mathcal{D}(J_{F,S})$, respectively $\mathcal{D}(C_F)$.

1.2. Into the method: orbital Schauder bases

Let $L(s, \chi)$ be the completed Hecke L-function of $\chi \in \widehat{C_F}$. We denote by $L_f(s, \chi)$ the finite Hecke L-function, i.e. for $s \in \mathbf{C}_{>1}$:

$$L_f(s, \chi) = \prod_{v \in \Sigma_{F,f}} L_v(\chi_v |\cdot|_v^s)$$

where $\Sigma_{F,f}$ is the set of all finite places of F.

The solution amounts to the following:

1.2.1. Fundamental Divisor Exact Sequence. Consider the following situation initially envisioned by Connes, but, contrary to what Connes does (who works, instead, with L^2-completions throughout) working with the Hausdorff locally convex topological $\mathbf{C}$-vector spaces:

$$0 \to |\cdot|_F^{1/2} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F) \to \mathscr{H}_F \to 0$$

and each of their C_F^1-isotypes, i.e.

$$0 \to |\cdot|_F^{1/2} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \to \mathscr{H}_F(\chi|_{C_F^1}) \to 0$$

these latter being nuclear Fréchet spaces.

The Mellin transform turns the second exact sequence into (forgetting about the uniformly Schwartz condition defining Schwartz-Bergman spaces):

$$0 \to L(s+1/2,\chi)((s+1/2)(s-1/2))^\delta \mathcal{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C}) \to \prod_{z:L(z,\chi)=0} \mathbf{C}^{N(z)} \to 0$$

where $N(\rho)$ is the multiplicity of ρ as a zero of $L(s,\chi)$. Here $\delta = 0$ if $\chi|_{C_F^1} \neq 1$, and $\delta = 1$ otherwise. This connection to the L-function requires all the Structure Theorems, and use of known analytic properties of the L-function (convexity bound, existence of a sequence of horizontal segments across the critical strip on which the reciprocal of $L_f(s,\chi)$ is defined and grows at most polynomially in the absolute ordinate).

This is the reason why I called this the "divisor exact sequence": it mirrors the global sections of

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

for the Stein space $X = \mathbf{C}$, and $D = \mathrm{div}(L(s+1/2,\chi)((s+1/2)(s-1/2))^\delta)$.

Considering these spaces systematically is already new. Only Meyer did so to some limited extent.

The following is completely new and was, instead, never done before.

(1) Take a Hardy-Sobolev completion of all spaces in the above linearly and topologically exact sequence of isotypes, to get:

$$0 \to W_I^{s,2}(|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))) \to W_I^{s,2}(C_F)(\chi|_{C_F^1}) \to W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1}) \to 0$$

where each space is now a Hilbert space.

Say:

(a) $I = [-a,a]$ for $a > 0$ fixed and arbitrary.
(b) $\varphi \in \mathcal{D}(C_F)$ is supported strictly over $C_F^{>1}$, i.e. over $C_F^{>1+\varepsilon}$ for some small enough $\varepsilon > 0$.

(2) To fix ideas we choose $s = 0$. The rightmost space gives the spectral side of the trace formula, which reads

$$\sum_{\rho, I} N(\rho) M(\varphi) \left(\chi|\cdot|_F^{\rho-1/2}\right)$$

where the sum is over those zeros ρ of $L(s,\chi)$ such that $\rho - 1/2$ is contained in the interior of $\mathbf{C}_I$. If $s > 0$, then zeros ρ such that $\rho - 1/2 \in \pm a + i\mathbf{R}$ also contribute to the above expression, but with a multiplicity function $N_s(\rho)$ defined as:

$$N_s(\rho) := \max\left\{N(\rho), \left[\frac{s-1}{2}\right]\right\}.$$

This is the trace of the effect of the convolution operator:

$$A(\varphi) : \psi \mapsto \int_{C_F} \varphi(g)\psi(g^{-1}x)\,\mathrm{d}\overline{\mu}(g)$$

on $\mathcal{S}_{\mathrm{E}}(C_F)$ and hence on its isotypes and their $W_I^{s,2}$-completions: this operator respects

$$W_I^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

and hence descends to $W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$. We call $A_{\mathrm{spec},I}(\varphi)$ this operator. It is nuclear, and the sum above is its trace.

This is computed using the Schauder basis on $\mathscr{H}_F(\chi|_{C_F^1})'_\beta$ given by the δ-distributions δ_z and their derivatives, for z varying over the zeros of $L(s,\chi)$ such that $z - 1/2 \in \mathbf{C}_I$: the calculation is completely explicit. By nuclearity of $A(\varphi)$ on $\mathscr{H}_F(\chi|_{C_F^1})$ and since $\mathscr{H}_F(\chi|_{C_F^1})$ is nuclear Fréchet, it is enough to compute the effect of the transpose $A(\varphi)'$ on $\mathscr{H}_F(\chi|_{C_F^1})'_\beta$ (the trace of $A_{\mathrm{spec},I}(\varphi)$ and its transpose are the same) on each basis element, to compute the matrix of $A_{\mathrm{spec},I}(\varphi)$ with respect to the basis: we get Jordan blocks, whose diagonal entries are all of the form $M(\varphi)\left(\chi|\cdot|_F^{\rho-1/2}\right)$, whose rank is exactly the multiplicity of ρ.

Explicitly, we can use the image in $W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$ of the Schauder basis on $\mathscr{H}_F(\chi|_{C_F^1})$ to write down the operator

$$A_{\mathrm{spec},I}(\varphi)$$

as a limit of finite rank operators. Like so:

$$A_{\mathrm{spec},I}(\varphi) = \sum_{z:L(z,\chi)=0, z-1/2\in\mathbf{C}_I} \sum_{0\le m\le N'(z)-1} \langle\cdot, A(\varphi)'\delta^{(m)}_{z-1/2}\rangle_{E,E'} \otimes \psi_{z,m}$$

with $N'(z)$ the multiplicity of z if $z-1/2$ is in the interior of $\mathbf{C}_I$, and $N'(z) = N_s(z)$ as defined above, otherwise. Here $E = W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$ (a Hilbert space!), where the sum is arranged choosing a labelling of the index zeros so that the absolute value of their ordinates is increasing with respect to such choice of ordering. Here$\{\psi_{z,m}\}$ is the image in $W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$ of the dual Schauder basis in $\mathscr{H}_F(\chi|_{C_F^1})$ to $\{\delta_z^{(m)}\}_{(z,m)}$.

The convergence of the above displayed sum is in the sense of trace-class operators, i.e. in the trace-norm on the Hilbert space $W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$.

Since $\delta_z^{(m)}$ is the functional obtained by integration (with respect to $\overline{\mu}$ on C_F) of functions in $W_I^{s,2}(C_F)$ against the functions $|\cdot|_F^{z-1/2}(\log|\cdot|_F)^m$, the finite rank kernels given by finite sums of $|\cdot|_F^{z-1/2}(\log|\cdot|_F)^m \otimes \psi_{z,m}$ are actual smooth functions of two variables on $C_F \times C_F$, and their sum converges in $L^2(C_F\times C_F, \nu\otimes\nu^\vee)$, where

$$\nu = \max\{|\cdot|_F^{-2a}, |\cdot|_F^{2a}\}(1+|\log|\cdot|_F|^2)^{s/2}\,\overline{\mu},\quad \nu^\vee = \frac{1}{\max\{|\cdot|_F^{-2a}, |\cdot|_F^{2a}\}(1+|\log|\cdot|_F|^2)^{s/2}}\,\overline{\mu}. \tag{1.2.1.1}$$

Such limit is, in particular, a measurable function of two variables on $C_F\times C_F$, which we call the "spectral" kernel and we denote by:

$$k_{\mathrm{spec},I,\varphi}(x,y).$$

In other words, we have

$$\int_{y\in C_F} k_{\mathrm{spec},I,\varphi}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y) = A_{\mathrm{spec},I}(\varphi)(\phi)$$

for all $\phi \in W_I^{s,2}\mathscr{H}_F(\chi|_{C_F^1})$.

We may lift the elements $\psi_{z,m}$ and take orthogonal projections onto the orthogonal complement of

$$W_I^{s,2}(|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))) \subset W_I^{s,2}(C_F)(\chi|_{C_F^1})$$

and rename the resulting kernel (still a trace-norm limit of the resulting finite rank kernels) as $k_{\mathrm{spec},I,\varphi}(x,y)$.

(3) The convolution operator $A(\varphi)$ on middle space $W_I^{s,2}(C_F)(\chi|_{C_F^1})$ has $\varphi(xy^{-1})$ for a kernel, and since we are assuming φ is strictly supported over $C_F^{\geq 1}$, this kernel vanishes along the diagonal.

(4) We call the difference of the two kernels:

$$\varphi(xy^{-1}) - k_{\mathrm{spec},I,\varphi}(x,y)$$

the *geometric kernel*, denoted

$$k_{\mathrm{geom},I,\varphi}(x,y).$$

It is smoothable along the diagonal, using the Hardy-Littlewood Maximal Theorem and Lebesgue's Differentiation Theorem, because $\varphi(xy^{-1})$ is and so is the spectral kernel, by nuclearity of $A_{\mathrm{spec},I}(\varphi)$.

We have the *main Trace Formula*:

$$\int_{C_F} k_{\mathrm{geom},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\int_{C_F} k_{\mathrm{spec},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\sum_{\rho,I} N(\rho)M(\varphi)\left(\chi|\cdot|_F^{\rho-1/2}\right).$$

The entire work is about computing $k_{\mathrm{geom},I,\varphi}$ and its integral along the diagonal *in another way*, i.e. other than that given to us by its very definition, in terms of the spectral kernel (again: this means we define these kernels along the diagonal using Lebesgue's Differentiation Theorem. We compute the integral of the resulting newly defined function of $(x,x) \in C_F \times C_F$).

1.2.2. Schauder bases. These are the main novelty of this work. We use them to split $k_{\mathrm{geom},I,\varphi}(x,y)$ into an infinite sum of pieces indexed by the places of F:

$$k_{\mathrm{geom},I,\varphi}(x,y) = k_{S,\varphi}(x,y) + \sum_{p\nmid S} k_{p,\varphi}(x,y)$$

and let the pieces $k_{p,\varphi}(x,y)$ act on certain subspaces of $W_I^{s,2}(|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)))$ to understand them.

The finding of Schauder bases on the corresponding subspaces of

$$|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$

enables us to write these "orbital kernels" $k_p(x,y)$ using the bases, as limits of finite ranks operators, in much the same way as for the spectral kernel. This will give us the value of each

$$\int_{C_F} k_{p,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x)$$

and they will be zero for all p (each $k_{p,\varphi}(x,y)$ is associated to contributions coming from all finite places above p, for each prime number p).

The choice of S depends both on the character and on φ through its support. See the sections below. This way we get $\int_{C_F} k_{p,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = 0$ for all prime numbers p such that all finite places of F whose residue characteristic is p, are outside of S.

1.2.3. Reduction to an S-local trace calculation. In light of the previous point, the main Trace Formula now reads as follows:

$$\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\int_{C_F} k_{\mathrm{spec},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\sum_{\rho,I} N(\rho)M(\varphi)\left(\chi|\cdot|_F^{\rho-1/2}\right).$$

In turn, $\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x)$ is then computed using a version of the Connes-Meyer "S-local trace formula", discussed in their papers.

Reducing to the S-local calculation of $\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x)$ is what this work is all about. Once the reduction is made, the problem is solved.

The ingredients in the S-local calculation (as present already in the papers of Connes and Meyer) are:

(1) The S-adelic Fourier transform: we need to transform $k_{S,\varphi}(x,y)$ in the y-variable. The construction will give that $k_{S,\varphi}$ factors through a kernel on $\mathrm{GL}_1(\mathbf{A}_{F,S})\times C_F$ with the same diagonal mass (skewered by the $\mathcal{O}_{F,S}^\times$-action. See sections below).

So the S-local additive Fourier transform becomes available, so long as we start off with

$$|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$

instead of

$$|\cdot|_F^{a}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$

for general $a\in\mathbf{R}_{>0}$.

This way, we will have that the product $|x|_{F,S}^a|y|_{F,S}^a$ arising by computing $k_{S,\varphi}(x,y)$, turns, along the diagonal, into $|x|_{F,S}^{2a}$. If $a=1/2$ exactly, then this is $|x|_{F,S}$, which will multiply the multiplicative Tamagawa measure μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ to give the additive measure ν_S, up to a positive multiplicative constant.

This forces the $1/2$ in the very statement of GRH. Connes' expectation on the S-adelic Fourier transform playing this role and being related to the necessity of the $1/2$ in the very statement of GRH, is therefore correct.

(2) A version of known calculations of Connes from his paper from 1998[1].

1.2.4. Poles of the S-idelic Zeta functions.

1.2.5. Recollections from Sarnak's equidistribution of long horocycles.

1.2.6. Periodic orbits and fixed points.

1.2.7. Orbital bases and orbital traces. From this moment onwards, all techniques, ideas and results, are original and never even partially present in prior work (as was the case in §2, apart from §2.3).

We explain where the Schauder bases are found, and how to think of them. We will make a brief analogy with Sarnak's paper on long horocycles to highlight periodic orbits and fixed points and their role. This leads to the identification of those elements of Deninger's Program that are correct, and why there is no 3-manifold, nor foliation, nor foliated cohomology theory.

1.2.8. A toy example. We start with a toy example.

Let $\gamma\in\mathbf{R}_{>0}$ such that $\gamma>1$.

[1]A. Connes. *Trace Formula in Noncommutative Geometry and the Zeros of the Riemann Zeta Function.* Selecta Mathematica. Vol.5, No.29, (1999).

Then $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ carries an absolute Schauder basis, which is a copy of the exponential system. The basis is the set of exponential functions

$$\exp\left(\frac{2\pi i m}{\log(\gamma)}\right).$$

The dual basis is the set of functionals given by integration against the conjugate exponential function, i.e. calling $z_m := \frac{2\pi i m}{\log(\gamma)} \in i\mathbf{R}$:

$$\delta_{z_m} : \phi \mapsto \frac{1}{\log(\gamma)} \int_{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}} \phi(t) e^{-\frac{2\pi i m \log(t)}{\log(\gamma)}} \, \mathrm{d}\overline{\mu}(t)$$

where $\overline{\mu}$ is the Haar measure induced on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ by the Haar measure $\mathrm{d}t/t$ on $\mathbf{R}_{>0}$ and the counting measure on $\gamma^{\mathbf{Z}}$.

We now recall that the averaging operator defined on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by:

$$\Phi_\gamma(\phi)(x) := \sum_{k \in \mathbf{Z}} \phi(\gamma^k x)$$

identifies $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ with the quotient $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/\ker(\Phi_\gamma)$, i.e.

$$0 \to \ker(\Phi_\gamma) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \to 0$$

is linearly and topologically exact. The Mellin transform $M : \phi \mapsto \int_{\mathbf{R}_{>0}} \phi(t) t^s \, \frac{\mathrm{d}t}{t}$ now identifies the above sequence with

$$0 \to (1-\gamma^{-s}) A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s}) A\mathcal{S}(\mathbf{C}) \to 0$$

therefore characterizing $\ker(\Phi_\gamma)$ in terms of the image of the Mellin transform in the Schwartz-Bergman space $A\mathcal{S}(\mathbf{C})$. The left term in the exact sequence above is reminiscent of a reciprocal Euler factor at a non-Archimedean place.

Now note that $\mathbf{R}_{>0}$ acts on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, on $\ker(\Phi_\gamma)$ and on the subquotient $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ by left-scaling by the inverse.

This is then turned into a convolution action of $\mathcal{D}(\mathbf{R}_{>0})$. In turns out that this convolution action sending φ to $A(\varphi)(\phi) := \int_{\mathbf{R}_{>0}} \varphi(t)\phi(t^{-1}x) \, \frac{\mathrm{d}t}{t}$ is nuclear on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ and the trace of $A(\varphi)$ on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is computed using the Schauder basis $\{\delta_{z_m}\}$.

Just work out what the (adjoint) scaling action on each δ_{z_m} is, and you get that it is by multiplication by $t^{\frac{2\pi i m}{\log(\gamma)}}$ and so the trace of $A(\varphi)$, which equals the trace of the transpose $A(\varphi)'$, is the sum of the eigenvalues:

$$\sum_{m \in \mathbf{Z}} M(\varphi)(z_m).$$

By the Poisson Summation Formula, this equals

$$\sum_{k \in \mathbf{Z}} \varphi(\gamma^k).$$

In particular, if the support of φ is strictly contained within $(1, \gamma)$ then $\mathrm{Trace}\, A(\varphi) = 0$ on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$!

This vanishing illustrates the origin of the vanishing of

$$\int_{C_F} k_{p,\varphi}(x,x) \, \mathrm{d}\overline{\mu}(x) = 0$$

for all but finitely many prime numbers p.

1.2.9. Orbital Schauder bases: non-Archimedean places. The linear subspace

$$\mathrm{PV}_{S_p} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

is a closed $\mathcal{D}(C_F)$-submodule. Recall that S_p is the finite set of finite places in F of residue characteristic p, and PV_{S_p} is the composition of the operators PV_v for each $v \in S_p$, each induced by sending $f \in \mathcal{S}(\mathbf{A}_F)$ to

$$f(x) - f(\pi_v^{-1}x)$$

for π_v a choice of uniformizer at $v \in S_p$. An analog of PV_v, PV_{S_p} is also available on

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})).$$

In the S-local case we assume $S_p \subset S$.

We call the resulting subquotients:

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) := \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

for the S-local one, and $\mathscr{E}_{S_p}(\chi|_{C^1_F})$ for the global one.

It turns out that the operator

$$p_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}_0(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$$

where the subscript $\mathcal{S}_0$ refers to the condition $f(0) = \widehat{f}(0) = 0$, required both globally and S-locally, identifies $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$ and $\mathscr{E}_{S_p}(\chi|_{C^1_F})$ linearly and topologically, even as continuous representations of C_F (which is identified with $C_{F,S} = C_F$ for S such that $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, which will be assumed to be the case. The action of $C_{F,S}$ on the former then factors through $J_{F,S}$).

Here comes the main observation: the Mellin transform turns PV_{S_p} into the multiplication operator by the finite product of (entire) reciprocal Euler factors:

$$\prod_{v \in S_p} L_v(\chi_v|\cdot|_v^s)^{-1}.$$

But then we are in a situation similar to the toy example!

$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$ carries an absolute Schauder basis. This is nothing fancy: it is a copy of the twisted exponential system that one finds on the quotients $\mathcal{S}(J_{F,S}/\pi_v^{\mathbf{Z}})$. Everything works as in the toy example. It's just long and notationally heavier. And to show that we indeed have a *copy* of the exponential Schauder basis on $\mathcal{S}(J_{F,S}/\pi_v^{\mathbf{Z}})$ (think $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$) one needs to use the general theory, explicit constructions, the Open Mapping Theorem for Fréchet as well as for LF-spaces.

Let us work globally to fix ideas. We omit mention of the character in the notation $\mathscr{E}_{S_p}(\chi|_{C^1_F})$.

For each $\varphi \in \mathcal{D}(C_F)$, upon denoting by $A_{\mathrm{geom},p,\chi}(\varphi)$ the operator induced on $\mathscr{E}_{S_p}$ by $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$, then $A_{\mathrm{geom},p,\chi}(\varphi)$ is nuclear. Its trace turns out to be:

$$\mathrm{Trace}(A_{\mathrm{geom},p,\chi}(\varphi)) = \sum_{\varepsilon_p} N(\chi|\cdot|_F^{\varepsilon_p}) \int_{C_F} \varphi(g)\chi(g)|g|_F^{\varepsilon_p}\, \mathrm{d}\overline{\mu}(g)$$

because we can use the Schauder basis on $\mathscr{E}_{S_p}$ to compute it!

Just as it is done in the toy example. The calculations are identical, and the Schauder basis is a close analog of the copy of the exponential system in the toy example.

The sum is over the complex numbers ε_p such that $\chi|\cdot|_F^{\varepsilon_p}$ is a zero of the entire function $\prod_{v\in S_p} L_v^{-1}$ (we call $N(\chi|\cdot|_F^{\varepsilon_p})$ its multiplicity) with $\chi|\cdot|_F^{\varepsilon_p}$ contained in the connected component $\Omega(C_F)(\chi|_{C^1_F})$.

Once again (as in the toy example) the Poisson Summation Formula also gives us that such trace equals:

$$\mathrm{Trace}(A_{\mathrm{geom},p,\chi}(\varphi)) = \sum_{v\in S_p} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \sum_{k\in\mathbf{Z}} \varphi(\pi_v^k)$$

and so if φ is supported strictly over $C_F^{(1,|\pi_v^{-1}|_v)}$ then this gives zero!

If φ is fixed and compactly supported over $(1,+\infty)$, then outside of finitely many places v, indeed φ will be supported on $C_F^{(1,|\pi_v^{-1}|_v)}$ because $|\pi_v^{-1}|_v = q_v$ grows to $+\infty$ as the residue characteristic of p grows, and so for all but finitely many v, the compact support of φ will be contained in the relatively compact regions $C_F^{(1,|\pi_v^{-1}|_v)}$.

In other words, for all but finitely many finite places v of F, the traces computed above will vanish if φ is supported over $(1,+\infty)$.

1.2.10. Schauder bases: Archimedean places. These will not be used, but worth mentioning with a view towards Deninger's Program.

There is an Archimedean analog of the Orbital Trace Formula from §3.2, and to the Archimedean places also correspond "orbital subquotients", both S-local and global. We fix ideas on the S-local ones for simplicity.

These are denoted

$$\mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}})$$

and they are the quotient of the $\chi|_{J^1_{F,S}}$-isotype of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ by the closed subspace

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}_\infty(\mathbf{A}_{F,S}))$$

where

$$\mathcal{S}_\infty(\mathbf{A}_{F,S}) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \otimes \bigotimes_{v\in S_f} \mathcal{S}(F_v)$$

and $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ is defined by requiring infinite order of vanishing along each of the coordinate "hyperplanes" in $\mathbf{A}_{F,S_\infty}$ (i.e. of equations $x_v = 0$ for v real, $z_v = 0$ for v complex. Infinite order of vanishing in the complex case means vanishing of all z_v- and $\overline{z}_v$-derivatives of any order at points with $z_v = 0$).

The linear subspace

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}_\infty(\mathbf{A}_{F,S})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

is a closed $\mathcal{D}(J_{F,S})$-submodule.

Let $\varphi \in \mathcal{D}(J_{F,S})$ be a test function. Assume φ is supported over $J^{>1}_{F,S}$.

Then there is an analog of the "orbital trace formula" at the non-Archimedean places.

Upon denoting by $A_{\mathrm{geom},\infty,\chi}(\varphi)$ the operator induced on $\mathscr{E}_{S,S_\infty}$ by $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, then $A_{\mathrm{geom},\infty,\chi}(\varphi)$ is a continuous operator whose matrix trace with respect to the Schauder basis has value:

$$\mathrm{Trace}(A_{\mathrm{geom},\infty,\chi}(\varphi)) = \sum_z N(\chi|\cdot|_F^z) \int_{C_F} \varphi(g)\chi(g)|g|_F^z \,\mathrm{d}\overline{\mu}(g)$$

where z varies over the zeros of the (entire) product of all the reciprocal Archimedean Euler factors.

These "local" trace formulas fulfill expectations of Deninger's from his paper on Lefschetz Trace Formulas and the Explicit Formulas: see the "local traces" contained therein[2].

1.2.11. The big picture: Deninger's Program and the role of the Mellin transform. Now take S saturated (i.e. it contains the whole S_p if it contains a finite place of residue characteristic p).

We may organize the nuclear Fréchet space

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

and its suquotients $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$ and $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ according to the following symbolism:

$$\begin{array}{ccccc}
\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S})) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \longrightarrow & \mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}}) \times \prod'_{p:S_p\subset S} \mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}}) \\
\downarrow & & \downarrow & & \downarrow \\
[D_{S,1}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] & \xrightarrow[j]{} & [D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] & \xleftarrow[i]{} & \coprod'_{v\in S}[D_{S,v}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] \\
\simeq\downarrow & & & & \downarrow\simeq \\
\underline{J_{F,S}/J^1_{F,S}} & & & & \left(\coprod_{v\in S_\infty} BU_v\right) \coprod \left(\coprod_{p:S_p\subset S} \coprod'_{v\in S_p} [\mathbf{R}_{>0}/F_v^\times]\right)
\end{array}$$

The top row in the diagram above is a short exact sequence (on the left and in the middle), both linearly and topologically.

The row below it indicates which loci of the quotient stack $[D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$ each nuclear Fréchet space is supported on. The space $D_{S,v}$ equals $H_v^\times$ defined below, $D_{S,1}$ equals $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and $D_{F,S}$ is the union of $D_{S,1}$ and $H_v^\times$ for all $v \in S$, the union being formed within $\mathbf{A}_{F,S}$ and endowed with the subspace topology.

The Orbital quotients $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$, respectivey $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ (where we omit mentioning the continuous character χ_S, identifying the χ_S-isotype of each such subquotient) are thought of as being supported over the fat points BU_v for each Archimedean place $v \in S_\infty$, respectively over the periodic orbits $[\mathbf{R}_{>0}/F_v^\times]$ (whose coarse spaces are, respectively, a loop $\mathbf{R}_{>0}/N(\mathfrak{p}_v)^{\mathbf{Z}}$ for each finite place $v \in S$).

We discussed what this means: the dual basis elements in $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ turn out to be supported over the union of "hyperplanes":

$$\bigcup_{v\in S_p} H_v^\times$$

where

$$H_v^\times = \{x \in \mathbf{A}_{F,S} : x_w \neq 0 \text{ for all } w \in S - \{v\},\ x_v = 0\}$$

but since these dual basis elements are derived from eigendistributions (as derivatives of an eigendistribution), their support "matters" only modulo the action of $\mathcal{O}_{F,S}^\times$.

This requires real work to show: the non-vanishing of $L(s,\chi)$ on the 1-line is used as an input, for example. It amounts to show that the basis elements in $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ can be arranged to be supported over such unions. Then we observe that the dual basis elements are all obtained as derivatives of eigendistributions, and so their support only "matters" modulo the $\mathcal{O}_{F,S}^\times$.

The eigen-property of the eigendistributions is with respect to the character $\overline{\chi}_S$. The support therefore really only "matters" modulo the action of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, not just $\mathcal{O}_{F,S}^\times$, and so we get the quotient stacks in the diagram.

This fulfills expectations of Deninger on periodic orbits and fixed points of a "flow". The flow here is the continuous action of $J_{F,S}/J^1_{F,S} \simeq \mathbf{R}_{>0}$ (restricted to the semigroup $\mathbf{R}_{>1}$ since all test functions we will be

[2]C. Deninger. *Lefschetz trace formulas and explicit formulas in analytic number theory.* Journal für die reine und angewandte Mathematik. 441 (1993).

using are only supported on $C_F^{\geq 1}$, hence descend to test functions supported on $J_{F,S}^{\geq 1}$ for all saturated S large enough).

With respect to the semigroup action $\mathbf{R}_{>1}$, the quotient stacks BU_v for v Archimedean are indeed "points of attraction" of the flow, as predicted by Deninger. The quotient stacks $[D_{S,v}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$ for v non-Archimedean are "periodic", as predicted by Deninger.

The fact that they serve as the "equivariant supports" (to be made precise in the paper in Chapter 8) of the dual basis elements on the orbital subquotients is in analogy with Sarnak's Theorem 2 in his paper on the equidistribution of long horocycles[3].

1.3. Orbital decomposition of the lifted convolution kernel, and the orbital pieces of the kernel

In this section we explain the main ingredients that enter the method to reduce the calculation of the diagonal mass of the global geometric kernel, to the calculation of the diagonal mass of an "S-local piece" of it. By "S-local" we will always be referring to the property of the kernel being supported over

$$\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}$$

and factoring through $\mathrm{GL}_1(\mathbf{A}_{F,S})$.

We are, for the moment, giving for granted that the kernel being referred to is a scalar-valued function, measurable, enjoying certain additional integrability properties inherited from the spectral kernel, of two variables on

$$\mathrm{GL}_1(\mathbf{A}_F) \times C_F$$

even though the global geometric kernel is given to us as a scalar-valued function on $C_F \times C_F$.

This is one of the aspects we will be discussing: how to lift such kernel with respect to the first of the two variables, from a scalar-valued function on $C_F \times C_F$ to a scalar-valued function on $\mathrm{GL}_1(\mathbf{A}_F) \times C_F$.

1.3.1. Indentations and skewered diagonal mass. In this subsection we illustrate how to "lift" a measurable kernel function

$$k(x, y) : X \times X \to \mathbf{C}$$

when X is a topological quotient space Y/H equipped with a regular Radon measure μ, and H is a group of measure-preserving transformations acting freely and properly by homeomorphisms.

We will be immediately specializing to the case $X = G$, for G a second countable Hausdorff locally compact abelian group, H a closed subgroup. For the purposes of this work, G will be $\mathrm{GL}_1(\mathbf{A}_F)$ or $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and H will be either $F^{\times}, \mathcal{O}^{\times}_{F,S}$, or $\mathrm{GL}_1(\mathbf{A}_F)^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

One key structure that the situation will come equipped with is that of fundamental domains. We start off with a brief review of the general recipe to lift functions from the quotient G/H to G preserving certain support conditions.

Let G be a second countable Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H. We call $\mu_{G/H}$ the quotient Haar measure on G/H. We denote by $\pi : G \to G/H$ the continuous quotient homomorphism. There exists a function $u_0 : G \to \mathbf{C}$ satisfying the following properties:

(1) u_0 is real-valued, nonnegative, continuous.
(2) Let K be an arbitrary compact subset of G. Then the intersection of the support of u_0 with $\pi^{-1}(\pi(K))$ is compact.

[3]P. Sarnak. *Asymptotic Behavior of Periodic Orbits of the Horocycle Flow and Eisenstein Series.* Communications on Pure and Applied Mathematics, Vol.XXXIV, pp.719-739 (1981).

(3) For all $x \in G$, the integral:

$$\int_H u_0(h^{-1}x)\,\mathrm{d}\mu_H(h) \tag{1.3.1.1}$$

converges, and upon denoting u_0^H its value function on G, we have that u_0^H is continuous on G, invariant under the continuous left-translation action by H on G, and identically 1 as a function on G/H.

For every continuous function $u \in C(G/H)$, we have that for all $x \in G$, the integral

$$\int_H (u_0 \cdot (u \circ \pi))(h^{-1}x)\,\mathrm{d}\mu_H(h) \tag{1.3.1.2}$$

converges, and its value is $u(\pi(x))$.

Functions such as u_0 are called throughout this work *indentations*. According to the conditions above, u_0 is a continuous indentation. We do not need to assume continuity, though, if all we wish to obtain is a function $u_0 : G \to \mathbf{C}$ satisfying (2), and such that the integral in (1.3.1.1) converges for all $x \in G$ to the constant function whose value is 1. We can find such u_0 such that it is measurable and such that the points of G at which u_0 is discontinuous has μ_G-measure zero. We will stick to continuous indentations, although in practice we will be using this more general type of indentation, to allow indicator functions of fundamental domains (when these exist) to qualify.

We need the following "two-variable" complements around H-indentations on G. It is useful to discuss it in general, to ease the notation.

Let G be a second countable Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H.

Assume H is discrete, and μ_H is the counting measure.

Let $\mu_{G/H}$ be the unique quotient measure on G/H.

We denote by $\pi : G \to G/H$ the continuous quotient homomorphism. Assume there exists a fundamental domain D for the continuous left-translation action by H on G.

Now recall that a general version of Lebesgue's Differentiation Theorem exists for Hausdorff locally compact abelian groups G. It rests upon an analog of the Hardy-Littlewood Maximal Theorem and the statement are directly analogous to those of the classical Theorems in the case of the real line and Euclidean spaces, so long as an analog of the closed balls that we find in Euclidean spaces can be found in G.

We call a sequence of Borel subsets $U_k \subset G$ that axiomatize properties satisfied by the closed balls in Euclidean spaces, a D'-sequence. Terminology borrowed from Hewitt-Ross[4]. The axioms defining D'-sequences are recalled in the main text. In our cases of interest, G will always carry such sequences by direct construction (using continuous sections to the idelic and S-idelic norms).

Say D is a fundamental domain for the H-action on G.

Assume that $\{U_k\}_{k\geq 0}$ is a D'-sequence for G such that $U_k \subset D$, and U_k is a relatively compact open symmetric unit-neighborhood for all integers $k \geq 0$. Then, upon defining $\overline{U}_k := \pi(U_k)$ for each integer $k \geq 0$, $\{\overline{U}_k\}_{k\geq 0}$ is a D'-sequence for G/H.

We define smoothenings over $G \times G$ and $G/H \times G/H$ using the D'-sequences $\{U_k \times U_k\}_{k\geq 0}$ for $G \times G$ and $\{\overline{U}_k \times \overline{U}_k\}_{k\geq 0}$ for $G/H \times G/H$. Namely, for $u : G \to \mathbf{C}$ a locally μ_G-integrable function on G, we define

$$\widetilde{u}(x) := \lim_{k\to+\infty} \frac{1}{\mu_G(U_k)} \int_{xU_k} u(y)\,\mathrm{d}\mu_G(y)$$

[4]E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis, Vol. II.* Springer-Verlag, Berlin - Gottingen - Heidelberg, 1970. See Ch.XI, §44.

whenever the limit exists. The Theorems imply that it exists for μ_G-almost all $x \in G$. The new function $\widetilde{u}$ is measurable and better behaved than u, hence the term "smoothening". We recall the standard terminology for those points of G at which the limit exists: we will be calling them "regular".

Let now u_0 be a continuous H-indentation on G. Then $u_0 \otimes u_0 \in C(G) \otimes C(G) \subset C(G \times G)$, a continuous function on $G \times G$, satisfies the following properties:

(1) The integral:

$$\int_H u_0(h^{-1}x)u_0(h^{-1}y)\,\mathrm{d}\mu_H(h) \tag{1.3.1.3}$$

converges pointwise everywhere on $G \times G$. The limit function $u_0^{\Delta}(x,y)$ is defined everywhere, continuous on $G \times G$, real-valued and nonnegative, and invariant under the continuous diagonal action of H on $G \times G$.

(2) For all $\mu_{G/H}$-measurable, locally $\mu_{G/H}$-integrable functions $u : G/H \times G/H \to \mathbf{C}$, the assignment:

$$\Psi_{u_0}^{\Delta} : u \mapsto u_0^{\Delta} \cdot (u \circ (\pi \times \pi)) \tag{1.3.1.4}$$

is valued in measurable complex-valued functions over $G \times G$. If $\mu_{G/H}$-almost every point $(x,x) \in G/H \times G/H$ is regular for u, then $\widetilde{u}(x,x)$ is $\mu_{G/H}$-measurable, defined $\mu_{G/H}$-almost everywhere. If it is absolutely $\mu_{G/H}$-integrable as a function on G/H, then the functions $\Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x,x)$ and $\Psi_{u_0}^{\Delta}(\widetilde{u})(x,h^{-1}x)$, defined for all $h \in H$, μ_G-almost all $x \in G$, and $(\mu_H \otimes \mu_G)$-measurable as functions of $(x,h) \in G \times H$, satisfy the following condition: the following two-fold integrals

$$\int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x), \quad \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{1.3.1.5}$$

are both absolutely convergent, and the following identities hold:

$$\int_{G/H} \widetilde{u}(x,x)\,\mathrm{d}\mu_{G/H}(x) = \int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x) \tag{1.3.1.6}$$

$$= \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{1.3.1.7}$$

$$= \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(x,h^{-1}x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{1.3.1.8}$$

$$= \int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(x,h^{-1}x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x). \tag{1.3.1.9}$$

(3) Let $\rho \in C_c(G/H \times G/H)$. Then:

$$\Psi_{u_0}^{\Delta}(\rho \cdot u)(x,y) = (\rho \circ (\pi \times \pi)) \cdot \Psi_{u_0}^{\Delta}(u)(x,y) \tag{1.3.1.10}$$

for $(\mu_G \otimes \mu_G)$-almost every $(x,y) \in G \times G$.

(4) We have, for $(\mu_G \otimes \mu_G)$-almost all $(x,y) \in G \times G$:

$$\Psi_{u_0}^{\Delta}(\widetilde{u})(x,y) = \lim_{k \to +\infty} \frac{1}{\mu_G(U_k)^2} \int_{xU_k \times yU_k} \Psi_{u_0}^{\Delta}(u)(y_1,y_2)\,\mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2). \tag{1.3.1.11}$$

The point $(x,y) \in G \times G$ is regular for $\Psi_{u_0}^{\Delta}(u)$ if and only if $(\pi(x),\pi(y)) \in G/H \times G/H$ is regular for u.

Point (2) teaches us how to compute $\int_{G/H} \widetilde{u}(x,x)\,\mathrm{d}\mu_{G/H}(x)$ in terms of integration over D, integration over H, and the values of the smoothening of

$$\Psi_{u_0}^{\Delta}(u)(x,y)$$

at the points of the skewered diagonals $(h^{-1}x,x)$, $(x,h^{-1}x)$, for $h \in H$, $x \in G$.

Part (3) teaches us how to multiply lifted functions after smoothening, by an invariant nice function on G (obtained by lifting a nice function on G/H along the quotient π). This will be used to apply truncations to lifted kernels, by multiplication by indicator functions of relatively compact subsets in the x-variable (mollified a little by convolving with a mollifier, to make them smooth).

Part (4) teaches us how to smoothen the function $\Psi_{u_0}^{\Delta}$ at points in $G \times G$: we smoothen u first on $G/H \times G/H$ at the image of $(x, y) \in G \times G$, and then apply $\Psi_{u_0}^{\Delta}$. The result is exactly the smoothening on $G \times G$ at (x, y).

We are now back to the situation of interest. We fix a continuous character $\chi \in \widehat{C_F}$. We are studying the geometric kernel function

$$k_{\mathrm{geom},I,\varphi} : C_F \times C_F \to \mathbf{C}$$

which is in turn defined as the difference

$$\varphi(xy^{-1}) - k_{\mathrm{spec},I,\varphi}(x, y)$$

where $\varphi \in \mathcal{D}(C_F)$ is smooth and compactly supported and $\varphi(xy^{-1})$ is the kernel of the convolution-by-φ operator on $W_I^{s,2}(C_F)(\chi|_{C_F^1})$, and $k_{\mathrm{spec},I,\varphi}(x, y)$ is the spectral kernel. As a function of (x, y), it is almost everywhere defined, measurable, square-integrable on $C_F \times C_F$ with respect to the product measure $\nu \otimes \nu^{\vee}$ defined using the weighted measures ν and $\nu^{\vee}$ (see (1.8.3.5)) from the definition of Hardy-Sobolev spaces, and $\overline{\mu}$-integrable along the diagonal in $C_F \times C_F$: the smoothening exists and it is integrable, that is. This follows from the very construction of the spectral kernel as the limit of finite rank kernels (each of which is explicit and even smooth on $C_F \times C_F$).

The key points of this subsection are the followin:

- *We lift $k_{\mathrm{geom},I,\varphi}$ to a kernel on* $\mathrm{GL}_1(\mathbf{A}_F) \times \mathrm{GL}_1(\mathbf{A}_F)$. This is done by choosing a smooth compactly supported function $\widetilde{\varphi}$ on $\mathrm{GL}_1(\mathbf{A}_F) \times \mathrm{GL}_1(\mathbf{A}_F)$ such that:

  $$x \mapsto \sum_{\xi \in F^{\times}} \widetilde{\varphi}(\xi x)$$

  is φ as a smooth function on C_F, by choosing an indentation u_0 (we will choose this using the fundamental domain $\tau(\mathbf{R}_{>0}) \cdot E_F^1$ for the $F^{\times}$-action on $\mathrm{GL}_1(\mathbf{A}_F)$, and by considering:

  $$K_{\mathrm{geom},I,\varphi}(x, y) := \widetilde{\varphi}(xy^{-1}) - \Psi_{u_0}^{\Delta}(k_{\mathrm{spec},I,\varphi})(x, y).$$

- *We compute its diagonal mass using the lifted kernel on skewered diagonals in* $\mathrm{GL}_1(\mathbf{A}_F) \times \mathrm{GL}_1(\mathbf{A}_F)$. The new kernel function $K_{\mathrm{geom},I,\varphi}(x, y)$ on $\mathrm{GL}_1(\mathbf{A}_F) \times \mathrm{GL}_1(\mathbf{A}_F)$ is invariant under the diagonal action of $F^{\times}$, and so the formulas in (12.2.2.4) all hold. We use the last one:

  $$\int_{C_F} \widetilde{k}_{\mathrm{geom},I,\varphi}(x, x)\, \mathrm{d}\overline{\mu}(x) = \int_D \sum_{\xi \in F^{\times}} \widetilde{K}_{\mathrm{geom},I,\varphi}(x, \xi x)\, \mathrm{d}\mu(x) \tag{1.3.1.12}$$

  where the superscripts $\widetilde{(\cdot)}$ refer to the smoothenings, computed using the chosen D'-sequence within $\mathrm{GL}_1(\mathbf{A}_F)$, and its image in C_F. These masses do not depend on any such choice, nor do they depend on $\widetilde{\varphi}$ and the indentation u_0.

The advantage of now working with the "lifted kernel" function $K_{\mathrm{geom},I,\varphi}$ instead of $k_{\mathrm{geom},I,\varphi}$ is that we can access the restricted product structure of the global ideles (which is lost in the quotient C_F by the diagonal action of the discrete subgroup $F^{\times} \subset \mathrm{GL}_1(\mathbf{A}_F)$) to "chop" the measurable function $K_{\mathrm{geom},I,\varphi}$ into component functions, by multiplying by indicator functions in the y-variable of explicit domains within $\mathrm{GL}_1(\mathbf{A}_F)$.

The formula in (1.3.1.12) teaches us how to compute the diagonal mass of the original kernel of interest $k_{\mathrm{geom},I,\varphi}$ using the lifted kernel. The idea is to cut $K_{\mathrm{geom},I,\varphi}$ into pieces and compute the right integral in (1.3.1.12) as a sum of analogs of the same type of integral for each of the pieces, by making the pieces explicit by letting them act on certain large subspaces of the global Schwartz space $\mathcal{S}(\mathbf{A}_F)$.

1.3.2. Divisorial stratification of the ideles. We start by identifying an interesting stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$, which is found only on $\mathrm{GL}_1(\mathbf{A}_F)$ as opposed to its quotient C_F, since it is defined by leveraging crucially the restricted product nature of $\mathrm{GL}_1(\mathbf{A}_F)$.

This stratification will adapt itself nicely to the structure of the Schwartz-Bruhat functions representing the global adelic θ-functions in

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)).$$

Recall that there exists a finite set of finite places S of F satisfying the following conditions:

(1) S contains all the ramified places of F and all the Archimedean places.
(2) S contains all the places of F at which χ is ramified.
(3) $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

such that any such θ-function ϕ can be realized as being of the form, as a function of $x \in C_F$:

$$\phi(x) = \Phi_{F,\chi|_{C_F^1}}(f)(x)$$

where

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}} \tag{1.3.2.1}$$

for some $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$.

This is the last of the five Structure Theorems, proved at the end of Chapter 5 of this work.

The stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ that we are about to define will weld itself perfectly well with the non-Archimedean "transversal" (i.e. tensor component) $\bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$ which all the global Schwartz-Bruhat functions on $\mathbf{A}_F$ representing the $\chi|_{C_F^1}$-homogeneous global θ-functions can be made to share in common.

The guiding principle is simple: since $\mathcal{O}_{F_v}^\times \subset F_v^\times$ is compact and open, hence so it is in F_v, and we have

$$1_{\mathcal{O}_{F_v}^\times} \cdot 1_{\mathcal{O}_{F_v}} = 1_{\mathcal{O}_{F_v}^\times}$$

as Schwartz-Bruhat functions on F_v. Under the local Mellin transform, $Z_v(1_{\mathcal{O}_{F_v}^\times}, c)$ and $Z_v(1_{\mathcal{O}_{F_v}}, c)$ are related by "Weil's Identity":

$$\mu_v(\mathcal{O}_{F_v}^\times) \cdot Z_v(1_{\mathcal{O}_{F_v}^\times}, \chi_v|\cdot|_v^s) = L_v(\chi_v|\cdot|_v^s)^{-1} Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$$

as meromorphic functions on $\Omega(F_v^\times)(\chi_v)$.

Multiplication by $1_{\mathcal{O}_{F_v}^\times}$ of $1_{\mathcal{O}_{F_v}}$ has the effect of replacing the local Z-function $Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$ with the product by the reciprocal local Euler factor $L_v(\chi_v|\cdot|_v^s)$.

Note also that the infinite product of the volumes $\mu_v(\mathcal{O}_{F_v}^\times)$ over all the finite places v of F is a finite product (these volumes are 1 for all but finitely many v's) and therefore converges to the square root of the absolute different.

We define

$$\rho_v := \bigotimes_{w \in S} 1_{F_w} \otimes \bigotimes_{w \notin S, w \neq v} 1_{F_v} \otimes 1_{\mathcal{O}_{F_v}^\times}.$$

This is a smooth function on $\mathrm{GL}_1(\mathbf{A}_F)$ (since $\mathcal{O}_{F_v}^\times \subset F_v$ is a compact open subset).

We define, for any finite subset T of finite places of F such that $S \cap T = \varnothing$:

$$\rho_T := \prod_{v \in T} \rho_v. \tag{1.3.2.2}$$

For any increasing chain of finite subsets T_k of finite places of F, as $k \geq 0$ varies over the nonnegative integers, and such that $T_{k+1} - T_k$ contains only one place, then the functions $\{\rho_{T_k}\}_{k \geq 0}$ define on $\mathrm{GL}_1(\mathbf{A}_F)$ a stratification by open and closed subsets, which we call the *divisorial stratification*.

Each stratum is:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : \rho_{T_k}(g) = 1\} = \{g \in \mathrm{GL}_1(\mathbf{A}_F)_{T_k} : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k\}$$

and the complement to each stratum into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} - \mathrm{GL}_1(\mathbf{A}_F)_{T_{k+1}} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in T_{k+1} - T_k\}.$$

At last, we will term the *primary divisorial stratification* the stratification $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ where we assume:

(1) S is saturated.
(2) Each finite set of finite places P_k is saturated, and obtained as a union of finitely many of the T_m's.
(3) $\{P_k\}_{k \geq 0}$ is the finest (under the refinement partial ordering) among such stratifications.

More directly, we define P_k to be the pre-image under the map $|\mathrm{Spec}(\mathcal{O}_F)| \to |\mathrm{Spec}(\mathbf{Z})|$ of the subset T_k, and therefore the stratum $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$ of the kth divisorial stratum on $\mathrm{GL}_1(\mathbf{A_Q})$. Explicitly, we have

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in \bigcup_{p \in T_k} S_p\}$$

where T_k denotes, here, the finite set of the first k rational primes. In other words, $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image of the kth stratum $\mathrm{GL}_1(\mathbf{A_Q})_{T_k}$ of the divisorial stratification on $\mathrm{GL}_1(\mathbf{A_Q})$ under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$.

The complement to each stratum of $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} - \mathrm{GL}_1(\mathbf{A}_F)_{P_{k+1}} = \bigcup_{Q \subset P_{k+1} - P_k} \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in P_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in Q\}$$

where Q varies over all non-empty finite subsets of $P_{k+1} - P_k$.

The primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$ adapts itself nicely to the Schwartz-Bruhat functions f on $\mathbf{A}_F$ of the form in (11.3.1.1). Indeed, upon calling:

$$\alpha_k := 1_{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}}$$

the indicator function of the kth stratum of the primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$, we have

$$\alpha_k \cdot (f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}) = f_S \otimes \bigotimes_{v \in S - P_k} 1_{\mathcal{O}_{F_v}} \otimes \bigotimes_{v \in P_k} 1_{\mathcal{O}_{F_v}^\times}$$

and hence multiplication by α_k replaces the Zeta function of f with the product of $Z(f, \chi|\cdot|_F^s)$ by the finitely many reciprocal Euler factors of $L(s, \chi)$ at the non-Archimedean places v of F indexed by P_k (the reciprocal of the product of the volumes of the local units is 1 because S is now assumed to contain all the ramified places of F!).

We call such multiplication process "restriction" from one stratum to the next, of the primary divisorial filtration.

In particular, and this is the main observation of this subsection: *the $(k+1)$st iteration of the restriction to the strata of the primary divisorial filtration is obtained from the kth, on the forward side of the Mellin transform, by multiplication by*

$$\prod_{v \in S_{p_{k+1}}} L_v(\chi_v|\cdot|_v^s)^{-1}.$$

If we let f (equivalently f_S) vary, we obtain that all the resulting global adelic θ-functions restricted to the kth stratum are contained in

$$\mathrm{PV}_{P_k} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

and all the global adelic θ-functions restricted to the $(k+1)$st stratum are contained in the yet smaller and closed subspace

$$\mathrm{PV}_{S_{p_{k+1}}}\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) = \mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)).$$

But now the subquotient:

$$\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$

is a copy of $\mathscr{E}_{S_p}(\chi|_{C_F^1})$, and it carries its own Orbital Schauder basis!

The idea is the following:

- Chop the lifted geometric kernel $K_{\mathrm{geom},I,\varphi}$ along the strata of the primary divisorial filtration on $\mathrm{GL}_1(\mathbf{A}_F)$ using the indicator functions α_k, $k \geq 0$. Consider the successive differences
$$(\alpha_{k+1} - \alpha_k)K_{\mathrm{geom},I,\varphi}$$
(this will be what we called $k_{S_{p_k},\varphi}(x,y)$) and write them down explicitly by letting them act on Schauder basis elements, writing them as limits of finite rank kernels.
- Compute traces and use the Poisson Summation Formula, as was done in the toy example, to get that all the partial kernels $k_{S_{p_k},\varphi}(x,y)$ have vanishing diagonal mass (after smoothening and using the formula from (1.3.1.12)).
- As $k \to +\infty$, we are left with a "remnant" of the global kernel $K_{\mathrm{geom},I,\varphi}$ entirely supported on
$$(\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}) \times \mathrm{GL}_1(\mathbf{A}_F)$$
and invariant under $\widehat{\mathcal{O}}^{\times}_{\Sigma_F-S}$ in the first variable (which is by convention the y-variable, i.e. it is listed as second). This, we will call $k_{S,\varphi}$, and we will be computing, by means of an analog of (1.3.1.12) and using calculations very close to those Connes performed in his "S-local trace formula" from his paper from 1998, its diagonal mass, leading to the Trace Formula which is the main subject of this work.

1.3.3. The Shell structure: degeneration from global to S-local. Observe that the orbital kernels only need to be evaluated over loci contained within

$$(\mathrm{GL}_1(\mathbf{A}_{F,S'}) \times \widehat{\mathcal{O}}^{\times}_{\Sigma_F - S'}) \times \mathrm{GL}_1(\mathbf{A}_F)$$

for S' large enough, when we look at their contributions to the trace. Then compute their effect on the S'-local θ-functions using the Shell Structure.[5]

1.3.4. Acting on the Paley-Wiener Levels. Introduce the Paley-Wiener levels and explain how to compute the orbital kernels by truncating them on the source and letting them act on the Paley-Wiener levels (after twisting the global geometric kernel by inversion on source and target, using Tate's Riemann-Roch Theorem).

Compute the orbital traces following the toy example.[6]

1.4. Putting everything together: Trace Formula and consequences

Putting everything together, we obtain the Trace Formula, valid on all $\varphi \in \mathcal{D}(C_F)$ satisfying the conditions listed below, and for the interval $I = [-a, a]$ for $a > 0$ arbitrary, kept fixed throughout the trace calculation (we choose the Sobolev degree $s = 0$ for simplicity):

[5] Finish!
[6] Finish!

$$\sum_{v\in S_\infty,\ v\ \text{real}} \frac{1}{2}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v) + \sum_{v\in S_\infty,\ v\ \text{complex}} \frac{1}{2\pi}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v)$$
$$+ \sum_{v\in\Sigma_{F,f}} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|_v}\,\mathrm{d}\mu_v(g_v)$$
$$= M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2}) - \sum_{\rho,I} N(\rho)M(\varphi)\left(c_\rho|\cdot|_F^{-1/2}\right)$$

where $q_v := |\pi_v|_v^{-1}$, for $v\in\Sigma_{F,f}$ and π_v any choice of uniformizer at v of the non-Archimedean local field F_v, and

$$M(\varphi)(c) := \int_{C_F} \varphi(g)c(g)\,\mathrm{d}\overline{\mu}(g)$$

is the Mellin transform of the test function φ, and $c_\rho\in\Omega(C_F)$ varies over the zeros of the Hecke L-function $c\mapsto L(c)$ contained in the interior of the relative vertical strip

$$\Omega(C_F)_I = \{c\in\Omega(C_F) : \sigma(c)\in I\}.$$

The nonnegative integer $N(\rho)$ is the multiplicity of the zero c_ρ of the Hecke L-function.

The assumptions on the test function are as follows:

(1) $\varphi\in\mathcal{D}(C_F)$, i.e. φ is smooth and compactly supported.
(2) $\varphi(C_F^1) = 0$.

1.4.1. Character by character. For φ homogeneous with respect to the right-translation action of C_F^1 on $\mathcal{D}(C_F)$, with weight $\overline{\chi}|_{C_F^1}$, for $\chi\in\widehat{C_F}$ fixed, and satisfying assumptions (a) and (b) above, the Trace Formula turns into:

$$\sum_{v\in S_\infty,\ v\ \text{real}} \frac{1}{2}\int_{\mathbf{R}_{>0}} \sum_{u\in\{\pm1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|}\,\frac{\mathrm{d}t}{t} + \sum_{v\in S_\infty,\ v\ \text{complex}} \frac{1}{2\pi}\int_{\mathbf{R}_{>0}}\int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|}\,2\mathrm{d}\theta\otimes\frac{\mathrm{d}t}{t}$$
$$+ \sum_{v\in\Sigma_{F,f}} \log(q_v)\sum_{m\geq1}\left(\frac{\chi(\pi_v^m)}{q_v^{m/2}}\varphi(q_v^m) + \frac{\overline{\chi}(\pi_v^m)}{q_v^{m/2}}\varphi(q_v^{-m})\right)$$
$$= M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2}) - \sum_{\rho,I} N(\rho)\int_{C_F}\chi(g)\varphi(g)|g|_F^{\rho-1/2}\,\mathrm{d}\overline{\mu}(g)$$

where $M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2})$ vanishes if φ is not C_F^1-invariant.

1.4.2. An explicit holomorphic family of test functions. The idea now is to consider the following holomorphic family of piece-wise smooth functions on C_F:

$$\varphi_z(t) := t^{-z}$$

for $t>1$, $\varphi_z(t) := 0$ for $t\in(0,1)$, $\varphi_z(1) = 1/2$ and approximate $\chi\cdot\varphi_z$ (viewed as a function on C_F by composition with the idelic norm) by smooth compactly supported functions on C_F satisfying (a) and (b), and C_F^1-homogeneous with respect to the conjugate of a chosen Hecke character.

For example, take $\varepsilon>0$ small, and ρ_ε the resulting mollifier as a function of $t\in\mathbf{R}_{>0}$. Take $h(t)\in\mathcal{D}(\mathbf{R}_{>0})$ be a bump function supported over $(1,+\infty)$ only (and valued in $[0,1]$).

Then

$$\varphi_{h,\varepsilon,z}(t) := h(t)(\rho_\varepsilon * \varphi_z)$$

tends to $\varphi_z(t)$ pointwise (for $t \neq 1$) as $\varepsilon \to 0$ and $h \to 1_{(1,+\infty)}$ pointwise, as h (which will be an explicit collection of bump functions) remains valued in $[0,1]$ and below $1_{(1,+\infty)}$, converging pointwise monotonically.

1.4.3. Montel's Theorem. Fix a compact subset $K \subset \mathbf{C}_{>\sup(I)}$ with non-empty interior intersecting with non-empty interior the half-plane $\mathbf{C}_{>1}$.

Write $\varphi = \chi \cdot (\varphi_{h,\varepsilon,z} \circ |\cdot|_F)$; note the dependence on $z \in \mathbf{C}$.

The integrals

$$\sum_{v \in S_\infty,\ v \text{ real}} \frac{1}{2} \int_{\mathbf{R}_{>0}} \sum_{u \in \{\pm 1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|} \frac{\mathrm{d}t}{t}, \quad \sum_{v \in S_\infty,\ v \text{ complex}} \frac{1}{2\pi} \int_{\mathbf{R}_{>0}} \int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|} 2\mathrm{d}\theta \otimes \frac{\mathrm{d}t}{t}$$

are bounded above by a common constant $A > 0$ independent of $z \in K$ and independent of the specific φ.

The boundedness follows from simple estimates and from the explicit form of the integrals (and the fact that they are only finitely many! Since they are indexed by the Archimedean places only. The integrals indexed by the non-Archimedean places are, instead, infinitely many, and they are the source of all the difficulties: their sum approximates $L_f'(s+1/2,\chi)/L_f(s+1/2,\chi)$).

By Montel's Theorem from complex analysis, we can extract a sequence of ε, h such that over such sequence, the convergence of the integrals in

$$\sum_{v \in S_\infty,\ v \text{ real}} \frac{1}{2} \int_{\mathbf{R}_{>0}} \sum_{u \in \{\pm 1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|} \frac{\mathrm{d}t}{t}, \quad \sum_{v \in S_\infty,\ v \text{ complex}} \frac{1}{2\pi} \int_{\mathbf{R}_{>0}} \int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|} 2\mathrm{d}\theta \otimes \frac{\mathrm{d}t}{t}$$

is uniform for z in any compact disc contained within the interior of K. Since each is a holomorphic function of $z \in K$, so is the limit.

A similar reasoning leads to the limit as $\varepsilon \to 0$ and $h \to 1_{(1,+\infty)}$ of the convergent sums

$$\sum_{\rho, I} N(\rho) \int_{C_F} \chi(g)\varphi_{h,\varepsilon,z}(g)|g|_F^{\rho-1/2}\, \mathrm{d}\overline{\mu}(g)$$

to be holomorphic as a function of z varying over the interior of K intersected with the complement of $\mathbf{C}_I$ (recall that the sum is only over those zeros ρ of $L(s,\chi)$ such that $\rho - 1/2 \in \mathbf{C}_I$, after parametrizing $\Omega(C_F)(\chi)$ as $\chi|\cdot|_F^s$).

1.4.4. The logarithmic derivative of $L_f(s,\chi)$. Now note that each $\varphi = \varphi_{h,\varepsilon,z}$ is, for all $z \in K$, supported over $(1,+\infty)$, and so $\varphi(q_v^{-m}) = 0$ for all integers $m > 0$.

The sum

$$\sum_{v \in \Sigma_{F,f}} \log(q_v) \sum_{m \geq 1} \left(\frac{\chi(\pi_v^m)}{q_v^{m/2}} \varphi(q_v^m) + \frac{\overline{\chi}(\pi_v^m)}{q_v^{m/2}} \varphi(q_v^{-m}) \right)$$

therefore tends, uniformly over the compact subsets of $\mathbf{C}_{>1}$, to

$$\frac{L_f'\left(s+\frac{1}{2},\chi\right)}{L_f\left(s+\frac{1}{2},\chi\right)}.$$

1.4.5. Putting everything together. Putting everything together from §5.1, we obtain that for $I = [-a,a]$ for a fixed real number $a > 0$, the logarithmic derivative of $L_f(s,\chi)$ satisfies, over the interior of K:

$$\text{holomorphic over } \mathrm{Int}(K) + \frac{L_f'(s+1/2,\chi)}{L_f(s+1/2,\chi)} = \delta \cdot \left(\frac{1}{s+1/2} + \frac{1}{s-1/2} \right) - \text{holomorphic over } \mathbf{C}_{>a}$$

with $\delta = 0$ if $\chi|_{C_F^1} \neq 1$ and $\delta = 1$ otherwise. This is, ultimately, the main consequence of the Trace Formula. Everything else follows from this.

We deduce that

$$\frac{L'_f(s+1/2,\chi)}{L_f(s+1/2,\chi)} - \delta \cdot \left(\frac{1}{s+1/2} + \frac{1}{s-1/2} \right)$$

is holomorphic over $\mathrm{Int}(K)$ for any $K \subset \mathbf{C}_{>a}$ by analytic continuation ($\mathbf{C}_{>1}$ versus $\mathbf{C}_{>a}$). And so $L'_f(s+1/2,\chi)/L_f(s+1/2,\chi)$ is holomorphic over $\mathbf{C}_{>a}$ with the sole possible exception of the point $s = 1/2$ (when $\chi|_{C_F^1} = 1$).

Since $a > 0$ is arbitrary, we deduce that $L'_f(s+1/2,\chi)/L_f(s+1/2,\chi)$ is holomorphic over $\mathbf{C}_{>0} - \{1/2\}$.

This implies that $L_f(s,\chi)$, and hence $L(s,\chi)$, has no zeros over $\mathbf{C}_{>1/2}$, and likewise for $L(s,\overline{\chi})$, so by the functional equation the same holds for $L(s,\chi)$ over $\mathbf{C}_{<1/2}$.

This completes the overview of how to deduce the main results on the location of the zeros of $L(s,\chi)$, from the Trace Formula.

1.4.6. Weil's Positivity Criterion is not used. This is reminiscent of the avoidance in Deligne's proof of the Weil Conjectures of the positivity aspect of the intersection pairing.

Positivity of the Weil functional from Weil's Explicit Formulas is not addressed: rather, the trace formula holds on a large enough class of test functions enabling us to approximate the holomorphic family of piece-wise smooth functions $\varphi_s(t) := t^s$ for $t > 1$, $\varphi_s(t) := 0$ for $t \in (0,1)$, and $\varphi(1/2) := 1/2$, to obtain information about

$$\frac{L'_f(s,\chi)}{L_f(s,\chi)}.$$

Indeed, the trace formula can only be proved for test functions φ on C_F (i.e. elements φ of $\mathcal{D}(C_F)$) such that $\varphi(C_F^1) = 0$. In fact, the trace calculation is performed for test functions supported over $[1+\varepsilon, +\infty)$ for some $\varepsilon > 0$ fixed, arbitrary. The Trace Formula is then extended to the test functions supported over $(0, 1-\varepsilon]$ for $0 < \varepsilon < 1$ fixed, arbitrary, and then further extended to the test functions such that $\varphi(C_F^1) = 0$ by a continuity argument.

The condition $\varphi(C_F^1) = 0$ cannot be removed before GRH is proved. The Trace Formula indeed is valid on any test function (even without this vanishing condition) but this can only be proved a posteriori, once novel information on the zeros, deduced from the Trace Formula when $\varphi(C_F^1) = 0$, is incorporated into Weil's Explicit Formulas.

1.4.7. Non-vanishing of the finite Hecke L-function $L_f(s,\chi)$ on the line $1 + i\mathbf{R}$ is used. This is analogous to when Deligne, in the process of proving the Riemann Hypothesis for the L-functions of smooth proper varieties over finite fields, had to prove the non-vanishing on the boundary of their critical strips.

For example, assume S is a finite set of places containing all the Archimedean places. Then the transition operator

$$p_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$$

induced by sending $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$ to

$$f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$$

is turned by the Mellin transform into the multiplication operator by

$$L^S(s,\chi) := L_f(s,\chi) \cdot \prod_{v \in S_f} L_v(\chi_v|\cdot|_v^s)^{-1}$$

for $L_v(\chi_v|\cdot|_v^s)$ the local Euler factor at $v \in S_f$.

To propagate the orbital Schauder bases to global ones, we consider, for any basis element ϕ (represented by an S-local θ-function) the global θ-function $p_S(\phi)$, and the non-vanishing of $L^S(s,\chi)$ on $1+i\mathbf{R}$ enables us to multiply the Mellin transform of $p_S(\phi)$ by the reciprocal

$$\frac{1}{L^S(s,\chi)}$$

to undo the effect of p_S and deduce global properties from S-local ones, using the Residue Theorem (on growing boxes with fixed horizontal width). To apply the Residue Theorem (and for other purposes too), *other properties of* $L_f(s,\chi)$ *are also used*:

- $L_f(s,\chi)$ grows at most polynomially uniformly over vertical strips of bounded width.
- There exists a sequence of ordinates $\{T_k\}_{k\in\mathbf{Z}}$ such that $\lim_{\to\pm\infty} T_k = \pm+\infty$, such that the reciprocal $1/L_f(s,\chi)$ is nonzero over $\mathrm{Im}(s)=T_k$ and grows at most polynomially with respect to $|T_k|$. This is seen by means of analogs of arguments of Titchmarsh on the Riemann Zeta function.
- $1/L_f(s,\chi)$ grows at most polynomially over $1+i\mathbf{R}$.

1.4.8. GRH is a topological phenomenon. It follows from the observation that the calculation of the *geometric side* of the Trace Formula can be performed intrinsically within

$$W_I^{s,2}(C_F)$$

for any $s\geq 0$ and $I=[-a,a]$ for any $a>0$ (as small as we please!).

Because then, once the geometric side is computed, we are allowed to continue performing the calculation of the *spectral side* too (!) by working intrinsically within $W_I^{s,2}(C_F)$. In so doing, the spectral side now reads (for Sobolev degree $s=0$):

$$\sum_{\rho,I} N(\rho)M(\varphi)(\rho-1/2)$$

where the sum is over those zeros of $L(z,\chi)$ such that $\rho-1/2$ is contained in the vertical strip $\mathbf{C}_{[-a,a]}$ and $N(\rho)$ is the multiplicity of the zero.

For Sobolev degree $s>0$, we get contributions from the zeros on the edge lines $\{\sigma=-a,a\}$, with multipliciity capped from above by a function of the Sobolev degree s.

The spectral side takes the form as above, because the point-evaluation functionals

$$\delta_z$$

at the points

$$z=\rho-1/2$$

as well as their distributional derivatives

$$\delta_z^{(m)}$$

(all continuous functionals on $\mathcal{S}_{\mathrm{E}}(C_F)$) do not extend by continuity to $W_I^{s,2}(C_F)$ unless

$$\rho-1/2\in\mathbf{C}_{[-a,a]}$$

(if the Sobolev degree $s=0$; otherwise the condition is $\rho-1/2\in\mathbf{C}_{[-a,a]}$ for $s>0$). This is reminiscent of Deligne's arguments shrinking the critical strip by enlarging zero-free regions by means of the tensor-power trick.

1.4.9. The Orbital Bases are the crux of the matter. They allow to bridge the spectral side to the S-local piece of the Trace Calculation.

Because they enable us to write down "on the basis", much as one does in linear algebra, the orbital pieces of the kernel (after one splits the global convolution kernel into a sum of pieces dictated by the stratification of the ideles by non-Archimedean local "units-versus-balls", as in §4, i.e. write their truncations down as

trace-class limits of explicit finite rank operators).

As the cutoff tends to $+\infty$, the limit value of the trace is computed by summing diagonal elements on Jordan blocks, and the Poisson Summation Formula gives the epxlict answer. For all but finitely many prime residue characteristics, the orbital traces associated to those primes are all zero!

The Trace Calculation then reduces to Connes' "S-local Trace Formula" from his paper from 1998 (something very similar).

This can all be done within $W^{s,2}_{[-a,a]}(C_F)$ for any fixed real number $a > 0$, and so we can compute the spectral side by staying within $W^{s,2}_{[-a,a]}(C_F)$ too.

1.4.10. The orbital bases are the principal tool to bridge the spectral side to the S-local piece of the geometric side. They reveal a massive cancellation phenomenon of the mass of the global kernel along the diagonal, concentrating all of the geometric side into the S-local piece, for S large enough and *fixed* (the calculation is done "character by character", and S depends on the test function φ and the character, and it is always assumed to satisfy $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$).

1.4.11. A version of this deconstruction method of mine resting on the bases. This went under the name of "infinite excision" approximately a decade ago, when we were about to visit Princeton. The ideas back then were immature in my head and it became clear that it would take much more time and work to bring all of it into focus.

1.5. On the programs of Connes and Deninger

1.5.1. Relation to Connes' and Meyer's work. The calculations in Connes' S-local trace formula are used in the end to compute the diagonal mass of $k_{S,\varphi}$. Observations from Connes' paper are also reviewed and much expanded. His "spectral interpretation" is recovered as the special case $I = \{0\}$ as discussed in the section on Notation (with our Sobolev degree s being Connes' δ in his paper).

Connes never tried to use the locally convex spaces of Schwartz-Bruhat functions and θ-functions that are the backbone of this work. In his paper, they are never considered.

Meyer considered $\mathcal{S}_I(G)$ and the spaces of S-local and global θ-functions, but only through their bornology (an axiomatization of bounded sets in a locally convex topological vector space). He never established anything on their isotypes, let alone defining and studying Schauder bases on the various S-local and global orbital subquotients.

Meyer's method of trace calculation rests upon operator-theoretic Euler products which only converge when the function spaces are given the bornologies associated to the locally convex topology on $\mathcal{S}_{\mathrm{E}}(C_F)$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, rather than coarser topologies associated to Sobolev or Hardy-Sobolev norms. It follows that in Meyer's work all the zeros, irrespective of their location, contribute to the spectral side of his trace formula the same way as they already do in Weil's own Explicit Formulas, i.e. through point-evaluation of the Mellin transform at the zero, shifted by $1/2$ to the left.

My method of trace calculation instead rests on leveraging the bases to compute the orbital kernels and get their traces, while identifying the effect each such kernel has by means of an approximate, weaker version of the Euler product convergence Meyer used (which for us comes down to a much simpler convergence assertion on relatively compact subsets of C_F, caused by the rapid decay condition on Schwartz-Bruhat functions on the

adeles). We get compensation from the bases, and the method works within $W_I^{s,2}(C_F)$ (isotype by isotype) instead of requiring to work with respect to the topology of $\mathcal{S}_{\mathrm{E}}(C_F)$.

The idea of bases to write down pieces of a deconstructed global kernel, reducing the calculation of its diagonal mass to the calculation of the diagonal mass of an S-local piece of the kernel, which is the computed using a version of Connes' S-local trace formula, is novel and was never considered in any form nor part, before.

1.5.2. Relation to Deninger's work. Periodic orbits and fixed points are confirmed to exist, their role clarified through the role of the Mellin transform: it is similar to that of the periodic cylinders featuring in Sarnak's paper on long horocycles.

The local traces that Deninger contemplated in his paper on Lefschetz trace formulas and the Explicit Formulas are indeed all realized as actual trace formulas using the orbital subquotients of the spaces of local and global θ-functions and the Schauder bases on each.

The visualization resting on the orbits helps to organize the trace calculation. Logically, it is not needed.

1.6. How this work was conducted, and why

1.6.1. The main ideas.

1.6.2. The role of Brian Conrad.

1.7. Excerpts from a letter to Peter Sarnak

1.7.1. Why "yes" for number fields, and "no" for function fields. The short answer is: because of the presence of Archimedean places (and, ultimately, the $\mathbf{Q}$-linear independence of the logarithms of the prime numbers, which plays a pervasive role sometimes explicitly and sometimes behind the scenes in many of the turns throughout my paper).

- They make all the function spaces involved far more interesting. At a basic level, as is well-known, they make the connected components of the Riemann surface of idelic quasi-characters be simply connected, contrary to the case of global function fields of positive characteristic where "vertical strips" are instead annuli of finite area.

  By far the most important difference directly affects the calculation in Caveat **??**: let $f \in \mathcal{S}(\mathbf{A}_F)$ be any Schwartz-Bruhat function, with F a number field. Assume $f(0) = \widehat{f}(0) = 0$ (so as to remove the poles of its idelic zeta integral in the sense of Tate's thesis, extended analytically using the functional equation [**CF**, Ch.XV,Thm.4.4.1]). Then the function

  $$\sum_{\xi \in F^\times} f(\xi g)$$

  of $g \in C_F$ obtained by periodizing f over the $F^\times$-orbits on the ideles $\mathrm{GL}_1(\mathbf{A}_F)$ turns out to be Schwartz-Bruhat on C_F (this holds also for F a global function field of positive characteristic) but if this function of g is not identically zero then it is *never* compactly supported.

  This is ultimately caused by the Γ-factors of the idelic zeta function associated to f (use Stirling's asymptotics combined with the convexity bound, and pass them through the lens of the Paley-Wiener Theorem).
- One instrument that is pervasive in the first seven chapters of my paper, and occurs frequently, is the use of diagonal quotients of the form:

(1.7.1.1) $$(\mathbf{R}^N \times A)/\Gamma_0$$

  for A an open subgroup of a locally profinite second-countable topological group and $\Gamma_0 \subset \mathbf{R}^N \times A$ a discrete subgroup.

These quotients (termed "solenoids" sometimes in the literature) have the advantage that one can use the diagonal action of Γ_0 (which can be either $F^\times$, or a discrete group of S-units, or subgroups of these, for example) to transfer displacements in the "A-variable" under group actions, to displacements in the Euclidean variables.

This is used all the time to make explicit constructions of functions with certain properties, to prove statements such as "the subquotient of the following topological vector space carries a topological basis satisfying the following conditions".

Quotients as in (1.7.1.1) are unavailable in the case F is a global function field, and the great advantages of having them throughout are lost.

- I would like to offer an analogy, to guide intuition as to why in the number field case the methods in my paper carry much more information than their positive characteristic analogs.

  For $F = k(X)$ with X a smooth projective geometrically irreducible curve over a finite field k, and G a connected reductive group over k, mere double coset spaces such as

$$G(F)\backslash G(\mathbf{A}_F)/K \tag{1.7.1.2}$$

  are certainly useful for some purposes (such as in the function field analogue of Tate's thesis, and the work of Godement-Jacquet on automorphic L-functions of unit groups of simple algebras that can be done uniformly over all global fields), but usually these spaces require richer geometric structure such as identifications of the form

$$\mathrm{Bun}_G(k) \simeq G(F)\backslash G(\mathbf{A}_F)/K$$

  (giving access to the moduli stack Bun_G of G-bundles on the curve X, and then function spaces as sections of ℓ-adic sheaves) to get deeper results on global L-functions (e.g., the works of Laurent and Vincent Lafforgue). Consider, specifically, L. Lafforgue's proof of Global Langlands for GL_n. Intersection cohomology and Deligne's methods from Weil II are used as an input in an essential manner. Weil II gives estimates on the Frobenius eigenvalues on the intersection cohomology of the moduli stack of Shtukas, in the process of accessing the Converse Theorem. The point that I am making is that deeper insight into automorphic L-functions and representations in the function field case, necessitates techniques from Algebraic Geometry, aimed at compensating for the much-too-poor geometric structure of the topological double quotients in (1.7.1.2).

  However, for F a number field, the double coset spaces in (1.7.1.2) are perfectly good! There is no need to imbue them with more structure.

  This is thanks to the Archimedean places. In the presence of Archimedean places, the double coset spaces from (1.7.1.2) already carry rich function spaces of smooth functions. Functional analysis and harmonic analysis are much less trivial and lead to flavorful results, all enabled by accessing differential geometry and Riemannian geometry through the Archimedean components of these spaces. With the non-Archimedean places serving the role of "transversals", much as in the book by Moore and Schochet [**MS**] on global analysis on foliated spaces and in the book *Geometric Asymptotics* by Guillemin and Sternberg [**GS**].

  A similar phenomenon is at play here: for F a global function field of positive characteristic, the function spaces that are directly analogous to those arising in my paper do not carry enough information to lead to anything more than an analog of the Lefschetz trace formula, which, as you know, even for curves is insufficient to establish the Weil bound. For F a number field, they are much more interesting and intricate, and several phenomena specific to number fields spring into action. One of them is the subject of the next bullet point.

- The $\mathbf{Q}$-linear independence of the logarithms of the prime numbers (a phenomenon that has no analog in positive characteristic, with comparable strength) pervades the choices of normed-topologies,

defining the completions in which the trace calculation that proves the main Theorem of the paper is carried out.

How? The following phenomenon gives you an example of how. It is a simple consequence of a result from my paper, and the **Q**-linear independence of the logarithms of the prime numbers is used in an essential manner throughout:

THEOREM 1.7.1.1 (On polar multiplicities - a consequence of a Theorem from Chapter 7). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous (unitary) character. Denote by $L(s,\chi)$ the Hecke L-function, and by $L_v(\chi_v|\cdot|_v^s)$ its local Euler factor at v, for any place v of F.*

(1) For v_1, v_2 two finite places of F with distinct residue characteristics, the local Euler factors $L_{v_1}(\chi_{v_1}|\cdot|_{v_1}^s)$ and $L_{v_2}(\chi_{v_2}|\cdot|_{v_2}^s)$ have at most one pole in common.

(2) If $\chi = \chi_0|\cdot|_F^{s_0}$ for $s_0 \in \mathbf{C}$ and χ_0 algebraic then assuming Schanuel's Conjecture, there exists at most one point s_1 of the complex s-plane (namely, $s_1 = -i\,\mathrm{Im}(s_0)$) such that s_1 is a pole of $L_v(\chi_v|\cdot|_v^s)$ for infinitely many finite places v of F.

(3) If χ is of finite order then the assertion in (2) holds unconditionally and $s_1 = 0$.

This Theorem has nothing to do with the zeros of the Hecke L-function, and is used to select normed-topologies on the function spaces of interest (such as Sobolev topologies of Sobolev degree at least $2[F:\mathbf{Q}]$) to ensure that certain "coefficient functionals" are continuous, thanks to results surrounding Theorem 1.7.1.1. These comprise the (dual bases of the) Schauder bases that Brian mentioned over the phone, and that enable the trace calculation that leads to the main Theorem of the paper.

Theorem 1.7.1.1 has no useful analog for F a function field of positive characteristic.

1.7.2. Highlights. The main highlights that I would like to convey, for then stating and outlining the main results face to face, are:

- **Weil's Positivity Criterion is not used.** The Trace Formula can only be proved under an additional assumption on the test function $\varphi \in \mathcal{D}(C_F)$:

$$\varphi(C_F^1) = 0. \tag{1.7.2.1}$$

  This assumption cannot be removed. It stays there until after GRH is deduced from the Trace Formula. It is forced on us by two divergence phenomena that can only be resolved by assuming that (1.7.2.1) holds.

  Weil's Positivity Criterion uses test functions of the form $\varphi = \psi_1 * \psi_2$, where $\psi_2(g) = \overline{\psi_1(g^{-1})}$, and the only φ of this form that satisfies (1.7.2.1) is identically zero.

  The fact that GRH does not follow from positivity of the Weil functional feels analogous to how Deligne through the use of families avoided having to address positivity of the intersection pairing.
- **Properties of the Hecke L-function on the 1-line (including the non-vanishing) are used as an input** throughout. Mostly to propagate Theorems from "S-local" (where we have access to additional flexibility; e.g., smooth compactly supported functions on $\mathrm{GL}_1(\prod_{v\in S} F_v)$ extend by zero to elements of $\mathcal{S}(\prod_{v\in S} F_v)$, which fails globally) to "global".

  This feels analogous to how Deligne too had to prove that the L-functions of a smooth projective geometrically irreducible variety over a finite field have no zeros at the edge of the critical strip, before employing the "tensor power trick" to enlarge the zero-free region and asymptotically maximize it.
- **The Geometric Side of the Trace Formula drives everything.** The Spectral Side is only computed at the very end, once the choice of normed-topology has been made, pinned down by the ingredients that are necessary to compute the Geometric Side. In other words, the zeros play no role in selecting the linear topological environment in which the Trace Formula is proved. They

are never used as an input anywhere, nor to make any construction. They remain "passive", so to speak. They are "victims" of the topology of the normed environment. In other words, we endow certain locally convex spaces arising from Tate's thesis with normed topologies **chosen by looking at what ingredients are truly necessary to perform the calculation of the Geometric Side of the Trace Formula. The zeros play no role here.**

The relevant topologies turn out to be the subspace topologies induced by each of the norms

$$\|f\|_{a,b} := \left(\int_G |f(g)| \, \max\{|g|_G^{2a}, |g|_G^{-2a}\} (1 + |\log|g|_G|^2)^{b/2} \, \mathrm{d}\mu_G(g) \right)^{1/2} \tag{1.7.2.2}$$

for $a > 0$ and $b \geq 0$ with $G = C_F$ the idele class group of the number field F, $|\cdot|_G = |\cdot|_F$ the idelic norm (these norms are studied briefly in the paper through the lens of the Mellin transform for $\mathbf{R}_{>0}$-modulated groups; i.e., Hausdorff locally compact abelian groups G equipped with a proper surjective continuous homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$).

I feel I need to repeat this: the choice of normed topologies $\|\cdot\|_{a,b}$ on these function spaces with $a > 0$ and $b \geq 0$ (I called them Hardy-Sobolev norms; the analogy is close enough, and the Mellin transform turns their completions into spaces of holomorphic functions on the vertical strip $\mathbf{C}_{(-a,a)} = \{|\sigma| < a\}$ with edge conditions of continuity and differentiability up to order $|b-1|/2)$ is made by the goal of finding normed topologies **such that the resulting completions contain all the barely necessary ingredients (i.e. the bare minimum) to perform the calculation of the Geometric Side of the Trace Formula (mostly, that certain collections of functionals be continuous and of a specific form, on certain closed subspaces derived from Schauder bases at the level of the locally convex spaces that sit densely in such normed completions).** In particular, **the zeros play no role in choosing the normed topologies** to which I am alluding.

The entire paper itself is about the Geometric Side of the Trace Formula only. It is about spelling out all the ingredients for the Geometric Side at the locally convex level first (the first seven chapters of the paper) and then passing them through the lens of coarser normed topologies. Only at the very end, once the calculation of the Geometric Side of the Trace Formula is performed by working intrinsically within the $\|\cdot\|_{a,b}$-completions and closures, we observe that, **therefore**, we can keep going and compute the Spectral Side too while remaining within the $\|\cdot\|_{a,b}$-completion. The normed topology of the environment now prevents some zeros from expressing themselves because their point-evaluation functionals and derivatives of those will not be continuous with respect to the $\|\cdot\|_{a,b}$-normed topology, depending on whether such functionals are supported on $\mathbf{C}_{(-a,a)}$, on the edges of $\mathbf{C}_{(-a,a)}$ (in which case they are not counted with their full multiplicity; only multiplicity up to the smallest integer less than or equal to $|b-1|/2)$, or outside $\mathbf{C}_{[-a,a]}$ altogether.

The takeaway is a collection of analogs of the Explicit Formulas of Weil in which the side involving the places of F (finite and Archimedean) and the poles (the Geometric Side) is identical to that from the Explicit Formulas, and the side involving the zeros (the Spectral Side) is not.

The degenerate case $a = 0$ does not lead to a Trace Formula anymore, but it does appear to lead to some information on the multiplicities of the zeros as outlined in the next section (so long as we choose $b \geq 2m_\chi$, where m_χ is the supremum over all prime numbers p of the highest pole-order of the poles of the product of all the local Euler factors at the finite places of residue characteristic p; in particular, $b \geq 2[F : \mathbf{Q}]$ qualifies). As Brian mentioned over the phone, conclusions on multiplicities cannot be obtained through the trace formula. In the next section, however, I do mention that from the extremal case $a = 0$ and $b \geq 2m_\chi$, we can extrapolate conclusions on the multiplicities at least in the form of expectations (which I had yet to discuss with Brian, and only briefly touched upon them with him about a year ago). These have yet to be fully worked out in writing, and are reserved for

sequel work (it takes time, it is unnecessary right now, and it is not a low-hanging fruit). I plan to return on this only after all is said and done. No need now.

1.7.3. Outline of a sanity check: the multiplicities of the zeros. As mentioned in the last bullet point of the previous section, the extremal case of the methods (the case $a = 0$ in (1.8.2.2)) leads to information on the multiplicities of the zeros.

This will not be a part of my paper, and it is reserved for a sequel. I worked out enough of it for myself to arrive at the conclusions below, aiming to get to conclusions that I could test against the existing literature. What follows therefore constitutes an example taken from a class of sanity checks that I put the methods through. Once again, allow me to be as clear as I can be: **I have not fully worked out all details for this part, but only enough to arrive at a conclusion in the form of expectations, so that I could test it against the literature. I can sketch what I have in mind for this, in person.** The takeaways are as follows:

- For $a = 0$, no trace formula can be proved for any $b \geq 0$ (see (1.8.2.2) for this notation). The obstruction is already contained in Theorem 1.7.1.1, as we now explain.

  Let F be a number field, $\chi \in \widehat{C_F}$ a continuous (unitary) character. We define, for each prime number p:

(1.7.3.1) $$m_{p,\chi}$$

  to be the maximum of the pole-orders (≥ 0) of

$$\prod_{v \in S_p} L_v(\chi_v |\cdot|_v^s)$$

  for S_p the finite set of non-Archimedean places of F whose residue characteristic is p. We further define:

(1.7.3.2) $$m_\chi := \sup_{p \text{ prime}} m_{p,\chi}.$$

  Note that m_χ is finite, bounded above by $[F : \mathbf{Q}]$, the degree of the number field.

  Then Theorem 1.8.2.2 alerts us to the following phenomenon: as S grows to encompass all places of F, the maximum of the pole-orders of the finite Euler products

$$\prod_{v \in S} L_v(\chi_v |\cdot|_v^s)$$

  grows unboundedly at at most one point of the connected component of the Riemann surface of idelic quasi-characters parametrized as $\chi|\cdot|_F^s$. Using the theory of Néron models of abelian varieties over F we can construct examples of algebraic Hecke characters χ_0 for which indeed such an "infinitely repeated pole" exists. As it happens, in such examples the central zero of $L(s, \chi_0|\cdot|_F^{-w/2})$, for w the weight of χ_0, has multiplicity strictly exceeding $[F : \mathbf{Q}]$.
- If a trace formula was provable for $a = 0$ and $b \geq 2m_\chi$ then it would also lead to a bound on the multiplicities of *all* zeros: their multiplicities would turn out to be bounded above by m_χ. In light of the previous bullet point, this is false by means of examples. In particular, we must expect to find an obstruction to the method of trace calculation!

  And indeed, we find it: the infinitely repeated pole to the finite Euler products turns out to be the obstruction. How, I will explain face to face. In other words, the fact that the trace formula cannot be proved in this extremal version of the methods is itself a sanity check.
- However, even without a full trace formula, we can still draw some conclusions about the multiplicities.

Although a full trace formula cannot be proved in the extremal situation $a = 0$ (not even if $b \geq 2m_\chi$, or $b \geq 2[F : \mathbf{Q}]$), the *method* of trace calculation still allows one to squeeze out some information.

The outcome that profiles itself is the following:

EXPECTATION 1.7.1 (On the multiplicities of the zeros). *Let F be a number field, χ_0 an algebraic Hecke character of weight w. Then:*

(1) Conditionally on Schanuel's Conjecture, all zeros of $L(s, \chi|\cdot|_F^{w/2})$ except at most the central zero $s = 1/2$ have multiplicity bounded above by m_χ.

(2) If χ_0 is of finite order (and hence $w = 0$) then, unconditionally, all zeros of $L(s, \chi_0)$ except at most the central zero $s = 1/2$ have multiplicity bounded above by $[F : \mathbf{Q}]$.

I looked into this extremal case of the methods precisely with the aim of challenging them. The outcome is:

- We expect an obstruction to the trace formula for $a = 0$ and any $b \geq 0$, even for $b \geq 2m_\chi$ or $b \geq 2[F : \mathbf{Q}]$, due to central zeros with high multiplicities (examples using CM abelian varieties through theory of Néron models. Numerical examples usually conditional on BSD), and we find one.
- Nonetheless, we are led to expectations on the multiplicities (the tentative argument will be outlined verbally to you face to face). I do not know that Expectation 1.7.1 was ever formulated before.

  The question becomes: does Expectation 1.7.1 hold against the existing literature? Close inspection (using work of Langlands and Deligne on Weil's generalization of Artin L-functions, to deal with point (1) of Expectation 1.7.1; point (2) is simpler, and relies on invariance of formation of the non-abelian L-function of F under induction down to $\mathbf{Q}$, up to inflating the finite order Hecke character to the Galois closure F' of F) shows that indeed Expectation 1.7.1 would follow from the Grand Simplicity Hypothesis, and it is therefore compatible with it (assuming that the L-functions of irreducible continuous finite dimensional complex representations of the Weil group of $\mathbf{Q}$ are automorphic, as is expected and is known to hold in special cases).

Note that part (2) of Expectation 1.7.1 for $F = \mathbf{Q}$ leads to $m_\chi = [F : \mathbf{Q}] = 1$.

1.8. A view from the sky: from a letter to Peter Sarnak

1.8.1. The missed trip to Princeton from a decade ago. Towards the end of my first year and the beginning of my second year as a Graduate Student at Stanford, inspired by p-adic Hodge Theory, the possibility occurred to me that one could use "Koszul complexes" of entire functions (a modern way to paraphrase what I was thinking) to make a trace calculation in a vein that would be hybrid between Connes' dream and Deninger's.

It all stemmed from the observation that if F is a number field (with idele class group C_F), $\chi : C_F \to \mathbf{C}^\times$ is a quasi-character, and S is a finite set of places of F containing all Archimedean places, all places ramified over $\mathbf{Q}$, and all places at which χ is ramified then for a generic Schwartz-Bruhat function of the form

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$$

with $f_S \in \mathcal{S}(\prod_{v \in S} F_v)$, if we replace $1_{\mathcal{O}_{F_v}}$ with $1_{\mathcal{O}_{F_v}^\times}$ one finite place outside S at a time we would slowly "degenerate" f into f_S in the sense that on the connected component $\Omega(\chi)$ of χ in the space of idelic quasi-characters the local contributions of $1_{\mathcal{O}_{F_v}^\times}$'s (for $v \notin S$) to the Mellin transform on $\Omega(\chi)$ are equal to 1 (when using the standard Haar measures).

At the S-local level, Connes already had a perfectly good technique to compute the relevant trace. His calculation is a reformulation of a calculation of Weil's from his papers on the Explicit Formulas, resting on the observation that a certain additive local Fourier transform of a logarithm function would be expressed as a principal value arising out of the Residue Theorem applied to the logarithmic derivative of the Hecke L-function.

In my eyes, a version of Connes' S-local calculation is definitive, i.e. at the "S-local level", it is the right scenario to hope for indeed. I have come to view parts of the calculations in Connes' S-local "trace formula" as his most important contribution (I have a feeling he would get annoyed, if told).

And so the strategy "degenerate something global into something S-local" seems a good rule of thumb to follow. This is exactly what Ralf Meyer does in his paper [**Me1**]: he uses operators which, under Mellin transform, turn into multiplication operators by finite reciprocal Euler factors. For his method to succeed, he needs them to converge to infinite reciprocal Euler products (with a few finitely many Euler factors removed).

This is possible only by working with very fine locally convex strict-LF topologies (or ensuing bornologies, as he does) and indeed his method of degeneration from global to S-local pans out, and leads to a trace formula in the vein that Connes was partly hoping for.

Such topologies, however, render *all* point-masses (and their derivatives) at all zeros of the Hecke L-function, continuous, and so the resulting trace formula is just a copy of Weil's own Explicit Formulas. This continuity aspect concerning the point-masses and their derivatives at all zeros, is what makes all the zeros (whatever their location and multiplicity may be) contribute to the spectral side of the trace formula on equal footing.

Now look at the situation with our chosen Schwartz-Bruhat function. We choose an enumeration $\{v_k\}_{k\geq 0}$ of all the places of F outside S.

Assume we have fixed a continuous (unitary) character $\chi \in \widehat{C_F}$, and that S contains all the Archimedean places, all the primes at which χ is ramified, and all the ramified places of F over $\mathbf{Q}$ (for other technical reasons it is also important in the main work to make S big enough to ensure $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, but the need for this isn't addressed here). Going from $f = f_S \otimes (\otimes_{v\notin S} 1_{\mathcal{O}_{F_v}})$ restricted to the idele group to

$$f' := f_S \otimes \otimes \bigotimes_{k\geq 0} 1_{\mathcal{O}^\times_{F_{v_k}}} \tag{1.8.1.1}$$

as a function on the idele group can either be done one step at a time, or infinitely many steps at a time. This latter is what Meyer does, roughly speaking.

Each time we replace $1_{\mathcal{O}_{F_{v_k}}}$ with $1_{\mathcal{O}^\times_{F_{v_k}}}$ for some k, the effect on the zeta integral of the Schwartz-Bruhat function on the connected component $\{\chi|\cdot|_F^s\}$ of χ is to remove the Euler factor at v_k, which is to say to multiply the zeta integral before such replacement by

$$1 - \chi(\mathfrak{p}_{v_k}) q_{v_k}^{-s}, \tag{1.8.1.2}$$

and so an infinite iteration of this process will converge as an operator (on the quotient of all $\chi|_{C_F^1}$-homogeneous adelic θ-functions) only when we endow the function space with a topology that is matched under Mellin transform by a topology at least as fine as the topology of uniform convergence on all vertical strips of bounded width of $\mathbf{C}_{>1} = \{\sigma > 1\}$ for $Z(f, \chi|\cdot|_F^s)$, and of $\mathbf{C}_{<0} = \{\sigma < 0\}$ for $Z(\widehat{f}, \overline{\chi}|\cdot|_F^{1-s})$.

This is, loosely, what Meyer does: he degenerates a global convolution operator using infinite Euler products, makes a trace calculation following Connes at the S-local level, and observes that by enlarging S by a finite amount the "part of the kernel" that he has left behind in the degeneration process contributes zero to the global trace. Convergence of infinite Euler products is crucial for him to prove this last fact.

The obstruction to repeating what Meyer does, but by working in normed completions such as the L^2-norm topology on $\mathcal{S}(C_F)$, is that infinite Euler products do not converge absolutely within the critical strip, by Mertens' Theorem.

The advantage of proving a global trace formula while working with respect to much coarser normed topologies as my work does, as opposed to the very fine limit-Fréchet topologies or bornologies that Meyer has to work with to make infinite Euler product operators converge, is enormous because the spectral side of such a global trace formula is affected by the choice of topology: not all zeros support point-masses (and their derivatives) that extend by continuity!

Mertens' Theorem is, in my eyes, a herald for GRH. If you clash against it, it means that you are entering forbidden territory: it means you have to come up with something new.

What occurred to me is the following: what if instead of doing this wished-for degeneration process from global to S-local by using an infinite iteration of $1_{\mathcal{O}_{F_{v_k}}} \mapsto 1_{\mathcal{O}_{F_{v_k}}^\times}$ to reach (1.8.1.1) (equivalently, to infinitely iterate multiplication by terms as in (1.8.1.2)), which is what Meyer does (I am paraphrasing a lot for him), *we did it one step at a time only?*

What if there is a way to keep track of the loss of information, *step by step*, instead of going all the way down straight away? And what if we could compute pieces of the trace step by step, until all that is left is at the "S-local level"? Imagine that f varies across all Schwartz-Bruhat functions on the global adeles (remove global poles by assuming $f(0) = \widehat{f}(0) = 0$) and follow me into the following toy example.

EXAMPLE 1.8.1.1. Denote by $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ the nuclear Fréchet space of *exponentially decaying* Schwartz-Bruhat functions on $\mathbf{R}_{>0}$, i.e. those $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ such that $|\cdot|^a\phi \in \mathcal{S}(\mathbf{R}_{>0})$ for every $a \in \mathbf{R}$. It is topologized in the evident manner with its projective locally convex topology, and the Mellin transform $M(\phi) := \int_{\mathbf{R}_{>0}} \phi(t)t^s \frac{\mathrm{d}t}{t}$ identifies it linearly and topologically with what I call the Schwartz-Hardy space $A\mathcal{S}(\mathbf{C})$ of those entire functions which are Schwartz uniformly on vertical strips of bounded width, topologized with the uniform Schwartz topology (equivalently, by the Residue Theorem, with the topology of uniform convergence on vertical strips of bounded width for functions and their products with all monomials s^N in the holomorphic s-variable).

Pick $\gamma > 1$ (think of the norm $N(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$). The space $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) = C^\infty(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ carries an absolute Schauder basis, namely the exponential system

$$\exp\left(\frac{2\pi i m}{\log(\gamma)}\right).$$

The dual basis is the set of functionals given by integration against the conjugate exponential function, i.e. calling $z_m := \frac{2\pi i m}{\log(\gamma)} \in i\mathbf{R}$:

$$\delta_{z_m} : \phi \mapsto \frac{1}{\log(\gamma)} \int_{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}} \phi(t) e^{-\frac{2\pi i m \log(t)}{\log(\gamma)}} \,\mathrm{d}\overline{\mu}(t)$$

where $\overline{\mu}$ is the Haar measure induced on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ by the Haar measure $\mathrm{d}t/t$ on $\mathbf{R}_{>0}$ and the counting measure on $\gamma^{\mathbf{Z}}$.

We now recall that the periodization operator defined on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by

$$\Phi_\gamma(\phi)(x) := \sum_{k \in \mathbf{Z}} \phi(\gamma^k x)$$

identifies $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ with the quotient $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/\ker(\Phi_\gamma)$, i.e.

$$0 \to \ker(\Phi_\gamma) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \to 0$$

is linearly and topologically exact. The Mellin transform $M : \phi \mapsto \int_{\mathbf{R}_{>0}} \phi(t)t^s \frac{\mathrm{d}t}{t}$ now identifies the above sequence with

$$0 \to (1-\gamma^{-s})A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C}) \to 0$$

therefore characterizing $\ker(\Phi_\gamma)$ in terms of the image of the Mellin transform in the Schwartz-Hardy space $A\mathcal{S}(\mathbf{C})$. The left term in the exact sequence above is reminiscent of a reciprocal Euler factor at a non-Archimedean place.

Now note that $\mathbf{R}_{>0}$ acts on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, on $\ker(\Phi_\gamma)$ and on the subquotient $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ by left-scaling by the inverse. This is then turned into a convolution action of $\mathcal{D}(\mathbf{R}_{>0})$. It turns out that this convolution action sending ϕ to $A(\varphi)(\phi) := \int_{\mathbf{R}_{>0}} \varphi(t)\phi(t^{-1}x)\,\frac{\mathrm{d}t}{t}$ is nuclear on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ and the trace of $A(\varphi)$ on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is computed using the Schauder basis $\{\delta_{z_m}\}$.

Upon computing what the (adjoint) scaling action on each δ_{z_m} is, you get that it is by multiplication by $t^{\frac{2\pi i m}{\log(\gamma)}}$ and so the trace of $A(\varphi)$, which equals the trace of the transpose $A(\varphi)'$, is the sum of the eigenvalues:

$$\sum_{m\in\mathbf{Z}} M(\varphi)(z_m).$$

By the Poisson Summation Formula, this equals

$$\log(\gamma)\sum_{k\in\mathbf{Z}} \varphi(\gamma^k).$$

In particular, if the support of φ is strictly contained within $(1,\gamma)$ then $\mathrm{Trace}\, A(\varphi) = 0$ on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$.

This toy example relates to the situation described in the discussion preceding it as follows. For $f \in \mathcal{S}(\mathbf{A}_F)$ (say satisfying $f(0) = \widehat{f}(0) = 0$), we have

$$Z(f', \chi|\cdot|_F^s) = (1 - \chi(\mathfrak{p}_{v_0})q_{v_0}^{-s})Z(f, \chi|\cdot|_F^s) \tag{1.8.1.3}$$

for

$$f = f_S \otimes \bigotimes_{v\notin S} 1_{\mathcal{O}_{F_v}} \text{ and } f' := f_S \otimes 1_{\mathcal{O}_{F_{v_0}}^\times} \otimes \bigotimes_{k\geq 1} 1_{\mathcal{O}_{F_{v_k}}}. \tag{1.8.1.4}$$

We can therefore measure the loss of information of the effect of replacing f with f', and more broadly of replacing $1_{\mathcal{O}_{F_v}}$ with $1_{\mathcal{O}_{F_v}^\times}$ for each place v outside S one at a time, using a Schauder basis of continuous functionals given by the point-masses at the zeros of the reciprocal local Euler factors

$$1 - \chi(\mathfrak{p}_{v_k})q_{v_k}^{-s}$$

for each $k \geq 0$. The trace calculation outlined in the toy example with large $\gamma = N(\mathfrak{p}_{v_k})$ finds its analog (incorporating twists by characters systematically), and depending on the *fixed* choice of test function $\varphi \in \mathcal{D}(C_F)$ (we also take S big enough depending on the support of such a fixed φ as well) these "partial traces" will be zero!

This is the key idea: follow the rough strategy of "degenerating a global convolution kernel from global to S-local", but instead of doing it using infinite Euler product operators, as Meyer instead does (paraphrasing), doing it step by step using Schauder bases of point-masses at the zeros of reciprocal local non-Archimedean Euler factors.

Shift the burden of the degeneration process a little onto the function space, as opposed to the Euler product operators of Meyer's, thus replacing this step in Meyer's trace formula with something that can work in coarser topologies too.

At this point the question is: can we make the calculation of the geometric side of the Trace Formula, while working solely inside a norm completion, such as those alluded to coming out of the Hardy-Sobolev norms from (1.8.2.2)?

The answer is "yes".

Long story short, the Hardy-Sobolev norms give us all we need to find, after completion, "enough" of the Schauder bases to pull off the step-wise deconstruction I have alluded to above (and specifically to write down

explicitly "pieces" of a global convolution kernel as trace-norm limits of finite rank kernels constructed by acting on basis elements), and reduce the calculation of the geometric side of the trace formula to an analog of the S-local trace calculation put forth by Connes, Meyer and, before them, Weil (in a way).

A version of the idea of the bases of point-masses at the zeros of reciprocal non-Archimedean Euler factors to compute a "piece" of the total trace is what came to me very early on.

Upon recognizing its value, the writing of the paper got started, and became all-consuming. A full-time endeavor taking 12 years to put it all together through efforts to find the right context (spanning functional analysis, harmonic analysis, and analytic number theory) and run various sanity checks. Back then, I became concerned that you might view it as a pipe-dream, one of many, had I shared a form of the ideas before such time as a paper would be fully or almost fully written up, so we stayed at Stanford.

We will now take a closer look.

1.8.2. Notation. We briefly summarize here notation for certain function spaces whose study is undertaken in the paper, both in the main text and in some of the appendices.

Throughout this subsection only, we will use the notation $k(x,y)$ to mean that k is a function of the two variables x and y (perhaps only defined almost everywhere, with x, y varying over some appropriate measure space). This notation does not refer to the value of k at (x,y). It is adopted only through this subsection, for simplicity (it is not employed anywhere in the paper).

We will be using the following terminology: the *backward side* of the Mellin transform, to mean its source function space (according to the situation). The *forward side* of the Mellin transform, to mean its target space, according to the situation.

1.8.2.1. *Spaces of Schwartz-Bruhat functions.*

- $\mathcal{S}(G)$ the space of Schwartz-Bruhat functions on a Hausdorff locally compact abelian group G. We will be usually assuming G is second-countable. The examples of main interest are $G = F_v, F_v^\times, \mathbf{A}_F, \mathbf{A}_{F,S} = \prod_{v\in S} F_v$, $C_F = \mathrm{GL}_1(\mathbf{A}_F)/F^\times$, $J_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$.
- Assume G is a second-countable $\mathbf{R}_{>0}$-modulated group. For $I \subset \mathbf{R}$ a nonempty interval, $\mathcal{S}_I(G)$ is the space of *I-exponentially decaying* Schwartz-Bruhat functions, i.e. $\phi \in C^\infty(G)$ such that $|\cdot|_G^a \phi \in \mathcal{S}(G)$ for all $a \in I$. We topologize $\mathcal{S}_I(G)$ with the locally convex inductive limit topology of the nuclear Fréchet topologies on $\mathcal{S}_I(k,G)$ for $k \subset G$ varying over the compact subgroups of G whose orthogonal $k^\perp \subset \widehat{G}$ is compactly generated. The topology on $\mathcal{S}_I(k,G)$ is the subspace topology induced by the linear injection

$$T : \mathcal{S}_I(k,G) \to \prod_{a\in I} \mathcal{S}(k,G) \tag{1.8.2.1}$$

  sending ϕ to $(|\cdot|_G^a \phi)_{a\in I}$. This is also $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1/k)$.
- For $I = \mathbf{R}$, we denote $\mathcal{S}_I(G)$ by $\mathcal{S}_{\mathrm{E}}(G)$, and call this the space of *exponentially decaying Schwartz-Bruhat functions on G*.
- Under the above assumptions on G, $\mathcal{S}_I(G)$ is a strict LF-space. Nuclear (this is used to get good properties of topological tensor products and to get a handle on continuous duals of quotient linear maps, such that they are linear topological embeddings; nuclearity often ensures that quotient maps lift bounded sets, which is the crux of the matter behind the good behavior of the continuous duals of quotient maps), complete (used for vector valued integration), Montel (used for the Heine-Borel property: to get constants in some estimates that are uniform on bounded sets of functions).

A version of these spaces makes an appearance in Meyer's paper [**Me1**, §4.1], although only through their bornologies. Structural results of their topologies (criteria for sets to be bounded, kernel theorems to describe them in terms of the special case $G = \mathbf{R}_{>0}$, and especially the study of function spaces of entire functions which are Schwartz uniformly on certain vertical strips of bounded width, topologized accordingly so as to

promote the Mellin transform into a linear topological isomorphism, are discussed in one of the appendices to my paper, and never appeared in prior work).

The first to ever introduce an incarnation of these "weighted" spaces, was Tate, who, without every initiating a systematic study, introduces an "L^1-version" in [**CF**, Ch.XV,§4.4,p.338]. Meyer's is a "smooth version" of Tate's definition, with no topology but only retaining a good theory of bounded sets, and in my paper its topology is defined and studied, both directly and through the lens of the Mellin transform.

1.8.2.2. *S-local and global θ-functions.* Recall that we denote by:

- $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ the quotient of $\mathcal{S}(\mathbf{A}_{F,S})$ by the kernel of the θ-function operator $\Phi_{F,S}$
$$\Phi_{F,S}(f)(g) := \sum_{\xi \in \mathcal{O}_{F,S}^{\times}} f(\xi g),\ g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$$
for $f \in \mathcal{S}(\mathbf{A}_{F,S})$. We endow $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ with the quotient topology. As a consequence of the Structure Theorems (a total of five, which occupy two of the early chapters of my paper), this is, among the other properties, a strict LF-space. We refer to $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ as the *space of S-local θ-functions*.
- $\chi \in \widehat{C_F}$ the arbitrary continuous character of C_F, and by χ_S, for S a finite set of places of F containing all the Archimedean places, and such that χ is unramified outside S, the unique continuous character of $J_{F,S}$ such that the composition of χ_S by the quotient
$$(\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}_{\Sigma_F - S}^{\times})/\mathcal{O}_{F,S}^{\times} \to J_{F,S}$$
is the restriction of χ. If $\chi|_{C_F^1} = 1$, we will be using $\chi = 1$ as unitary base point for the connected component $\Omega(C_F)(\chi)$ of the Riemann surface $\Omega(C_F)$, rather than $\chi = |\cdot|_F^{ix_0}$ for some $x_0 \in \mathbf{R}$.
- $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ the quotient of $\mathcal{S}(\mathbf{A}_F)$ by $\ker(\Phi_F)$, with Φ_F the global θ-function operator defined by
$$\Phi_F(f)(g) := \sum_{\xi \in F^{\times}} f(\xi g),\ g \in \mathrm{GL}_1(\mathbf{A}_F).$$
As a consequence of the Structure Theorems, $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ is itself a strict LF-space. We refer to $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ as the *space of global θ-functions*.
- If $f(0) = \widehat{f}(0) = 0$, then $\Phi_F(f)$ is an exponentially decaying Schwartz-Bruhat function on C_F, i.e. for any integer $N \geq 1$, $|\Phi_F(f)(g)| \ll e^{-N|\log|g|_F|}$ as $|\log|g|_F| \to +\infty$. We denote the closed subspace $\ker(\delta_0) \cap \ker(\widehat{\delta_0})$ by $\mathcal{S}_0(\mathbf{A}_F)$, i.e. the closed subspace of those global adelic Schwartz-Bruhat functions f such that $f(0) = \widehat{f}(0) = 0$. We denote by $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}_0(\mathbf{A}_F))$ its image in $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ (itself a closed subspace, because both δ_0 and its adelic Fourier transform are $F^{\times}$-invariant.
- We denote by $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J_{F,S}^1}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F))$ the isotypes of the spaces of θ-functions. These are nuclear Fréchet spaces. All equipped with a continuous left-translation action of $J_{F,S}$, respectively C_F, and hence a convolution action by the elements of $\mathcal{D}(J_{F,S})$, respectively $\mathcal{D}(C_F)$.

We omit further info on the Structure Theorems. We only mention here that they are mostly about the spaces of S-local θ-functions, and then known properties of Hecke L-functions propagate the properties of the S-local spaces to the global ones (for example, the convexity bound, giving at-most-polynomial growth for the uncompleted Hecke L-function uniformly over vertical strips of bounded width).

One of the consequences of the Structure Theorems is that, for $\chi \in \widehat{C_F}$ fixed, and S large enough (finite and containing all the Archimedean places, containing all the places at which χ is ramified, and satisfying

$\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. No condition on S containing the places of F ramified over $\mathbf{Q}$), the S-local to global operator:

$$p_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

is a linear topological isomorphism (compatible with continuous actions of $C_{F,S} := (\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^\times_{\Sigma_F - S})/\mathcal{O}^\times_{F,S}$ on source and target. On the source, the $C_{F,S}$-action factors through a continuous action of $J_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}^\times_{F,S}$ by invariance).

In a nutshell, within the $\chi|_{C^1_F}$-isotype $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$, all elements ϕ can be written as the image of some element in $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ of the special form:

$$\Phi_F\left(f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}\right)$$

for some $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$.

1.8.2.3. *Hardy-Sobolev spaces.* Let $I \subset \mathbf{R}$ be a compact interval of the form $I = [-a, a]$ for $a \in \mathbf{R}_{>0}$ (we omit the general case of an arbitrary compact interval I).

For G a second-countable Hausdorff locally compact abelian group and an $\mathbf{R}_{>0}$-modulated group, we denote by

$$W^{b,2}_I(G)$$

the completion of $\mathcal{S}_{\mathrm{E}}(G)$ (since I is compact, it is also the completion of $\mathcal{S}_I(G)$) with respect to the linear topology generated by the norm:

$$\|\phi\|_{a,b} := \left(\int_G |\phi(g)|^2 \max\{|g|_G^{-2a}, |g|_G^{2a}\}(1 + |\log|g|_G|^2)^{b/2}\, \mathrm{d}\mu(g)\right)^{1/2} \tag{1.8.2.2}$$

with μ a fixed choice of Haar measure on G.

The study of these Hilbert spaces, even in the special case $b = 0$, makes no appearance in the literature. In particular, they never featured in work of Connes nor Meyer's.

We refer to $W^{b,2}_I(G)$ as the *Hardy-Sobolev space* of G with base I.

If $I = \{0\}$, then $W^{b,2}_I(G)$ is referred to as the Sobolev space $W^{b,2}(G)$ of Sobolev degree b in the paper (in this special case, i.e. for $a = 0$, indeed the Hilbert spaces $W^{b,2}(G)$ appeared, for $G = C_F$ the idele class group of a number field F, in Connes' paper from 1998, and specifically in [Co,]. They are, in the end, never used in what he was proposing. In fact, on [Co,] Connes declares his intent of devoting much of this paper to the task of getting rid of the Sobolev weight $(1 + |\log|g|_F|^2)^{b/2}$, b denoted δ therein). The extremal case $a = 0$ is never used in my paper, with the sole exception of outlining at the end of the paper the sanity check involving the multiplicities of the zeros, which I have mentioned earlier in §4 of this letter.

Assume $b \geq 0$ is an even integer for simplicity. Then the topology on $W^{b,2}_I(G)$ is also generated by the finitely many semi-norms:

$$\|\phi\|_{\alpha,k} := \|\partial^k_z M(\phi)(z)\|_{L^2(\alpha + i\mathbf{R}, \mathrm{d}x)},\ \alpha = -a, a \tag{1.8.2.3}$$

for all $0 \leq k \leq b/2$, i.e. by integration of the Mellin transform $M(\phi)$ and some of its holomorphic derivatives along the edge of the vertical strip $\mathbf{C}_I$ of bounded width $2a$.

1.8.3. The route to the Trace Formula. Let $L(s, \chi)$ be the completed Hecke L-function of $\chi \in \widehat{C_F}$. We denote by $L_f(s, \chi)$ the finite Hecke L-function, i.e. for $s \in \mathbf{C}_{>1}$:

$$L_f(s, \chi) = \prod_{v \in \Sigma_{F,f}} L_v(\chi_v| \cdot |_v^s)$$

where $\Sigma_{F,f}$ is the set of all finite places of F.

We outline some of the main ingredients behind the Trace Formula that is the main subject of the paper.

1.8.3.1. *Fundamental Divisor Exact Sequence.* Consider the following situation, initially envisioned by Connes, but, contrary to what Connes does (who works, instead, with L^2-completions throughout) working with the Hausdorff locally convex topological $\mathbf{C}$-vector spaces:

$$0 \to |\cdot|_F^{1/2} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F) \to \mathscr{H}_F \to 0$$

and each of their C_F^1-isotypes, i.e.

$$0 \to |\cdot|_F^{1/2} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \to \mathscr{H}_F(\chi|_{C_F^1}) \to 0 \tag{1.8.3.1}$$

these latter being nuclear Fréchet spaces.

The presence of the quasi-character $|\cdot|_F^{1/2}$ in front of the space of global adelic θ-functions means the following: we consider $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ as a subspace of $\mathcal{S}_{\mathrm{E}}(C_F)$. It is a *closed* subspace, and a continuous sub-representation of C_F. This is a consequence of the five Structure Theorems alluded to earlier. The space $|\cdot|_F^{1/2} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is the image of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ under the multiplication operator by the quasi-character $|\cdot|_F^{1/2}$ (this multiplication operator is a linear topological automorphism on $\mathcal{S}_{\mathrm{E}}(C_F)$. The Mellin transform turns it into a shift by $1/2$ to the left, on the level of entire functions, Schwartz uniformly on vertical strips of bounded width).

We parametrize the Riemann surface Ω of all idelic quasi-characters as $\{\chi|\cdot|_F^s\}$ as χ varies over a choice of unitary base point for the connected components of Ω (which we choose to be the of finite order the connected component contains a quasi-character with finite image).

The Mellin transform turns the second exact sequence into:

$$0 \to L(s+1/2,\chi)((s+1/2)(s-1/2))^\delta \mathcal{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C}) \to V_{\chi|_{C_F^1}} \to 0 \tag{1.8.3.2}$$

where $N(\rho)$ is the multiplicity of ρ as a zero of $L(s,\chi)$. Here $\delta = 0$ if $\chi|_{C_F^1} \neq 1$, and $\delta = 1$ otherwise.

The quotient $V_{\chi|_{C_F^1}}$ carries a Schauder basis satisfying several good properties: its dual basis is given by the point-masses at the zeros of $L(s+1/2,\chi)$ and their derivatives.

This connection to the L-function requires all the Structure Theorems, and use of known analytic properties of the L-function (convexity bound, existence of a sequence of horizontal segments across the critical strip on which the reciprocal of $L_f(s,\chi)$ is defined and grows at most polynomially in the absolute ordinate. In the process of describing the image of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ under the Mellin transform, we must also draw from Bohr's theory of almost periodic functions, taken to a fuller expression in Bagchi's Ph.D. thesis, to prove that finite Euler products of non-Archimedean local factors admit a sequence of horizontal segments whose ordinates tend to $\pm\infty$ respectively, and on which they are bounded, in order to perform asymptotic contour integration. This can also be done using the Kronecker-Weyl Theorem).

The sequence (1.8.3.2) is the reason why I titled this subsection the "divisor exact sequence": (1.8.3.2) mirrors the global sections of

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

for the Stein space $X = \mathbf{C}$, and $D = \mathrm{div}(L(s+1/2,\chi)((s+1/2)(s-1/2))^\delta)$.

1.8.3.2. *Normed completions.* One of the key steps on the way to the Trace Formula, is to pass the sequences (7.2.0.1) and (1.8.3.2) (linearly and topologically exact) through the lens of coarser normed completions.

Why Hardy-Sobolev completions, specifically? They do something for us: they make point-mass functionals supported on the imaginary line, together with their derivatives of arbitrary order, continuous. To this end, giving ourself a little bit of interior around the imaginary line (which is what the Hardy-Sobolev norms do, if

you rethink them through the Mellin transform, i.e. you re-interpret the topology they, each, define, as also obtained by a finite family of seminorms defined by integration along the edges of the strip $\mathbf{C}_I$ as in (1.8.2.3)). This feature of the Hardy-Sobolev normed topologies (promoting point-masses supported on the imaginary line along with their derivatives, to continuous functionals with respect to such normed topologies) provides us with one of the ingredients that we necessitate, in order to make the calculation of the *Geometric Side* of the Trace Formula (note that the zeros are playing no role in how we select such topologies. It will be the Geometric Side of the Trace Formula, to dictate what such normed topologies must do for us, in order for the calculations to work within such normed environments too).

We will briefly return to these points later. Ultimately, we will be doing the following:

(1) Call $I := [-a, a]$. Take a Hardy-Sobolev completion of all spaces in the above linearly and topologically exact sequence of isotypes (7.2.0.1), to get:

$$0 \to W_I^{b,2}(|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))) \to W_I^{b,2}(C_F)(\chi|_{C_F^1}) \to W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1}) \to 0$$

where each space is now a Hilbert space.

(2) Now tame the continuous left-translation action of C_F on all such function spaces, into a convolution action by an arbitrary, fixed, test function $\varphi \in \mathcal{D}(C_F)$.

We are going to assume that φ is supported "strictly over $C_F^{\geq 1}$" (this will be because certain closed subspaces of $W_I^{b,2}(C_F)$ on which we will let the convolution kernel act, will only be stable under the action of the semigroup $C_F^{\geq 1}$ as opposed to C_F, and so this choice of support on φ lets C_F act only through its sub-semigroup $C_F^{\geq 1}$, as required), i.e. over $C_F^{>1+\varepsilon}$ for some small enough $\varepsilon > 0$.

(3) To fix ideas we choose $b = 0$ (which, for the purposes of my paper, is sufficient anyway). The rightmost space gives the spectral side of the trace formula, which reads

$$\sum_{\rho, I} N(\rho) M(\varphi) \left(\chi|\cdot|_F^{\rho - 1/2}\right) \tag{1.8.3.3}$$

where the sum is over those zeros ρ of $L(s, \chi)$ such that $\rho - 1/2$ is contained in the interior of $\mathbf{C}_I$. If $b > 0$, then zeros ρ such that $\rho - 1/2 \in \pm a + i\mathbf{R}$, i.e. located on the edges of $\mathbf{C}_I$ would also contribute to the above expression, but with a multiplicity function $N_b(\rho)$ defined as:

$$N_b(\rho) := \max\left\{N(\rho), \left[\frac{s-1}{2}\right]\right\}.$$

This is the trace of the effect of the convolution operator:

$$A(\varphi) : \psi \mapsto \int_{C_F} \varphi(g)\psi(g^{-1}x)\,\mathrm{d}\overline{\mu}(g)$$

on the quotient $\mathscr{H}_F$ of $\mathcal{S}_{\mathrm{E}}(C_F)$ by $|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and hence on its isotypes and their $W_I^{b,2}$-completions: this operator respects

$$W_I^{b,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

and hence descends to $W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1})$. We call $A_{\mathrm{spec},I}(\varphi)$ this last operator. It is nuclear, and the sum in (1.8.3.4) above is its trace.

Note that on $\mathscr{H}_F$ and $\mathscr{H}_F(\chi|_{C_F^1})$, the former a nuclear strict LF-space, the latter a nuclear Fréchet space, the convolution-by-φ operator is also nuclear, and its trace is identical to the "spectral side" (the part involving the zeros only) of Weil's Explicit Formulas, i.e. it is (on $\mathscr{H}_F(\chi|_{C_F^1})$ for

simplicity)

$$\sum_{\rho: L(\rho,\chi)=0} N(\rho) M(\varphi) \left(\chi|\cdot|_F^{\rho-1/2}\right). \tag{1.8.3.4}$$

You can already appreciate how the topology on the environment affects the spectral side of what will be the Trace Formula: working within the nuclear Fréchet quotient topology $\mathscr{H}_F(\chi|_{C_F^1})$ of the topology on $\mathcal{S}_{\mathrm{E}}(C_F)$, which in turn makes the space of global θ-functions a *closed* subspace with its own topology inherited from $\mathcal{S}(\mathbf{A}_F)$(!), all point-masses (and their derivatives) at *all* zeros, are continuous, and they offer dual Schauder basis elements, which can be used to write the transpose of the convolution-by-φ operator as a limit of explicit finite rank operators, to get its trace.

Working within the normed $W_I^{b,2}$-environment, instead, only those point-masses (and their derivatives) which extend by continuity, turn out to play this same role, and lead to nonzero contributions to the spectral side of the Trace Formula, which therefore takes the form in (1.8.3.4).

(4) The convolution operator $A(\varphi)$ on middle space $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ has $\varphi(xy^{-1})$ for a kernel, and since we are assuming φ is strictly supported over $C_F^{>1}$, this kernel (which is already defined along the diagonal) vanishes along the diagonal.

(5) I will be explaining face to face how to show that there is a measurable kernel function

$$k_{\mathrm{spec},I,\varphi} : C_F \times C_F \to \mathbf{C}$$

which:

(a) acts continuously and trivially (i.e. as zero) on the space of global θ-functions

$$|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \subset W_I^{b,2}(C_F)(\chi|_{C_F^1})$$

(b) is "smoothable along the diagonal" (this concept is defined using a version of the Hardy-Littlewood Maximal Theorem and the Lebesgue Differentiation Theorem, leveraging the trace class property of the convolution-by-φ operator on $W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1})$), such that, writing

$$\widetilde{k}_{\mathrm{spec},I,\varphi}(x,x)$$

for its smoothening along the diagonal (an almost-everywhere convergent average of the kernel itself, the convergence being ensured by the trace-class property. Here is where the Maximal Theorem is used) its mass along the diagonal (with respect to the Haar measure $\overline{\mu}$ on C_F) equals the trace from (1.8.3.4).

(c) Its effect, as an operator on $W_I^{b,2}(C_F)(\chi|_{C_F^1})$, when composed with the quotient onto $W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1})$, factors through the convolution-by-φ operator on $W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1})$.

The construction of $k_{\mathrm{spec},I,\varphi}$ is simple. It is about lifting basis elements from $W_I^{b,2}\mathscr{H}_F(\chi|_{C_F^1})$, making explicit estimates for singular values using the rapid decay of the Mellin transform and its derivatives, using basic facts about the number of zeros within a closed box cutting all across the critical strip at heights $\pm T$ being $O(\log(T))$, with the implied constant depending on the unitary base point. These are the same ingredients about the zeros that are already used to prove the Explicit Formulas. *Nothing more about the zeros will ever be needed* other than the non-vanishing of the Hecke L-function on the 1-line, which is used throughout the study of the Schauder bases on quotients of the spaces of θ-functions, to which I alluded to in the earlier sections.

The kernel function $k_{\mathrm{spec},I,\varphi}$ turns out to be, as a consequence of its construction and the estimates, an element of $L^2(C_F \times C_F, \nu \otimes \nu^\vee)$, where

$$\nu = \max\{|\cdot|_F^{-2a}, |\cdot|_F^{2a}\}(1+|\log|\cdot|_F|^2)^{b/2}\,\overline{\mu}, \quad \nu^\vee = \frac{1}{\max\{|\cdot|_F^{-2a}, |\cdot|_F^{2a}\}(1+|\log|\cdot|_F|^2)^{b/2}}\,\overline{\mu}. \tag{1.8.3.5}$$

(6) From (3),(4) and (5), we can show that there exists a measurable function

$$k_{\mathrm{geom},I,\varphi} : C_F \times C_F \to \mathbf{C}$$

which we will be referring to as the *geometric kernel*, which is also smoothable along the diagonal, whose smoothening $\widetilde{k}_{\mathrm{geom},I,\varphi}(x,x)$ (notation to mean it is a function defined almost everywhere on C_F) along the diagonal is integrable with respect to the Haar measure $\overline{\mu}$, and its mass satisfies the "proto-Trace Formula":

$$\int_{C_F} \widetilde{k}_{\mathrm{geom},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\int_{C_F} \widetilde{k}_{\mathrm{spec},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\sum_{\rho,I} N(\rho)M(\varphi)\left(\chi|\cdot|_F^{\rho-1/2}\right). \tag{1.8.3.6}$$

The effect of the kernel function $k_{\mathrm{geom},I,\varphi}$ on the elements of $|\cdot|_F^{1/2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ itself, turns out to agree with the convolution action by φ inherited from $\mathcal{S}_{\mathrm{E}}(C_F)$. To convey what $k_{\mathrm{geom},I,\varphi}$ is, I will say this: it turns out that, upon calling $A_I(\varphi)$ the convolution-by-φ operator on $W_I^{b,2}(C_F)(\chi|_{C_F^1})$ and by P_θ the orthogonal projection within $W_I^{b,2}(C_F)(\chi|_{C_F^1})$ onto the closure

$$W_I^{b,2}(|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)))$$

of $|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, it turns out that the operator $A_I(\varphi)\circ P_\theta$ is an integral operator (although neither trace class, nor Hilbert-Schmidt), and $k_{\mathrm{geom},I,\varphi}$ is a representative for its integral kernels (this is all accomplished by means of simple, direct constructions).

The entire work is about computing pieces of $k_{\mathrm{geom},I,\varphi}$ explicitly, to compute the mass along the diagonal of the smoothening $\widetilde{k}_{\mathrm{geom},I,\varphi}$ *in a different way*, i.e. other than the one provided by (1.8.3.6).

1.8.3.3. *Divisorial stratification of the ideles.* The next step now is to split $k_{\mathrm{geom},I,\varphi}(x,y)$ into an infinite sum of pieces indexed by the places of F:

$$k_{\mathrm{geom},I,\varphi}(x,y) = k_{S,\varphi}(x,y) + \sum_{p\nmid S} k_{p,\varphi}(x,y)$$

and let the pieces $k_{p,\varphi}(x,y)$ act on certain subspaces of $W_I^{b,2}(|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)))$ to understand them.

This subsection is aimed at conveying the main ideas on how to accomplish this, i.e. how to split the geometric kernel.

We start by identifying an interesting stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$, which is found only on $\mathrm{GL}_1(\mathbf{A}_F)$ as opposed to its quotient C_F, since it is defined by leveraging crucially the restricted product nature of $\mathrm{GL}_1(\mathbf{A}_F)$.

This stratification will adapt itself nicely to the structure of the Schwartz-Bruhat functions representing the global adelic θ-functions in

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)).$$

Recall that there exists a finite set of finite places S of F satisfying the following conditions:

(1) S contains all the ramified places of F and all the Archimedean places.
(2) S contains all the places of F at which χ is ramified.
(3) $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

such that any such θ-function ϕ can be realized as being of the form, as a function of $x \in C_F$:

$$\phi(x) = \Phi_{F,\chi|_{C_F^1}}(f)(x)$$

where

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}} \tag{1.8.3.7}$$

for some $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$.

This is the last of the five Structure Theorems, proved at the end of Chapter 5 of this work.

The stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ that we are about to define will weld itself perfectly well with the non-Archimedean "transversal" (i.e. tensor component) $\bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$ which all the global Schwartz-Bruhat functions on $\mathbf{A}_F$ representing the $\chi|_{C_F^1}$-homogeneous global θ-functions can be made to share in common.

The guiding principle is simple: since $\mathcal{O}_{F_v}^\times \subset F_v^\times$ is compact and open, hence so it is in F_v, and we have

$$1_{\mathcal{O}_{F_v}^\times} \cdot 1_{\mathcal{O}_{F_v}} = 1_{\mathcal{O}_{F_v}^\times}$$

as Schwartz-Bruhat functions on F_v. Under the local Mellin transform, $Z_v(1_{\mathcal{O}_{F_v}^\times}, c)$ and $Z_v(1_{\mathcal{O}_{F_v}}, c)$ are related by "Weil's Identity":

$$\mu_v(\mathcal{O}_{F_v}^\times) \cdot Z_v(1_{\mathcal{O}_{F_v}^\times}, \chi_v|\cdot|_v^s) = L_v(\chi_v|\cdot|_v^s)^{-1} Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$$

as meromorphic functions on $\Omega(F_v^\times)(\chi_v)$.

Multiplication by $1_{\mathcal{O}_{F_v}^\times}$ of $1_{\mathcal{O}_{F_v}}$ has the effect of replacing the local Z-function $Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$ with the product by the reciprocal local Euler factor $L_v(\chi_v|\cdot|_v^s)$.

Note also that the infinite product of the volumes $\mu_v(\mathcal{O}_{F_v}^\times)$ over all the finite places v of F is a finite product (these volumes are 1 for all but finitely many v's) and therefore converges to the square root of the absolute different.

We define

$$\rho_v := \bigotimes_{w \in S} 1_{F_w} \otimes \bigotimes_{w \notin S, w \neq v} 1_{F_v} \otimes 1_{\mathcal{O}_{F_v}^\times}.$$

This is a smooth function on $\mathrm{GL}_1(\mathbf{A}_F)$ (since $\mathcal{O}_{F_v}^\times \subset F_v$ is a compact open subset).

We define, for any finite subset T of finite places of F such that $S \cap T = \varnothing$:

$$\rho_T := \prod_{v \in T} \rho_v. \tag{1.8.3.8}$$

For any increasing chain of finite subsets T_k of finite places of F, as $k \geq 0$ varies over the nonnegative integers, and such that $T_{k+1} - T_k$ contains only one place, then the functions $\{\rho_{T_k}\}_{k \geq 0}$ define on $\mathrm{GL}_1(\mathbf{A}_F)$ a stratification by open and closed subsets, which we call the **divisorial stratification**.

Each stratum is:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : \rho_{T_k}(g) = 1\} = \{g \in \mathrm{GL}_1(\mathbf{A}_F)_{T_k} : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k\}$$

and the complement to each stratum into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} - \mathrm{GL}_1(\mathbf{A}_F)_{T_{k+1}} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in T_{k+1} - T_k\}.$$

At last, we will term the **primary divisorial stratification** the stratification $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ where we assume:

(1) S is saturated (it contains all the finite places of F of a given residue characteristic, if it contains one of them).
(2) Each finite set of finite places P_k is saturated, and obtained as a union of finitely many of the T_m's.
(3) $\{P_k\}_{k \geq 0}$ is the finest (under the refinement partial ordering) among such stratifications.

More directly, we define P_k to be the pre-image under the map $|\mathrm{Spec}(\mathcal{O}_F)| \to |\mathrm{Spec}(\mathbf{Z})|$ of the subset T_k, and therefore the stratum $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$ of the kth divisorial stratum on $\mathrm{GL}_1(\mathbf{A_Q})$. Explicitly, we have

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in \bigcup_{p \in T_k} S_p\}$$

where T_k denotes, here, the finite set of the first k rational primes. In other words, $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image of the kth stratum $\mathrm{GL}_1(\mathbf{A_Q})_{T_k}$ of the divisorial stratification on $\mathrm{GL}_1(\mathbf{A_Q})$ under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$.

The complement to each stratum of $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} - \mathrm{GL}_1(\mathbf{A}_F)_{P_{k+1}} = \bigcup_{Q \subset P_{k+1} - P_k} \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in P_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in Q\}$$

where Q varies over all nonempty finite subsets of $P_{k+1} - P_k$.

The primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$ adapts itself nicely to the Schwartz-Bruhat functions f on $\mathbf{A}_F$ of the form in (11.3.1.1). Indeed, upon calling:

$$\alpha_k := 1_{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}}$$

the indicator function of the kth stratum of the primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$, we have

$$\alpha_k \cdot (f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}) = f_S \otimes \bigotimes_{v \in S - P_k} 1_{\mathcal{O}_{F_v}} \otimes \bigotimes_{v \in P_k} 1_{\mathcal{O}_{F_v}^\times}$$

and hence multiplication by α_k replaces the Zeta function of f with the product of $Z(f, \chi|\cdot|_F^s)$ by the finitely many reciprocal Euler factors of $L(s, \chi)$ at the non-Archimedean places v of F indexed by P_k (the reciprocal of the product of the volumes of the local units is 1 because S is now assumed to contain all the ramified places of F!).

We call such multiplication process "restriction" from one stratum to the next, of the primary divisorial filtration.

In particular, and this is the main observation of this subsection: **the $(k+1)$st iteration of the restriction to the strata of the primary divisorial filtration is obtained from the kth, on the forward side of the Mellin transform, by multiplication by**

$$\prod_{v \in S_{p_{k+1}}} L_v(\chi_v|\cdot|_v^s)^{-1}.$$

Now recall the main idea: "degenerate" a global adelic θ-function into an S-local one, by representing it as the image of a global Schwartz-Bruhat function

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$$

and then replacing each $1_{\mathcal{O}_{F_v}}$ by $1_{\mathcal{O}_{F_v}^\times}$. Here S is large enough (it satisfies the assumptions we stated so far throughout this subsection, and it is fixed). Note that replacing $1_{\mathcal{O}_{F_v}}$ by $1_{\mathcal{O}_{F_v}^\times}$ has the effect of sending f to the function of $x \in \mathbf{A}_F$:

$$x \mapsto f(x) - f(\pi_v^{-1} x)$$

with π_v any choice of uniformizer at the non-Archimedean place v. We repackage this operation into an operator PV_v, which turns out to descend to the space of global adelic θ-functions.

We denote by PV_{P_k} the composition of the PV_v as v varies over the places in P_k.

If we let f (equivalently f_S) vary, we obtain that all the resulting global adelic θ-functions restricted to the kth stratum are contained in

$$\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$

and all the global adelic θ-functions restricted to the $(k+1)$st stratum are contained in the yet smaller and closed subspace

$$\mathrm{PV}_{S_{p_{k+1}}}\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))=\mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)).$$

But now the subquotient:

$$\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) \tag{1.8.3.9}$$

carries a Schauder basis that is analogous to the exponential system from the toy example (see Example 1.8.1.1)!

Long story short:

- Using the indicator functions α_k, $k\geq 0$ of the strata of the divisorial filtration on the global ideles, we chop the kernel function $k_{\mathrm{geom},I,\varphi}$ along the strata of the primary divisorial filtration on $\mathrm{GL}_1(\mathbf{A}_F)$. Consider the successive differences

  $$(\alpha_{k+1}-\alpha_k)k_{\mathrm{geom},I,\varphi}$$

  (this will be what we called $k_{p_k,\varphi}(x,y)$) and write them down explicitly by letting them act on (lifts of the) Schauder basis elements on the subquotients (11.3.1.3), upon incorporating the effect of multiplication by $|\cdot|_F^{1/2}$ (here I am omitting several items. The subquotients used to this effect incorporate a support condition on the Fourier transform of the comprising global Schwartz-Bruhat functions f. They carry such Schauder bases because they are all realized as copies of twisted forms of the exponential system itself, and we can always arrange such support condition to hold (by compactness of the quotients $C_F/\pi_v^{\mathbf{Z}}$ for each $v\notin S$, in a nutshell). The version of the subquotients from (11.3.1.3) incorporating such support condition is termed *Paley-Wiener Levels* in the paper. The support condition aims at making multiplication by $|\cdot|_F^a$ for any $a>0$, continuous on the closure of the subspace generated by the Schwartz-Bruhat functions f satisfying such support condition on their Fourier transform. We are going to omit all of this from this overview. These elements are not difficult, and they are quite natural: they only make for a very long story).
- Smoothen the "partial kernels" $k_{p_k,\varphi}$ along the diagonal (using a version of the Hardy-Littlewood Maximal Theorem) and compute their own mass along the diagonal, by using their explicit form as a limit of finite rank operators, obtained by using the basis.

  The answer we get is:

  $$\sum_{\varepsilon_p}N(\chi|\cdot|_F^{\varepsilon_p})\int_{C_F}\varphi(g)\chi(g)|g|_F^{\varepsilon_p-1/2}\,\mathrm{d}\overline{\mu}(g)$$

  where the sum is over the complex numbers ε_p such that $\chi|\cdot|_F^{\varepsilon_p}$ is a zero of the entire function $\prod_{v\in S_p}L_v^{-1}$ (we call $N(\chi|\cdot|_F^{\varepsilon_p})$ its multiplicity) with $\chi|\cdot|_F^{\varepsilon_p}$ contained in the connected component $\Omega(C_F)(\chi|_{C_F^1})$.

Once again (as in the toy example) the Poisson Summation Formula also gives us that such trace equals:

$$\sum_{v\in S_p}\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\sum_{k\in\mathbf{Z}}\varphi(\pi_v^k)$$

and so if φ is supported strictly over $C_F^{(1,|\pi_v^{-1}|_v)}$ then this gives zero!

If φ is fixed and compactly supported over $(1,+\infty)$, then outside of finitely many places v, indeed φ will be supported on $C_F^{(1,|\pi_v^{-1}|_v)}$ because $|\pi_v^{-1}|_v = q_v$ grows to $+\infty$ as the residue characteristic of p grows, and so for all but finitely many v, the compact support of φ will be contained in the relatively compact regions $C_F^{(1,|\pi_v^{-1}|_v)}$.

In other words, for all but finitely many finite places v of F, the "partial kernels" $k_{p,\varphi}$ contribute zero to the mass of the geometric kernel.

- We are left with a "remnant" of the global kernel $k_{\mathrm{geom},I,\varphi}$ entirely "supported" (meaning to be explained, in terms of the effect of the various kernels on the global adelic θ-functions) on

$$(\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}) \times \mathrm{GL}_1(\mathbf{A}_F)$$

and invariant under $\widehat{\mathcal{O}}^{\times}_{\Sigma_F-S}$ in the first variable (which is by convention the y-variable, i.e. it is listed as second). This, we will call $k_{S,\varphi}$, and we will be computing, by means of an analog of (1.3.1.12) and using calculations very close to those Connes performed in his "S-local trace formula" from his paper from 1998, reprised by Meyer, its diagonal mass, leading to the Trace Formula which is the main subject of this work.

In light of this, the main Trace Formula now reads as follows:

$$\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\int_{C_F} k_{\mathrm{spec},I,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x) = -\sum_{\rho,I} N(\rho)M(\varphi)\left(\chi|\cdot|_F^{\rho-1/2}\right).$$

In turn, $\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x)$ is then computed using a version of the Connes-Meyer "S-local trace formula", discussed in their papers.

Reducing to the S-local calculation of $\int_{C_F} k_{S,\varphi}(x,x)\,\mathrm{d}\overline{\mu}(x)$ is what this work is all about. Once the reduction is made, the problem is solved.

The ingredients in the S-local calculation (as present already in the papers of Connes and Meyer) are:

(1) The S-adelic Fourier transform: we need to transform $k_{S,\varphi}(x,y)$ in the y-variable. The construction will give that $k_{S,\varphi}$ factors through a kernel on $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times C_F$ with the same diagonal mass (skewered by the $\mathcal{O}^{\times}_{F,S}$-action. See sections below).

So the S-local additive Fourier transform becomes available, so long as we start off with

$$|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

instead of

$$|\cdot|_F^{a}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

for general $a \in \mathbf{R}_{>0}$.

This way, we will have that the product $|x|_{F,S}^a |y|_{F,S}^a$ arising by computing $k_{S,\varphi}(x,y)$, turns, along the diagonal, into $|x|_{F,S}^{2a}$. If $a = 1/2$ exactly, then this is $|x|_{F,S}$, which will multiply the multiplicative Tamagawa measure μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ to give the additive measure ν_S, up to a positive multiplicative constant.

This forces the $1/2$ in the very statement of GRH. Connes' expectation on the S-adelic Fourier transform playing this role and being related to the necessity of the $1/2$ in the very statement of GRH, is therefore correct.

(2) A version of known calculations of Connes from his paper from 1998[7].

[7] A. Connes. *Trace Formula in Noncommutative Geometry and the Zeros of the Riemann Zeta Function.* Selecta Mathematica. Vol.5, No.29, (1999).

1.8.4. Putting everything together: Trace Formula and consequences. Putting everything together, we obtain the Trace Formula, valid on all $\varphi \in \mathcal{D}(C_F)$ satisfying the conditions listed below, and for the interval $I = [-a, a]$ for $a > 0$ arbitrary, kept fixed throughout the trace calculation (we choose the Sobolev degree $s = 0$ for simplicity):

$$\begin{aligned}\sum_{v\in S_\infty,\ v \text{ real}} \frac{1}{2}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v) + &\sum_{v\in S_\infty,\ v \text{ complex}} \frac{1}{2\pi}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v)\\ &+ \sum_{v\in\Sigma_{F,f}} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int_{F_v^\times}{}' \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|_v}\,\mathrm{d}\mu_v(g_v)\\ &= M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2}) - \sum_{c,I} N(c)M(\varphi)\left(c|\cdot|_F^{-1/2}\right)\end{aligned}$$

where $q_v := |\pi_v|_v^{-1}$, for $v \in \Sigma_{F,f}$ and π_v any choice of uniformizer at v of the non-Archimedean local field F_v, and

$$M(\varphi)(c) := \int_{C_F} \varphi(g)c(g)\,\mathrm{d}\overline{\mu}(g)$$

is the Mellin transform of the test function φ, and $c \in \Omega(C_F)$ varies over the zeros of the Hecke L-function $c \mapsto L(c)$ contained in the interior of the relative vertical strip

$$\Omega(C_F)_I = \{c \in \Omega(C_F) : \sigma(c) \in 1/2 + I\}.$$

The nonnegative integer $N(c)$ is the multiplicity of the zero c of the Hecke L-function.

The assumptions on the test function are as follows:

(1) $\varphi \in \mathcal{D}(C_F)$, i.e. φ is smooth and compactly supported.
(2) $\varphi(C_F^1) = 0$.

1.8.4.1. *Character by character.* For φ homogeneous with respect to the right-translation action of C_F^1 on $\mathcal{D}(C_F)$, with weight $\overline{\chi}|_{C_F^1}$, for $\chi \in \widehat{C_F}$ fixed, and satisfying assumptions (a) and (b) above, the Trace Formula turns into:

$$\begin{aligned}\sum_{v\in S_\infty,\ v \text{ real}} \frac{1}{2}\int_{\mathbf{R}_{>0}} \sum_{u\in\{\pm1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|}\,\frac{\mathrm{d}t}{t} + &\sum_{v\in S_\infty,\ v \text{ complex}} \frac{1}{2\pi}\int_{\mathbf{R}_{>0}}\int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|}\,2\mathrm{d}\theta\otimes\frac{\mathrm{d}t}{t}\\ &+ \sum_{v\in\Sigma_{F,f}} \log(q_v)\sum_{m\geq 1}\left(\frac{\chi(\pi_v^m)}{q_v^{m/2}}\varphi(q_v^m) + \frac{\overline{\chi}(\pi_v^m)}{q_v^{m/2}}\varphi(q_v^{-m})\right)\\ &= M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2}) - \sum_{\rho,I} N(\rho)\int_{C_F}\chi(g)\varphi(g)|g|_F^{\rho-1/2}\,\mathrm{d}\overline{\mu}(g)\end{aligned}$$

where $M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2})$ vanishes if φ is not C_F^1-invariant.

1.8.4.2. *An explicit holomorphic family of test functions.* The idea now is to consider the following holomorphic family of piece-wise smooth functions on C_F:

$$\varphi_z(t) := t^{-z}$$

for $t > 1$, $\varphi_z(t) := 0$ for $t \in (0, 1)$, $\varphi_z(1) = 1/2$ and approximate $\chi \cdot \varphi_z$ (viewed as a function on C_F by composition with the idelic norm) by smooth compactly supported functions on C_F satisfying (a) and (b), and C_F^1-homogeneous with respect to the conjugate of a chosen Hecke character.

For example, take $\varepsilon > 0$ small, and ρ_ε the resulting mollifier as a function of $t \in \mathbf{R}_{>0}$. Take $h(t) \in \mathcal{D}(\mathbf{R}_{>0})$ be a bump function supported over $(1, +\infty)$ only (and valued in $[0, 1]$).

Then

$$\varphi_{h,\varepsilon,z}(t) := h(t)(\rho_\varepsilon * \varphi_z)$$

tends to $\varphi_z(t)$ point-wise (for $t \neq 1$) as $\varepsilon \to 0$ and $h \to 1_{(1,+\infty)}$ pointwise, as h (which will be an explicit collection of bump functions) remains valued in $[0, 1]$ and below $1_{(1,+\infty)}$, converging point-wise monotonically.

1.8.4.3. *Montel's Theorem.* Fix a compact subset $K \subset \mathbf{C}_{>\sup(I)}$ with nonempty interior intersecting with nonempty interior the half-plane $\mathbf{C}_{>1}$.

Write $\varphi = \chi \cdot (\varphi_{h,\varepsilon,z} \circ |\cdot|_F)$; note the dependence on $z \in \mathbf{C}$.

The integrals

$$\sum_{v \in S_\infty,\ v \text{ real}} \frac{1}{2} \int_{\mathbf{R}_{>0}} \sum_{u \in \{\pm 1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|} \frac{\mathrm{d}t}{t}, \quad \sum_{v \in S_\infty,\ v \text{ complex}} \frac{1}{2\pi} \int_{\mathbf{R}_{>0}} \int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|} 2\mathrm{d}\theta \otimes \frac{\mathrm{d}t}{t}$$

are bounded above by a common constant $A > 0$ independent of $z \in K$ and independent of the specific φ.

The boundedness follows from simple estimates and from the explicit form of the integrals (and the fact that they are only finitely many! Since they are indexed by the Archimedean places only. The integrals indexed by the non-Archimedean places are, instead, infinitely many, and they are the source of all the difficulties: their sum approximates $L'_f(s + 1/2, \chi)/L_f(s + 1/2, \chi)$).

By Montel's Theorem from complex analysis, we can extract a sequence of ε, h such that over such sequence, the convergence of the integrals in

$$\sum_{v \in S_\infty,\ v \text{ real}} \frac{1}{2} \int_{\mathbf{R}_{>0}} \sum_{u \in \{\pm 1\}} \frac{\chi_v(tu)\varphi(tu)|t|^{1/2}}{|1-t|} \frac{\mathrm{d}t}{t}, \quad \sum_{v \in S_\infty,\ v \text{ complex}} \frac{1}{2\pi} \int_{\mathbf{R}_{>0}} \int_{\mathbf{T}} \frac{\chi_v(te^{i\theta})\varphi(te^{i\theta})|t|^{1/2}}{|1-te^{i\theta}|} 2\mathrm{d}\theta \otimes \frac{\mathrm{d}t}{t}$$

is uniform for z in any compact disc contained within the interior of K. Since each is a holomorphic function of $z \in K$, so is the limit.

A similar reasoning leads to the limit as $\varepsilon \to 0$ and $h \to 1_{(1,+\infty)}$ of the convergent sums

$$\sum_{\rho, I} N(\rho) \int_{C_F} \chi(g)\varphi_{h,\varepsilon,z}(g)|g|_F^{\rho-1/2} \, \mathrm{d}\overline{\mu}(g)$$

to be holomorphic as a function of z varying over the interior of K intersected with the complement of $\mathbf{C}_I$ (recall that the sum is only over those zeros ρ of $L(s, \chi)$ such that $\rho - 1/2 \in \mathbf{C}_I$, after parametrizing $\Omega(C_F)(\chi)$ as $\chi|\cdot|_F^s$).

1.8.5. The logarithmic derivative of $L_f(s, \chi)$. Now note that each $\varphi = \varphi_{h,\varepsilon,z}$ is, for all $z \in K$, supported over $(1, +\infty)$, and so $\varphi(q_v^{-m}) = 0$ for all integers $m > 0$.

The sum

$$\sum_{v \in \Sigma_{F,f}} \log(q_v) \sum_{m \geq 1} \left(\frac{\chi(\pi_v^m)}{q_v^{m/2}} \varphi(q_v^m) + \frac{\overline{\chi}(\pi_v^m)}{q_v^{m/2}} \varphi(q_v^{-m}) \right)$$

therefore tends, uniformly over the compact subsets of $\mathbf{C}_{>1}$, to

$$\frac{L'_f\left(s + \frac{1}{2}, \chi\right)}{L_f\left(s + \frac{1}{2}, \chi\right)}.$$

1.8.5.1. *Putting everything together.* Putting everything together from §5.1, we obtain that for $I = [-a, a]$ for a fixed real number $a > 0$, the logarithmic derivative of $L_f(s, \chi)$ satisfies, over the interior of K:

$$\text{holomorphic over } \mathrm{Int}(K) + \frac{L_f'(s+1/2,\chi)}{L_f(s+1/2,\chi)} = \delta \cdot \left(\frac{1}{s+1/2} + \frac{1}{s-1/2}\right) - \text{holomorphic over } \mathbf{C}_{>a} \tag{1.8.5.1}$$

with $\delta = 0$ if $\chi|_{C_F^1} \neq 1$ and $\delta = 1$ otherwise. This is, ultimately, the main consequence of the Trace Formula. Everything else follows from this.

We deduce that

$$\frac{L_f'(s+1/2,\chi)}{L_f(s+1/2,\chi)} - \delta \cdot \left(\frac{1}{s+1/2} + \frac{1}{s-1/2}\right)$$

is holomorphic over $\mathrm{Int}(K)$ for any $K \subset \mathbf{C}_{>a}$ by analytic continuation ($\mathbf{C}_{>1}$ versus $\mathbf{C}_{>a}$). And so $L_f'(s + 1/2, \chi)/L_f(s + 1/2, \chi)$ is holomorphic over $\mathbf{C}_{>a}$ with the sole possible exception of the point $s = 1/2$ (when $\chi|_{C_F^1} = 1$).

Since $a > 0$ is arbitrary, we deduce that $L_f'(s + 1/2, \chi)/L_f(s + 1/2, \chi)$ is holomorphic over $\mathbf{C}_{>0} - \{1/2\}$.

This implies that $L_f(s, \chi)$, and hence $L(s, \chi)$, has no zeros over $\mathbf{C}_{>1/2}$, and likewise for $L(s, \overline{\chi})$, so by the functional equation the same holds for $L(s, \chi)$ over $\mathbf{C}_{<1/2}$.

This completes the overview of how to deduce the main results on the location of the zeros of $L(s, \chi)$, from the Trace Formula.

CHAPTER 2

Adelic θ-functions

We fix a number field F, writing $\mathcal{O}_F$ for its ring of integers.

In [**CF**, Ch.XV,§4,Def.4.4.1], idelic zeta integrals, i.e. integrals over $\mathrm{GL}_1(\mathbf{A}_F)$ of the form

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)} f(g)c(g)\,\mathrm{d}\mu(g)$$

for f a suitable measurable function on the Hausdorff locally compact abelian topological group $\mathrm{GL}_1(\mathbf{A}_F)$, μ a fixed choice of Haar measure on $\mathrm{GL}_1(\mathbf{A}_F)$, and as c varies over a suitable subset of all continuous quasi-characters of $\mathrm{GL}_1(\mathbf{A}_F)$ satisfying $c(F^\times) = 1$, are defined and shown to be well behaved for test functions f from the complex vector space $\mathfrak{z}$ introduced in [**CF**, Ch.XV,§4.4, $\mathfrak{z}_1$-$\mathfrak{z}_3$].

The set of continuous quasi-characters of $\mathrm{GL}_1(\mathbf{A}_F)$ satisfying $c(F^\times) = 1$ is equipped with the structure of a Riemann surface Ω. Such functions are designed to ensure convergence of the integrals from [**CF**, Ch.XV,Def.4.4.1] over $\mathrm{GL}_1(\mathbf{A}_F)$ over large domains within Ω, and the validity of the adelic "Riemann-Roch Theorem" from [**CF**, Ch.XV,Thm.4.2.1].

In this Chapter we study properties of the Hausdorff locally convex topological complex vector space $\mathcal{S}(\mathbf{A}_F)$ of Schwartz-Bruhat functions on the adele ring $\mathbf{A}_F$.

A consequence of the results recorded here is the well known fact that Schwartz-Bruhat functions on $\mathbf{A}_F$ are examples of functions from $\mathfrak{z}$.

The topological properties of $\mathcal{S}(\mathbf{A}_F)$ and its strong dual $\mathcal{S}(\mathbf{A}_F)'_\beta$ will enable us to study the continuous representation of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}(\mathbf{A}_F)$ and its homological and co-homological invariants, leading to precise information on Hecke L-functions and their divisor of zeros and poles.

The main goals of this Chapter are as follows:

(1) Analyze the decay properties of functions on the idele class group C_F of F constructed by averaging under the continuous left-translation action by $F^\times$, Schwartz-Bruhat functions on $\mathbf{A}_F$ (restricted to $\mathrm{GL}_1(\mathbf{A}_F)$ under composition with the continuous map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{A}_F$).

(2) Discussing S-local analogs of the constructions and results from (1), for S a finite set of places of F containing all the Archimedean places.

(3) Revisiting [**CF**, Ch.XV,§4.4] of Tate's dissertation, with the specific purpose of rewriting Tate's idelic zeta integrals as Mellin transforms of smooth functions on $\mathbf{R}_{>0}$ with rapid decay at infinity, such functions being linearly associated to the Schwartz-Bruhat functions on $\mathbf{A}_F$, respectively $\mathbf{A}_{F,S}$.

(4) Studying continuity properties of the formation of the functions constructed in (2), together with the interaction with directional derivation along continuous 1-parameter subgroups of $\mathrm{GL}_1(\mathbf{A}_F)$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S})$.

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2.1. Schwartz-Bruhat functions

Distribution theory rests upon the construction of Hausdorff locally convex topological vector spaces of test functions, usually satisfying favorable topological properties, such as completeness, nuclearity, barrelledness and the Heine-Borel property (these latter two properties defining the class of Montel Hausdorff locally convex topological vector spaces; see [**Jar**, Part II,Ch.11,§11.5]).

Distributions are ordinarily defined as the strong dual of a Hausdorff locally convex topological complex vector space of suitable test functions. Such spaces of distributions often receive a linear topological embedding from the space of test functions itself, enabling us to think of distributions as generalized functions.

For the concept of *strong dual* we refer the reader to Definition C.2.1.5(3): for a Hausdorff locally convex topological $\mathbf{C}$-vector space E, its strong dual is the continuous dual endowed with the topology of uniform convergence on bounded subsets, and denoted as E'_β.

In this section we recall the notion of tempered distribution on the real line, briefly reviewing its generalization to tempered distributions on Hausdorff locally compact abelian groups, specializing to the case of the Hausdorff locally compact abelian group $\mathbf{A}_F$ of the adeles of a number field.

The general theory of Schwartz-Bruhat functions on Hausdorff locally compact abelian groups G is reworked in Appendix D, with the aim at providing proofs of the main structural results on $\mathcal{S}(G)$, relying on Appendix C, where we discuss complements and background material from functional analysis.

For the purposes of this Chapter, we will be recalling basic facts on Schwartz-Bruhat functions as the need arises, and in the specific cases of interest only.

2.1.1. Schwartz-Bruhat functions. We begin by reviewing the example of the space of Schwartz functions on the real line.

2.1.1.1. *Schwartz functions on the real line.* Let f_1, f_2 be smooth complex valued functions on the real line. Deciding how "near" f_1 and f_2 are is a task to be addressed depending on the behavior of $f_1(x), f_2(x)$ as $|x| \to +\infty$, since $\mathbf{R}$ is not compact.

The uniform norm provides one notion of nearness among smooth complex valued functions on $\mathbf{R}$, so long as the functions are bounded over $\mathbf{R}$. The resulting topology on the $\mathbf{C}$-vector space of bounded smooth complex valued functions is too coarse to allow a rich strong dual. Even on smooth functions with compact support, the resulting linear topology is uninteresting.

Example 2.1.1.1. The space of distributions on the real line is built as the strong dual (see [**Sch-a**, Ch.III,§3,p.71]) of the Hausdorff locally convex topological complex vector space of smooth compactly supported functions $C_c^\infty(\mathbf{R})$, endowed with the locally convex (strict) inductive limit topology (see [**Scha**, Ch.II,§6,p.57] and [**Jar**, Part I,Ch.4,§4.6], [**Sch-a**, Ch.III,§1,Thm.II]) promoting the linear inclusions $C_K^\infty(\mathbf{R}) \subset C_c^\infty(\mathbf{R})$ to continuous linear operators (in fact, linear topological embeddings).

Here $C_K^\infty(\mathbf{R})$ denotes the complex vector space of smooth functions on $\mathbf{R}$ with support contained within the compact subset $K \subset \mathbf{R}$ (see [**Sch-a**, p.24]), endowed with the topology of uniform convergence of functions and their derivatives of any order. We refer the reader to [**Sch-a**, Ch.III,§1].

Before continuing, we caution the reader on the difference between inductive limit topologies and locally convex inductive limit topologies.

Remark 2.1.1.2 (Cautionary remark on the difference of inductive limit and locally convex inductive limit topology). The locally convex inductive limit topology on $C_c^\infty(\mathbf{R})$ differs from the inductive limit topology, which is generally not locally convex. A counterexample can be constructed from [**Shi**, Lemme 4] and its proof.

The inductive limit topology of a sequence of topological spaces equipped with continuous transition maps (satisfying good properties, such as being topological embeddings with closed image) may be unwieldy. It is difficult to write down examples of open neighborhoods.

For locally convex inductive limits of filtered families of Hausdorff locally convex topological $\mathbf{C}$-vector spaces, instead, we can write down a 0-neighborhood base as follows. Let I be a filtered set (i.e. equipped with a partial ordering) and $\{E_i\}_{i\in I}$ an I-indexed collection of Hausdorff locally convex topological $\mathbf{C}$-vector spaces.

Assume given injective transition maps $u_{ij} : E_i \to E_j$ for every $j \geq i$ in I, and let E be the filtered colimit linear space. We call $u_i : E_i \to E$ the induced linear inclusion. A 0-neighborhood base for the locally convex inductive limit topology on E is given by the convex hulls

$$\mathrm{ch}(\bigcup_{i\in I} u_i(V_i))$$

for V_i varying through a given fundamental system of 0-neighborhoods in E_i. We refer the reader to [**Gro-f**, Ch.4,§1,Prop.1(2),p.136].

Locally convex inductive limits have a tendency to be large and do not satisfy the second-countability condition (in particular, they are usually non-metrizable).

For example the $\mathbf{N}$-indexed direct sum $\bigoplus_{\mathbf{N}} \mathbf{C}$ is non-metrizable because it does not have a countable 0-neighborhood base.

The space of *tempered* distributions on the real line is, instead, built as the strong dual of the Hausdorff locally convex topological complex vector space of Schwartz functions on $\mathbf{R}$.

Example 2.1.1.3. The complex vector space $\mathcal{S}(\mathbf{R})$ of Schwartz functions on $\mathbf{R}$ is comprised of the smooth functions $f \in C^\infty(\mathbf{R})$ such that

$$\|f\|_{m,N} := \sup_{x\in\mathbf{R}} (1 + |x|^N)\, |\partial_x^m f(x)| < +\infty$$

where $\partial_x^m f$ is the order-m derivative of f, for all integers $N, m \geq 0$, topologized with the locally convex topology generated by the family $\{\|\cdot\|_{m,N}\}_{m,N\geq 0}$ of seminorms.

The space of *tempered distributions* on $\mathbf{R}$ is defined to be the Hausdorff locally convex topological vector space $\mathcal{S}(\mathbf{R})'_\beta$: the strong dual of $\mathcal{S}(\mathbf{R})$. This is the continuous dual of $\mathcal{S}(\mathbf{R})$ endowed with the topology of uniform convergence (for continuous linear functionals) on the bounded subsets of $\mathcal{S}(\mathbf{R})$.

2.1.1.2. *Schwartz functions on Euclidean spaces.* Schwartz functions can be similarly defined for Euclidean spaces, which we now briefly recall. We fix an integer $m \geq 0$ and consider the real Euclidean space $\mathbf{R}^m$. We parametrize $\mathbf{R}^m$ with its standard coordinates $x = (x_i)_{1\leq i\leq m}$.

We define, for every $\alpha \in \mathbf{N}^m$, the $\mathbf{C}$-linear operator

$$D_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} : C^\infty(\mathbf{R}^m) \to C^\infty(\mathbf{R}^m) \tag{2.1.1.1}$$

with $|\alpha| := \sum_{i=1}^m \alpha_i$. We write, for any $x \in \mathbf{R}^m$ and $\alpha \in \mathbf{N}^m$

$$x^\alpha := \prod_{i=1}^m x_i^{\alpha_i}. \tag{2.1.1.2}$$

Definition 2.1.1.4. Let $f : \mathbf{R}^m \to \mathbf{C}$ be a smooth function. We say f is of *rapid decay at infinity* if and only if for every $\alpha, \beta \in \mathbf{N}^m$, the function

$$\|\cdot\|_{\alpha,\beta} : C^\infty(\mathbf{R}^m) \to [0, +\infty] \tag{2.1.1.3}$$

defined as

$$\|f\|_{\alpha,\beta} : \sup_{x\in\mathbf{R}^m} (1+|x^\alpha|)\,|D_x^\beta f(x)| \tag{2.1.1.4}$$

is finite on f.

We denote by $\mathcal{S}(\mathbf{R}^m)$ the $\mathbf{C}$-vector space of all smooth functions on $\mathbf{R}^m$ of rapid decay at infinity. We topologize $\mathcal{S}(\mathbf{R}^m)$ with the locally convex topology generated by the family $\{\|\cdot\|_{\alpha,\beta\in\mathbf{N}^m}\}$ of seminorms. We refer to the locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{R}^m)$ as the *Schwartz space* of $\mathbf{R}^m$.

REMARK 2.1.1.5. There are several families of explicit seminorms on $\mathcal{S}(\mathbf{R}^m)$ generating the same locally convex topology. For instance, $p_{\alpha,\beta} : \mathcal{S}(\mathbf{R}^m) \to \mathbf{R}_{\geq 0}$, for each $\alpha,\beta \in \mathbf{N}^m$, given by $p_{\alpha,\beta}(f) := \sup_{x\in\mathbf{R}^m} |x^\alpha D_x^\beta f(x)|$. Another important such family is

$$p_{\alpha,\beta,r} := \|x^\alpha D_x^\beta f(x)\|_{L^r(\mathbf{R}^m,\mathrm{d}x)}$$

for $r \geq 1$ a fixed choice of real number. Other such families generating an equivalent topology on $\mathcal{S}(\mathbf{R}^m)$ are given by the same assignments, upon replacing x^α with $u(x)$ as u varies over all polynomial functions on $\mathbf{R}^m$ with complex coefficients. We refer the reader to [**KG**, Part I,Ch.III,(3)].

We shall next record a result on properties of the linear topology on $\mathcal{S}(\mathbf{R}^m)$, but first we must clarify the role of boundedness in the definition of strong duals. The β-topology, or *strong* topology, on the continuous dual E' of a Hausdorff locally convex topological complex vector space E is defined in [**Jar**, Part II,Ch.8,§8.4,Example 3.A,p.153] as the topology of uniform convergence on all subsets of E that are $\sigma(E,E')$-bounded, i.e. bounded for the weakened topology on E. However, the subsets of E that are bounded with respect to the weakened topology are exactly the subsets of E that are bounded for the intrinsic topology on E by [**Jar**, Part II,Ch.8,§8.3,Thm.4,p.151] (see also [**NB**, Thm.8.8.7]). Definition C.2.1.5 is therefore equivalent to the definition in [**Jar**, Part II,Ch.8,§8.4,Example 3.A,p.153], and it is [**Scha**, Ch.III,§3,Example 4(d),p.81].

PROPOSITION 2.1.1.6. *Let $m \geq 0$ be an integer. The locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{R}^m)$ is nuclear, Fréchet and reflexive. The strong dual $\mathcal{S}(\mathbf{R}^m)'_\beta$ is complete, nuclear and Montel.*

REMARK 2.1.1.7. We recall that a Hausdorff locally convex topological complex vector space E is *semi-Montel* if and only if it satisfies the Heine-Borel property: if $B \subset E$ is a bounded subset, then B is relatively compact in E; see [**Jar**, Part II,§11]. By [**Jar**, Part II,§11,Prop.1], if E is semi-Montel, then it is barrelled (see [**Jar**, Part II,§11.1]) if and only if it is reflexive, and semi-Montel spaces satisfying any of these additional properties are, by definition, Montel spaces.

PROOF OF PROPOSITION 2.1.1.6. Nuclearity of $\mathcal{S}(\mathbf{R}^m)$ is [**Tre**, Part III,Ch.51,Corollary,p.530]. The family of seminorms $\|\cdot\|_{\alpha,\beta}$ from (2.1.1.4) is separating. The locally convex topology on $\mathcal{S}(\mathbf{R}^m)$ is therefore Hausdorff.

The seminorms from (2.1.1.4) are countably many, and hence $\mathcal{S}(\mathbf{R}^m)$is, by [**Kothe-I**, Ch.4,§18,(2),p.205], metrizable. It is Fréchet so long as it is complete. Since it is metrizable (because (2.1.1.3) is a countable fundamental family of seminorms), it is enough to show convergence of any Cauchy *sequence* $\{f_k\}_{k\geq 0}$ in $\mathcal{S}(\mathbf{R}^m)$ with respect to the topology generated by the given seminorms.

We first observe that $\{f_k\}_{k\geq 0}$ converges with respect to the uniform norm topology, since $C_0(\mathbf{R}^m)$ is Banach (see [**Fol**, Prop.4.35]). We call its limit f.

The functions $g_{\alpha,k} := D_x^\alpha f_k(x)$ also define a Cauchy sequence $\{g_{\alpha,k}\}_{k\geq 0}$ for each $\alpha \in \mathbf{N}^m$ with respect to the uniform norm topology on $C_0(\mathbf{R}^m)$, converging to some f_α, which is seen to be $D_x^\alpha f$ since the convergence is, in particular, uniform on compact subsets for $\{D_x^\alpha f_k\}_{k\geq 0}$ for each $\alpha \in \mathbf{N}^m$, and f_α is therefore of the desired form on every compact subset of $\mathbf{R}^m$.

The function f is therefore smooth and rapidly decaying at infinity. We need to show that the convergence of $\{f_k\}_{k\geq 0}$ to f is in the sense of the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{R}^m)$.

For each $\beta \in \mathbf{N}^m$, $\{x^\beta g_{\alpha,k}\}_{k\geq 0}$ is a Cauchy sequence for the uniform topology on $C_0(\mathbf{R}^m)$, and its limit $f_{\alpha,\beta}$ is, by uniqueness, $x^\beta D_x^\alpha f$, since this can be checked locally. It follows that $x^\beta D_x^\alpha f_k \to x^\beta D_x^\alpha f$ with respect to the uniform topology on $C_0(\mathbf{R}^m)$, and since this holds for every $\alpha, \beta \in \mathbf{N}^m$, we deduce that $f_k \to f$ in $\mathcal{S}(\mathbf{R}^m)$.

Since $\mathcal{S}(\mathbf{R}^m)$ is nuclear, then every bounded subset in $\mathcal{S}(\mathbf{R}^m)$ is relatively compact by [**Tre**, Part III,Ch.50,Prop.50.2(5.12)]. By completeness, $\mathcal{S}(\mathbf{R}^m)$ has the Heine-Borel property, i.e. it is semi-Montel as per [**Jar**, Part II,Ch.11,§11.5,p.230].

By [**NB**, Thm.13.2.13], its strong dual $\mathcal{S}(\mathbf{R}^m)'_\beta$, the space of tempered distributions on $\mathbf{R}^m$, is complete, and it is nuclear by [**Tre**, Part III,Ch.50,Prop.50.6]. Both $\mathcal{S}(\mathbf{R}^m)$ and $\mathcal{S}(\mathbf{R}^m)'_\beta$ are reflexive (the latter by [**Tre**, Part II,Prop.36.10]). We further refer the reader to [**Sch-a**, Ch.VII,§3].

By reflexivity and [**Jar**, Part II,Ch.11,§11.5,Prop.1], $\mathcal{S}(\mathbf{R}^m)$ is Montel, and hence so is $\mathcal{S}(\mathbf{R}^m)'_\beta$ by [**Jar**, Part II,Ch.11,§11.5,Prop.4(f)]. □

2.1.1.3. *Schwartz-Bruhat functions on Hausdorff locally compact abelian groups.* Definition 2.1.1.4 bootstraps to the case of elementary abelian Lie groups, i.e. Hausdorff locally compact abelian groups isomorphic to $\mathbf{R}^m \times (\mathbf{R}/\mathbf{Z})^n \times \mathbf{Z}^k \times F$ for integers $m, n, k \geq 0$, and a finite abelian group F, and, from this case, to the case of arbitrary Hausdorff locally compact abelian groups through their classification theory. This is done in Bruhat's paper [**Bru**], and specifically in [**Bru**, §9]. Such functions are termed *Schwartz-Bruhat* throughout this work.

By Lemma D.1.3.7(2), the Hausdorff locally convex topological $\mathbf{C}$-vector space of Schwartz-Bruhat functions on *elementary* Lie groups (i.e. Hausdorff locally compact abelian groups of the form $\mathbf{R}^m \times (\mathbf{R}/\mathbf{Z})^n \times \mathbf{Z}^k \times F$) is linearly and topologically isomorphic to $\mathcal{S}(\mathbf{R}^m) \mathbin{\widehat{\otimes}_\pi} C^\infty((\mathbf{R}/\mathbf{Z})^n) \mathbin{\widehat{\otimes}_\pi} \mathbf{s}_k \otimes_\pi \mathbf{C}^F$, where $\mathcal{S}(\mathbf{R}^m)$ was studied in Proposition 2.1.1.6, $C^\infty((\mathbf{R}/\mathbf{Z})^n)$ is endowed with the topology of uniform convergence on functions and their derivatives of arbitrary order, and $\mathbf{s} = \mathcal{S}(\mathbf{Z})$ is the space of rapidly decaying $\mathbf{Z}$-indexed sequences $(\{a_k\}_{k\in\mathbf{Z}}$ such that $a_k = O(|k|^{-N})$ for every integer $N \geq 0$ as $|k| \to +\infty)$, with $\mathbf{s}_k$ the k-th projective tensor power of $\mathbf{s}$.

Recall that an arbitrary Hausdorff locally compact abelian group G is the direct limit, as a topological abelian group, of its open compactly generated subgroups (i.e. generated as a group by a compact neighborhood of 1).

Any compact subgroup $k \subset G$ is called *good* (following [**Bru**, §1]) if and only if its open annihilator $k^\perp$ is ompactly generated as a subgroup of the dual group $\widehat{G}$. Any open compactly generated subgroup $\Gamma \subset G$ containing k is such that Γ/k is an elementary Lie group (seen upon combining, for instance, [**DE**, Cor.4.2.5] and [**DE**, Cor.4.2.6] with [**DE**, Prop.3.6.1(c)]).

We now briefly recall how to define and topologize $\mathcal{S}(G)$ (we refer the reader to [**Bru**, §9]). To construct $\mathcal{S}(G)$, we will work with pairs (k, Γ) where $k \subset G$ is a good subgroup and $\Gamma \subset G$ is an open compactly generated subgroup containing k. Since Γ/k is an abelian Lie group, we have already discussed the space $\mathcal{S}(\Gamma/k)$ of Schwartz-Bruhat functions. These can be viewed as special k-invariant functions on Γ, and we topologize this space as discussed above. The collection of all such pairs (k, Γ) is filtered and $\mathcal{S}(G)$ is defined to be the filtered colimit of the $\mathcal{S}(\Gamma/k)$'s as a $\mathbf{C}$-linear space, linearly topologized with the locally convex inductive limit topology.

The topological vector space $\mathcal{S}(G)$ is Hausdorff for general G (see Theorem D.1.5.7, resting, as far as the Hausdorff property is concerned, on Lemma D.1.5.6). If G is separable, $\mathcal{S}(G)$ is complete, nuclear and Montel, and so is its strong dual $\mathcal{S}(G)'_\beta$. We refer the reader to Theorem D.1.5.7 for a proof.

As a special case, for $G = \mathbf{R}$ the classical Schwartz space $\mathcal{S}(\mathbf{R})$ is recovered, and it is complete, nuclear and Montel, with strong dual $\mathcal{S}(\mathbf{R})'_\beta$ sharing the same properties. Throughout this work, we will be referring to $\mathcal{S}(G)$ as the *Schwartz-Bruhat space* of G, and to its topology as the *Schwartz-Bruhat topology*.

2.1.2. Local Schwartz-Bruhat functions. The main cases of Hausdorff locally compact abelian groups G of interest to this work are $G = \mathbf{A}_F$ for a number field F, $G = \mathbf{A}_{F,S}$ for S a finite set of places of F containing the Archimedean places, $G = C_F = \mathrm{GL}_1(\mathbf{A}_F)/F^\times$, the idele class group of F, and its S-local subgroup $C_{F,S} = (\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times)/\mathcal{O}_{F,S}^\times$, and $G = J_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$, the quotient of $C_{F,S}$ by $1 \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times$.

We will require, later in Chapter 3, the Hausdorff locally convex topological $\mathbf{C}$-vector spaces of Schwartz-Bruhat functions on the idele class group C_F, and the S-idele class group $C_{F,S}$, along with its quotient $J_{F,S}$. These will be introduced, and their properties reworked, when the need arises.

In this subsection, we treat the case $G = F_v^\times$, for v a non-Archimedean place of F.

2.1.2.1. *Smooth functions on non-Archimedean fields.* We recall that for G a Hausdorff locally compact abelian group, a function $f : G \to \mathbf{C}$ is called *smooth* if and only if it is k-invariant for some compact subgroup $k \subset G$ whose open annihilator $k^\perp$ is compactly generated as a subgroup of the dual group $\widehat{G}$ and for every open compactly generated subgroup $\Gamma \subset G$ containing k the induced function on the quotient Γ/k (an elementary Lie group!) is smooth in the sense of differential geometry.

We denote by $C^\infty(G)$ the $\mathbf{C}$-algebra of smooth complex-valued functions on G, and denote by $C_c^\infty(G)$ the subspace of those elements of $C^\infty(G)$ that are also compactly supported. When G is locally profinite, $C_c^\infty(G)$ coincides with the $\mathbf{C}$-linear space of complex-valued locally constant functions with compact support.

Let F be a number field, v a finite place of F. We have, as $\mathbf{C}$-linear spaces,

$$C_c^\infty(F_v) = \mathcal{S}(F_v).$$

Remark 2.1.2.1. We caution the reader that the containment $C_c^\infty(F_v^\times) \subset \mathcal{S}(F_v^\times)$ is strict. Indeed, via $\mathcal{O}_{F_v}^\times$-invariance, $\mathcal{S}(\pi_v^{\mathbf{Z}}) \simeq \mathcal{S}(\mathbf{Z})$ (for π_v a uniformizer at v) is a subspace of $\mathcal{S}(F_v^\times)$, and $\mathcal{S}(\mathbf{Z})$ is the space of rapidly decaying $\mathbf{Z}$-indexed sequences.

2.1.2.2. *The locally convex topological complex vector spaces $\mathcal{S}(F_v)$ for $v \in \Sigma_{F,f}$.* In this subsection we recall how to explicitly describe the elements of the Schwartz-Bruhat space of the local fields F_v for v non-Archimedean.

As a Hausdorff locally compact abelian group, we recall that F_v is the filtered union of compact open subgroups $K_v \subset F_v$, and for two compact open subgroups $K_{v,1} \subset K_{v,2}$, by openness of $K_{v,1}$ in $K_{v,2}$ and compactness of $K_{v,2}$ the quotient $K_{v,2}/K_{v,1}$ is a finite abelian group.

As per the very construction of $\mathcal{S}(G)$ for G a Hausdorff locally compact abelian group, the basic Hausdorff locally convex topological vector spaces $\mathcal{S}(k, \Gamma)$ (see [**Bru**, §1]) arising in the construction are all of the form $\mathcal{S}(K_{v,2}/K_{v,1})$, which is a finite dimensional complex vector space. As such, it carries a unique Hausdorff locally convex topology (i.e. the Euclidean topology) compatible with the Euclidean topology on the complex numbers, and $\mathcal{S}(F_v)$ is a locally convex inductive limit of finite dimensional complex vector spaces, each endowed with its natural Euclidean topology.

Remark 2.1.2.2. We inform the reader concerned about the possibility of uncountably-indexed locally convex inductive limits arising, that the locally convex inductive systems of interest throughout this work will always be equivalent to countable ones. They will often arise from topological vector spaces related through formation of quotients of spaces of Schwartz-Bruhat functions on second-countable Hausdorff locally compact

abelian groups. For example, $\mathcal{S}(F_v)$ is countably generated, since F_v has a countable 0-neighborhood base of compact open subgroups.

REMARK 2.1.2.3 (On the necessity of topologizing $\mathcal{S}(F_v)$). Let F be a number field. The aim of the next few Chapters of this work is to collect certain averages over $F^\times$-orbits on $\mathrm{GL}_1(\mathbf{A}_F)$ of Schwartz-Bruhat functions on $\mathbf{A}_F$, into topological vector spaces, for then studying them through the lens of the Mellin transform. Several subquotients of such topological vector spaces will be essential in performing the trace calculation that is the main subject of study of this work.

Their topology will be of great importance, because it will be one of the purposes of the first half of this work to identify Schauder bases on such subquotients, and Schauder bases are sensitive to the topology of their linear environment. The defining conditions of a Schauder basis is that every element of the ambient Hausdorff locally convex topological vector space, should be expressible as a convergent countably infinite linear combination of basis elements in a unique way, with continuous coefficient functionals.

To build such topological vector spaces, we will be starting off from the Schwartz-Bruhat topology on $\mathcal{S}(\mathbf{A}_F)$ and $\mathcal{S}(\mathbf{A}_{F,S})$. It renders itself necessary to topologize $\mathcal{S}(F_v)$ for every non-Archimedean place as well, as simple as such topology is: it is the finest locally convex topology, equivalently the topology rendering any linear form continuous.

We deduce the following two structural results on $\mathcal{S}(F_v)$ and the spaces $\mathcal{S}(G)$ for $G = \prod_{v\in S} F_v$ with S a finite set of finite places of the number field F, and $G = \widehat{\prod}_{v\notin S}(F_v, \mathcal{O}_{F_v})$ for S a finite set of places of F containing all the Archimedean places:

PROPOSITION 2.1.2.4. *Let F be a number field, v a finite place.*

(1) *$\mathcal{S}(F_v)$ is countably $\mathbf{C}$-linearly generated, and it carries the finest locally convex topology. In particular, it is a strict LF-space.*
(2) *$\mathcal{S}(F_v)$ is Hausdorff locally convex, complete, nuclear, Montel, and its strong dual is a nuclear Fréchet space.*
(3) *Every $\mathbf{C}$-linear subspace of $\mathcal{S}(F_v)$ is linearly and topologically complemented.*

Recall that a subspace V of a Hausdorff locally convex topological $\mathbf{C}$-vector space E is called *linearly and topologically complemented* if and only if there exists a subspace $W \subset E$ such that $V \oplus W \to E$ is a homeomorphism. In particular, V and W are closed, since E is Hausdorff.

PROOF. A cofinal sequence of compact open subgroups of F_v is given by $\{\pi_v^m \mathcal{O}_{F_v}\}_{m\in\mathbf{Z}}$, for π_v any choice of generator of the maximal ideal of the complete discrete valuation ring of $\mathcal{O}_{F_v}$.

As a $\mathbf{C}$-linear space, $\mathcal{S}(F_v)$ is therefore the countable inductive limit of the countably many finite dimensional vector spaces $\mathcal{S}(\pi_v^N \mathcal{O}_{F_v}/\pi_v^m \mathcal{O}_{F_v})$ for $N \leq m$, where the finite group $\pi_v^N \mathcal{O}_{F_v}/\pi_v^m \mathcal{O}_{F_v}$ is endowed with the discrete topology since $\pi_v^m \mathcal{O}_{F_v}$ is open in $\pi_v^N \mathcal{O}_{F_v}$. As such, it is countably linearly generated. By Proposition C.2.5.1(6), the rest of the assertions hold. □

EXAMPLE 2.1.2.5. Every Schwartz-Bruhat function on $\mathbf{Q}_p$, for p an arbitrary prime number, is smooth and compactly supported, hence locally constant with compact support, i.e. we have $\mathcal{S}(\mathbf{Q}_p) = C_c^\infty(\mathbf{Q}_p)$. Since every open subset in $\mathbf{Q}_p$ is a finite disjoint union of open p-adic balls, i.e. subsets of the form $a + p^N\mathbf{Z}_p$, for some $N \in \mathbf{Z}^+$ and $a \in \mathbf{Q}_p$, then, upon writing $1_{a+p^N\mathbf{Z}_p}$ for the indicator function of $a + p^N\mathbf{Z}_p$, each $f \in \mathcal{S}(\mathbf{Q}_p)$ is of the form:

$$f = \sum_{i=1}^{n} c_i 1_{a_i+p^{N_i}\mathbf{Z}_p}$$

for nonzero integers N_i and $a_i \in \mathbf{Q}_p$, and for $c_i \in \mathbf{C}$.

EXAMPLE 2.1.2.6. Similarly to Example 2.1.2.5, every Schwartz-Bruhat function on F_v, v a non-Archimedean place of the number field F, is smooth and compactly supported, hence locally constant with compact support. Every open subset in F_v is a finite disjoint union of subsets of the form $a+\pi_v^N\mathcal{O}_{F_v}$, for π_v a uniformizer at v, for some $N\in\mathbf{Z}^+$ and $a\in F_v$, then, upon writing $1_{a+\pi_v^N\mathcal{O}_v}$ for the indicator function of $a+\pi_v^N\mathcal{O}_{F_v}$, each $f\in\mathcal{S}(F_v)$ is of the form:

$$f=\sum_{i=1}^{n}c_i 1_{a_i+\pi_v^{N_i}\mathcal{O}_{F_v}} \tag{2.1.2.1}$$

for nonzero integers N_i and $a_i\in F_v$, and for $c_i\in\mathbf{C}$.

REMARK 2.1.2.7. For any finite place v of F, $\mathcal{S}(F_v)$ is topologized with the locally convex inductive limit induced by the unique topology of the finite dimensional $\mathbf{C}$-vector spaces comprising the inductive limit presentation of the $\mathbf{C}$-linear space $\mathcal{S}(F_v)$.

PROPOSITION 2.1.2.8. *Let F be a number field, S_1 a finite set of non-Archimedean places of F, S_2 a finite set of places of F containing all Archimedean places. Let $G_1:=\prod_{v\in S_1}F_v$ and $G_2:=\widehat{\prod}_{v\notin S_2}(F_v,\mathcal{O}_{F_v})$.*

(1) *The Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are countably $\mathbf{C}$-linearly generated, and carry the finest locally convex topology. In particular, they are strict LF-spaces, and hence bornological.*
(2) *$\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are Hausdorff locally convex, complete, nuclear, Montel, and their strong duals are nuclear Fréchet spaces.*
(3) *Every $\mathbf{C}$-linear subspace of $\mathcal{S}(G_1)$, respectively $\mathcal{S}(G_2)$ is linearly and topologically complemented.*

REMARK 2.1.2.9. We note that Proposition 2.1.2.8 applies, in particular, to $G=\mathbf{A}_F^f$. More generally, if the Hausdorff locally compact abelian group G is locally profinite, $\mathcal{S}(G)$ carries the finest locally convex topology, and Proposition C.2.5.1 describes the properties of the finest locally convex topology.

If G is, in addition, separable, then $\mathcal{S}(G)$ is countably $\mathbf{C}$-linearly generated, and Proposition C.2.5.1(6) applies. In particular, for G locally profinite and separable, $\mathcal{S}(G)$ is complete, nuclear, Montel, and its strong dual $\mathcal{S}(G)'_\beta$ is nuclear Fréchet.

PROOF. We call $\widehat{\mathcal{O}}_{F,S}$ the compact open subgroup $\prod_{v\in S_1}\mathcal{O}_{F_v}\subset G_1$, and $\widehat{\mathcal{O}}_{F,\Sigma_F-S_2}$ the compact open subgroup $\prod_{v\notin S_2}\mathcal{O}_{F_v}\subset G_2$.

A cofinal sequence of compact open subgroups of G_1, respectively G_2, is given by $\{N\widehat{\mathcal{O}}_{F,S_1}\}_{N\in\mathbf{Z}^+}$, respectively $\{N\widehat{\mathcal{O}}_{F,\Sigma_F-S_2}\}_{N\in\mathbf{Z}^+}$.

As a $\mathbf{C}$-linear space, $\mathcal{S}(G_1)$ is therefore the countable inductive limit of the countably many finite dimensional vector spaces $\mathcal{S}(N'\widehat{\mathcal{O}}_{F,S_1}/N\widehat{\mathcal{O}}_{F,S_1})$ for $0<N'\leq N$. Similarly, as a $\mathbf{C}$-linear space, $\mathcal{S}(G_2)$ is the countable inductive limit of the countably many finite dimensional vector spaces $\mathcal{S}(N'\widehat{\mathcal{O}}_{F,\Sigma_F-S_2}/N\widehat{\mathcal{O}}_{F,\Sigma_F-S_2})$.

As such, $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are both countably linearly generated. By Proposition C.2.5.1(6), the rest of the assertions hold. □

2.1.2.3. *Uniform sequential continuity of the continuous left $F_v^\times$-module $\mathcal{S}(F_v)$.* The Hausdorff locally compact abelian group $F_v^\times$ acts on F_v by left-translation. Its compact open subgroup $\mathcal{O}_{F_v}^\times$ acts on F_v accordingly.

We denote by:

$$\rho_v:F_v^\times\times\mathcal{S}(F_v)\to\mathcal{S}(F_v) \tag{2.1.2.2}$$

such action, assigned by $\rho_v(g,f)(x):=f(g^{-1}x)$ as functions of $x\in F_v$.

Let S_1 be a finite set of non-Archimedean places of F, and S_2 a finite set of places of F containing all the Archimedean places.

The Hausdorff locally compact abelian groups $\mathrm{GL}_1(\prod_{v\in S_1} F_v)$ and $\mathrm{GL}_1(\widehat{\prod}_{v\notin S_2}(F_v, \mathcal{O}_{F_v}))$ act continuously on $\prod_{v\in S_1} F_v$, respectively $\widehat{\prod}_{v\notin S_2}(F_v, \mathcal{O}_{F_v})$, by left-translation. We denote by

$$\rho_{F,S_1}, \quad \rho_{F,\Sigma_F - S_2}$$

such actions.

We denote by $\widehat{\mathcal{O}}_{F,S_1}$ the compact open subgroup of $\prod_{v\in S_1} F_v$ defined as $\prod_{v\in S_1} \mathcal{O}_{F_v}$, and by $\widehat{\mathcal{O}}_{F,\Sigma_F - S_2}$ the compact open subgroup of $\widehat{\prod}_{v\notin S_2}(F_v, \mathcal{O}_{F_v})$.

The continuous left-translation action of $\mathcal{O}_{F_v}^\times$, respectively of $\widehat{\mathcal{O}}_{F,S_1}^\times$, $\widehat{\mathcal{O}}_{F,\Sigma_F - S_2}^\times$, is "uniformly continuous" in the following sense:

LEMMA 2.1.2.10. *Let F be a number field.*

(1) *Let v a finite place of F. Let $\{f_m\}_{m\geq 0}$ be a convergent Cauchy sequence in $\mathcal{S}(F_v)$, with limit f. Let p be any continuous seminorm on $\mathcal{S}(F_v)$. Then we have:*

$$\lim_{m\geq 0} \sup_{g\in \mathcal{O}_{F_v}^\times} p(\rho_v(g, f_m - f)) = 0. \tag{2.1.2.3}$$

(2) *Let S_1 be a finite set of non-Archimedean places of F. Let $\{f_m\}_{m\geq 0}$ be a convergent Cauchy sequence in $\mathcal{S}(\prod_{v\in S_1} F_v)$. Let p be a continuous seminorm on $\mathcal{S}(\prod_{v\in S_1} F_v)$. Then we have:*

$$\lim_{m\geq 0} \sup_{g\in \widehat{\mathcal{O}}_{F,S_1}^\times} p(\rho_{F,S_1}(g, f_m - f)) = 0. \tag{2.1.2.4}$$

(3) *Let S_2 be a finite set of places of F containing all the Archimedean places. Let $\{f_m\}_{m\geq 0}$ be a convergent Cauchy sequence in $\mathcal{S}(\widehat{\prod}_{v\notin S_2}(F_v, \mathcal{O}_{F_v}))$, with limit f. Then we have:*

$$\lim_{m\geq 0} \sup_{g\in \widehat{\mathcal{O}}_{F,\Sigma_F - S_2}} p(\rho_{F,\Sigma_F - S_2}(g, f_m - f)) = 0. \tag{2.1.2.5}$$

PROOF. (1) : By Proposition 2.1.2.4(1), $\mathcal{S}(F_v)$ carries the finest locally convex topology, and it is countably linearly generated.

Every convergent Cauchy sequence $\{f_m\}_{m\geq 0}$ in $\mathcal{S}(F_v)$ is bounded (see Remark C.4.3.4).

By Proposition C.2.5.1(4), there exist integers $N \leq m$ and a bounded subset $B \subset \mathcal{S}(\pi_v^N \mathcal{O}_{F_v}/\pi_v^m \mathcal{O}_{F_v})$ whose image in $\mathcal{S}(F_v)$ contains $\{f_m\}_{m\geq 0}$ and its limit f. Since the compact open subgroups $\pi_v^N \mathcal{O}_{F_v}, \pi_v^m \mathcal{O}_{F_v} \subset F_v$ are $\mathcal{O}_{F_v}^\times$-stable, the continuous left-translation action by $\mathcal{O}_{F_v}^\times$ preserves the finite dimensional subspace V obtained as the image in $\mathcal{S}(F_v)$ of $\mathcal{S}(\pi_v^N \mathcal{O}_{F_v}/\pi_v^m \mathcal{O}_{F_v})$. Since $\mathcal{O}_{F_v}^\times$ is profinite, its action factors through a finite quotient. The supremum in (2.1.2.3) is therefore realized as a maximum over a finite set of representatives in $\mathcal{O}_{F_v}^\times$ for such finite quotient coset space, and (2.1.2.3) therefore holds by continuity of the action of $\mathcal{O}_{F_v}^\times$ on $\mathcal{S}(F_v)$.

(2) and (3) follow from a similar argument, using the cofinal sequences of compact open subgroups $N\widehat{\mathcal{O}}_{F,S_1} \subset \prod_{v\in S_1} F_v$, respectively $N\widehat{\mathcal{O}}_{F,\Sigma_F - S_2} \subset \widehat{\prod}_{v\notin S_2}(F_v, \mathcal{O}_{F_v})$, as N varies over the positive integers, and the fact that both $\widehat{\mathcal{O}}_{F,S_1}^\times$ and $\widehat{\mathcal{O}}_{F,\Sigma_F - S_2}^\times$ are profinite. □

REMARK 2.1.2.11. The conclusions of Lemma 2.1.2.10 hold for most of the actions of the compact groups G arising throughout this work. We will say the continuous action ρ_G of G on the Hausdorff locally convex topological $\mathbf{C}$-vector space E is *uniformly continuous*, for any convergent net $u_i \to u$, we have $\sup_{g\in G} p(\rho_G(g, u_i - u)) \to 0$ for any continuous seminorm p on E.

This equivalently means that for any convergent Cauchy net $f_i \to f$ in E, $i \in I$ for I a directed set, the vector valued functions $u_i : G \to E$ defined as $\rho_G(g, f_i)$ converge to $\rho_G(g, f)$ in $C(G, E)$ equipped with the topology of uniform convergence (i.e. generated by the family of seminorms $\sup_{g\in G} p(\cdot)$ as p varies over a fundamental set of continuous seminorms on E).

We term this condition *sequential uniform continuity*, if it holds for convergent sequences as opposed to general nets; see Proposition D.1.9.2.

REMARK 2.1.2.12. Let F be a number field, v a finite place of F. Let S be a finite set of places containing all the Archimedean places.

We denote by $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F-S}$ the profinite group $\prod_{v\notin S}\mathcal{O}^{\times}_{F_v}$, and by $\widehat{\prod}_{v\notin S}(F_v,\mathcal{O}_{F_v})$ the restricted product of the additive groups of the non-Archimedean local fields F_v constrained by the compact open subgroups $\mathcal{O}_{F_v}$, for each non-Archimedean place $v\notin S$.

We know from Proposition 2.1.2.4 and Proposition 2.1.2.8, that $\mathcal{S}(F_v)$ and $\mathcal{S}(\widehat{\prod}_{v\notin S}(F_v,\mathcal{O}_{F_v}))$ are bornological and complete, since they both carry the finest locally convex topology.

We know from Lemma 2.1.2.10 that the continuous left-translation action by $\mathcal{O}^{\times}_{F_v}$, respectively $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F-S}$, on each, is sequentially uniformly continuous. The assumptions of Lemma 4.3.3.1 in Chapter 4 are therefore satisfied, and the Lemma applies upon choosing E to be $\mathcal{S}(F_v)$, respectively $\mathcal{S}(\widehat{\prod}_{v\notin S}(F_v,\mathcal{O}_{F_v}))$, or a quotient by a (necessarily closed) subspace, and upon choosing $G=\mathcal{O}^{\times}_{F_v}$, respectively $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F-S}$. We will be recording this observation here for later reference.

2.1.3. Adelic Schwartz-Bruhat functions. We refer the reader to §2.1 for more on Schwartz-Bruhat functions on a general Hausdorff locally compact abelian group G, following Bruhat's work [**Bru**] and resting on the structure theory of Hausdorff locally compact abelian groups. We often refer, throughout this work, to the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(G)$ of Schwartz-Bruhat functions on G, as the *Schwartz-Bruhat space* of G.

2.1.3.1. *Kernel Theorem for the Adelic Schwartz-Bruhat spacee.* In this subsection, we focus on the case $G=\mathbf{A}_F,\mathbf{A}_{F,S}$ for F a number field and S a finite set of places of F containing all the Archimedean places.

We begin with the special case $F=\mathbf{Q}$.

EXAMPLE 2.1.3.1. We recall that we have an isomorphism $\mathbf{A_Q}=\mathbf{R}\times\mathbf{A}^f_{\mathbf{Q}}$ of Hausdorff locally compact abelian groups, where $\mathbf{A}^f_{\mathbf{Q}}$ is the topological ring of the finite adeles of $\mathbf{Q}$. See Definition A.1.1.3(2). As an additive topological group, $\mathbf{A}^f_{\mathbf{Q}}$ is itself a Hausdorff locally compact abelian group, totally disconnected and such that a neighborhood base at 0 is given by the subsets of the form $N\widehat{\mathbf{Z}}$ for $N\in\mathbf{Z}^+$.

Any compact open subgroup of $\mathbf{A}^f$ contains one of the form:

$$K=\prod_p K_p$$

with finite index, where $K_p=\mathbf{Z}_p$ for all but finitely many primes, and $K_p\subset\mathbf{Q}_p$ is a compact open subgroup of $\mathbf{Q}_p$ for all p. As such, $K_p\subset p^{-N_p}\mathbf{Z}_p$ for some integer $N_p\in\mathbf{Z}$.

As for the case of the p-adic rationals, and, in fact, as a consequence of the description of $\mathcal{S}(\mathbf{Q}_p)$ from Example 2.1.2.5, the $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}^f_{\mathbf{Q}})$ agrees with the $\mathbf{C}$-vector space of all smooth compactly supported functions on $\mathbf{A}^f_{\mathbf{Q}}$. The $\mathbf{C}$-linear space $\mathcal{S}(\mathbf{A}^f_{\mathbf{Q}})$ is topologized in a manner similar to $\mathcal{S}(\mathbf{Q}_p)$ (see Proposition 2.1.2.8(1) applied to $G_2=\mathbf{A}^f_{\mathbf{Q}}$).

There is a canonical linear isomorphism

$$\mathcal{S}(\mathbf{R})\otimes\mathcal{S}(\mathbf{A}^f_{\mathbf{Q}})\to\mathcal{S}(\mathbf{A_Q}) \tag{2.1.3.1}$$

leading to a description of the generic element of $\mathcal{S}(\mathbf{A_Q})$: it is a finite linear combination of functions of the form:

$$f=f_\infty\otimes\bigotimes_p f_p$$

where $f_\infty \in \mathcal{S}(\mathbf{R})$ and, for each prime number p, f_p is the indicator function of a subset of the form $a + p^N \mathbf{Z}_p$ for $N \in \mathbf{Z}$ and $a \in \mathbf{Q}_p$, and $f_p = 1_{\mathbf{Z}_p}$ for all but finitely many primes.

In fact, the linear isomorphism from (2.1.3.1) can be promoted to a topological one, by topologizing the algebraic tensor product as the *inductive* topological tensor product: it is the finest locally convex topology promoting the canonical bi-linear map $\mathcal{S}(\mathbf{R}) \times \mathcal{S}(\mathbf{A}_{\mathbf{Q}}^f) \to \mathcal{S}(\mathbf{R}) \otimes_\iota \mathcal{S}(\mathbf{A}_{\mathbf{Q}}^f)$ to a separately continuous bi-linear map. We will be proving this in Proposition 2.1.3.2 below. In particular, $\mathcal{S}(\mathbf{R}) \otimes_\iota \mathcal{S}(\mathbf{A}_{\mathbf{Q}}^f)$ is thereby complete, this completeness ultimately stems from expressing that topological tensor product as a strict countable inductive limit of Fréchet spaces (such as a product of finitely many copies of $\mathcal{S}(\mathbf{R})$), which is complete by [**Scha**, Ch.II,6.6,p.59].

We refer the reader to the discussion following (C.2.3.22) from §C.2.3, and to the references contained therein, for more on inductive tensor products. Proposition C.2.3.13 provides further basic properties, as well as more on other tensor topologies (projective and injective) which will be of relevance throughout this work.

Suffices here to say that inductive tensor products behave well with respect to inductive constructions (such as locally convex inductive limits) and they lend themselves to kernel theorems (see Lemma D.1.8.1) for spaces of test functions. We further refer the reader to Remark 2.1.3.3 below.

LEMMA 2.1.3.2. *Let F be a number field. The locally convex topological complex vector spaces $\mathcal{S}(\mathbf{A}_F^f)$ and $\mathcal{S}(\mathbf{A}_F)$ satisfy the following conditions:*

(1) *They are strict locally convex inductive limits of nuclear Fréchet spaces, and hence strict LF-spaces. They are Hausdorff, complete, nuclear and Montel, and their strong duals satisfy the same properties. Moreover, $\mathcal{S}(\mathbf{A}_F^f)'_\beta$ is a nuclear Fréchet space.*

(2) *The canonical linear map*

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f) \to \mathcal{S}(\mathbf{A}_F) \tag{2.1.3.2}$$

is a linear topological isomorphism. In particular, the inductive tensor product on the left side of (2.1.3.2) *is complete.*

(3) *For any $f \in \mathcal{S}(\mathbf{A}_F)$, there exist:*
   (1) *an integer $r \geq 1$.*
   (2) *Functions $\{f_i\}_{1 \leq i \leq r} \subset \mathcal{S}(\mathbf{A}_F)$ such that for each $1 \leq i \leq r$, f_i has the form $f_i = f_{\infty,i} \otimes f_{0,i}$ where $f_{\infty,i} \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_{0,i} = \prod_{v \in \Sigma_{F,f}} 1_{K_{v,i}}$ where $K_{v,i} \subset F_v$ is a compact subgroup with $K_{v,i} = \mathcal{O}_{F_v}$ for all but finitely many $v \in \Sigma_{F,f}$.*
   (3) *Complex numbers $\{c_i\}_{1 \leq i \leq r} \subset \mathbf{C}$.*

   such that $f = \sum_{1 \leq i \leq r} c_i f_i$.

REMARK 2.1.3.3 (Cautionary remark on inductive and projective tensor topologies). We refer to [**Scha**, Ch.II,§5] for a discussion of projective topologies. For E, F two locally convex complex vector spaces, there exists on the algebraic tensor product $E \otimes F$ a unique locally convex topology such that for any locally convex complex vector space G, continuous linear maps $E \otimes F \to G$ correspond bijectively to *continuous* (i.e. not merely separately continuous) bi-linear maps $E \times F \to G$ under the canonical correspondence given by composing a continuous linear map $E \otimes F \to G$ with the canonical continuous linear map $E \times F \to E \otimes F$.

This topology is referred to as the *projective* tensor product topology on $E \otimes F$, due to its projective nature, and also termed *π-topology*. A fundamental system of neighborhoods of 0 for this topology on $E \otimes F$ is given by the convex circled hulls of the sets of the form

$$U_i \otimes V_j := \{u \otimes v : u \in U_i,\ v \in V_j\}$$

where U_i and V_j are fundamental systems of neighborhoods of 0 in E, respectively F.

In this work, the projective topology is often used in combination with the inductive tensor topology. The completed inductive tensor product is denoted by $E \mathbin{\overline{\otimes}} F$, and it is most suitable to inductive constructions

and "kernel theorems" (results expressing a class of functions on a product space as a suitable completed tensor product of function spaces on the factors). Lemma D.1.8.1 gives examples of kernel theorems with Schwartz spaces and their strong duals in the setting of locally compact abelian groups.

We refer the reader to §C.2.3 of Appendix C, to [**Gro-c**, Thm.1] (and the discussion preceding the Theorem), [**Scha**, Ch.II,§5] and [**Jar**, Part III,§15.1] for a discussion of the projective (or π-)topology on an algebraic tensor product of two Hausdorff locally convex topological vector spaces, and to (C.2.3.23) and particularly to Proposition C.2.3.13 for a discussion and basic properties of inductive tensor products.

We observe here that the canonical continuous linear map:

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \mathcal{S}(\mathbf{A}_F^f)$$

is a continuous linear bijection. It is not, however, a linear topological isomorphism, because $\mathcal{S}(\mathbf{A}_F^f)$ is a strict locally convex inductive limit of finite dimensional Euclidean spaces (in particular, a strict locally convex inductive limit of Fréchet spaces) and the transition maps are *strict inclusions*. Moreover, $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a non-normable Fréchet space. Under such circumstances, the continuous linear bijection from the inductive to the projective tensor product is not a linear topological embedding, hence not a linear topological isomorphism. This is a consequence of [**Gro-b**, Ch.I,§1,No.3,Rem.1,p.47].

We further refer the reader to Remark 4.5.1.3 where a similar situation is discussed.

We further refer the reader to Remark 2.1.3.4 below, where the three main tensor topologies occurring throughout this work, are compared on $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \mathcal{S}(\mathbf{A}_F^f)$.

Proof of Proposition 2.1.3.2. Both $\mathbf{A}_F$ and $\mathbf{A}_F^f$ are Hausdorff locally compact abelian groups, and both are separable, being second-countable (see Lemma **??**). As such, [**Bru**, §9,Prop.11] applies and gives that $\mathcal{S}(\mathbf{A}_F)$ and $\mathcal{S}(\mathbf{A}_F^f)$ are Hausdorff, complete, nuclear and Montel with their strong duals satisfying the same properties.

We now provide a direct argument.

The locally convex complex topological vector space $\mathcal{S}(\mathbf{A}_F^f)$ is a strict inductive limit of finite dimensional $\mathbf{C}$-vector spaces (each endowed with its unique linear topology compatible with the Euclidean topology on the complex numbers).

Indeed, since $\mathbf{A}_F^f$ is the filtered union compact open subgroups of the form $K_v = \prod_{v \in \Sigma_{F,f}} H_v$, with $H_v \subset F_v$ compact open and $H_v = \mathcal{O}_{F_v}$ for all but finitely may places $v \in \Sigma_{F,f}$, $\mathcal{S}(\mathbf{A}_F^f)$ is the locally convex inductive limit of the finite dimensional complex vector spaces $\mathcal{S}(K_v'/K_v)$, as (K_v, K_v') varies across pairs of compact open subgroups in $\mathbf{A}_F^f$ with $K_v \subset K_v'$. The set of such pairs contains a cofinal filtered subset that is countable, since $\mathbf{A}_F^f$ is second-countable.

It follows that $\mathcal{S}(\mathbf{A}_F^f)$ carries the finest locally convex topology, and Proposition C.2.5.1(6) describes its properties. Alternatively, see Proposition 2.1.2.8 applied to $G_2 = \mathbf{A}_F^f$.

By Proposition 2.1.2.8(6), $\mathcal{S}(\mathbf{A}_F^f)'_\beta$ is a nuclear Fréchet space, hence Montel: by [**Tre**, Part III,Ch.50,Prop.50.2(5.12)], it is semi-Montel, as per [**Jar**, Part II,Ch.11,§11.5,p.230], and since it is barrelled, it is Montel (see [**Tre**, Part III,Ch.50,Cor.3,p.520]). This finishes the proof of (1) for $\mathcal{S}(\mathbf{A}_F^f)$.

We call $\{V_m\}_{m \geq 0}$ a choice of sequence of finite dimensional $\mathbf{C}$-linear spaces, with $\alpha_m : V_m \to V_{m+1}$ the injective $\mathbf{C}$-linear transition operators, such that $\varinjlim_{m \geq 0}(V_m, \alpha_m) = \mathcal{S}(\mathbf{A}_F^f)$.

The induced $\mathbf{C}$-linear operators between inductive tensor products

$$\alpha_{\mathcal{S},m} : \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_{m+1} \tag{2.1.3.3}$$

are injective, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a $\mathbf{C}$-linear space and the $\mathbf{C}$-linear spaces underlying the tensor products are algebraic tensor products, and continuous by [**Jar**, Part III,Ch.15,§15.2,Prop.1].

Each $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m$ is Hausdorff and complete since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is by Proposition 2.1.1.6, and V_m is finite dimensional. Moreover, each $\alpha_{\mathcal{S},m}$ is a topological embedding by [**Bou-GT**, Ch.I,§3.5,Prop.9], since it admits

a continuous section $s_m : \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_{m+1} \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m$ induced by any choice of $\mathbf{C}$-linear section to the injective $\mathbf{C}$-linear map α_m. Alternatively, since $\alpha_m(V_m)$ is complemented in V_{m+1}, $\alpha_{\mathcal{S},m}$ splits linearly and topologically by [**Jar**, Part III,Ch.15,§15.2,Prop.3(b)].

As such, the sequence $\{\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m, \alpha_{\mathcal{S},m}\}_{m\geq 0}$ is strict in the sense of [**Jar**, Part I,Ch.4,§4.6], and the locally convex inductive limit topology on $\varinjlim_{m\geq 0}(\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi V_m)$ is Hausdorff by [**Jar**, Ch.4,§4.6,Thm.1] and complete by [**Jar**, Ch.4,§4.6,Thm.4].

The injective continuous linear operators $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f)$ are topological embeddings, since the continuous linear map $V_m \to \mathcal{S}(\mathbf{A}_F^f)$ splits linearly by Proposition C.2.5.1(3), and hence so does $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f)$ by [**Jar**, Part III,Ch.15,§15.2,Prop.3(b)].

It follows that the topology induced on $\varinjlim_{m\geq 0}(\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m)$ by the canonical linear operator $\varinjlim_{m\geq 0}(\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f)$ is the locally convex inductive limit topology, by Proposition C.2.3.13(2), i.e. the canonical injective linear operator $\varinjlim_{m\geq 0}(\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f)$ is a linear topological embedding.

It is surjective since the linear spaces underlying the topological tensor products are the respective algebraic tensor products, and hence it is a linear topological isomorphism.

We deduce that the canonical linear operator from (2.1.3.2) is a surjective linear topological embedding, and hence $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f)$ and $\mathcal{S}(\mathbf{A}_F)$ are Hausdorff, complete, nuclear and Montel, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is by Proposition 2.1.1.6, each V_m is, being finite dimensional, $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is, as these properties are preserved by finite linear topological direct sums, and since they are preserved by strict inductive limits.

This proves (1) and (2): indeed, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a nuclear Fréchet space, so is each $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m \simeq \mathcal{S}(\mathbf{A}_{F,S_\infty})^{\oplus N_m}$ by [**Scha**, Ch.III,§7,Thm.7.4], and so $\mathcal{S}(\mathbf{A}_F)$ is a strict LF-space and a strict locally convex inductive limit of nuclear Fréchet spaces. Moreover, since each $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota V_m$ is a reflexive Hausdorff locally convex topological $\mathbf{C}$-vector space, and the transition maps split linearly and topologically, Lemma C.1.6.1 applies, to give that $\mathcal{S}(\mathbf{A}_F)'_\beta$ is a countable projective limit of Hausdorff complete nuclear locally convex topological $\mathbf{C}$-vector spaces, and hence Hausdorff, complete (see [**Scha**, Ch.II,§5,5.3,p.52]), nuclear (see [**Scha**, Ch.III,§7,Corollary,p.103]) and hence Montel (by [**Scha**, Ch.III,§7,Cor.2,p.101]).

We deduce that $\mathcal{S}(\mathbf{A}_F^f)$ is isomorphic, as a complex vector space, to the $\mathbf{C}$-linear subspace of those continuous functions that are finite linear combinations of functions of the form $\prod_{v\in\Sigma_{F,f}} 1_{K_v}$, where $K_v \subset F_v$ is a compact subset, with $K_v = \mathcal{O}_{F_v}$ for all but finitely many $v \in \Sigma_{F,f}$, and (3) follows from (2). □

Remark 2.1.3.4 (Comparison of the main tensor topologies). Upon endowing the algebraic tensor product $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \mathcal{S}(\mathbf{A}_F^f)$ with the injective tensor product topology (see [**Tre**, Ch.43], [**Gro-b**, Ch.I], [**Scha**, Ch.III,§5] and [**Scha**, Ch.II,§6], and particularly [**Gro-b**, Ch.I,§3,n.3,p.89,Def.5]), the linear map

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\varepsilon \mathcal{S}(\mathbf{A}_F^f) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \mathcal{S}(\mathbf{A}_F^f)$$

induced by [**Jar**, Part III,§16.1,Prop.3] is a linear topological isomorphism since, by nuclearity of $\mathcal{S}(\mathbf{A}_F^f)$, the injective and projective tensor product topologies on $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \mathcal{S}(\mathbf{A}_F^f)$ coincide (see [**Gro-b**, Ch.I,§5,Prop.35]).

In light of Remark 2.1.3.3, the inductive tensor topology on $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \mathcal{S}(\mathbf{A}_F^f)$ is strictly finer than the injective and projective topologies, while the latter two are the same.

Remark 2.1.3.5. We note that Schwartz-Bruhat functions of the form $f = \bigotimes_v f_v$, with $f_v \in \mathcal{S}(F_v)$ for *every* place v of F, and $f_v = 1_{\mathcal{O}_{F_v}}$ for all but finitely many finite places of F, linearly generate a dense (and even sequentially dense) linear subspace of $\mathcal{S}(\mathbf{A}_F)$ (for instance, from [**Tre**, Thm.51.6,(51.3)]).

They are linear generators of $\mathcal{S}(\mathbf{A}_F)$ only for F having a single Archimedean place: $\mathbf{Q}$ and imaginary quadratic fields.

2.1.4. Inductive structure of $\mathcal{S}(\mathbf{A}_F)$. We present here a structural result on $\mathcal{S}(\mathbf{A}_F)$, exhibiting $\mathcal{S}(\mathbf{A}_F)$ as a locally convex inductive limit of the S-adelic Schwartz-Bruhat spaces $\mathcal{S}(\mathbf{A}_{F,S})$.

PROPOSITION 2.1.4.1. *Let F be a number field, S a finite set of places of F containing all of the Archimedean places. We denote by S_f the subset of S of all the finite places of F contained in S. Then:*

(1) *The canonical linear map:*

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \bigotimes_{v\in S_f} \mathcal{S}(F_v) \to \mathcal{S}(\mathbf{A}_{F,S}) \tag{2.1.4.1}$$

the topological tensor product being inductive, is a linear topological isomorphism, whose formation is compatible, in the sense of (2.1.4.2) below, with extensions $S \subset S'$ of the finite set of places S of F.

(2) *$\mathcal{S}(\mathbf{A}_{F,S})$ is a strict locally convex inductive limit of nuclear Fréchet spaces, and hence a strict LF-space. It is Hausdorff, complete, nuclear and Montel, and its strong dual satisfies the same properties.*

(3) *If S' is another finite set of places of F, with $S \subset S'$, then the canonical linear map:*

$$\pi_{S,S'} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S'}) \tag{2.1.4.2}$$

defined by the assignment:

$$\pi_{S,S'}(f) := f \otimes (\prod_{v\in S'_f - S_f} 1_{\mathcal{O}_{F_v}}) \tag{2.1.4.3}$$

is continuous, injective, and induces on $\mathcal{S}(\mathbf{A}_{F,S})$ its intrinsic Hausdorff locally convex topology as the pre-image of the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S'})$.

(4) *The maps $\pi_{S,S'}$ from (2.1.4.2) are topological embeddings.*

(5) *The canonical linear map:*

$$\pi : \operatorname*{colim}_{S\subset\Sigma_F} \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_F) \tag{2.1.4.4}$$

induced by the maps:

$$\pi_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_F) \tag{2.1.4.5}$$

each defined by $\pi_S(f) := f \otimes 1_K$ where $K \subset \widehat{\prod}_{v\notin S}(F_v, \mathcal{O}_{F_v})$ is the compact subset $\prod_{v\notin S} \mathcal{O}_{F_v}$, is a linear topological isomorphism, upon endowing the inductive limit with its locally convex inductive limit topology.

PROOF. (1) and (2) : Since each $\mathcal{S}(F_v)$, for v any finite place of F, is the locally convex inductive limit of finite dimensional $\mathbf{C}$-vector spaces with injective $\mathbf{C}$-linear transition maps, so is the tensor product $\bigotimes_{v\in S_f} \mathcal{S}(F_v)$, since the tensor product of finitely many injective $\mathbf{C}$-linear maps between finite dimensional $\mathbf{C}$-vector spaces is an injective $\mathbf{C}$-linear map between finite dimensional $\mathbf{C}$-vector spaces, and a $\mathbf{C}$-linear subspace of a finite dimensional $\mathbf{C}$-vector space, endowed with the Euclidean topology, always inherits the subspace topology, and as such it is closed.

By Proposition 2.1.2.4 and Lemma C.2.5.2, $\bigotimes_{v\in S_f} \mathcal{S}(F_v)$ is indeed the locally convex inductive limit of a sequence of finite dimensional $\mathbf{C}$-vector spaces with injective $\mathbf{C}$-linear transition maps. It is nuclear by [**Jar**, Part III,Ch.21,§21.2,Cor.3] and the leftmost side in (2.1.4.1) is itself the locally convex inductive limit of a sequence of Hausdorff locally convex topological complex vector spaces of the form $\mathcal{S}(\mathbf{A}_{F,S_\infty})^{\oplus N}$ for varying integers $N \geq 1$, with injective transition maps, each a topological embedding by [**Jar**, Part II,Ch.16,§15.2,Cor.2(b),p.348] (using nuclearity of $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ to identify injective and projective tensor topologies, together with Corollary C.2.3.12, using that $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a Fréchet space, as is any finite dimensional $\mathbf{C}$-vector space).

By [**Scha**, Ch.II,§6,6.6], $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \bigotimes_{v\in S_f} \mathcal{S}(F_v)$ (and by Proposition C.2.3.13(2)) is Hausdorff and complete, and by [**Jar**, Part III,Ch.21,§21.2,Cor.3], it is nuclear (being a countable locally convex inductive limit of nuclear Fréchet spaces). By [**Jar**, Part II,Ch.11,§11.5,Prop.4(e)], it is Montel.

The linear map in (2.1.4.1) is a continuous bijection.

We consider the closed subspace

$$\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota V\subset \mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota \bigotimes_{v\in S_f}\mathcal{S}(F_v) \tag{2.1.4.6}$$

for $V\subset\bigotimes_{v\in S_f}\mathcal{S}(F_v)$ a finite dimensional $\mathbf{C}$-vector subspace.

The inductive tensor product $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota V$ is a Fréchet space, and the composition of the linear topological embedding in (2.1.4.6) followed by the continuous linear operator from (2.1.4.1), is continuous, injective, and its image is a $\mathbf{C}$-linear subspace W of $\mathcal{S}(\mathbf{A}_{F,S})$ spanned by the Schwartz-Bruhat functions of the form $f_\infty\otimes\bigotimes_{v\in S_f}1_{K_{v,i_v}}$ for $f_\infty\in\mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $K_{v,i_v}\subset F_v$ are finitely many compact subsets. W is therefore closed and metrizable, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a Fréchet space by Proposition 2.1.1.6.

By [**NB**, Thm.14.4.5], the restriction of the continuous linear operator from (2.1.4.1) to $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota V$ is therefore a linear topological embedding. Since the topologies on $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota\bigotimes_{v\in S_f}\mathcal{S}(F_v)$ and $\mathcal{S}(\mathbf{A}_{F,S})$ are locally convex strict inductive limits of the topologies on subspaces of the form $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes V$ and W, compatibly with the continuous linear operator from (2.1.4.1) (see Proposition C.2.3.13(2)), then such operator is a linear topological embedding, hence a linear topological isomorphism.

We deduce that $\mathcal{S}(\mathbf{A}_{F,S})$ is Hausdorff, complete, nuclear and Montel, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota\bigotimes_{v\in S_f}\mathcal{S}(F_v)$ is.

Since $\mathcal{S}(\mathbf{A}_{F,S})$ is Montel, then the strong dual $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ is Hausdorff and Montel by [**Scha**, Ch.IV,§5,5.9] (see also [**MV**, Part IV,Ch.24,Thm.24.25,p.286]). It is complete by [**Scha**, Ch.IV,§6,6.1]. Since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a strict inductive limit of nuclear Fréchet spaces (of the form $\mathcal{S}(\mathbf{A}_{F,S_\infty})^{\oplus N}$ for integers $N\geq 1$) with topologically split transition maps, its strong dual is a projective limit (see Lemma C.1.6.1) of nuclear spaces, hence nuclear by [**Gro-b**, Ch.II,Thm.9].

Compatibility of the linear topological isomorphism in (2.1.4.1) with enlargement of the finite set of places S is a consequence of the construction.

This proves (1) and (2), noting that $\mathcal{S}(\mathbf{A}_{F,S})$ is a strict locally convex inductive limit of subspaces of the form $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\iota V\simeq\mathcal{S}(\mathbf{A}_{F,S_\infty})^{\oplus N_V}$ for a finite dimensional $\mathbf{C}$-vector space V (N_V being the dimension of V), and each is therefore nuclear and Fréchet by [**Scha**, Ch.III,§7,Thm.7.4].

(3) follows from (1).

(4) : By (3), the inductive system $\{\mathcal{S}(\mathbf{A}_{F,S})\}$ is strict.

(5) : The linear operator in (2.1.4.4) is a continuous bijection as a consequence of the definition of Schwartz-Bruhat functions on $\mathbf{A}_F$. Since each π_S is a linear topological embedding, then π is a linear topological isomorphism since by (4) the inductive system $\{\mathcal{S}(\mathbf{A}_{F,S})\}_{S\subset\Sigma_F}$ is strict. □

Corollary 2.1.4.2. *Let F be a number field, S,S' finite sets of places of F containing all Archimedean places, with $S\subset S'$. The canonical linear maps:*

$$\mathcal{S}(\mathbf{A}_{F,S})\otimes_\iota\mathcal{S}(G)\to\mathcal{S}(\mathbf{A}_F),\ \ \mathcal{S}(\mathbf{A}_{F,S})\otimes_\iota\mathcal{S}(\prod_{v\in S'-S}F_v)\to\mathcal{S}(\mathbf{A}_{F,S'}) \tag{2.1.4.7}$$

with $G:=\widehat{\prod}_{v\notin S}(F_v,\mathcal{O}_{F_v})$, are linear topological isomorphisms. In particular, both inductive tensor products are complete.

Proof. In light of Proposition 2.1.4.1(1) and Proposition 2.1.3.2(2), it is enough to show that the canonical linear map:

$$\bigotimes_{v\in S_f}\mathcal{S}(F_v)\otimes_\iota\mathcal{S}(G)\to\mathcal{S}(\mathbf{A}_F^f) \tag{2.1.4.8}$$

the topological tensor product on the left side of (2.1.4.8) being inductive, is a linear topological isomorphism. Since it is a bijection, then Proposition C.2.5.1 implies it is a linear topological isomorphism. Alternatively, this holds as a consequence of Lemma D.1.8.1.

Similarly, the second operator in (2.1.4.7) is also a linear topological isomorphism. □

LEMMA 2.1.4.3. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Then:*

(1) *There exists a continuous linear map*

$$\Pi_S : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F) \tag{2.1.4.9}$$

whose image is $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$ and is the identity on $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$, for π_S as in (2.1.4.5)*, and satisfies $\Pi_S^2 = \Pi_S$. In particular, the operator π from* (2.1.4.4) *has topologically complemented image.*

(2) *For any finite set S' of places of F containing S, the continuous linear map*

$$\Pi'_{S,S'} : \mathcal{S}(\mathbf{A}_{F,S'}) \to \mathcal{S}(\mathbf{A}_F) \tag{2.1.4.10}$$

defined as

$$\Pi'_{S,S'} := \Pi_S \circ \pi_{S'} \tag{2.1.4.11}$$

whose image is $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$, factors through a continuous linear map

$$\Pi_{S,S'} : \mathcal{S}(\mathbf{A}_{F,S'}) \to \mathcal{S}(\mathbf{A}_{F,S'}) \tag{2.1.4.12}$$

whose image is $\pi_{S,S'}(\mathcal{S}(\mathbf{A}_{F,S}))$, satisfying $\Pi_{S,S'}^2 = \Pi_{S,S'}$.

In particular, the map $\pi_{S,S'}$ from (2.1.4.3) *has linearly and topologically complemented image.*

(3) *For two finite sets S', S'' of places of F such that $S \subset S' \subset S''$, we have:*

$$\Pi_{S,S'} = \Pi_{S,S''} \circ \pi_{S',S''} \tag{2.1.4.13}$$

where $\pi_{S,S'}$ is as in (2.1.4.2)*.*

PROOF. (1) : We have

$$\pi_S(\mathcal{S}(\mathbf{A}_{F,S})) = \mathcal{S}(\mathbf{A}_{F,S}) \otimes 1_K$$

where $K \subset \widehat{\prod}_{v \notin S}(F_v, \mathcal{O}_{F_v})$ is the compact open subset $\prod_{v \notin S} \mathcal{O}_{F_v}$ and 1_K is its indicator function.

For $G := \widehat{\prod}_{v \notin S}(F_v, \mathcal{O}_{F_v})$, its Schwartz-Bruhat space $\mathcal{S}(G)$ is Hausdorff locally convex as a topological $\mathbf{C}$-vector space, and the line spanned by 1_K is complemented by the Hahn-Banach Theorem. Indeed, for the functional $\mathbf{C} \cdot 1_K \to \mathbf{C}$ defined as $\varphi : c1_K \mapsto c$, there exists a continuous extension $\widetilde{\varphi} : \mathcal{S}(G) \to \mathbf{C}$ by [**NB**, Cor.7.3.3], satisfying $\widetilde{\varphi}|_{\mathbf{C} \cdot 1_K} = \varphi$. We define

$$\Pi(f) := \widetilde{\varphi}(f) 1_K$$

to obtain a continuous operator $\Pi : \mathcal{S}(G) \to \mathcal{S}(G)$ such that $\Pi^2 = \Pi$ and the image of Π is $\mathbf{C} \cdot 1_K$.

We now define:

$$\Pi'_S := \mathrm{id}_{\mathcal{S}(\mathbf{A}_{F,S})} \otimes \Pi : \mathcal{S}(\mathbf{A}_{F,S}) \otimes_\iota \mathcal{S}(G) \to \mathcal{S}(\mathbf{A}_{F,S}) \otimes_\iota \mathcal{S}(G).$$

However, since we have $\mathbf{A}_F \simeq \mathbf{A}_{F,S} \times G$, we also have, by Corollary 2.1.4.2, a linear topological isomorphism $\mathcal{S}(\mathbf{A}_F) \simeq \mathcal{S}(\mathbf{A}_{F,S}) \otimes_\iota \mathcal{S}(G)$, where the target is already complete since it is the strict inductive limit of a sequence of complete Hausdorff locally convex spaces, as $\mathcal{S}(G)$ is the strict inductive limit of a sequence of finite dimensional $\mathbf{C}$-vector spaces.

The continuous linear operator Π'_S therefore induces a unique continuous linear operator $\Pi_S : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$ satisfying $\Pi_S^2 = \Pi_S$. Its image is the image under the linear topological isomorphism $\mathcal{S}(\mathbf{A}_{F,S}) \otimes_\pi \mathcal{S}(G)$, of the subspace $\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathbf{C} \cdot 1_K$, which is $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$. Moreover, by design of Π'_S, we have, for every $f \in \pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$, i.e. $f = f_0 \otimes 1_K$ for unique $f_0 \in \mathcal{S}(\mathbf{A}_{F,S})$:

$$(\Pi_S \circ \pi_S)(f) = \Pi_S(f_0 \otimes 1_K) = (\mathrm{id}_{\mathcal{S}(\mathbf{A}_{F,S})}(f_0) \otimes \Pi(1_K)) = f_0 \otimes 1_K = f.$$

(2) and (3) follow from (1) and the identity $\pi_{S''} \circ \pi_{S',S''} = \pi_{S'}$ as continuous linear operators $\mathcal{S}(\mathbf{A}_{F,S'}) \to \mathcal{S}(\mathbf{A}_F)$. □

For E a Hausdorff locally convex $\mathbf{C}$-vector space, we denote by $\mathscr{L}(E) := \mathscr{L}(E,E)$ the $\mathbf{C}$-linear space of continuous linear endomorphisms of E, and write $\mathscr{L}_\beta(E)$ for $\mathscr{L}(E,E)$ endowed with the topology of uniform convergence on all bounded subsets of E (see Definition C.2.1.5).

The following Corollary will not be used throughout this work. It is however indicative of how the S-local Schwartz-Bruhat spaces $\mathcal{S}(\mathbf{A}_{F,S})$ encompass large pieces of the global Schwartz-Bruhat space $\mathcal{S}(\mathbf{A}_F)$, up to a closed subspace ("smaller and smaller" as S grows large) that splits off linearly and topologically. This will be a general pattern to follow throughout the later Chapters of this work, where we will be analyzing large subquotients of $\mathcal{S}(\mathbf{A}_F)$ by studying their S-local analogs for all large enough S: the fact that such analysis is possible rests on properties of Hecke L-functions, to be harnessed through the systematic use of the Mellin transform (idelic, in the spirit of Tate's dissertation from [**CF**, Ch.XV], and S-idelic).

COROLLARY 2.1.4.4. *Let F be a number field, S a finite set of places of F containing all the Archimedean places of F. Let:*

$$\Pi_S : \mathcal{S}(\mathbf{A}_F) \to \pi_S(\mathcal{S}(\mathbf{A}_{F,S})) \tag{2.1.4.14}$$

be any continuous projection onto the linear image of π_S from (2.1.4.5) *satisfying $\Pi_S \circ \pi_S = \pi_S$. Then the net*

$$\{\Pi_S\}_{S \subset \Sigma_F} \subset \mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_F))$$

indexed by the finite subsets of Σ_F containing S_∞, converges to the identity on $\mathcal{S}(\mathbf{A}_F)$ with respect to the topology of uniform convergence on bounded susbsets.

PROOF. To see this, we let $B \subset \mathcal{S}(\mathbf{A}_F)$ be a bounded subset. By Proposition 2.1.4.1(4), and by [**Scha**, Ch.II,§6,6.5], there exits $B_0 \subset \mathcal{S}(\mathbf{A}_{F,S_0})$ bounded, for S_0 large enough, such that $B = \pi_{S_0}(B_0)$.

For any $S \subset \Sigma_F$ finite and containing S_∞, such that $S_0 \subset S$, we therefore have $\Pi_S = \mathrm{id}$ over B. We call $I(S_0)$ their set, and we call I the set of all finite subsets $S \subset \Sigma_F$ containing S_∞. I is filtered by inclusion and contains $I(S_0)$ as a filtered subset.

$I(S_0)$ is cofinal in I, since for every $S \in I$, the set $S_0 \cup S$ is in $I(S_0)$ and contains S. In particular, the net $I \to \mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_F))$ defined as $T_i := \Pi_S$ where $S = i$ in I, converges to id uniformly over the bounded subset B of $\mathcal{S}(\mathbf{A}_F)$. Indeed, this requires the existence, for every $\varepsilon > 0$, of an element $i_0 \in I$ such that for every $i \geq i_0$:

$$\sup_{b \in B} p(T_i(b) - b) < \varepsilon$$

with p varying over any fundamental family of continuous seminorms on $\mathcal{S}(\mathbf{A}_F)$. We may choose $i_0 := S_0$, and this indeed holds (for any $\varepsilon > 0$).

Note that the seminorms $p_B : \mathscr{L}(\mathcal{S}(\mathbf{A}_F)) \to \mathbf{R}_{\geq 0}$, as B varies over all the bounded subsets of $\mathcal{S}(\mathbf{A}_F)$, defined as $p_B(T) := \sup_{b \in B} p(T(b))$ generate the topology β on $\mathscr{L}(\mathcal{S}(\mathbf{A}_F))$ of uniform convergence on bounded subsets (see [**Scha**, Ch.III,§3,p.81], as p varies over any fundamental family of continuous seminorms on $\mathcal{S}(\mathbf{A}_F)$, $\mathfrak{S}$ the set of all bounded subsets of $\mathcal{S}(\mathbf{A}_{F,S})$; see also (C.2.1.4)). We have just shown that for each bounded subset $B \subset \mathcal{S}(\mathbf{A}_F)$, we have $\lim_{S \in I} p_B(\Pi_S - \mathrm{id}) = 0$.

This means $\Pi_S \to \mathrm{id}$ uniformly on the bounded subsets of $\mathcal{S}(\mathbf{A}_F)$, i.e. $\Pi_S \to \mathrm{id}$ in $\mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_F))$. □

2.2. Global adelic θ-functions

Let F be a number field. In this section we will be defining and studying the linear map associating to a Schwartz-Bruhat function $f : \mathbf{A}_F \to \mathbf{C}$ its restriction to $\mathrm{GL}_1(\mathbf{A}_F)$ averaged over the continuous translation action by the discrete subgroup $F^\times$. We will show that such averages are themselves continuous as functions on the idele class group C_F of F, and decaying rapidly at infinity in a precise sense that can be controlled using the idelic norm.

Such $F^\times$-averaged Schwartz-Bruhat functions are called *global adelic θ-functions*. We will study them systematically throughout the rest of Chapter 2, and then collect them into a topological vector space in the next Chapter.

2.2.1. Averaging over a full lattice. In this subsection we record a convergence result concerning averages of norm functions on Euclidean spaces under the translation action by a full lattice.

Let $N \geq 1$ be an integer and consider $\Gamma \subset \mathbf{R}^N$ a discrete subgroup. If Γ is co-compact, then Γ is a finite free abelian group of rank N, i.e. a full lattice.

We denote by $\| \cdot \|_\infty : \mathbf{R}^N \to \mathbf{R}_{\geq 0}$ the norm $\|x\|_\infty := \max\{|x_i|_{\mathbf{R}}, 1 \leq i \leq N\}$.

LEMMA 2.2.1.1. *Let $N \geq 1$. Let $\Gamma \subset \mathbf{R}^N$ be a full lattice. For every real number $r \geq 1$, the set*

$$X_r := \{x \in \mathbf{R}^N : \|\xi\|_\infty \in [r, r+1)\} \subset \mathbf{R}^N \tag{2.2.1.1}$$

is bounded, Lebesgue measurable, and it satisfies:

$$|\Gamma \cap X_r| = O_{\Gamma,N}(r^{N-1}) \tag{2.2.1.2}$$

as $r \to +\infty$, the implied constant depending N and on Γ.

REMARK 2.2.1.2. We note that the implied constant in (2.2.1.2) depends on N on itself, and also on N through Γ, through its successive minima: for a full lattice $\Gamma \subset \mathbf{R}^N$, the successive minima $r_1, \ldots, r_N$ are defined as, for each integer $1 \leq k \leq N$, the infimum over all real numbers $r \geq 0$ such that $B_0(r) \cap \Gamma$ contains k vectors which are $\mathbf{R}$-linearly independent (the closed ball $B_0(r)$ of radius $r > 0$ and center the origin $0 \in \mathbf{R}^N$ being defined using the standard Euclidean norm); see [**Schm**, §2,p.347].

The method to estimate the error term for $|\Gamma \cap X_r| - \frac{\mathrm{vol}(X_r)}{\mathrm{vol}(\mathbf{R}^N/\Gamma)}$, where volumes are computed in $\mathbf{R}^N$ with respect to the Lebesgue measure, normalized so as to give the hypercube $[0,1)^N$ volume 1, follows [**Schm**, Lemma 1,p.347] and springs from the proof of [**Davenport-1**, §2,Thm.p.180].

PROOF. We denote, for every real number $r \geq 1$, by $X_r \subset \mathbf{R}^N$ the subset $X_r := \{x \in \mathbf{R}^N : \|x\|_\infty \in [r, r+1)\}$. Since $\| \cdot \|_\infty : \mathbf{R}^N \to \mathbf{R}_{\geq 0}$ is continuous, surjective and proper, and X_r is the pre-image of the bounded interval $[r, r+1)$, we deduce that X_r is measurable and relatively compact, and so is its image under the orthogonal projection onto any linear subspace of $\mathbf{R}^N$. Indeed, such image is a convex polyhedron within such subspace.

Let $L \subset \mathbf{R}^N$ be a line. Then $L \cap X_r$ is either empty or the union of at most two segments. Indeed, for $x, y \in \mathbf{R}^N$ and such that $L = x + \mathbf{R} \cdot y$, we have

$$x + ty \in X_r$$

if and only if

$$f(t) := \sup_{1 \leq k \leq N} |x_k + ty_k| \in [r, r+1).$$

Note that y is necessarily nonzero.

The function $f : \mathbf{R} \to \mathbf{R}_{\geq 0}$ is continuous, surjective and proper. Moreover, its fibre over $r \in \mathbf{R}_{\geq 0}$ is the pre-image under the map $t \mapsto x + ty$, of the intersection of the closed surface $S_r := \{\|x\|_\infty = r\}$, the boundary of the N-dimensional hypercube in $\mathbf{R}^N$ with edge-length r, with L. The fibres of f are therefore finite of size at most 2, since the interior of S_r is convex.

The subset X_r is obtained from the interior of S_{r+1} by removing the interior of S_r, and so the intersection of L with the interior of S_{r+1}, which, by the foregoing, is either a point of a segment, intersects X_r in the complement of the intersection of L with the interior of S_r, again a point of a segment. This shows that $L \cap X_r$ is either empty, a segment or two segments.

It follows that [**Schm**, Lemma 1,p.347] (see also [**Gao**, p.15]) applies. Indeed, X_r is of class 2, following the terminology of loc.cit., and for $\lambda_1, \ldots, \lambda_N$ the successive minima of Γ (for each $k \geq 1$, each λ_k is the

infimum among all positive real numbers λ such that Γ contains k linearly independent vectors in $\mathbf{R}^N$ of Euclidean norm at most λ) we have $\lambda_{N-1} \leq r$ for all large enough $r > 0$, since λ_N does not depend on r.

Then we have:

$$\left| |\Gamma \cap X_r| - \frac{\mathrm{vol}(X_r)}{\mathrm{vol}(\mathbf{R}^N/\Gamma)} \right| \leq (1 + \frac{2}{\sqrt{3}} 3^N N)^N \max_{1 \leq k \leq N-1} \frac{r^k}{\prod_{1 \leq i \leq k} \lambda_k},$$

where $\mathrm{vol}(X_r) = (2(r+1))^N - (2r)^N = 2^N N r^{N-1} + O_N(r^{N-2})$ as $r \to +\infty$.

We deduce that (2.2.1.2) holds, the implied constant depending on N and on Γ both through its co-volume and through its successive minima. □

We need the following convergence result:

PROPOSITION 2.2.1.3. *Let $N \geq 1$ be an integer, $\Gamma \subset \mathbf{R}^N$ a full lattice. Let $\| \cdot \| : \mathbf{R}^N \to \mathbf{R}_{\geq 0}$ be a norm on $\mathbf{R}^N$ inducing the Euclidean topology. Let $K \subset \mathbf{R}^N$ be a fixed arbitrary compact subset. Let $\sigma > 0$ be a real number.*

Then the series

$$\sum_{\xi \in \Gamma - \Gamma \cap K} \|\xi + x\|^{-\sigma} \tag{2.2.1.3}$$

converges for $\sigma > N$, uniformly with respect to x varying over K, and, for every $\varepsilon > 0$, uniformly for $(\sigma, x) \in [N + \varepsilon, +\infty) \times K$.

PROOF. There exists a real number $C > 0$ such that $\| \cdot \|_\infty \leq C \| \cdot \|$ as real-valued continuous functions on $\mathbf{R}^N$.

We therefore have the inequality

$$\sum_{\xi \in \Gamma - \Gamma \cap K} \|\xi + x\|^{-\sigma} \leq \frac{1}{C^\sigma} \sum_{\xi \in \Gamma - \Gamma \cap K} \|\xi + x\|_\infty^{-\sigma}. \tag{2.2.1.4}$$

We now call

$$Z_m := \{(\xi, x) \in (\Gamma - \Gamma \cap K) \times K : \|\xi + x\|_\infty \in [m, m+1)\} \subset (\Gamma - \Gamma \cap K) \times K \tag{2.2.1.5}$$

for $m \in \mathbf{N}$.

For each $x \in K$, we denote $Z_{m,x} \subset Z_m$ the fibre of $Z_m \to K$ over $\{x\}$. Since, for each $x \in \mathbf{R}^N$ fixed, $\{y \in \mathbf{R}^N : \|x + y\|_\infty \in [m, m+1)\}$ is relatively compact and $\Gamma \subset \mathbf{R}^N$ is discrete, the fibre $Z_{m,x}$ is finite.

We note that $Z_{m,0}$ is non-empty.

We have $|Z_{m,0}| \ll m^{N-1}$ by Lemma 2.2.1.1. For $x \in K$ fixed, upon combining $|\|\xi\|_\infty - \|x\|_\infty| \leq \|\xi + x\|_\infty$ and $\|x + \xi\|_\infty \leq \|x\|_\infty + \|\xi\|_\infty$, we deduced that if $(\xi, x) \in Z_{m,x}$ is fixed, then we have

$$\|\xi\|_\infty \in [m - \|x\|_\infty, m + \|x\|_\infty + 1).$$

For $m(x)$ the least nonnegative integer such that $\|x\|_\infty \leq m(x)$, we have

$$\xi \in \bigcup_{m - m(x) \leq k \leq m + m(x)} Z_{k,0}$$

and hence

$$|Z_{m,x}| \ll_{x,N} \sum_{1, m - m(x) \leq k \leq m + m(x)} k^{N-1} \ll_{x,N} (m + m(x))^{N-1}.$$

Since K is compact, there exists an integer $m(K) \geq 0$ such that $m(x) \leq m(K)$ for every $x \in K$.

It follows that we have

$$|Z_{m,x}| \ll_{K,N} (m + m(K))^{N-1} \ll_{K,N} m^{N-1}$$

the implied constant depending only on N and K.

We have

$$\sum_{\xi\in\Gamma-\Gamma\cap K}\|\xi+x\|_\infty^{-\sigma}\ll_{K,N}\sum_{m\geq 1}m^{N-1-\sigma}$$

and hence the series in (2.2.1.3) is absolutely convergent, uniformly with respect to $x\in K$, for each $\sigma>N$, and the convergence is uniform for $(\sigma,x)\in[N+\varepsilon,+\infty)\times K$ for every $\varepsilon>0$. □

2.2.2. Averaging over $\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F$ under $(F^\times\times F)$-translation. For F a number field and $f\in\mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function on the adeles of F, we can restrict f to $\mathrm{GL}_1(\mathbf{A}_F)$, i.e. consider the pre-composition of f with the continuous identity map $\mathrm{GL}_1(\mathbf{A}_F)\to\mathbf{A}_F^\times$, with $\mathbf{A}_F^\times$ endowed with the adelic topology, followed by the topological embedding $\mathbf{A}_F^\times\subset\mathbf{A}_F$.

Such restriction is a continuous map $\mathrm{GL}_1(\mathbf{A}_F)\to\mathbf{C}$, and the continuous left-translation action by the discrete subgroup $F^\times\subset\mathrm{GL}_1(\mathbf{A}_F)$ leads to the formation of the average of f along each $F^\times$-orbit in $\mathrm{GL}_1(\mathbf{A}_F)$, thought of as a function on C_F.

2.2.2.1. *The main convergence Lemma.* The following Lemma establishes basic properties of such averages and their formation. A large part of the materials covered in this subsection is discussed in a similar form also in [**RV**, Ch.7,§7.2,Lemma 7.6].

We denote by

$$\tau:\mathbf{R}_{>0}\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$$

a choice of continuous homomorphic section to the continuous group homomorphism $|\cdot|_\infty:=\prod_{v\in S_\infty}|\cdot|_v$ and call

$$T_\infty:=\tau(\mathbf{R}_{>0}).$$

We further call E_∞ the image under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F)\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ of the idelic fundamental domain E_F^1 from (A.2.3.14), which is, in turn, the finite union of $E_0\times\widehat{\mathcal{O}}_F^\times$ and of its g_i-translates, for $\{g_i\}_{1\leq i\leq h_F}$ a finite set of representatives in $\mathrm{GL}_1(\mathbf{A}_F)^1$ of the h_F ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$. Therefore, E_∞ is, concretely, the finite union of E_0 and the $g_{i,\infty}$-translates of E_0 within $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, where $g_{i,\infty}$ is the image of g_i into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ for each $1\leq i\leq h_F$.

By Lemma A.2.3.11, $T_\infty\subset\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is a closed (Lie) subgroup, and by Lemma A.2.3.10, E_∞ is relatively compact.

Lemma 2.2.2.1 (Averaging over $F^\times$-orbits). *Let F be a number field. We let $f\in\mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function over $\mathbf{A}_F$. Then the following series:*

$$\Psi_F(f)(g,x):=\sum_{\xi\in F^\times}f(g(\xi+x)),\quad(g,x)\in\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F \tag{2.2.2.1}$$

converges absolutely and uniformly over the compact subsets of $\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F$. As a function of $(g,x)\in\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F$, this satisfies the following conditions:

(1) *$\Psi_F(f)(g,x)$ is a continuous function of $(g,x)\in\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F$, and the function*

$$\Psi_F(f)(g,x)+f(gx)$$

(adding in the term $f(gx)$ for $\xi=0$) is, as a function of (g,x), invariant under the left translation action of $1\times F\subset\mathrm{GL}_1(\mathbf{A}_F)\times\mathbf{A}_F$.

(2) *$\Psi_F(f)(g,0)$ is a continuous function of $g\in\mathrm{GL}_1(\mathbf{A}_F)$, invariant under the left translation action of $F^\times\subset\mathrm{GL}_1(\mathbf{A}_F)$.*

(3) *If $f\in\mathcal{S}(\mathbf{A}_F)$ has the special form*

$$f=f_\infty\prod_{v\in\Sigma_{F,f}}f_v \tag{2.2.2.2}$$

where $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_v = 1_{K_v}$ for $K_v \subset \mathcal{O}_{F_v}$ a compact open subset for all finite places v of F, with $K_v = \mathcal{O}_{F_v}$ for all but finitely many finite places v of F, then there exists a fractional ideal $I_f \subset F$ such that

$$\sum_{\xi \in F^\times} |f(\xi g)| \leq \sum_{\xi \in I_f - \{0\}} |f_\infty(\xi g_\infty)| \tag{2.2.2.3}$$

where $g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, the image of g under the projection $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, is allowed to vary over the subset $T_\infty \cdot E_\infty \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

REMARK 2.2.2.2. We note that no constraint on (g, x) is assumed in Lemma 2.2.2.1(1),(2), whereas g_∞ in (3) is assumed to be constrained to vary over $T_\infty \cdot E_\infty$. The assertions in (1) and (2) are local in nature and can be checked by reducing to the case of f as in (3), leveraging (2.2.2.3) over $T_\infty \cdot E_\infty$, insofar as checking the relevant assertions over general compact subsets of $\mathrm{GL}_1(\mathbf{A}_F) \times \mathbf{A}_F$ can be reduced to the case of the single class of compact subsets $(K \cdot \overline{E}_F^1) \times \overline{D}_F$, where E_F^1 is the idelic fundamental domain from (A.2.3.14), D_F is the adelic fundamental domain from (A.2.1.2), and K varies over the compact subsets of $T := s_F(\mathbf{R}_{>0})$, where s_F is the canonical section to the idelic norm (see (A.2.2.9)).

We refer the reader to the details within the argument below.

PROOF. We write f as a finite $\mathbf{C}$-linear combination

$$f = \sum_{k=1}^{m} c_k f_k$$

where each f_k is of the form

$$f_k(x) = f_\infty(x_\infty) \prod_{v \in \Sigma_{F,f}} 1_{K_v}(x_v),\ x = (x_\infty, (x_v)_{v \in \Sigma_{F,f}}) \in \mathbf{A}_F \tag{2.2.2.4}$$

with $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$, where $x_\infty = (x_v)_{v \in S_\infty} \in \prod_{v \in S_\infty} F_v \simeq \mathbf{A}_{F,S_\infty}$, with $K_v \subset F_v$ a compact subset for all finite places v of F and $K_v = \mathcal{O}_{F_v}$ for all but finitely many finite places v (since finite $\mathbf{C}$-linear combinations of the functions of the form in (2.2.2.4) span $\mathcal{S}(\mathbf{A}_F)$, as per Proposition 2.1.3.2(3)).

Since all the statements of the Lemma are stable under finite $\mathbf{C}$-linear combinations of Schwartz-Bruhat functions, we may rename any of the f_k as f and reduce to the case f is of the form in (2.2.2.4).

Any compact subset in $\mathrm{GL}_1(\mathbf{A}_F) \times \mathbf{A}_F$ is contained in a finite union of $(F^\times \times F)$-translates of subsets of the form $Z = (s_F([a,b]) \cdot E_F^1) \times \overline{D}_F$, for E_F^1 the idelic fundamental domain of (A.2.3.14), $s_F : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ the canonical section to the idelic norm, and $[a,b] \subset \mathbf{R}_{>0}$ a sufficiently large (dependent on the given compact subset in $\mathrm{GL}_1(\mathbf{A}_F) \times \mathbf{A}_F$) compact interval, and $\overline{D}_F$ the compact subset $\mathbf{A}_F$ obtained as the closure of the fundamental domain D_F from (A.2.1.2) for the additive translation action of F on $\mathbf{A}_F$. By Proposition A.2.3.10(1), Z is indeed compact in $\mathrm{GL}_1(\mathbf{A}_F) \times \mathbf{A}_F$.

In order to show the locally uniform convergence of (2.2.2.1), at the cost of replacing the function f with an $F^\times$-translate and applying an F-translation to the summation index in $\sum_{\xi \in F}$, it is enough to prove uniform convergence over compact subsets of the form $Z_0 \times \overline{D}_F$ for a compact subset $Z_0 \subset \mathrm{GL}_1(\mathbf{A}_F)$ of the form $s_F([a,b]) \cdot \overline{E}_F^1$.

For the moment, let $(g, x) \in Z$ be fixed. Then the set of those $\xi \in F$ such that $f(g(\xi + x))$ is nonzero, satisfies the following condition: for every finite place v of F, $\xi \in -x + g^{-1}K_v \subset F_v$ with ξ viewed as an element of F_v and $\xi \in -x + g^{-1}\mathcal{O}_{F_v}$ for all but finitely many v.

We call

$$F_{(g,x)} := \{\xi \in F : f(g(\xi + x)) \neq 0\} \subset F \tag{2.2.2.5}$$

and observe that for every finite place v of F, the v-adic absolute value of any $\xi \in F_{(g,x)}$ satisfies

$$0 \leq |\xi|_v \leq \max\{|x|_v, C_v\,|g_v|_v^{-1}\} \leq \max\{1, C_v\,|g_v|_v^{-1}\} \tag{2.2.2.6}$$

with $C_v := |K_v|_v \geq 0$, which is finite being K_v compact and $|\cdot|_v : F_v \to \mathbf{R}_{\geq 0}$ continuous. We have used that for every $x \in \overline{D}_F$, $|x_v|_v \leq 1$ for every finite place v (see (A.2.1.2)).

We call S the finite set of finite places of F such that $K_v \neq \mathcal{O}_{F_v}$.

We have:

(1) If $K_v = \mathcal{O}_{F_v}$, we have $|\xi|_v \leq \max\{1, |g_v|_v^{-1}\}$. This is because $|\mathcal{O}_v| \leq 1$ for every finite place v of F, and $K_v = \mathcal{O}_v$ for all such finite places v.

Moreover, there exists a finite set of finite places S_g of F such that $|g_w|_w = 1$ for all $w \in \Sigma_{F,f} - S_g$, since g is an idele. It follows that for all $v \in \Sigma_{F,f} - (S_g \cup S)$, we have $|\xi|_v \leq 1$.

(2) For $v \in S$, $|\xi|_v$ is bounded above by a positive constant C independent of a, b, but possibly depending on F, S. Namely, we call

$$C := \max_{1 \leq i \leq h_F} \max\{C_v |g_{i,v}|_v^{-1} : v \in S\} \tag{2.2.2.7}$$

where $g_1, \ldots, g_{h_F} \in \mathrm{GL}_1(\mathbf{A}_F)^1$ is the choice of representatives for the h_F ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$, chosen in the construction of E_F^1.

We now let (g, x) vary over the compact subset $Z = Z_0 \times \overline{D}_F$ of $\mathrm{GL}_1(\mathbf{A}_F) \times \mathbf{A}_F$.

We observe that the chosen subset $\bigcup_{g \in Z_0} S_g$ of the set of finite places of F, is itself finite.

Indeed, by (A.2.3.16), given the structure of E_F^1 from (A.2.3.14) and the fact that $s_F([a,b])$ is contained in the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times 1 \subset \mathrm{GL}_1(\mathbf{A}_F)$, the non-Archimedean coordinates of the points in Z_0 are contained in a subset of $\mathrm{GL}_1(\mathbf{A}_F^f)$ of the following form:

$$\overline{g_1}\widehat{\mathcal{O}}_F^\times \cup \ldots \cup \overline{g_{h_F}}\widehat{\mathcal{O}}_F^\times \tag{2.2.2.8}$$

where h_F is the class number of F, and with each $\overline{g_i} \in \mathrm{GL}_1(\mathbf{A}_F^f)$ the projection onto $\mathrm{GL}_1(\mathbf{A}_F^f)$ of an element $g_i \in \mathrm{GL}_1(\mathbf{A}_F)^1$.

It follows that we have

$$\bigcup_{g \in Z_0} S_g \subset S_{g_1} \cup \ldots \cup S_{g_{H_\Gamma}} \tag{2.2.2.9}$$

a finite set.

Upon renaming $S \cup \bigcup_{(g,x) \in Z} S_g = S \cup \bigcup_{g \in Z_0} S_g$ as S, again a finite set of finite places of F, we may and do assume that we have, calling

$$F_Z := \bigcup_{(g,x) \in Z} F_{(g,x)} = \{\xi \in F : f(g(\xi + x)) \neq 0 \,\text{for some}\, (g,x) \in Z\} \subset F \tag{2.2.2.10}$$

that for every $\xi \in F_Z$, the following conditions hold:

(1) If $v \in \Sigma_{F,f} - S$, we have $|\xi|_v \leq 1$.
(2) For $v \in S$, we have $|\xi|_v \leq C$.

In other words, upon defining

$$U := \{(u_v) \in \mathbf{A}_F^f : |u_v|_v \leq C \text{ for all } v \in S,\ |u_v|_v \leq 1 \text{ for all } v \in \Sigma_{F,f} - S\} \subset \mathbf{A}_F^f$$

a compact open subgroup of $\mathbf{A}_F^f$ since S is finite, we have $F_Z \subset U$, with F diagonally embedded into $\mathbf{A}_F^f$.

For each $v \in S$, we call n_v a nonnegative integer such that $|\pi_v|_v^{n_v} \geq C$. Since S is finite, then the subset $I_f := \prod_{v \in S} \mathfrak{p}_v^{n_v} \subset F$ is a fractional ideal, and $I_f\,\mathbf{A}_F^f$ contains U.

We therefore have, for every $(g,x)\in Z$:

$$\sum_{\xi\in F^\times}|f(g(\xi+x))|\le\sum_{\xi\in I_f-\{0\}}|f_\infty(g_\infty(\xi+x_\infty))|\left|\prod_{v\in S}f_v(g_v(\xi+x_v))\right|\le\sum_{\xi\in I_f-\{0\}}|f_\infty(g_\infty(\xi+x_\infty))| \tag{2.2.2.11}$$

where $x_\infty=(x_v)_{v\in S_\infty}\in\mathbf{A}_{F,S_\infty}$ is the image of $x\in\overline{D}_F$ into $\mathbf{A}_{F,S_\infty}$ under the projection $\mathbf{A}_F\to\mathbf{A}_{F,S_\infty}$, and $g_\infty\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is the image of $g\in Z_0\subset\mathrm{GL}_1(\mathbf{A}_F)$ under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F)\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

Here we have used that $|f_v|\le 1$ for every $v\in\Sigma_{F,f}$, because, regardless of whether $v\in S$ or not, f_v is an indicator function, hence valued in $\{0,1\}$.

Upon choosing $x=0$, then the leftmost side of (2.2.2.11) is invariant under left translation by $F^\times$ on $g\in\mathrm{GL}_1(\mathbf{A}_F)$, if convergent. In particular, the inequality in (2.2.2.11) with $x=0$ holds for all $g\in F^\times(s_F([a,b])\cdot\overline{E}_F^1)=s_F([a,b])\cdot\mathrm{GL}_1(\mathbf{A}_F)^1$ and as g_∞ varies over $s_F([a,b])\cdot\overline{E}_\infty$.

Moreover, since formation of I_f does not depend on a,b, g is allowed to vary over $\bigcup_{a<b}s_F([a,b])\cdot\mathrm{GL}_1(\mathbf{A}_F)^1=\mathrm{GL}_1(\mathbf{A}_F)$, and the inequality in (2.2.2.11) with $x=0$ still holds as g_∞ varies over $T_\infty\cdot\overline{E}_\infty$, which is what we wanted to show. This proves (3), provided the rightmost side in (2.2.2.11) converges uniformly over our choice of Z. (Note that formation of I_f depends on our choice of Z.)

We now study the uniform convergence of the right side of (2.2.2.11) over $Z=Z_0\times\overline{D}_F$ where

$$Z_0=s_F([a,b])\cdot\left(\bigcup_{1\le i\le h_F}g_i(\overline{E}_0\times\widehat{\mathcal{O}}_F^\times)\right)$$

for $g_i\in\mathrm{GL}_1(\mathbf{A}_F)^1$ representives of $\mathrm{Pic}(\mathcal{O}_F)$ (see Proposition A.2.3.9). It is enough to replace Z with $s_F([a,b])\cdot(\overline{E}_0\times\widehat{\mathcal{O}}_F^\times)$, with $E_0\subset\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ defined in (A.2.3.13).

We therefore let the Archimedean component g_∞ of $g\in\mathrm{GL}_1(\mathbf{A}_F)$, vary over the compact subset $s_{F,\infty}([a,b])\cdot\overline{E}_0$ of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and we let x_∞, the Archimedean component of x, vary over $\overline{D}_{F,\infty}$, the compact closure of $D_{F,\infty}$ from (A.2.1.1). Recall that $s_{F,\infty}$ is the continuous one-parameter subgroup of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ induced by the canonical section s_F to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$, being its image contained in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})\times 1\subset\mathrm{GL}_1(\mathbf{A}_F)$ (see (A.2.2.9)).

For each integer $N\ge 1$, there exist a compact subset $B\subset\mathbf{A}_{F,S_\infty}\simeq\prod_{v\in S_\infty}F_v$ containing the origin, a real number $C_{N,B}>0$, such that $|f_\infty(y_\infty)|\le C_{N,B}\,\|y_\infty\|^{-N}$ for all $y_\infty\in\mathbf{A}_{F,S_\infty}-B$, where $\|\cdot\|$ is the standard Euclidean norm on $\mathbf{A}_{F,S_\infty}$ induced by the absolute values $|\cdot|_v$ for each infinite place v of F, defined by $\|y_\infty\|^2=\sum_{v\in S_\infty}|y_v|_v^{N_v}$ where $N_v=2$ if v is real, and $N_v=1$ if v is complex, and with $y_\infty=(y_v)_{v\in S_\infty}$.

We define $Z_\infty:=(s_{F,\infty}([a,b])\cdot\overline{E}_0)\times\overline{D}_{F,\infty}$.

For (g_∞,x_∞) varying over Z_∞, the condition on $\xi\in I_f-\{0\}$:

$$g_\infty(\xi+x_\infty)\in B$$

reads

$$\xi\in-\overline{D}_{F,\infty}+(s_{F,\infty}([a,b])\cdot\overline{E}_0)^{-1}B.$$

Since $I_f\subset\mathbf{A}_{F,S_\infty}$ is discrete, its intersection with the compact subset

$$Y:=-\overline{D}_{F,\infty}+(s_{F,\infty}([a,b])\cdot\overline{E}_0)^{-1}B$$

of $\mathbf{A}_{F,S_\infty}$ is finite and the convergence of the rightmost side of (2.2.2.11) is equivalent to the convergence of the series

$$\sum_{\xi\in I_f-Y\cap I_f}|f_\infty(g_\infty(\xi+x_\infty))| \tag{2.2.2.12}$$

uniformly over Z_∞.

This is in turn implied by that of the series

$$\sum_{\xi\in I_f - Y\cap I_f} \|g_\infty(\xi + x_\infty)\|^{-N} \tag{2.2.2.13}$$

over Z_∞.

Since $s_{F,\infty}([a,b])\cdot \overline{E}_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is compact, and g_∞ varies over a compact subset of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, hence away from the origin, the convergence of the series in (2.2.2.13) uniformly over Z_∞ is implied by that of the series

$$\sum_{\xi\in I_f - B\cap I_f} \|\xi + x_\infty\|^{-N} \tag{2.2.2.14}$$

uniformly over $\overline{D}_{F,\infty}$. This holds for $N > [F:\mathbf{Q}] = \dim_{\mathbf{R}}(\mathbf{A}_{F,S_\infty})$ locally uniformly with respect to x_∞ by Proposition 2.2.1.3 applied to (2.2.2.14) with $\Gamma = I_f$ up to $\mathbf{R}$-linear isomorphism of Euclidean spaces.

The inequality in (2.2.2.11) shows that the series in (2.2.2.1) converges absolutely and uniformly on the compact subset $Z \subset \mathrm{GL}_1(\mathbf{A}_F)\times \mathbf{A}_F$.

Local uniformity of the convergence and continuity of f implies that the sum of the series is itself continuous as a function of $(g,x)\in \mathrm{GL}_1(\mathbf{A}_F)\times \mathbf{A}_F$.

Both invariance conditions (1) and (2) hold by inspection, given the absolute convergence of the series. □

Remark 2.2.2.3. To later study the functions arising as

$$\Psi_F(f)(g,0) = \sum_{\xi\in F^\times} f(\xi g)$$

we will need to know that for f as in part (3) of the Lemma, formation of I_f depends on f only through its non-Archimedean tensor component, and through the choice of E_F^1, but not through f_∞.

What the argument is saying is that for each choice of compact subset $Z \subset \mathrm{GL}_1(\mathbf{A}_F)\times \mathbf{A}_F$, at least one such I_f exists. The existence of I_f can be reduced to the case $Z = Z_0 \times \overline{D}_F$ with Z_0 of the form $K\cdot E_F^1$, for some compact subset K of $s_F(\mathbf{R}_{>0})$.

By inspection of the proof, formation of I_f will not depend on K. It does rest upon the structure of E_F^1, and so the I_f constructed above will do for the choice $Z = (K\cdot E_F^1)\times \overline{D}_F$, for any compact subset $K \subset s_F(\mathbf{R}_{>0})$.

2.2.3. The θ-function operator $\Phi_F : \mathcal{S}(\mathbf{A}_F) \to C(C_F)$. In this section we study the linear operator

$$\Phi_F : \mathcal{S}(\mathbf{A}_F) \to C(C_F)$$

defined by sending a Schwartz-Bruhat function on $\mathbf{A}_F$ to its restriction to the idele group $\mathrm{GL}_1(\mathbf{A}_F)$, averaged over the orbits under the action of the discrete subgroup $F^\times$.

The operator Φ_F is a special case of Ψ_F defined in Lemma 2.2.2.1. The Riemann-Roch Theorem in [**CF**, Thm.4.2.1] will enable sharp estimates for the resulting functions over the idele class group C_F, over *both* loci $C_F^{\gg 1}$ and $C_F^{\ll 1}$, i.e. the subsets of those idele classes whose idelic norm is very large, respectively very small.

By Lemma 2.2.2.1(2), for any $f\in \mathcal{S}(\mathbf{A}_F)$ the continuous function $\Psi_F(g,0)$ of $g\in \mathrm{GL}_1(\mathbf{A}_F)$ is invariant under the left translation action by the discrete subgroup $F^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$.

Definition 2.2.3.1 (Global adelic θ-functions). Let F be a number field, $f\in \mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function.

We denote by

$$\Phi_F(f)(g) := \Psi_F(f)(g',0) = \sum_{\xi\in F^\times} f(\xi g') \tag{2.2.3.1}$$

the continuous function of $g \in C_F$ obtained from f by averaging the restriction of f to $\mathrm{GL}_1(\mathbf{A}_F)$ over the $F^\times$-orbits, where g' is any choice of lift of g under the continuous surjective homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to C_F$.

We refer to the function $\Phi_F(f)$ on C_F as the *global adelic θ-function* of f.

The definition of Φ_F in (2.2.3.1) is independent of such choice by $F^\times$-invariance of $\Psi_F(g, 0)$.

The assignment Φ_F on $\mathcal{S}(\mathbf{A}_F)$ is valued in $C(C_F)$ and its formation is $\mathbf{C}$-linear. We will now proceed to study the function $\Phi_F(f)(g)$ of $g \in C_F$, and its asymptotic behavior as $|g|_F \to 0^+$ and as $|g|_F \to +\infty$.

We refer to $\Phi_F(f)$ as "θ-function", with a non-capitalized θ, because the true direct analog of the Jacobi Θ-function from Riemann's paper on the number of prime numbers below a given magnitude, is the function defined by the full sum

$$\Theta_F(f)(g) := \sum_{\xi \in F} f(\xi g)$$

i.e. not merely the sum over the translates by *nonzero* elements of F. The construction under consideration is indeed the one in use throughout this work, i.e. we will be using $\Phi_F(f)$ throughout this work, as opposed to $\Theta_F(f)$.

We have taken a habit of using the letter Φ to denote the adelic θ-functions thus defined, because we will be showing that the operator Φ_F is valued in the Schwartz-Bruhat space $\mathcal{S}(C_F)$ of the idele class group, and we will be denoting with the letter ϕ the generic element of $\mathcal{S}(C_F)$.

2.2.3.1. *Estimates over the idele class group.* The following Theorem is Tate's "Riemann-Roch Theorem", for which we refer the reader to [**CF**, Ch.XV,§4,Thm.4.2.1].

THEOREM 2.2.3.2 (Adelic Riemann-Roch Theorem). *Let F be a number field, $f \in \mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function. As continuous functions over C_F, the following identity holds:*

$$\Phi_F(f)(g) = |g|_F^{-1} \Phi_F(\widehat{f})(g^{-1}) + |g|_F^{-1} \widehat{f}(0) - f(0). \tag{2.2.3.2}$$

REMARK 2.2.3.3 (On the volume normalization and the global Fourier Transform). We note that the choice of Haar measure on $\mathbf{A}_F$, respectively $\mathbf{A}_{F,S}$, in force throughout this work, is the unique choice of Haar measure such that, upon endowing the discrete subgroups $F \subset \mathbf{A}_F$, respectively $\mathcal{O}_{F,S} \subset \mathbf{A}_{F,S}$, with the counting measure, the unique measure determined as per Theorem D.1.2.1 on the compact Hausdorff quotients $\mathbf{A}_F/F$, respectively $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$, gives such quotients volume 1.

We refer the reader to Remark A.3.4.2, and to Remark A.3.2.5, and the references contained therein.

This choice of measure is the same being made in [**CF**, Ch.XV,§4] and in [**Co**, Appendix I,(34),p.57], and removes volume factors from the statement of Theorem 2.2.3.2 (since they equal 1); see the beginning of [**CF**, Ch.XV,§4,p.334].

PROOF. This is a direct consequence of [**CF**, Ch.XV,Thm.4.2.1] and Lemma 2.2.2.1, which shows that the Theorem applies. We have, by [**CF**, Ch.XV,Thm.4.2.1]:

$$\Psi_F(g, 0) - |g|_F^{-1} \widehat{f}(0) + f(0) = |g|_F^{-1} \Psi_F(\widehat{f})(g^{-1}, 0) \tag{2.2.3.3}$$

as continuous functions on $\mathrm{GL}_1(\mathbf{A}_F)$. Since $|\cdot|_F$ is continuous and $F^\times$-invariant, then the identity in (2.2.3.3) reads, as an identity of continuous functions of $g \in C_F$:

$$\Phi_F(f)(g) - |g|_F^{-1} \widehat{f}(0) + f(0) = |g|_F^{-1} \Phi_F(\widehat{f})(g^{-1})$$

which is (2.2.3.2). □

The following is the main result of this subsection:

THEOREM 2.2.3.4 (Estimates over the Idele Classes - I). *Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function on $\mathbf{A}_F$. The continuous function $\Phi_F(f)$ on C_F, satisfies the following growth conditions:*

(1) *for every integer $N \geq 0$, there exists a real number $C_N > 0$ such that for every $g \in C_F$, we have*

$$|\Phi_F(f)(g)| \leq C_N |g|_F^{-N} \tag{2.2.3.4}$$

as $g \in C_F$ is such that $|g|_F \to +\infty$.

(2) *For every integer $N \geq 0$, there exists a real number $C_N > 0$ such that for every $g \in C_F$, we have*

$$|\Phi_F(f)(g) - |g|_F^{-1} \widehat{f}(0) + f(0)| \leq C_N |g|_F^{N} \tag{2.2.3.5}$$

as $g \in C_F$ is such that $|g|_F \to 0$.

(3) *Assume $f(0) = \widehat{f}(0) = 0$. For every integer $N \geq 0$, there exists a real number $C_N > 0$ such that for every $g \in C_F$, we have*

$$|\Phi_F(f)(g)| \leq C_N e^{-N\,|\log|g|_F|} \tag{2.2.3.6}$$

as $g \in C_F$ is such that $|\log|g|_F| \to +\infty$ (equivalently, $|g|_F \to 0, +\infty$).

(4) *Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_F)$ be a bounded subset. For every integer $N \geq 0$, there exist real numbers $C_{\mathcal{B},N}, T_{\mathcal{B},N} > 0$ such that* (2.2.3.4), (2.2.3.5) *and* (2.2.3.6) *hold for $C_N = C_{\mathcal{B},N}$, and for all $g \in C_F$ such that $|g|_F > T_{\mathcal{B},N}$, respectively $|g|_F < T_{\mathcal{B},N}$, respectively $e^{-|\log|g|_F|} < T_{\mathcal{B},N}$.*

(5) *For every integer $N \geq 0$, the assertions in* (4) *all hold with $T_{\mathcal{B},N} = 1$ and with a possibly larger but still uniform $C_{\mathcal{B},N} > 0$.*

Proof. By Lemma 2.2.2.1, the series (2.2.2.1) defining $\Psi_F(f)(g,0) = \sum_{\xi \in F^\times} f(\xi g)$ converges absolutely and uniformly on the compact subsets of $\mathrm{GL}_1(\mathbf{A}_F)$ to a continuous function of $g \in \mathrm{GL}_1(\mathbf{A}_F)$.

By Theorem 2.2.3.2, we have, as continuous functions on C_F:

$$\Phi_F(f)(g) - |g|_F^{-1} \widehat{f}(0) + f(0) = |g|_F^{-1} \Phi_F(\widehat{f})(g^{-1}). \tag{2.2.3.7}$$

The change of variable $g \mapsto g^{-1}$ on C_F therefore reduces (2.2.3.5) for f to (2.2.3.4) for $\widehat{f}$, and since (3) follows from the combination of (1) and (2), it is enough to verify that (1) holds.

We write $f = \sum_{k=1}^m c_k f_k$ with $m \geq 1$ an integer, $c_k \in \mathbf{C}$ and $f_k \in \mathcal{S}(\mathbf{A}_F)$ of the special form $f_k = f_{k,\infty} \prod_{v \nmid \infty} f_{k,v}$, with $f_{k,\infty} \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_{k,v} \in \mathcal{S}(F_v)$ with $f_{k,v} = 1_{\mathcal{O}_{F_v}}$ for all but finitely many finite places v of F. Since formation of Φ_F in (2.2.3.4) is $\mathbf{C}$-linear with respect to f, we may rename any of f_k as f and reduce to the case f is of this special form.

We claim that (2.2.3.4) follows from the rapid decay condition on f_∞. By replacing g with an $F^\times$-translate, we can assume g varies over $T_\infty \cdot E_\infty$ (T_∞ and E_∞ being defined in the discussion preceding Lemma 2.2.2.1). The continuous homomorphic section τ to $\prod_{v \in S_\infty} |\cdot|_v$ and therefore $|\cdot|_F$, that we are choosing, is the diagonal section $s_{F,\delta}$, constructed in (A.2.2.11), a choice that will interact well with the supremum norm on the Euclidean space $\mathbf{A}_{F,S_\infty} \simeq \mathbf{R}^{r_1+2r_2}$.

Thus Lemma 2.2.2.1(3) applies, and we have

$$\sum_{\xi \in F^\times} |f(\xi g)| \leq \sum_{\xi \in I_f - \{0\}} |f_\infty(\xi g_\infty)| \tag{2.2.3.8}$$

for a fractional ideal I_f depending on f, where g_∞ is the image of g under the continuous projection $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and g_∞ is allowed to vary over the subset $T_\infty \cdot E_\infty \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

Since $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$, then for any integer $N' \geq 0$, there exists a large enough compact subset $B \subset \mathbf{A}_{F,S_\infty} \simeq \prod_{v \in S_\infty} F_v$ containing the origin, and a real number $C_{B,N'} > 0$, such that for all $x_\infty \in \mathbf{A}_{F,S_\infty} - B$ we have

$$|f_\infty(x_\infty)| \leq C_{B,N'} \|x_\infty\|^{-N'}$$

where $\|\cdot\|$ is the Euclidean supremum norm on $\mathbf{A}_{F,S_\infty}$ induced by the Archimedean absolute value on each F_v, defined by $\|x_\infty\| := \sup_{v \in S_\infty} |x_v|_v^{N_v}$ where $N_v = 1$ if v is real, and $N_v = 1/2$ if v is complex.

We define $T = s_{F,\delta}(\mathbf{R}_{>0})$ for $s_{F,\delta}$ the diagonal section to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$ from (A.2.2.11), and E_F^1 denotes the fundamental domain of the $F^\times$-action on $\mathrm{GL}_1(\mathbf{A}_F)^1$ from (A.2.3.14). Since $I_f \subset \mathbf{A}_{F,S_\infty}$ is discrete, there exists a large enough real number $C > 0$ such that for all $t \in T$ for which $|t|_F > C$ we have, upon calling E_∞ the image of E_F^1 under the continuous projection $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and t_∞ the image of t:

$$B \cap (t_\infty \cdot (I_f - \{0\}) \cdot E_\infty) = \varnothing. \tag{2.2.3.9}$$

Note that the subset E_∞ is bounded. We claim that there exists a constant $C > 0$ such that for any $t \in T$ for which $|t|_F > C$, $B \cap t_\infty \xi E_\infty = \varnothing$ for all $\xi \in I_f - \{0\}$. The proof rests on the inequality

$$\|t_\infty \xi E_\infty\| \geq A\|t_\infty \xi\| = A|t|_F\|\xi\| \tag{2.2.3.10}$$

for a real number $A > 0$ depending only on E_∞ and $\|B\|$ is a compact subset of $\mathbf{R}_{>0}$. Indeed, assume there exists an element $x \in B \cap t_\infty \xi E_\infty$, so $\|x\|$ is contained in the intersection of $[0, \max \|B\|]$ and $[A|t|_F\|\xi\|, +\infty)$. We will now show that these latter two intervals do not overlap (thereby getting a contradiction from the assumed existence of such x) when $|t|_F$ is sufficiently large and ξ is arbitrary in $I_f - \{0\}$.

Since $I_f \subset \mathbf{A}_{F,S_\infty}$ is a lattice, let $M := \min_{\xi \in I_f - \{0\}} \|\xi\| > 0$. We define $C := (AM)^{-1} \max \|B\|$. For any $\xi \in I_f - \{0\}$, from (2.2.3.10) we have:

$$\|t_\infty \xi E_\infty\| \geq AM|t|_F \tag{2.2.3.11}$$

and hence the intersection $B \cap t_\infty \xi E_\infty$ is empty for all $\xi \in I_f - \{0\}$ and when $|t|_F > C$.

We therefore have, for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$ of the form $g = s_{F,\delta}(t_0)g_0$ for unique $t_0 \in \mathbf{R}_{>0}$, and $g_0 \in E_F^1$, with respective projections t_∞ and $g_{0,\infty}$ onto $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$

$$|f_\infty(\xi t_\infty g_{0,\infty})| \leq C_{B,N'} \|\xi t_\infty g_{0,\infty}\|^{-N'} = C_{B,N'}\, t_0^{-N'} \|\xi g_{0,\infty}\|^{-N'} \tag{2.2.3.12}$$

for any $\xi \in I_f - \{0\}$ and $t_0 \in \mathbf{R}_{>0}$ with $t_0 > C$, equivalently $|s_{F,\delta}(t_0)|_F > C$.

The series $\sum_{\xi \in I_f - \{0\}} \|\xi g_{0,\infty}\|^{-N'}$ converges for all $N' > r_1 + 2r_2 = \dim_{\mathbf{R}}(\mathbf{A}_{F,S_\infty})$ by Proposition 2.2.1.3 and uniformly with respect to $g_{0,\infty}$ varying over E_∞ (since its closure in $\mathbf{A}_{F,S_\infty}$ is compact and contained in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$). Moreover, this sum is bounded on the compact subset $\overline{E}_\infty$ by continuity with respect to $g_{0,\infty}$. Thus, if we choose $N' = 1 + \max\{r_1 + 2r_2, N\}$ then there exists a constant $C_N > 0$ such that

$$\sum_{\xi \in I_f - \{0\}} |f_\infty(\xi t_\infty g_{0,\infty})| \leq C_N\, |t|_F^{-N} = C_N\, |g|_F^{-N}, \tag{2.2.3.13}$$

where we have used that $|g|_F > C > 1$, and hence $|g|_F^{-N'} \leq |g|_F^{-N}$.

(4) : By Proposition 2.1.2.4 and Proposition 2.1.2.8(1), $\mathcal{S}(\mathbf{A}_F^f)$ carries the finest locally convex topology (see also the proof of Proposition 2.1.3.2), and hence each bounded subset of $\mathcal{S}(\mathbf{A}_F^f)$ is contained in a finite dimensional subspace of $\mathcal{S}(\mathbf{A}_F^f)$ by Proposition C.2.5.1(4).

Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_F)$ be a bounded subset. By Proposition 2.1.3.2(2), the canonical linear operator $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f) \to \mathcal{S}(\mathbf{A}_F)$ is a linear topological isomorphism, and hence by Proposition C.2.3.13(2), and writing $\mathcal{S}(\mathbf{A}_F^f)$ as the strict locally convex inductive limit of a countable family of finite dimensional $\mathbf{C}$-vector subspaces, the bounded subset $\mathcal{B}$ in $\mathcal{S}(\mathbf{A}_F)$ is contained, by [**Scha**, Ch.II,§6,6.5,p.59], in one which is the image of

$$\mathrm{cch}(\mathcal{B}_\infty \otimes \mathcal{B}_f)$$

the convex circled hull of the subset $\mathcal{B}_\infty \otimes \mathcal{B}_f$, with $\mathcal{B}_\infty \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$, and $\mathcal{B}_f$ bounded in a finite dimensional $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_F^f)$ (we refer the reader to [**Jar**, Part III,Ch.15,§15.1,p.324] for the analogous assertion for the projective tensor topology and its own bounded subsets).

There therefore exist finitely many $f_1, \ldots, f_m \in \mathcal{S}(\mathbf{A}_F^f)$ whose linear span contains $\mathcal{B}_f$. In particular, all the functions in $\mathcal{B}_f$ are supported on a large enough compact open subset $K \subset \mathbf{A}_F^f$.

Every element of $\mathcal{B}$ is therefore a finite absolutely convex $\mathbf{C}$-linear combination (i.e. one whose coefficients have complex Euclidean absolute values that sum to a real number bounded above by 1; see [**Jar**, Part I,Ch.6,§6.1,p.102] for $r = 1$; see also [**NB**, Thm.4.2.11]) of Schwartz-Bruhat functions of the form $f_\infty \otimes f_0$ with each of the f_0 supported on the common K whose existence was proved above. Moreover, each f_0 is obtained as a finite linear combination of indicator functions of compact open subsets all contained in K, with complex coefficients of complex absolute value bounded above by a positive real number depending only on $\mathcal{B}_f$ (since $\mathcal{B}_f$ is a bounded subset of a finite dimensional Euclidean $\mathbf{C}$-vector space, and hence, within such finite dimensional Euclidean $\mathbf{C}$-vector space, upon choosing a $\mathbf{C}$-linear basis $\{f_1, \ldots, f_m\}$ as in the foregoing, $\mathcal{B}_f$ is contained within a compact box).

By construction of C_N from (1), this constant therefore depends on the element of $\mathcal{B}_\infty \otimes \mathcal{B}_f$ only through the $\mathcal{B}_\infty$-tensor factor, through K, and through the positive real constant bounding from above the complex absolute values of the coefficients of the $\mathbf{C}$-linear expansion of the generic element $f_0 \in \mathcal{B}_f$ with respect to the chosen $\mathbf{C}$-linear basis of indicator functions $\{f_1, \ldots, f_m\}$.

Specifically, the dependence through K is through the formation of the fractional ideal $I_f \subset F$.

That the real number $T_N > 0$ such that (2.3.1.3) holds if $|g|_F > T_N$, and whose formation is uniform as f varies over $\mathcal{B}$, exists, is seen by observing that formation of B from the foregoing depends on f through f_∞, for f of the special form $f = f_\infty \otimes f_0 \in \mathcal{B}_\infty \otimes \mathcal{B}_f$.

In particular, the same ball $B \subset \mathbf{A}_{F,S_\infty}$ qualifies to give, through the argument from the foregoing, the estimate in (2.2.3.8) as f_∞ varies over the bounded subset $\mathcal{B}_\infty \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$.

We have now shown that (2.3.1.3) holds for each f of the special form $f_\infty \otimes f_0$ with $f_\infty \in \mathcal{B}_\infty$ and $f_0 \in \mathcal{B}_f$, with C_N and T_N existing and depending only on $\mathcal{B}_\infty$ and $\mathcal{B}_f$.

By linearity of Φ_F and the design of the convex circled hull (see [**Jar**, Part I,Ch.6,§6.1,p.102] for $r = 1$; see also [**NB**, Thm.4.2.11]), the estimate in (2.2.3.4) propagates with the same C_N and T_N to the elements of $\mathrm{cch}(\mathcal{B}_\infty \otimes \mathcal{B}_f)$, and hence to all the elements of the bounded subset $\mathcal{B}$.

We rename C_N as $C_{\mathcal{B},N}$, and T_N as $T_{\mathcal{B},N}$, and these constants satisfy the requirements of (4).

The case of (2.2.3.5) and (2.2.3.6) follows from the foregoing (at the cost of changing $T_{\mathcal{B},N}$, obtaining another constant that still depends only on $\mathcal{B}$ and N, and this is possible since the Fourier transform on $\mathcal{S}(\mathbf{A}_F)$ is a linear topological automorphism, hence preserves bounded subsets in both directions.

(5) : By (4), there exist, for each integer $N \geq 0$, positive real numbers $C_{\mathcal{B},N}, T_{\mathcal{B},N} > 0$ such that the assertions in (1),(2) and (3) hold, respectively, for all $f \in \mathcal{B}$, upon replacing T_N with $T_{\mathcal{B},N}$ and C_N with $C_{\mathcal{B},N}$.

We prove that there exists $C_{\mathcal{B}} > 0$, such that for all $f \in \mathcal{B}$, we have:

$$|\Phi_F(f)(g)| \leq C_{\mathcal{B}} \tag{2.2.3.14}$$

for all $g \in C_F$ satisfying $|g|_F \in [1/t_{\mathcal{B},N}, t_{\mathcal{B},N}]$, where $t_{\mathcal{B},N} := \max\{1/T_{\mathcal{B},N}, T_{\mathcal{B},N}\}$.

If so, then we rename $\max\{C_{\mathcal{B}}, C_{\mathcal{B},N}\}$ as $C_{\mathcal{B},N}$, and (5) follows upon replacing $T_{\mathcal{B},N}$ with 1, and with this choice of $C_{\mathcal{B},N}$.

As in the argument for (4), there exists a bounded subset $\mathcal{B}_\infty \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and a bounded subset $\mathcal{B}_f \subset \mathcal{S}(\mathbf{A}_F^f)$ entirely contained in a finite dimensional $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_F^f)$, such that $\mathcal{B} \subset \mathrm{cch}(\mathcal{B}_\infty \otimes \mathcal{B}_f)$, the convex circled hull of the subset $\mathcal{B}_\infty \otimes \mathcal{B}_f \subset \mathcal{S}(\mathbf{A}_F)$.

The generic element of $\mathrm{cch}(\mathcal{B}_\infty \otimes \mathcal{B}_f)$ is therefore an absolutely convex finite $\mathbf{C}$-linear combitation of elements of the form $f_\infty \otimes f_0$, with $f_\infty \in \mathcal{B}_\infty$, and, upon choosing a $\mathbf{C}$-linear basis $\{f_1, \ldots, f_m\}$ for a fixed large enough finite dimensional $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_F^f)$ containing $\mathcal{B}_f$ (which exists by [**Scha**, Ch.II,§6,6.5,p.59]), f_0 is in turn a finite $\mathbf{C}$-linear combination of $f_1, \ldots, f_m$ whose complex coefficients have complex absolute value bounded above by a positive real number which depends only on $\mathcal{B}_f$.

To obtain the estimate from (2.2.3.14) with a constant $C_{\mathcal{B}}$ that does not depend on $f \in \mathcal{B}$, but only on $\mathcal{B}$, it is therefore enough to find such constant when f is of the form $f_\infty \otimes f_k$, for each $1 \leq k \leq m$. We let $K \subset \mathbf{A}_F^f$ be a compact open subset containing the supports of all the finitely many $f_1, \ldots, f_m$.

There exists a fractional ideal $I \subset F$ depending only on K, such that:

$$|\Phi_F(f_\infty \otimes 1_K)(g)| \leq \sum_{\xi \in I - \{0\}} |f_\infty(\xi g_\infty)| \tag{2.2.3.15}$$

for all $g \in \tau(\mathbf{R}_{>0}) \cdot E_F^1$, with g_∞ the image of $g \in \mathrm{GL}_1(\mathbf{A}_F)$ into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ under the continuous homomorphic projection $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, where τ is a choice of continuous homomorphic section to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$, and E_F^1 is the idelic fundamental domain from (A.2.3.14); see Lemma 2.2.2.1(3).

The series on the right side of (2.2.3.15) converges uniformly on the compact subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and hence its limit is a continuous function on the image of $\tau(\mathbf{R}_{>0}) \cdot E_F^1$ into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$. Since $\tau([1/t_{\mathcal{B},N}, t_{\mathcal{B},N}]) \cdot E_F^1$ is relatively compact and with non-empty interior, and I does not depend on f_∞, we deduce that:

$$\alpha : f_\infty \mapsto \sup_{g \in \tau([1/t_{\mathcal{B},N}, t_{\mathcal{B},N}]) \cdot E_F^1} \sum_{\xi \in I - \{0\}} |f_\infty(\xi g_\infty)|$$

is a continuous function on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$. Continuity is seen as follows: for any $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and any sequence $\{f_{\infty,j}\}_{j \geq 0} \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$ converging to f_∞ in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, for each fixed integer $r \geq 1$ there a compact subset $Z \subset \mathbf{A}_{F,S_\infty}$ containing the origin and such that, for every $j \geq 0$, $|f_{\infty,j}(x)| \leq C\|x\|^{-r}$ for every $x \in \mathbf{A}_{F,S_\infty} - Z$, for a real number $C > 0$ independent of $j \geq 0$, and upon choosing a norm $\|\cdot\|$ on $\mathbf{A}_{F,S_\infty}$ inducing the Euclidean topology on $\mathbf{A}_{F,S_\infty}$.

The sum $\sum_{\xi \in I - \{0\}} |f_{\infty,j}(\xi g_\infty)|$ is therefore split into the sum of a finite sum, whose finite index is independent of $j \geq 0$ but only depends on Z and I, and of a series of positive real numbers bounded above by a series of the form $C \sum_{\xi \in I - \{0\}} \|\xi g_\infty\|^{-r}$, which converges over $\mathbf{A}_{F,S_\infty} - Z$ uniformly on compact subsets (see Proposition 2.2.1.3), upon choosing $r > [F : \mathbf{Q}]$.

By Dominated Convergence, the limit as $j \to +\infty$ of $\sum_{\xi \in I - \{0\}} |f_{\infty,j}(\xi g_\infty)|$ is $\sum_{\xi \in I - \{0\}} |f_\infty(\xi g_\infty)|$, proving sequential continuity of the function $f_\infty \mapsto \sum_{\xi \in I - \{0\}} |f_\infty(\xi g_\infty)|$. Since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is metrizable, it is a sequential topological space, and hence the function $f_\infty \mapsto \sum_{\xi \in I - \{0\}} |f_\infty(\xi g_\infty)|$ is continuous, being sequentially continuous.

Since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is Montel by Theorem 2.1.3.2, it has the Heine-Borel property, and hence the closure of $\mathcal{B}_\infty$ is compact in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$.

The continuous function α is therefore bounded on $\mathcal{B}_\infty$, by a positive real number $C_{\mathcal{B}_\infty} > 0$. Upon maximizing $C_{\mathcal{B}_\infty}$ finitely many times, one per each choice of f_k, for $1 \leq k \leq m$, and then multiplying it by the maximum of the positive real number and its reciprocal, which bounds from above the complex coefficients of each $\mathbf{C}$-linear expansion of each element of $\mathcal{B}_f$ with respect to the $\mathbf{C}$-linear basis $\{f_1, \ldots, f_m\}$ of the finite dimensional Euclidean space of $\mathcal{S}(\mathbf{A}_F^f)$ containing $\mathcal{B}_f$, we obtain a real number $C_{\mathcal{B}} > 0$ such that (2.2.3.14) holds for all $f \in \mathcal{B}_\infty \otimes \mathcal{B}_f$, and therefore for all the elements of its convex circled hull, and hence for all $f \in \mathcal{B}$. □

2.2.4. Compatibility with directional derivatives: global adelic θ-functions. We fix a choice $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ of continuous homomorphic section to the idelic norm. This is a continuous 1-parameter subgroup, which, we thought of as such, we denote by $\gamma_\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$. It makes sense to form derivatives along γ_τ of smooth functions on $\mathrm{GL}_1(\mathbf{A}_F)$ and C_F.

Before we state and prove the main results of this subsection, we classify the images of the continuous 1-parameter subgroups of $\mathrm{GL}_1(\mathbf{A}_F)$, i.e. continuous homomorphisms $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ of Hausdorff locally compact abelian groups.

LEMMA 2.2.4.1. *Let F be a number field. Let $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ be a continuous 1-parameter subgroup.*

Then $\gamma(\mathbf{R}_{>0})$ is contained in the closed subgroup:

$$\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \mathrm{GL}_1(\mathbf{A}_F^f) = \mathrm{GL}_1(\mathbf{A}_F). \tag{2.2.4.1}$$

Then γ satisfies exactly one of the following two conditions:

(1) *The composition $|\cdot|_F \circ \gamma$ is a smooth automorphism of the abelian Lie group $\mathbf{R}_{>0}$, and there exists a unique nonzero real number α such that $|\gamma(t)|_F = t^\alpha$ for all $t \in \mathbf{R}_{>0}$.*
(2) *The image of γ is contained in the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \{1\}$ of $\mathrm{GL}_1(\mathbf{A}_F)$.*

Proof. Since $\mathrm{GL}_1(\mathbf{A}_F) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \mathrm{GL}_1(\mathbf{A}_F^f)$ as Hausdorff locally compact topological abelian groups, and $\mathrm{GL}_1(\mathbf{A}_F^f)$ is totally disconnected, then, as $\mathbf{R}_{>0}$ is connected, $\gamma(\mathbf{R}_{>0})$ is connected and hence contained in the identity component of $\mathrm{GL}_1(\mathbf{A}_F)$, which is indeed contained within the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\}$ from (2.2.4.1).

The composition $|\cdot|_F \circ \gamma$ is a continuous group endomorphism of $\mathbf{R}_{>0}$, and it is therefore smooth (by the Closed Subgroup Theorem: a closed subgroup of a Lie group is an embedded Lie group), being $\mathbf{R}_{>0}$ a Lie group. Its tangent map is multiplication by a real number α. If $\alpha \neq 0$, then it is smoothly invertible and hence a Lie group automorphism. Since $\mathbf{R}_{>0}$ is simply connected, it is of the form $t \mapsto t^\alpha$.

If $\alpha = 0$, then $\gamma(\mathbf{R}_{>0}) \subset \mathrm{GL}_1(\mathbf{A}_F)^1 \cap (\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\}) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \{1\}$. □

Lemma 2.2.4.2. *Let F be a number field, $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ a continuous 1-parameter subgroup, $f \in \mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function.*

Then the limit:

$$(\partial_\gamma f)(x) := \lim_{t \to 1} \frac{f(\gamma(t)x) - f(x)}{t - 1} \tag{2.2.4.2}$$

exists for every $x \in \mathbf{A}_F$, and it is a Schwartz-Bruhat function on $\mathbf{A}_F$. Moreover, $\partial_\gamma : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$ is continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_F)$.

Proof. Since the continuous linear operator from (2.1.3.2) is a linear topological isomorphism by Proposition 2.1.3.2, for each Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_F)$, there exist Schwartz-Bruhat functions $f_1, \ldots, f_m \in \mathcal{S}(\mathbf{A}_F)$, each of the form:

$$f_k = f_{k,\infty} \otimes \bigotimes_{v \in \Sigma_{F,f}} f_{k,v}$$

with $f_{k,v} \in \mathcal{S}(F_v)$ for each finite place v, $f_v = 1_{\mathcal{O}_{F_v}}$ for all but finitely many v, and $f_{k,\infty} \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ for each $1 \leq k \leq m$, and complex numbers $c_1, \ldots, c_m$, such that $f = \sum_{1 \leq k \leq m} c_k f_k$.

Since formation of the limit in (2.2.4.2) is $\mathbf{C}$-linear with respect to f, the existence of the limit is reduced to the case $f = f_k$ for some $1 \leq k \leq m$, i.e. f is of the form $f = f_\infty \otimes \bigotimes_{v \in \Sigma_{F,f}} f_v$, for f_v the indicator function of some compact open subset $K_v \subset F_v$ for each $v \in S$, S the minimal finite set of places containing all the Archimedean places of F, such that for every finite place $v \notin S$, $f_v = 1_{\mathcal{O}_{F_v}}$.

In this special case, since each f_v is unaffected by the dilation under the idele $\gamma(t)$ for each $t \in \mathbf{R}_{>0}$ in light of Lemma 2.2.4.1, we are further reduced to the case $S = S_\infty$, where the derivation operator ∂_γ is the directional derivative along the continuous, hence smooth, 1-parameter subgroup $\gamma : \mathbf{R}_{>0} \to \mathbf{A}_{F,S_\infty}$.

The Schwartz function space $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is stable under directional derivation, since it is so under partial derivation, and the partial derivative operators ∂_{x_i}, for x_i a choice of smooth coordinates on the real Euclidean space $\mathbf{A}_{F,S_\infty}$, are continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$.

We now observe that the composition of γ_∞ with each projection $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to F_v^\times$ for each Archimedean place v, is a connected subgroup of $F_v^\times$, acting on F_v by multiplication by a nonzero scalar $\gamma_v(t) \in F_v^\times$ for each $t \in \mathbf{R}_{>0}$.

The scalar $\gamma_v(t)$ is of the form t^{α_v}, for $\alpha_v \in \mathbf{R}$ if v is real, and for $\alpha_v \in \mathbf{C}$ for v complex. We define

$$\alpha(y) := \gamma_\infty(e^y) \cdot x \tag{2.2.4.3}$$

a smooth curve in the Euclidean space $\mathbf{A}_{F,S_\infty}$, such that $\alpha(0) = x$.

We have:

$$\lim_{y\to 0} \frac{f_\infty(\alpha(y)) - f_\infty(\alpha(0))}{y} = \lim_{t\to 1} \frac{f_\infty(\gamma_\infty(t)x) - f_\infty(x)}{t-1} \tag{2.2.4.4}$$

since $\lim_{y\to 0} \frac{e^y-1}{y} = 1$.

Upon fixing smooth coordinates $x_v : \mathbf{R} \to \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \simeq \mathbf{A}_{F,S_\infty}$ for v real, and holomorphic and anti-holomorphic $z_v, \overline{z}_v : \mathbf{C} \to \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \simeq \mathbf{A}_{F,S_\infty}$, for v complex, we write the left side of (2.2.4.4) as:

(2.2.4.5)

$$\lim_{y\to 0} \frac{f_\infty(\alpha(y)) - f_\infty(\alpha(0))}{y} = \sum_{v \text{ real}} \alpha_v x_v \partial_{x_v} f_\infty(x) + \sum_{v \text{ complex}} \{(\partial_{z_v} + \partial_{\overline{z}_v}) f_\infty(x) \operatorname{Re}(\alpha_v z_v) - i\,(-\partial_{z_v} + \partial_{\overline{z}_v}) f_\infty(x) \operatorname{Im}(\alpha_v z_v)\}$$

which exhibits the limits in (2.2.4.4) as existing and being valued in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, defining the linear operator ∂_{γ_∞}, which is in turn exhibited as a finite linear combination of continuous linear endomorphisms of $\mathcal{S}(\mathbf{A}_{F,S_\infty})$.

We have just shown that the composition of ∂_γ with the continuous linear operator $\pi_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_F)$ from (2.1.4.5) is continuous for each finite S containing all the Archimedean places of F. It follows, by Proposition 2.1.4.1, since $\mathcal{S}(\mathbf{A}_F)$ is a strict locally convex inductive limit of the $\mathcal{S}(\mathbf{A}_{F,S})$ as S varies over the filtered set of finite subsets of Σ_F containing S_∞ (see [**Scha**, Ch.II,§6,pp.58-59]), that $\partial_\gamma : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$ is continuous.

The foregoing implies that under the linear topological isomorphism:

$$T : \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\mathbf{A}_F^f) \to \mathcal{S}(\mathbf{A}_F)$$

from (2.1.3.2) of Proposition 2.1.3.2(2), we have

$$T \circ (\partial_{\gamma_\infty} \otimes \mathrm{id}_{\mathcal{S}(\mathbf{A}_F^f)}) \circ T^{-1} = \partial_\gamma \tag{2.2.4.6}$$

as linear endomorphisms of $\mathcal{S}(\mathbf{A}_F)$. Since T and T^{-1} are continuous, and ∂_{γ_∞}, the directional derivative on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ along the continuous, hence smooth, 1-parameter subgroup γ_∞ induced by the composition γ_∞ of γ by the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, is continuous, then so is ∂_γ.

The identity in (2.2.4.6) as $\mathbf{C}$-linear endomorphisms of $\mathcal{S}(\mathbf{A}_F)$, holds because the composition γ_∞ of γ with the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ remains injective by Lemma 2.2.4.1. □

LEMMA 2.2.4.3 (Existence of directional derivatives - global adelic θ-functions). *Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function on $\mathbf{A}_F$. Let $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ be a continuous 1-parameter subgroup. The directional derivative $\partial_\gamma \Phi_F(f)$ exists, it is continuous, and we have*

$$\Phi_F(\partial_\gamma f) = \partial_\gamma \Phi_F(f) \tag{2.2.4.7}$$

as continuous functions on C_F.

PROOF. By linearity of both ∂_γ and Φ_F, we may and do assume $f = f_\infty \otimes \bigotimes_{v\in\Sigma_{F,f}} f_v$ for $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$, $f_v \in \mathcal{S}(F_v)$ for all $v \in \Sigma_{F,f}$, and $f_v = 1_{K_v}$ for a compact open subset $K_v \subset F_v$ for all finite places v (see Proposition 2.1.3.2), with $K_v = \mathcal{O}_{F_v}$ for all but finitely many v.

We choose a continuous homomorphic section τ to the idelic norm, and we call $T := \tau(\mathbf{R}_{>0})$ the closed (by Lemma **??**) subgroup of $\mathrm{GL}_1(\mathbf{A}_F)$.

We denote by $T_\infty \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ the image of T under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$. The subgroup T_∞ is closed since it is the image of $\mathbf{R}_{>0}$ under the injective continuous group homomorphism induced by the section τ (see Lemma A.2.3.11).

Lemma 2.2.2.1(3) provides a fractional ideal $I_f \subset F$ such that, for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$ and for $g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ the image of g under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, under the constraint $g_\infty \in T_\infty \cdot E_\infty$ (where E_∞ is the image of the idelic fundamental domain E_F^1 from (A.2.3.14) under

the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$), the following inequality holds:

$$\sum_{\xi\in F^\times} |f(\xi g)| \leq \sum_{\xi\in I_f-\{0\}} |f_\infty(\xi g_\infty)|. \tag{2.2.4.8}$$

The fractional ideal I_f depends on f only through $\bigotimes_{v\in\Sigma_{F,f}} f_v$ (and it depends on the idelic fundamental domain $E_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1$: see Remark 2.2.2.3). In particular, the two Schwartz-Bruhat functions f and $\gamma(t)^*f$, the latter being the function $x \mapsto f(\gamma(t)x)$ on the adeles $\mathbf{A}_F$ of F (where the idele $\gamma(t)$ acts on $\mathbf{A}_F$ continuously by left multiplication) share the same non-Archimedean component $\bigotimes_{v\in\Sigma_{F,f}} f_v$, and their Archimedean components are f_∞, respectively $\gamma(t)^*f_\infty$. The latter is the function on $\mathbf{A}_{F,S_\infty}$ defined by $x_\infty \mapsto f_\infty(\gamma(t)x_\infty)$, where $\gamma(t)x_\infty$ is succint notation to denote the left multiplication by the projection of $\gamma(t) \in \mathrm{GL}_1(\mathbf{A}_F)$ onto $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, on the element $x_\infty \in \mathbf{A}_{F,S_\infty}$.

For $t \neq 1$, the inequality in (2.2.4.8) therefore remains valid for

$$\frac{\gamma(t)^*f - f}{t-1} - \partial_\gamma f \tag{2.2.4.9}$$

replacing f, and correspondingly

$$\frac{\gamma_\infty(t)^*f_\infty - f_\infty}{t-1} - \partial_{\gamma_\infty} f_\infty \tag{2.2.4.10}$$

replacing f_∞, with the same $I := I_f$, since this fractional ideal provided by Lemma 2.2.2.1(3) qualifies for $\gamma(t)^*f$, $\partial_\gamma f$ and f (its formation only depends on the non-Archimedean components of $\gamma(t)^*f$, $\partial_\gamma f$ and f and on the idelic fundamental domain E_F^1, which in turn does not depend on f).

From (2.2.4.8) applied to (2.2.4.9) in the role of f and (2.2.4.10) in the role of f_∞, we have

$$\left|\frac{\Phi_F(\gamma(t)^*f)(g) - \Phi_F(f)(g)}{t-1} - \Phi_F(\partial_\gamma f)\right| \leq \sum_{\xi\in I-\{0\}} \left|\frac{\gamma_\infty(t)^*f_\infty(\xi g_\infty) - f_\infty(\xi g_\infty)}{t-1} - \partial_{\gamma_\infty} f_\infty(\xi g_\infty)\right| \tag{2.2.4.11}$$

for all $g \in T \cdot E_F^1$, with $g_\infty \in T_\infty \cdot E_\infty$.

Since $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$, its rapid decay at infinity ensures that uniformly over the compact subsets of $T_\infty \cdot E_\infty$, the series on the right side of (2.2.4.11) converges. We will now set out to proving this convergence assertion.

Since f_∞ is smooth, the function, for $t \neq 1$

$$u(t, g_\infty) := \frac{\gamma_\infty(t)^*f_\infty(g_\infty) - f_\infty(g_\infty)}{t-1} - \partial_{\gamma_\infty} f_\infty(g_\infty)$$

extends to a continuous function over $\mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ vanishing at $t = 1$ (note that continuity follows from uniform differentiability of Schwartz functions on Euclidean spaces, i.e. the incremental ratios converge uniformly), and the complex valued function ψ over

$$\mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$$

defined as

$$\phi(t, y, g_\infty) := u(t, yg_\infty) \tag{2.2.4.12}$$

is also continuous.

We now fix two compact intervals $[a, b], [a', b'] \subset \mathbf{R}_{>0}$, and assume $1 \in (a, b)$. The function defined as

$$\widetilde{\phi}(y) := \sup_{(t,g_\infty)\in[a,b]\times\tau([a',b'])\cdot\overline{E}_\infty} |\phi(t, y, g_\infty)|$$

is continuous over $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ since $[a, b] \times \tau([a', b']) \cdot \overline{E}_\infty$ is compact Hausdorff and metrizable (for a continuous function $u : X \times Y \to \mathbf{C}$ with X, Y metric spaces, and Y compact, $\sup_{y\in Y} u(x, y)$ is a continuous function of $x \in X$).

We now prove that for each integer $N \geq 1$ we can find a large enough compact subset $B \subset \mathbf{A}_{F,S_\infty}$ depending on N such that for all $\xi \in I$ not contained in B

$$\widetilde{\phi}(\xi) \leq C_{B,N}\|\xi\|^{-N} \tag{2.2.4.13}$$

for some constant $C_{B,N} > 0$.

To see this, we proceed as follows. The assignment $u : t \mapsto u(t,\cdot)$ is a continuous 1-parameter family of Schwartz functions on the Euclidean space $\mathbf{A}_{F,S_\infty}$, i.e. it is a continuous map:

$$u : \mathbf{R}_{>0} \to \mathcal{S}(\mathbf{A}_{F,S_\infty}).$$

Here we have used Lemma 2.2.4.2 to deduce that $\partial_{\gamma_\infty} f_\infty$ is itself a Schwartz function.

Since $[a,b]$ is compact Hausdorff, so is $\mathcal{B} := u([a,b])$, and hence $\mathcal{B}$ is bounded in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$. The assignment

$$p : u \mapsto \sup_{x_\infty \in \mathbf{A}_{F,S_\infty}} \|x_\infty\|^N \, |u(x_\infty)|$$

is a continuous seminorm on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, and hence $p(\mathcal{B}) \subset [0,r] \subset \mathbf{R}$ for a real number $r \geq 0$.

There therefore exists a compact subset $B \subset \mathbf{A}_{F,S_\infty}$ such that:

$$\sup_{x_\infty \in \mathbf{A}_{F,S_\infty} - B} |u(t,x_\infty)| \leq r\|x_\infty\|^{-N}. \tag{2.2.4.14}$$

From (2.2.4.14), since r is independent of t, y, g_∞ (reprising the notation from (2.2.4.12)) we deduce that

$$\widetilde{\phi}(y) \leq A_N\|y\|^{-N} \tag{2.2.4.15}$$

for all $y \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ satisfying the largeness condition:

$$y \notin (\gamma([a,b]) \cdot \tau([a',b']) \cdot \overline{E}_\infty)^{-1} \cdot B \tag{2.2.4.16}$$

with A_N obtained from $r\|\gamma(t) y g_\infty\|^{-N}$ by maximizing with respect to $\gamma(t)g_\infty$ varying over the compact subset $\gamma([a,b]) \cdot \tau([a',b']) \cdot \overline{E}_\infty$ (which, note, does not contain the origin of the Euclidean space $\mathbf{A}_{F,S_\infty}$).

We rename $(\gamma([a,b]) \cdot \tau([a',b']) \cdot \overline{E}_\infty)^{-1} \cdot B$ as B, and A_N as $C_{B,N}$. We have shown that for these choices of B and $C_{B,N}$, (2.2.4.13) holds.

The discreteness of $I \subset \mathbf{A}_{F,S_\infty}$ ensures that $(I - \{0\}) \cap B$ is finite. For $t \in [a,b]$ and $g_\infty \in E_\infty$, the sum in the right side of (2.2.4.11) is bounded above by

$$\sum_{\xi \in (I-\{0\}) \cap B} \left| \frac{\gamma_\infty(t)^* f_\infty(\xi g_\infty) - f_\infty(\xi g_\infty)}{t-1} - \partial_{\gamma_\infty} f_\infty(\xi g_\infty) \right| + C_{B,N} \sum_{\xi \in I - \{0\} : \xi \notin B} \|\xi\|^{-N}. \tag{2.2.4.17}$$

By Proposition 2.2.1.3, the series $\sum_{\xi \in I-\{0\}:\xi \notin B} \|\xi\|^{-N}$ is absolutely convergent so long as we choose $N > [F : \mathbf{Q}]$. Since I is discrete within $\mathbf{A}_{F,S_\infty}$, for every $\varepsilon > 0$ we may choose a compact subset $B_\varepsilon \subset \mathbf{A}_{F,S_\infty}$ large enough so as to have:

$$C_{B,N} \sum_{\xi \in I - \{0\} : \xi \notin (B_\varepsilon \cup B)} \|\xi\|^{-N} < \varepsilon/2. \tag{2.2.4.18}$$

Since we will be wanting to let t be arbitrarily close to 1, we note that $B \cup B_\varepsilon$ is independent of $t \in [a,b]$. Continuity of $u(t, g_\infty)$ (which, recall, is implied by uniform differentiability of the Schwartz function f_∞) now allows us to choose $\delta > 0$ such that for all $t \in \mathbf{R}_{>0}$ satisfying $0 < |t-1| < \delta$ we have

$$\left| \frac{\gamma_\infty(t)^* f_\infty(\xi g_\infty) - f_\infty(\xi g_\infty)}{t-1} - \partial_{\gamma_\infty} f_\infty(\xi g_\infty) \right| \leq \frac{\varepsilon}{2|(I-\{0\}) \cap (B \cup B_\varepsilon)|}, \tag{2.2.4.19}$$

which implies

$$\sum_{\xi \in (I-\{0\}) \cap (B_\varepsilon \cup B)} \left| \frac{\gamma_\infty(t)^* f_\infty(\xi g_\infty) - f_\infty(\xi g_\infty)}{t-1} - \partial_{\gamma_\infty} f_\infty(\xi g_\infty) \right| < \frac{\varepsilon}{2}. \tag{2.2.4.20}$$

Upon combining (2.2.4.17), (2.2.4.20) and (2.2.4.19), we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $t \in \mathbf{R}_{>0}$ satisfying $0 < |t-1| < \delta$, we have:

$$\left| \frac{\Phi_F(\gamma(t)^* f)(g) - \Phi_F(f)(g)}{t-1} - \Phi_F(\partial_\gamma f)(g) \right| < \varepsilon \tag{2.2.4.21}$$

for all $g \in \tau([a',b']) \cdot E_F^1$.

This shows that

$$\lim_{t \to 1} \frac{\Phi_F(f)(\gamma(t)g) - \Phi_F(f)(g)}{t-1} \tag{2.2.4.22}$$

exists for all $g \in \tau([a',b']) \cdot E_F^1$, and it equals $\Phi_F(\partial_\gamma f)(g)$. Since $[a',b'] \subset \mathbf{R}_{>0}$ is arbitrary, we see that the limit in (2.2.4.22) exists for all $g \in T \cdot E_F^1$ and the convergence is uniform on the compact subsets of $T \cdot E_F^1$, and hence for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$ since the two continuous functions $\Phi_F(\gamma(t)^* f)$ and $\Phi_F(f)$ are invariant under the left-translation action by $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$ for all $t \in \mathbf{R}_{>0}$.

Since the limit in (2.2.4.22), if it exists, is the directional derivative $\partial_\gamma \Phi_F(f)$ as a function on C_F, we see that (2.2.4.7) holds pointwise over C_F. By Lemma 2.2.4.1 we know $\partial_\gamma f$ is a Schwartz-Bruhat function over $\mathbf{A}_F$, so $\Phi_F(\partial_\gamma f)$ is continuous as a function over C_F by Lemma 2.2.2.1 and hence so is $\partial_\gamma \Phi_F(f)$. □

2.2.4.1. *Directional derivatives and the global polar hyperplanes.* Throughout the remainder of this subsection we study the compatibility of formation of directional derivatives along continuous 1-parameter subgroups $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$, and formation of the two polar hyperplanes $\ker(\delta_0)$ and $\ker(\widehat{\delta_0})$ on $\mathcal{S}(\mathbf{A}_F)$.

Definition 2.2.4.4 (Global polar hyperplanes). Let F be a number field.

We define:

$$\mathcal{S}_0(\mathbf{A}_F) := \ker(\delta_0) \cap \ker(\widehat{\delta_0}) \tag{2.2.4.23}$$

where $\delta_0 : \mathcal{S}(\mathbf{A}_F) \to \mathbf{C}$ is the continuous functional $\delta_0(f) := f(0)$, and $\widehat{\delta_0}$ is its adelic Fourier transform, i.e. $\widehat{\delta_0}(f) := \widehat{f}(0)$.

We will be explaining the role of the two hyperplanes $\ker(\delta_0)$ and $\ker(\widehat{\delta_0})$ in Chapter 3 (see Theorem 3.2.2.1 and Example 3.2.2.4).

Lemma 2.2.4.5. *Let F be a number field. Let $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ be a continuous 1-parameter subgroup. Then:*

(1) *The continuous linear endomorphism*

$$\partial_\gamma : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F) \tag{2.2.4.24}$$

preserves the closed subspace $\mathcal{S}_0(\mathbf{A}_F)$.

(2) *For*

$$T : \mathcal{S}(\mathbf{A}_F)/\mathcal{S}_0(\mathbf{A}_F) \to \mathbf{C} \oplus \mathbf{C} \tag{2.2.4.25}$$

the linear isomorphism $(\delta_0, \widehat{\delta_0})$, the induced linear map

$$\partial_\gamma : \mathcal{S}(\mathbf{A}_F)/\mathcal{S}_0(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)/\mathcal{S}_0(\mathbf{A}_F) \tag{2.2.4.26}$$

satisfies:

$$T \circ \partial_\gamma \circ T^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_\gamma \end{pmatrix} \tag{2.2.4.27}$$

where α_γ is the unique real number such that $|\cdot|_F \circ \gamma$, a smooth endomorphism of $\mathbf{R}_{>0}$, is of the form $t \mapsto t^{\alpha_\gamma}$.

(3) *For $f \in \mathcal{S}(\mathbf{A}_F)$, the pointwise limit*

$$\lim_{t\to 1} \frac{1}{t-1}\left\{|\gamma(t)|_F^{-1}\gamma(t)^*\widehat{f} - \widehat{f}\right\} \tag{2.2.4.28}$$

as functions on $\mathbf{A}_F$ (where $\gamma(t)^*u(x) = u(\gamma(t)^{-1}x)$ as functions on $\mathbf{A}_F$) exists and is given by

$$\lim_{t\to 1} \frac{1}{t-1}\left\{|\gamma(t)|_F^{-1}\gamma(t)^*\widehat{f} - \widehat{f}\right\} = \widehat{\partial_\gamma f}, \tag{2.2.4.29}$$

so it belongs to $\mathcal{S}(\mathbf{A}_F)$. Explicitly, its value at any $x \in \mathbf{A}_F$ is equal to

$$\int_{\mathbf{A}_F} f(y)\overline{\psi}(xy)\left\{-\alpha_\gamma + 2\pi i\overline{\Lambda(\gamma'(1)xy)}\right\}\,\mathrm{d}\nu(y). \tag{2.2.4.30}$$

REMARK 2.2.4.6. The mode of convergence of the limit in (2.2.4.28) is uniform on bounded subsets in $\mathcal{S}(\mathbf{A}_F)$, i.e. it converges in $\mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_F))$. This is due to the fact that $\mathcal{S}(\mathbf{A}_F)$ is barrelled and complete, hence quasi-complete. Indeed, as t varies over a compact neighborhood of 1, the proof of (3) shows that

$$\frac{1}{t-1}\left\{|\gamma(t)|_F^{-1}\gamma(t)^*\widehat{f} - \widehat{f}\right\}$$

is bounded in $\mathcal{S}(\mathbf{A}_F)$, being the image of a compact Hausdorff space under a continuous map, hence equicontinuous by Uniform Boundedness (see [**Scha**, Ch.III,§3.4,Corollary,p.82] and [**Scha**, Ch.III,Thm.4.2,p.83]).

PROOF. (1) and (3) : If $f \in \mathcal{S}(\mathbf{A}_F)$, we note that by definition

$$(\partial_\gamma f)(0) = \lim_{t\to 1}\frac{f(\gamma(t)0) - f(0)}{t-1} = 0 \tag{2.2.4.31}$$

since $\gamma(t)0 = 0$ within $\mathbf{A}_F$.

We now want to show that we have

$$\widehat{\partial_\gamma f}(0) = 0.$$

We will check this by proving (3) first, from which it follows by choosing $x = 0$. Indeed, (2.2.4.29) reads

$$\widehat{(\partial_\gamma f)}(x) = \lim_{t\to 1}\frac{1}{t-1}\left\{|\gamma(t)|_F^{-1}\widehat{f}(\gamma(t)^{-1}x) - \widehat{f}(x)\right\}$$

which gives

$$\widehat{(\partial_\gamma f)}(0) = \lim_{t\to 1}\frac{1}{t-1}\left\{|\gamma(t)|_F^{-1}\widehat{f}(\gamma(t)^{-1}0) - \widehat{f}(0)\right\} = 0$$

since $\gamma(t)^{-1}0 = 0$.

In the calculations that follow, setting $x = 0$ from the outset leads to substantial simplifications. The reader is welcome to assume $x = 0$ throughout, leading to the formula in (2.2.4.29) in this special case only.

We now assume $f \in \mathcal{S}(\mathbf{A}_F)$ is of the form

$$f = f_\infty \otimes \bigotimes_{v\in\Sigma_{F,f}} f_v$$

for $f_v \in \mathcal{S}(F_v)$ with $f_v = 1_{K_v}$ for $K_v \subset F_v$ a compact open subset for all $v \in \Sigma_{F,f}$, and $K_v = \mathcal{O}_{F_v}$ for all but finitely many $v \in \Sigma_{F,f}$. We further assume f_∞ is *compactly supported*.

By [**RV**, Prop.5.6(iii)], we have

$$\widehat{(\partial_\gamma f)}(x) = \int_{\mathbf{A}_F}(\partial_\gamma f)(y)\overline{\psi(xy)}\,\mathrm{d}\nu(y) = \left(\int_{\mathbf{A}_{F,S_\infty}}(\partial_{\gamma_\infty}f_\infty)(y_\infty)\prod_{v\in S_\infty}\overline{\psi_v(x_vy_v)}\,\mathrm{d}\nu_\infty(y_\infty)\right)\prod_{v\in\Sigma_{F,f}}\int_{F_v} f_v(x_v)\overline{\psi_v(x_vy_v)}\,\mathrm{d}\nu_v(y_v) \tag{2.2.4.32}$$

for $\psi : \mathbf{A}_F \to \mathrm{U}(1)$ the standard additive character (see [**RV**, Ch.7,§7.1,p.253]), where ν is the additive global Tamagawa measure on $\mathbf{A}_F$ from Definition A.3.1.1, $\psi_v : F_v \to \mathrm{U}(1)$ is the standard additive character

(see [RV, Ch.7,§7.1,pp.249-253]), ν_v is the additive local Tamagawa measure on F_v for each $v \in \Sigma_F$, and $\nu_\infty := \bigotimes_{v\in S_\infty} \nu_v$.

We analyze the integral:

$$\int_{\mathbf{A}_{F,S_\infty}} (\partial_{\gamma_\infty} f_\infty)(y_\infty) \prod_{v\in S_\infty} \overline{\psi_v(x_v y_v)}\, \mathrm{d}\nu_\infty(y_\infty) \tag{2.2.4.33}$$

Let $[a,b] \subset \mathbf{R}_{>0}$ an arbitrary compact interval such that $1 \in (a,b)$. We define $K := \gamma_\infty([a,b]) \cdot \mathrm{supp}(f_\infty) \subset \mathbf{A}_{F,S_\infty}$. Since $\mathrm{supp}(f_\infty)$ is compact, so is K as a subset of $\mathbf{A}_{F,S_\infty}$.

By Dominated Convergence, applied to a bounded absolutely ν_∞-integrable majorant to

$$|(f_\infty(\gamma_\infty(t)y_\infty) - f_\infty(y_\infty))/(t-1)|$$

for $y_\infty \in \mathrm{supp}(f_\infty)$ and $t \in [a,b]$ varying over a sequence tending to 1 (which exists as a consequence of smoothness of f_∞ over $\mathbf{A}_{F,S_\infty}$ and Hadamard's Lemma), we may write the integral in (2.2.4.33) as the following limit with t varying through a sequence:

$$\lim_{t\to 1} \frac{1}{t-1} \left\{ \int_{\mathbf{A}_{F,S_\infty}} f_\infty(\gamma_\infty(t)y_\infty) \prod_{v\in S_\infty} \overline{\psi_v(x_v y_v)}\, \mathrm{d}\nu_\infty(y_\infty) - \int_{\mathbf{A}_{F,S_\infty}} f_\infty(y_\infty) \prod_{v\in S_\infty} \overline{\psi_v(x_v y_v)}\, \mathrm{d}\nu_\infty(y_\infty) \right\}. \tag{2.2.4.34}$$

Upon defining $\psi_\infty : \mathbf{A}_{F,S_\infty} \to \mathrm{U}(1)$ as $\psi_\infty := \prod_{v\in S_\infty} \psi_v$, this limit is the same as

$$\lim_{t\to 1} \frac{1}{t-1} \left\{ |\gamma_\infty(t)|_\infty^{-1} \int_{\mathbf{A}_{F,S_\infty}} f_\infty(y_\infty) \prod_{v\in S_\infty} \overline{\psi_\infty(x_\infty \gamma_\infty(t)^{-1} y_\infty)}\, \mathrm{d}\nu_\infty(y_\infty) - \int_{\mathbf{A}_{F,S_\infty}} f_\infty(y_\infty) \prod_{v\in S_\infty} \overline{\psi_v(x_v y_v)}\, \mathrm{d}\nu_\infty(y_\infty) \right\}. \tag{2.2.4.35}$$

By Lemma 2.2.4.1, we have $|\gamma_\infty(t)|_\infty = |\gamma(t)|_F$ for every $t \in \mathbf{R}_{>0}$, where $|\cdot|_\infty = \prod_{v\in S_\infty} |\cdot|_v$.

Upon combining (2.2.4.35) and (2.2.4.32), then by [RV, Prop.5.6(iii)] we have:

$$\widehat{(\partial_\gamma f)}(x) = \lim_{t\to 1} \frac{1}{t-1} \left\{ |\gamma(t)|_F^{-1} \widehat{f}(\gamma(t)^{-1}x) - \widehat{f}(x) \right\} \tag{2.2.4.36}$$

for each $x \in \mathbf{A}_F$ where again $t \to 1$ through a sequence, and therefore with $t \to 1$ in general. This proves (3) for the special class of f considered above (especially with f_∞ compactly supported).

Formation of both sides of (2.2.4.36) is $\mathbf{C}$-linear with respect to f. It follows that (2.2.4.36) holds for all Schwartz-Bruhat functions $f \in \mathcal{S}(\mathbf{A}_F)$ that are finite $\mathbf{C}$-linear combinations of functions of the form $f_\infty \otimes \bigotimes_{v\in\Sigma_{F,f}} f_v$, with $f_v \in \mathcal{S}(F_v)$ for every $v \in \Sigma_{F,f}$, and $f_v = 1_{K_v}$ for all finite places v of F, with $K_v \subset F_v$ a compact open subset, and $K_v = \mathcal{O}_{F_v}$ for all but finitely many finite places, and f_∞ with *compact support*.

The goal is to now show that formation of the limit

$$\lim_{t\to 1} \frac{1}{t-1} \left\{ |\gamma(t)|^{-1} \widehat{f}(\gamma(t)^{-1}x) - \widehat{f}(x) \right\}$$

is sequentially continuous in f for each x. Since the linear span of the class of functions on which we have proved that (2.2.4.28) holds is sequentially dense in $\mathcal{S}(\mathbf{A}_F)$, sequential continuity of both sides of the identity at each x will propagate the identity to the whole $\mathcal{S}(\mathbf{A}_F)$, proving (2.2.4.29) of (3) in general.

The expression in (2.2.4.28) is not well-suited to analyzing sequential continuity in f at each x. Since it is a limit of integrals, our aim is to use the Dominated Convergence Theorem to rewrite it as a single integral so as to better understand its continuity properties in f.

Since

$$\frac{|\gamma(t)|_F^{-1} \widehat{f}(\gamma(t)^{-1}x) - \widehat{f}(x)}{t-1} = \int_{\mathbf{A}_F} f(y) \left\{ \frac{|\gamma(t)|_F^{-1} \overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi(xy)}}{t-1} \right\} \mathrm{d}\nu(y) \tag{2.2.4.37}$$

for $t \neq 1$ by definition of the Fourier transform, and f has rapid decay, to apply Dominated Convergence we aim to bound the integrand factor

$$\frac{|\gamma(t)|_F^{-1}\overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi(xy)}}{t-1} \tag{2.2.4.38}$$

for t near 1 and uniformly in $y \in \mathbf{A}_F$ with each fixed x (we will even obtain such a bound with uniformity in x).

We have

$$\frac{|\gamma(t)|_F^{-1}\overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi(xy)}}{t-1} = \left\{\frac{|\gamma(t)|_F^{-1}-1}{t-1}\overline{\psi(\gamma(t)^{-1}xy)}\right\} + \frac{\overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi}(xy)}{t-1} \tag{2.2.4.39}$$

with $|\overline{\psi(\gamma(t)^{-1}xy)}| = 1$, it is enough to bound the function

$$\frac{\psi(\gamma(t)^{-1}xy) - \psi(xy)}{t-1} \tag{2.2.4.40}$$

for t near 1 uniformly with respect to y.

We can rewrite this difference quotient as

$$\frac{\phi(t) - \phi(1)}{t-1}$$

where $\phi(t) := \psi(\gamma(t^{-1})xy) - \psi(xy) = e^{2\pi i\Lambda(\gamma(t^{-1})xy)}$ where $\Lambda(u) = \sum_{v\in S_\infty}\Lambda_v(u_v)$, for Λ_v the standard local additive character at v (see Remark A.3.2.5).

We have

$$\frac{\phi(t)-\phi(1)}{t-1} = \frac{1}{t-1}\int_1^t \phi'(u)\,\mathrm{d}u = \int_0^1 \phi'(1+u(t-1))\,\mathrm{d}u.$$

We compute (for $v \neq 0$)

$$\phi'(v) = 2\pi i\, e^{2\pi i\Lambda(\gamma(v^{-1})xy)}\,\Lambda\left(\frac{\mathrm{d}}{\mathrm{d}v}\left(\gamma(v^{-1})\right)xy\right) = 2\pi i\, e^{2\pi i\Lambda(\gamma(v^{-1})xy)}\,\Lambda\left(\frac{\gamma'(v^{-1})}{-v^2}xy\right)$$

and so

$$|\phi'(v)| \leq 2\pi\left|\Lambda\left(\frac{\gamma'(v^{-1})}{-v^2}xy\right)\right| \leq 2\pi C\|x\|\,\|y\|\,\frac{\|\gamma'(v^{-1})\|}{v^2} \tag{2.2.4.41}$$

for some constant $C > 0$ (via continuity of Λ), for $v = 1+u(t-1)$ with $0 \leq u \leq 1$ and $-1/2 \leq t-1 \leq 1/2$ we have $1/2 \leq v \leq 3/2$, and hence

$$|\phi'(v)| \leq 2\pi\left|\Lambda\left(\frac{\gamma'(v^{-1})}{-v^2}xy\right)\right| \leq 2\pi C'\|x\|\,\|y\| \tag{2.2.4.42}$$

where the constant C' is $4C\max_{v\in[1/2,3/2]}\|\gamma'(v^{-1})\|$.

It follows that the function from (2.2.4.38) is bounded above by $u(x,y) := 2\pi C'\|x\|\|y\|$, and so by Dominated Convergence (since $f \in \mathcal{S}(\mathbf{A}_F)$) applied to the integral in (2.2.4.37), we deduce that we can exchange the order of formation of the limit as $t \to 1$ and of the integral, to obtain

$$\lim_{t\to 1}\int_{\mathbf{A}_F} f(y)\left\{\frac{|\gamma(t)|_F^{-1}\overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi(xy)}}{t-1}\right\}\mathrm{d}\nu(y) = \int_{\mathbf{A}_F} f(y)\lim_{t\to 1}\left\{\frac{|\gamma(t)|_F^{-1}\overline{\psi(\gamma(t)^{-1}xy)} - \overline{\psi(xy)}}{t-1}\right\}\mathrm{d}\nu(y).$$

The right side is finite and it equals

$$\int_{\mathbf{A}_F} f(y)\overline{\psi}(xy)\left\{-\alpha_\gamma + 2\pi i\overline{\Lambda(\gamma'(1)xy)}\right\}\mathrm{d}\nu(y),$$

(where α_γ is the unique real number such that $|\gamma(t)|_F = t^{\alpha_\gamma}$ for all $t \in \mathbf{R}_{>0}$) which is visibly sequentially continuous in f.

Since the adelic Fourier transform is a linear topological automorphism of $\mathcal{S}(\mathbf{A}_F)$ and ∂_γ is a continuous linear endomorphism of $\mathcal{S}(\mathbf{A}_F)$ by Lemma 2.2.4.2, the left side of (2.2.4.36) is also continuous as a $\mathbf{C}$-linear endomorphism on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, and hence we see that both sides of (2.2.4.36) are sequentially continuous $\mathbf{C}$-linear endomorphisms of $\mathcal{S}(\mathbf{A}_F)$, which agree on a sequentially dense $\mathbf{C}$-linear subspace. We deduce that the identity in (2.2.4.36) holds for all $f \in \mathcal{S}(\mathbf{A}_F)$. This proves (3) and finishes the proof of (1).

(2) : From (1) we know $\delta_0(\partial_\gamma f) = 0$. From (2.2.4.29) we deduce $\widehat{\delta}_0(\partial_\gamma f) = -\alpha_\gamma\, \widehat{\delta}_0(f)$. □

2.2.5. The homogeneous θ-function operators $\Phi_{F,\omega} : \mathcal{S}(\mathbf{A}_F) \to C(C_F/C_F^1)$. In this subsection we define a version of the θ-function operator Φ_F which is ω-homogeneous with respect to the continuous left-translation action by C_F^1, the closed subgroup of C_F of norm-one idele classes.

We recall that we choose on C_F^1 the Haar measure $\overline{\mu}_1$ induced as per Theorem D.1.2.1 by the choices of the counting measure on $F^\times$ and the Haar measure μ_1 on $\mathrm{GL}_1(\mathbf{A}_F)^1$ constructed in Definition A.3.2.1.

The Haar measure μ_1 was constructed upon choosing a continuous homomorphic section to the idelic norm. The Haar measure $\overline{\mu}_1$, however, does not depend on such choice, but only on the volume it assigns to the compact Hausdorff abelian group C_F^1.

Definition 2.2.5.1 (ω-homogeneous global adelic θ-functions). Let F be a number field, $\omega \in \widehat{C_F^1}$ a continuous character. We define the $\mathbf{C}$-linear operator

$$\Phi_{F,\omega} : \mathcal{S}(\mathbf{A}_F) \to C(C_F) \tag{2.2.5.1}$$

by

$$\Phi_{F,\omega}(f)(g) := \int_{C_F^1} \Phi_F(f)(hg)\omega(h)\,\mathrm{d}\overline{\mu}_1(h) \tag{2.2.5.2}$$

for $f \in \mathcal{S}(\mathbf{A}_F)$ and $g \in C_F$. We refer to $\Phi_{F,\omega}(f)$ as the *ω-homogeneous global adelic θ-function* of f.

The (absolute) convergence of the above integral is immediate from the compactness of C_F^1 but the continuity in g requires an argument since f is not compactly supported on $\mathrm{GL}_1(\mathbf{A}_F)$ (unless $f = 0$). This continuity is part of Lemma 2.2.5.2(2).

2.2.5.1. *Properties of the homogeneous θ-function operator.* We prove that formation of $\Phi_{F,\omega}$ is linear and valued in continuous functions on C_F, ω-homogeneous under the left-translation action of C_F^1 on $C(C_F)$.

Lemma 2.2.5.2. *Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function, and $\omega \in \widehat{C_F^1}$ a continuous character. Then, upon denoting $\widetilde{\omega} : \mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{U}(1)$ the composition of ω with the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_F)^1 \to C_F^1$:*

(1) *The integral*

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)^1} f(hg)\widetilde{\omega}(h)\,\mathrm{d}\mu_1(h) \tag{2.2.5.3}$$

is absolutely convergent and its formation is continuous as a function of $g \in \mathrm{GL}_1(\mathbf{A}_F)$. The resulting continuous function is $\widetilde{\omega}$-homogeneous under the left-translation action by $\mathrm{GL}_1(\mathbf{A}_F)^1$.

(2) *We have, as functions of $g \in \mathrm{GL}_1(\mathbf{A}_F)$:*

$$\Phi_{F,\omega}(f)(g) = \int_{\mathrm{GL}_1(\mathbf{A}_F)^1} f(hg)\widetilde{\omega}(h)\,\mathrm{d}\mu_1(h). \tag{2.2.5.4}$$

In particular, $\Phi_{F,\omega}(f)$ is a continuous function on C_F.

Proof. (1) : By (2.2.2.3) from Lemma 2.2.2.1, we have, for every $g \in C_F$ and $g' \in \mathrm{GL}_1(\mathbf{A}_F)$ any lift of g to $\mathrm{GL}_1(\mathbf{A}_F)$, whose image in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$

we denote g_∞:

$$\sum_{\xi\in F^\times}|f(\xi g')|\leq\sum_{\xi\in I_f-\{0\}}|f_\infty(\xi g_\infty)| \tag{2.2.5.5}$$

where $I_f\subset F$ is a fractional ideal depending on f, and $f_\infty\in\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is the Schwartz function on the Euclidean space $\mathbf{A}_{F,S_\infty}$ that serves as the Archimedean component of f.

We consider the idelic fundamental domain E_F^1 from (A.2.3.14) for the continuous $F^\times$-action on $\mathrm{GL}_1(\mathbf{A}_F)^1$. The subset $E_F^1\subset\mathrm{GL}_1(\mathbf{A}_F)^1$ is, recall from (A.2.3.16), of the form

$$E=(E_0\times\widehat{\mathcal{O}}_F^\times)\cup(g_{2,\infty}E_0\times g_{2,f}\widehat{\mathcal{O}}_F^\times)\cup\ldots\cup(g_{h_F,\infty}E_0\times g_{h_F,f}\widehat{\mathcal{O}}_F^\times)\subset\mathrm{GL}_1(\mathbf{A}_F)^1 \tag{2.2.5.6}$$

where $g_i=(g_{i,\infty},g_{i,f})$ is an idele for each integers $1\leq i\leq h_F$ with $g_1=1$ and $|g_i|_F=1$ for every $1\leq i\leq h_F$.

Since the measurable function $\sum_{\xi\in F^\times}|f(\xi g')|$ on $\mathrm{GL}_1(\mathbf{A}_F)$ is $F^\times$-invariant and continuous, being the series in the right side of (2.2.5.5) convergent uniformly on the compact subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, the integral

$$\int_{C_F^1}\sum_{\xi\in F^\times}|f(\xi g')|\,\mathrm{d}\overline{\mu}_1(g) \tag{2.2.5.7}$$

where $\overline{\mu}_1$ is the unique Haar measure on C_F^1 prescribed by Theorem D.1.2.1 by the choices of the counting measure as the Haar measure on $F^\times$, and μ_1 as the Haar measure on $\mathrm{GL}_1(\mathbf{A}_F)^1$ from Definition A.3.2.1, under the canonical isomorphism of abelian topological groups $C_F^1\simeq\mathrm{GL}_1(\mathbf{A}_F)^1/F^\times$.

The integral in (2.2.5.7) equals

$$\int_{E_F^1}\sum_{\xi\in F^\times}|f(\xi g')|\,\mathrm{d}\mu_1(g') \tag{2.2.5.8}$$

by Proposition D.1.2.4 (see also [**Bou-II**, INT VII.52,§3,Thm.4(c)], applied to $n=1$, since the action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$ is free, and to $k(g)=\sum_{\xi\in F^\times}|f(\xi g')|$ as a function of $g\in\mathrm{GL}_1(\mathbf{A}_F)^1/F^\times$, which is measurable and integrable over the compact quotient $\mathrm{GL}_1(\mathbf{A}_F)^1/F^\times$, being k continuous). Since E_F^1 is relatively compact by Lemma A.2.3.8(1), and the function $g'\mapsto\sum_{\xi\in F^\times}|f(\xi g')|$ is continuous on $\mathrm{GL}_1(\mathbf{A}_F)^1$, the integral (2.2.5.8) is finite, proving the absolute convergence of the integral (2.2.5.3) in light of Lemma D.1.2.3 (see also [**DE**, Cor.1.5.4]). By Dominated Convergence, formation of the integral in (2.2.5.3) is continuous with respect to $g\in C_F$.

By invariance of the Haar measure μ_1 on $\mathrm{GL}_1(\mathbf{A}_F)$, formation of the continuous function on $\mathrm{GL}_1(\mathbf{A}_F)$ whose value at $g\in\mathrm{GL}_1(\mathbf{A}_F)$ is given by the integral in (2.2.5.3), is $\widetilde{\omega}$-homogenous with respect to the left translation action by $\mathrm{GL}_1(\mathbf{A}_F)^1$.

We verify this explicitly. We note that by hg within the integral over C_F^1 defining $\Phi_{F,\omega}(f)$ in (2.2.5.2), we mean the image of $g\in\mathrm{GL}_1(\mathbf{A}_F)$ into C_F, translated by h^{-1} on the left using the continuous left-translation action of C_F^1 on C_F, being C_F^1 a subgroup of C_F. In particular, for each $g\in\mathrm{GL}_1(\mathbf{A}_F)$, $\Phi_F(f)(hg)$ makes sense for every $h\in C_F^1$ as a function of $g\in\mathrm{GL}_1(\mathbf{A}_F)$ invariant under the left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$.

By invariance of the Haar measure $\overline{\mu}_1$, we have that for any $h_0 \in C_F^1$ and any choice of lift $\widetilde{h}_0 \in \mathrm{GL}_1(\mathbf{A}_F)^1$ of h_0, and $\widetilde{h}$ of $h \in C_F^1$, and any fixed choice $\chi \in \widehat{C_F}$ of continuous character whose restriction to C_F^1 is ω:

$$\begin{aligned}\Phi_{F,\omega}(f)(\widetilde{h}_0^{-1}g) &= \int_{\mathrm{GL}_1(\mathbf{A}_F)^1} f(\widetilde{h}_0^{-1}\widetilde{h}g)\chi(\widetilde{h})\,\mathrm{d}\mu_1(\widetilde{h}) \\ &= \chi(\widetilde{h}_0)\int_{\mathrm{GL}_1(\mathbf{A}_F)^1} f(\widetilde{h}g)\chi(\widetilde{h})\,\mathrm{d}\mu_1(\widetilde{h}) \\ &= \omega(h_0)\int_{C_F^1} \Phi_F(f)(hg)\omega(h)\,\mathrm{d}\overline{\mu}_1(h) \\ &= \omega(h_0)\Phi_{F,\omega}(f)(g)\end{aligned}$$

where we have used Theorem D.1.2.1 to justify both the first and last of the above identities, valid for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$. Since $\Phi_F(f)$ is $F^\times$-invariant, as a function on $\mathrm{GL}_1(\mathbf{A}_F)$, so is $\Phi_{F,\omega}(f)$, and hence $\Phi_{F,\omega}(f)(\widetilde{h}_0^{-1}g)$ only depends on $\widetilde{h}_0^{-1}g$ through the image of this element of $\mathrm{GL}_1(\mathbf{A}_F)$ in C_F, which is $h_0^{-1}g$. We have shown that we have:

$$\Phi_{F,\omega}(f)(h_0^{-1}g) = \omega(h_0)\Phi_{F,\omega}(f)(g)$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$, and hence, $\Phi_{F,\omega}(f)$ is, as a continuous function on C_F, ω-homogeneous under the continuous left-translation action by C_F^1 on $C(C_F)$.

(2) : By (1) and Theorem D.1.2.1 (see also [**DE**, Thm.1.5.3]), the identity in (2.2.5.4) holds for every $g \in C_F$, and since both sides are continuous functions of $g \in C_F$, it holds as continuous functions on C_F. □

2.2.5.2. *Decay estimates over C_F/C_F^1.* Lemma 2.2.5.2(1) justifies the terminology "ω-homogeneous" from Definition 2.2.5.1.

We have yet to collect the global adelic θ-functions into a $\mathbf{C}$-linear space and endow it with a Hausdorff locally convex topology, and with this aim in mind we construct the ω-homogeneous θ-functions directly first, proving their properties below.

Remark 2.2.5.3. Lemma 4.3.3.1 from Chapter 3 provides a systematic method to produce elements in appropriate Hausdorff locally convex topological $\mathbf{C}$-vector spaces equipped with a suitable continuous action of a compact Hausdorff abelian group G, that are ω-homogeneous, for a fixed choice of continuous character $\omega \in \widehat{G}$.

In this subsection we provide an analog of the estimates from Theorem 2.2.3.4.

Theorem 2.2.5.4 (Estimates over the idele classes - II). *We let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function, and $\omega \in \widehat{C_F^1}$ a continuous character. The function $\Phi_{F,\omega}(f)$ on C_F satisfies the following conditions:*

(1) *$\Phi_{F,\omega}(f)$ is a continuous function on C_F, ω-homogeneous under the left translation action by C_F^1.*

(2) *It satisfies the following growth condition: for every integer $N \geq 0$, there exists a real number $C_N > 0$ depending on N and f, such that for every $g \in C_F$, we have*

$$|\Phi_{F,\omega}(f)(g)| \leq C_N |g|_F^{-N} \tag{2.2.5.9}$$

as $|g|_F \to +\infty$. For every integer $N \geq 0$, there exists a real number $C_N > 0$ depending on N and f, such that

$$|\Phi_{F,\omega}(f)(g) - (|g|_F^{-1}\,\overline{\mu}_1(C_F^1)\,\widehat{f}(0) - \overline{\mu}_1(C_F^1)\,f(0))\,\delta_\omega| \leq C_N\,|g|_F^N \tag{2.2.5.10}$$

as $|g|_F \to 0$, where $\delta_\omega = 0$ if ω is non-trivial on C_F^1, and $\delta_\omega = 1$ if ω is trivial on C_F^1.

(3) *For any continuous 1-parameter subgroup $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$, we have:*

$$\Phi_{F,\omega}(\partial_\gamma f) = \partial_\gamma \Phi_{F,\omega}(f). \tag{2.2.5.11}$$

In particular, $\Phi_{F,\omega}(f)$, composed with the quotient $\mathrm{GL}_1(\mathbf{A}_F) \to C_F$, is a smooth in the Archimedean directions.

(4) *If either $f(0) = \widehat{f}(0) = 0$ or $\omega|_{C_F^1}$ is non-trivial, then for every integer $N \geq 0$, there exists a real number $C_N > 0$ depending on N and f, such that*

$$|\Phi_{F,\omega}(f)(g)| \leq C_N e^{-N\,|\log|g|_F|} \tag{2.2.5.12}$$

as $|\log|g|_F| \to +\infty$. In particular, $\overline{\omega}\Phi_{F,\omega}(f)$, as a function of $\log|g|_F$, is a Schwartz function on $\mathbf{R}$ with exponential decay at infinity, together with all of its derivatives.

(5) *Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_F)$ be a bounded subset. For every integer $N \geq 0$, there exist real numbers $C_{\mathcal{B},N}, T_{\mathcal{B},N} > 0$, such that* (2.2.5.9), (2.2.5.10), (2.2.5.11), (2.2.5.12), *hold with $C_N = C_{\mathcal{B},N}$ and for all $\log|g|_F > T_{\mathcal{B},N}$, respectively $\log|g|_F^{-1} > T_{\mathcal{B},N}$, respectively $|\log|g|_F| > T_{\mathcal{B},N}$, and for all $f \in \mathcal{B}$.*

(6) *For every integer $N \geq 0$, the assertions in* (5) *all hold with $T_{\mathcal{B},N} = 1$, upon possibly replacing the constant $C_{\mathcal{B},N} > 0$ with a larger, still uniform, constant.*

Proof. Parts (1), (2), and (5) follow from Lemma 2.2.5.2 and Theorem 2.2.3.4 since C_F^1 is compact Hausdorff and $\overline{\mu}_1$ is Radon, hence finite on C_F^1, and integration over C_F^1 preserves the growth conditions from (2.2.3.4) and (2.2.3.5), from which (4) follows.

(3) : We use Lemma 2.2.4.3 to prove (2.2.5.11) to show that ∂_γ commutes with integration over C_F^1 with respect to the finite Radon measure $\overline{\mu}_1$. More specifically, we need to prove that for each $g \in C_F$ and t varying in a compact neighborhood $[a,b]$ of 1

$$\frac{\Phi_F(f)(\gamma(t^{-1})gh)\omega(h) - \Phi_F(f)(\gamma(t^{-1})gh)\omega(h)}{t-1} \tag{2.2.5.13}$$

as a function of $h \in C_F^1$ (a compact space) is bounded by a constant independent of t (but possibly depending on g) for all $t \in [a,b]$, $t \neq 1$. This will justify the use of Dominated Convergence to pass the limit in t of (2.2.5.13) through the $\overline{\mu}_1$-integral over C_F^1.

We have the inequality

$$\left|\frac{\Phi_F(\gamma(t^{-1})gh)\omega(h) - \Phi_F(gh)\omega(h)}{t-1}\right| \leq \sup_{t_0 \in [a,b]} \left|\frac{\mathrm{d}}{\mathrm{d}t}\Phi_F(\gamma(t^{-1})gh)\right|_{t=t_0} \tag{2.2.5.14}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_F(\gamma(t^{-1})gh)|_{t=t_0} = \frac{1}{t_0}\partial_\gamma\Phi_F(f)(\gamma(t_0^{-1})gh) = \frac{1}{t_0}\Phi_F(\partial_\gamma(f))(\gamma(t_0^{-1})gh) \tag{2.2.5.15}$$

by Lemma 2.2.4.3. That same lemma gives that $\Phi_F(\partial_\gamma f)$ is continuous over C_F. It follows that as (t_0, h) varies in the compact $[a,b] \times C_F^1$ the right side in (2.2.5.14) is bounded above by a positive real number A (dependent on g). By varying γ, $\Phi_{F,\omega}(f)$ composed back to $\mathrm{GL}_1(\mathbf{A}_F)$ is a smooth function in the Archimedean directions (note that a general derivative at a point along a 1-parameter subgroup γ can be computed in terms of the derivative along γ at 1 for a translate of the function). This finishes the proof of (3).

(4) : From (2) it follows that since the idelic norm is a submersion $\Phi_{F,\omega}$ is smooth as a function of $\log|g|_F$ and the growth condition in (2.2.5.12) implies that $\Phi_{F,\omega}(f)$ decays faster than any positive integer power of $e^{-|\log|g|_F|}$, and so do its derivatives by repeated use of (2.2.5.11).

(6) : The assertion follows from Theorem 2.2.3.4(5), by replacing $C_{\mathcal{B},N}$ in that result with its product against the $\overline{\mu}_1$-volume of C_F^1. □

2.3. S-local adelic θ-functions

Let F be a number field. Throughout this section, we fix a finite set S of places containing all of the Archimedean places of F. We refer the reader to §A.1.5 of Appendix A, on S-local adeles, ideles and class groups, and the S-local analogs of the respective fundamental domains within $\mathbf{A}_F$ and $\mathrm{GL}_1(\mathbf{A}_F)$.

We further refer the reader to Definition A.3.4.1 for the construction of the S-local additive and multiplicative Tamagawa measures ν_S on $\mathbf{A}_{F,S}$, μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, $\overline{\mu}_S$ on $C_{F,S}$, as well as $\mu_{S,1}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and $\overline{\mu}_{S,1}$ on $J^1_{F,S}$.

Recall that we denote by $\mathbf{A}_{F,S}$ the product $\prod_{v\in S} F_v$, where F_v is the additive topological group of the local field F_v.

The Hausdorff locally compact abelian group $\mathrm{GL}_1(\mathbf{A}_{F,S})$ is therefore isomorphic to the product $\prod_{v\in S} F_v^\times$, on which the S-units $\mathcal{O}^\times_{F,S}$ acts properly and freely as a discrete closed subgroup. The quotient is denoted by $J_{F,S}$, a Hausdorff locally compact abelian group which, equipped with the proper continuous homomorphism $|\cdot|_{F,S} : J_{F,S} \to \mathbf{R}_{>0}$, in particular, its kernel $J^1_{F,S}$ is compact.

In the previous sections we studied the "global averaging" of Schwartz-Bruhat functions on the adeles $\mathbf{A}_F$ over the orbits in $\mathrm{GL}_1(\mathbf{A}_F)$ of the $F^\times$-action, leading to the global adelic θ-functions of Definition 2.2.3.1. In this section, we define S-local analogs of the θ-function operators Φ_F and $\Phi_{F,\omega}$, for ω now a continuous character of $J^1_{F,S}$.

The reason we seek S-local analogues is as follows. Eventually we will collect the *global* adelic θ-functions discussed so far into a topological vector space, and study several large quotients of that space (for example, proving that such quotients admit Schauder bases with certain good properties for later use, to compute traces). The study of the topological vector spaces of global θ-functions rests on the following principle: we "exhaust" them by more manageable S-local analogs, where explicit constructions are possible and several additional tools available. One illustration of this is that smooth compactly supported functions on $\mathrm{GL}_1(\mathbf{A}_F)$ do *not* extend by zero to global Schwartz-Bruhat functions on $\mathbf{A}_F$ (the ideles have additive measure zero within the adeles), but they do at the S-local level.

We further note that there is a relation between continuous characters of C^1_F and of $J^1_{F,S}$, as will be discussed in the next Chapter. For $\omega \in \widehat{C^1_F}$, we may restrict ω to $C^1_{F,S}$, where $C_{F,S}$ is the quotient of $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \prod_{v\in\Sigma_F - S} \mathcal{O}^\times_{F_v}$ by the diagonal action of the S-units. For S large enough so as to have $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, we have that the continuous injective open group homomorphism $C_{F,S} \to C_F$ is an isomorphism of Hausdorff locally compact abelian groups. Clearly $J_{F,S}$ is also the quotient of $C_{F,S}$ by the compact Hausdorff closed subgroup defined as the image in $C_{F,S}$ of $\Gamma := \{1\} \times \prod_{v\in\Sigma_F - S} \mathcal{O}^\times_{F_v}$.

If ω is unramified away from S, i.e. the restriction of ω to $\mathcal{O}^\times_{F_v}$ is trivial for all $v \notin S$, then ω is Γ-invariant, and it uniquely determines a continuous character of $J^1_{F,S}$. Conversely, a continuous character of $J^1_{F,S}$ uniquely determines a continuous character of $C^1_{F,S}$, which for S large enough so as to have $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ uniquely extends to a continuous character of C^1_F that is unramified outside S.

2.3.1. S-local θ-function operators. In this section, we study S-local analogs of the θ-function operators Φ_F and $\Phi_{F,\omega}$.

NOTATION 2.3.1.1. For F a number field, and S a finite set of places of F containing all the Archimedean places, we will be denoting by S_f the intersection:

$$S_f := \Sigma_{F,f} \cap S$$

the finite set of all the finite places of F contained in S.

DEFINITION 2.3.1.1 (S-local adelic θ-functions). Let F be a number field, S a finite set of places containing all the Archimedean places of F. We define:

$$\Phi_{F,S} : \mathcal{S}(\mathbf{A}_{F,S}) \to C(\mathrm{GL}_1(\mathbf{A}_{F,S})) \tag{2.3.1.1}$$

to be the linear operator determined by the assignment, for every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$:

$$\Phi_{F,S}(f)(g) := \sum_{\xi\in\mathcal{O}^\times_{F,S}} f(\xi g) \tag{2.3.1.2}$$

as functions of $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$. We refer to $\Phi_{F,S}(f)$ as the *S-local adelic θ-function.*

2.3.1.1. *Estimates over $J_{F,S}$.* A consequence of the Proposition below is that indeed formation of $\Phi_{F,S}$ is well-defined, linear and valued in continuous functions on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and is invariant under the continuous left-translation action by the discrete abelian group $\mathcal{O}_{F,S}^{\times}$.

PROPOSITION 2.3.1.2 (Estimates over the S-idele classes - I). *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

(1) *For every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the series defining $\Phi_{F,S}(f)$ in* (2.3.1.1) *converges absolutely and uniformly over the compact subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S})$. The sum is a continuous function over $\mathrm{GL}_1(\mathbf{A}_{F,S})$, invariant under the continuous left-translation action by the discrete subgroup $\mathcal{O}_{F,S}^{\times}$.*
(2) *For every integer $N \geq 1$, there exists a real number $C_N > 0$ such that:*

$$|\Phi_{F,S}(f)(g)| \leq C_N |g|_{F,S}^{-N} \tag{2.3.1.3}$$

as $|g|_{F,S} \to +\infty$.
(3) *Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a bounded subset. Then for every integer $N \geq 0$, there exist real numbers $C_{\mathcal{B},N} > 0$ and $T_{\mathcal{B},N} > 0$, such that such that* (2.3.1.3) *holds with $C_N = C_{\mathcal{B},N}$ for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} > T_{\mathcal{B},N}$.*
(4) *For every integer $N \geq 0$, the assertions in* (3) *all hold with $T_{\mathcal{B},N} = 1$, upon replacing the constant $C_{\mathcal{B},N} > 0$ with a possibly larger, still uniform, constant.*

We refer the reader to the end of [**Co**, §VII,p.30] and the beginning of [**Co**, §VII,p.31], from the proof of [**Co**, §VII,Lemma 1(a),p.30], for a method similar to the one for the proof of (1) and (2) below, which applies to prove (1) and (2) of Proposition 2.3.1.2 too.

See also the beginning of [**Co**, §VII,p.30] (where $\Phi_{F,S}(f)$ is denoted $E_0(f)$).

PROOF. We denote by S_f the intersection $S_f := S \cap \Sigma_{F,f}$. Since the linear map

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \bigotimes_{v \in S_f} \mathcal{S}(F_v) \to \mathcal{S}(\mathbf{A}_{F,S})$$

is an isomorphism (see (2.1.4.1) from Proposition 2.1.4.1), for each Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$ there exist Schwartz-Bruhat functions $f_1, \ldots, f_m \in \mathcal{S}(\mathbf{A}_{F,S})$, each of the form:

$$f_k = f_{k,\infty} \otimes \bigotimes_{v \in S_f} f_{k,v}$$

with $f_{k,v} \in \mathcal{S}(F_v)$ for each $v \in S_f$ and $f_{k,\infty} \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ for each $1 \leq k \leq m$, and complex numbers $c_1, \ldots, c_m$, such that $f = \sum_{1 \leq k \leq m} c_k f_k$.

If (1) and (2) hold for each f_k, then they hold for f (with different constants C_N). We may rename any of the f_k as f and reduce to the case $f = f_\infty \otimes \bigotimes_{v \in S_f} f_v$ for $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_v \in \mathcal{S}(F_v)$ for each $v \in S_f$. Without loss of generality, we may assume $f_v = 1_{K_v}$ for each $v \in S_f$, with K_v a compact open subset of F_v.

We choose a continuous homomorphic section $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ to the S-idelic norm.

(1) : For the purpose of studying the $\mathcal{O}_{F,S}^{\times}$-invariant sum (2.3.1.2) near a point $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, by Proposition A.2.4.4 we can replace g with an $\mathcal{O}_{F,S}^{\times}$-translate so that $g \in \tau(\mathbf{R}_{>0}) \cdot \overline{E}_{F,S}^1$. We recall that $E_{F,S}^1$ is the S-idelic fundamental domain from (A.2.4.4), for the continuous left-translation action of the discrete subgroup $\mathcal{O}_{F,S}^{\times}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

We have

$$\sum_{\xi\in\mathcal{O}_{F,S}^{\times}}|f(\xi g)|\leq\left(\prod_{v\in S_f}\|f_v\|_u\right)\sum_{\xi\in I}|f_\infty(\xi g_\infty)| \tag{2.3.1.4}$$

where $g_\infty\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is the image of $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$ under the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S})\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ induced by the containment $S_\infty\subset S$ and I is a discrete subset of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ constructed as follows.

For each $v\in S_f$, we define $C_v:=\max|K_v|_v>0$. We fix $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, and the summand $f(\xi g)$ in the sum defining $\Phi_{F,S}$ from (2.3.1.2) is nonzero if and only if $\xi\in g_v^{-1}K_v$ for every $v\in S_f$. If $g\in\tau(\mathbf{R}_{>0})\cdot\overline{E}^1_{F,S}$, then by compactness of $\overline{E}^1_{F,S}$ we have $|g_v^{-1}|_v\leq C$ for C a positive real number independent of $v\in S_f$. It follows that $\xi\in\mathcal{O}_{F,S}^{\times}$ satisfies:

$$|\xi|_v\leq CC_v$$

for all $v\in S_f$ and upon renaming $C\cdot\max_{v\in S_f}C_v$ as C we obtain that $\xi\in\mathcal{O}_{F,S}^{\times}$ satisfies $|\xi|_v\leq C$ for all $v\in S$ if $f(\xi g)\neq 0$ for some $g\in\tau(\mathbf{R}_{>0})\cdot\overline{E}^1_{F,S}$. In other words, ξ is contained in the cross-product:

$$\{x\in\mathbf{A}_{F,S}:|x_v|\leq C\text{ for all }v\in S_f\}\times_{\mathbf{A}_{F,S}}\mathcal{O}_{F,S}^{\times}. \tag{2.3.1.5}$$

The fractional ideal $J:=\prod_{v\in S_f}\mathfrak{p}_v^{n_v}\subset F$, for n_v the least integer such that $|\pi_v|_v^{n_v}\leq C$, with π_v any choice of local uniformizer at v, contains the subset in (2.3.1.5). Note that if S_f is empty then $J=\mathcal{O}_F$ (a consequence of the definition).

We define $I:=J-\{0\}$. We aim at applying Proposition 2.2.1.3 to deduce that the series in the right side of (2.3.1.4) converges, uniformly on compact subsets of the image of $\tau(\mathbf{R}_{>0})\cdot\overline{E}_{S,0}$, and hence so does the series defining $\Phi_{F,S}(f)$ in (2.3.1.2), absolutely and uniformly over compact subsets of $\tau(\mathbf{R}_{>0})\cdot\overline{E}_{S,0}$. We refer the reader to (A.2.4.5) for the definition of $E_{S,0}$ in relation to $E^1_{F,S}$.

In order to do so, we proceed as follows. Let $[a,b]\subset\mathbf{R}_{>0}$ be an arbitrary compact interval, and we call $E_{S,\infty}$ the image in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ of the relatively compact subset $E_{S,0}$ of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

We define:

$$\widetilde{\psi}(y):=\sup_{t\in[a,b],g_\infty\in\overline{E}_{S,\infty}}|\psi(t,y,g_\infty)| \tag{2.3.1.6}$$

where $\psi(t,y,g_\infty):=f_\infty(\tau(t)yg_\infty)$ as a function of $(t,y,g_\infty)\in\mathbf{R}_{>0}\times\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})\times\overline{E}_{S,\infty}$. The function $\widetilde{\psi}:\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})\to\mathbf{R}_{\geq 0}$ is continuous, since ψ is and $[a,b]\times\overline{E}_{S,\infty}$ is Hausdorff with compact closure and ψ is defined and bounded on such closure for every $y\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

Since $f_\infty\in\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is Schwartz, for every integer $N\geq 1$ there exists a large enough closed ball $B\subset\mathbf{A}_{F,S_\infty}$ containing the origin and a real number $C_{B,N}>0$ such that $|f_\infty(x_\infty)|\leq C_{B,N}\|x_\infty\|^{-N}$ for all $x_\infty\in\mathbf{A}_{F,S_\infty}-B$ and $\|\cdot\|$ is any choice of quasi-norm on $\mathbf{A}_{F,S_\infty}$ (it satisfies the triangle inequality up to a universal constant) inducing the Euclidean topology (noting that the constant $C_{B,N}$ and the closed ball B itself depend on such choice).

It follows that for all $(t,y,g_\infty)\in\mathbf{R}_{>0}\times\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})\times\overline{E}_{S,\infty}$ such that $\tau(t)yg_\infty\notin B$, we have

$$|\psi(t,y,g_\infty)|\leq C_{B,N}\|\tau(t)yg_\infty\|^{-N}. \tag{2.3.1.7}$$

For every $y\notin(\tau([a,b])\cdot\overline{E}_{S,\infty})^{-1}\cdot B\subset\mathbf{A}_{F,S_\infty}$ and for every $(t,g_\infty)\in[a,b]\times\overline{E}_{S,\infty}$, (2.3.1.7) therefore holds, and it remains valid upon maximizing over $(t,g_\infty)\in[a,b]\times\overline{E}_{S,\infty}$. We claim that there exists a real number $C_{B,S,N,a,b}>0$ (depending on $C_{B,N}$, on $\overline{E}_{S,\infty}$, on a,b and B) such that

$$\widetilde{\psi}(\xi)\leq C_{B,S,N,S,a,b}\|\xi\|^{-N} \tag{2.3.1.8}$$

for every $\xi\in I$ away from $(\tau([a,b])\cdot\overline{E}_{S,\infty})^{-1}\cdot B$.

We now obtain the constant $C_{B,N,S,a,b}$. Multiplication by $\tau(t)g_\infty$ is a linear automorphism A_{t,g_∞} of the vector space $\mathbf{A}_{F,S_\infty}$ of dimension $[F:\mathbf{Q}]$. We equip $\mathbf{A}_{F,S_\infty}$ with the vector space quasi-norm $\|\cdot\| := \sup_{v\in S_\infty}\{|\cdot|_v^{1/N_v}\}$ with $N_v = 1$ if v is real, and $N_v = 2$ if v is complex and observe that A_{t,g_∞} is bounded below, i.e. for every $y \in \mathbf{A}_{F,S_\infty}$, we have $\|A_{t,g_\infty}y\| \geq C_0(t,g_\infty)\|y\|$ for a positive real number $C_0(t,g_\infty) > 0$. For example

$$C_0(t,g_\infty) := \min_{v\in S_\infty} |\tau(t)_v g_{\infty,v}|_v^{1/N_v}$$

qualifies. As a function of (t,g_∞) over the compact subset $[a,b]\times\overline{E}_{S,\infty}$, $C_0(t,g_\infty)$ is bounded below by a positive real number A because $[a,b]\times\overline{E}_{S,\infty}$ is compact.

We observe that we have $1/|\tau(t)_v g_{\infty,v}|_v^{N/N_v} \leq C_0(t,g_\infty)^{-N} \leq A^{-N}$ for every $(t,g_\infty)\in[a,b]\times\overline{E}_{S,\infty}$. We then define

$$C_{B,S,N,a,b} := C_{B,N}\cdot A^{-N}.$$

The constant $C_{B,S,N,a,b}$ works because of the following inequality:

$$|\psi(t,y,g_\infty)| \leq C_{B,N}\|\tau(t)yg_\infty\|^{-N} \leq C_{B,N}\cdot A^{-N}\|y\|^{-N} = C_{B,S,N,a,b}\|y\|^{-N}$$

for every $y \notin (\tau([a,b])\cdot\overline{E}_{S,\infty})^{-1}\cdot B$.

We choose a closed ball B' containing $(\tau([a,b])\cdot\overline{E}_{S,\infty})^{-1}\cdot B$. Since I is discrete, $I\cap B'$, and so convergence of the series in the right side of (2.3.1.4) holds so long as the series $\sum_{\xi\in I-B'}\|\xi\|^{-N}$ is convergent. We now choose $N > [F:\mathbf{Q}] = \dim_{\mathbf{R}}(\mathbf{A}_{F,S_\infty})$, and all the constants in the foregoing accordingly, and apply Proposition 2.2.1.3 (with $x=0$) to obtain the desired convergence.

(2) : Since $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ for each integer $N\geq 1$ there exists $C_{N,0}>0$ such that $|f_\infty(x_\infty)| \leq C_{N,0}\|x_\infty\|_\infty^{-N}$ for all $x_\infty \in \mathbf{A}_{F,S_\infty}$ away from a large enough closed ball $B_0 \subset \mathbf{A}_{F,S_\infty}$, where $\|\cdot\|_\infty$ is the sup-norm on the Euclidean space $\mathbf{A}_{F,S_\infty}$ defined upon fixing coordinates for each Archimedean place v of F, choosing an isomorphism $\mathbf{A}_{F,S_\infty}\simeq\mathbf{R}^{r_1+2r_2}$.

We fix a continuous homomorphic section $\tau : \mathbf{R}_{>0}\to \mathrm{GL}_1(\mathbf{A}_{F,S})$ to the S-idelic norm $|\cdot|_{F,S}$. Since $\mathrm{GL}_1(\mathbf{A}_{F,S})\subset\mathbf{A}_{F,S}$ is an open topological embedding, $E_{S,0}$ is relatively compact in $\mathbf{A}_{F,S}$. Thus, for every real number $T>1$ we see that $\tau([1/T,T])\cdot E_{S,0}$ is relatively compact in $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and $\mathbf{A}_{F,S}$, with non-empty interior.

For any closed ball $B\subset\mathbf{A}_{F,S_\infty}$ containing the origin, there exists a real number $T_0>0$ such that B contains the image under $\mathrm{GL}_1(\mathbf{A}_{F,S})\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ of $\tau([1/T_0,T_0])\cdot E_{S,0}$, and it does not contain the image of $\tau([1/T,T])\cdot E_{S,0}$ for any $T>T_0$.

We choose τ to be the section into the copy of $\mathbf{R}_{>0}$ contained within $F_{v_0}^\times$, for a choice of $v_0\in S_\infty$ (defining τ as the composition of s_{F,v_0} from (A.2.2.5) with the continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F)\to\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$), to obtain, upon enlarging T_0 if necessary, that for B so large as to contain $B_0\cup\tau([1/T_0,T_0])\cdot E_{S,0}$, there exists a constant $C_{0,N}>0$, such that the following inequality holds

$$|f_\infty(\tau(t)\xi g_\infty)| \leq C_{N,0}\|\tau(t)\xi g_\infty\|_\infty^{-N} = C_{N,0}t^{-N}\|\xi g_\infty\|_\infty^{-N} \tag{2.3.1.9}$$

for all $t>T_0$, $g_\infty\in\overline{E}_\infty$ (the closure of the image of $E^1_{F,S}$ into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$) and $\xi\in I$. Upon maximizing the function of $(x,g_\infty)\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})\times\overline{E}_\infty$ given by $\|xg_\infty\|_\infty^{-N}$ over $g_\infty\in\overline{E}_\infty$ (thus obtaining a continuous function of x) by a method similar as in the proof of (1) we see that there exists a real number $C_{N,1}>0$ such that

$$\sup_{g_\infty\in\overline{E}_\infty}\|xg_\infty\|_\infty^{-N} \leq C_{N,1}\|x\|_\infty^{-N}$$

for all $x\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ with $\|x\|_\infty$ large enough. Up to enlarging B, we may assume that for every $\xi\in I$ with $\xi\notin B$, we have $\|\xi\|_\infty$ is as sufficiently large as required.

Renaming $C_{N,0} \cdot C_{N,1}$ as C_N we see that upon choosing $N > r_1 + 2r_2$ the sum $\sum_{\xi \in I} \|\xi\|_\infty^{-N}$ converges by Proposition 2.2.1.3 and upon absorbing its value into the constant C_N we obtain that indeed

$$\sum_{\xi \in I} |f_\infty(\xi g_\infty \tau(t))| \leq C_N t^{-N}$$

for all $g_\infty \in \overline{E}_\infty$ and all $t > T_0$. This implies that (2.3.1.3) holds.

(3) : Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a bounded subset. By Proposition 2.1.4.1(1), the canonical linear operator $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \bigotimes_{v \in S_f} \mathcal{S}(F_v) \to \mathcal{S}(\mathbf{A}_{F,S})$ is a linear topological isomorphism, and hence by Proposition C.2.3.13(2), a bounded subset $\mathcal{B}$ in $\mathcal{S}(\mathbf{A}_{F,S})$ is contained in one which is the image of

$$\mathrm{cch}(\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w)$$

the convex circled hull of the subset $\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w$.

Since, by Lemma 2.1.2.4(1), $\mathcal{S}(F_v)$ carries the finest locally convex topology, $\mathcal{B}_w$ is, for each $w \in S_f$, contained in a finite dimensional subspace of $\mathcal{S}(F_v)$ by Proposition C.2.5.1(4).

There therefore exist finitely many $f_{w,1}, \ldots, f_{w,m_w} \in \mathcal{S}(F_w)$ whose linear span contains $\mathcal{B}_w$. In particular, all the functions in $\mathcal{B}_w$ are supported on a large enough compact open subset $K_w \subset F_w$, and this is so for each $w \in S_f$.

We deduce that there exists a large enough compact open subset $K \subset \prod_{v \in S_f} F_v$ such that every element of $\mathcal{B}$ is a finite absolutely convex $\mathbf{C}$-linear combination (i.e. one whose coefficients have complex Euclidean absolute values that sum to a real number bounded above by 1; see [**Jar**, Part I,Ch.6,§6.1,p.102] for $r = 1$; see also [**NB**, Thm.4.2.11]) of Schwartz-Bruhat functions of the form $f_\infty \otimes \bigotimes_{w \in S_f} f_w$ with each of the f_w supported on the common K whose existence was proved above.

By construction of C_N from (2), this constant depends on $f \in \mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w$, and similarly in light of the estimate from (2.3.1.4), the constants $C_{B,N}$, $C_{B',N,S,a,b}$, $C_{N,0}$ and C_N depend on f through f_∞ and through K only.

Specifically, the dependence through K is through the formation of the discrete subset $I \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$: I depends only on the compact open subset K of $\prod_{v \in S_f} F_v$ obtained by projecting the support of the element f onto $\prod_{v \in S_f} F_v$ (upon inspection of the construction of the subset of $\mathbf{A}_{F,S}$ defined in (2.3.1.5)), and as f varies over the image in $\mathcal{S}(\mathbf{A}_{F,S})$ of $\mathrm{cch}(\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w)$ such projection is contained in K, and hence I can be made to depend on f only through K, therefore obtaining an I that qualifies for the same role in the argument from the foregoing, for all f varying over $\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w$.

That the real number $T_N > 0$ such that (2.3.1.3) holds if $|g|_{F,S} > T_N$, and whose formation is uniform as f varies over $\mathcal{B}$, exists, is seen by observing that formation of B_0 depends on f through f_∞, for f of the special form $f = f_\infty \otimes \bigotimes_{v \in S_f} f_v \in \mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w$.

In particular, the same ball $B_0 \subset \mathbf{A}_{F,S_\infty}$ qualifies to give, through the argument from the foregoing, the estimate in (2.3.1.9) as f_∞ varies over the bounded subset $\mathcal{B}_\infty \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$.

We have now shown that (2.3.1.3) holds for each f of the special form $f_\infty \prod_{w \in S_f} f_w$ with $f_\infty \in \mathcal{B}_\infty$ and $f_w \in \mathcal{B}_w$ for each $w \in S_f$, with C_N and T_N existing and depending only on $\mathcal{B}_\infty$ and $\bigotimes_{w \in S_f} \mathcal{B}_w$.

By linearity of $\Phi_{F,S}$ and the design of the convex circled hull (see [**Jar**, Part I,Ch.6,§6.1,p.102] for $r = 1$; see also [**NB**, Thm.4.2.11]), the estimate in (2.3.1.3) propagates with the same C_N and T_N to the elements of $\mathrm{cch}(\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w)$, and hence to all the elements of the bounded subset $\mathcal{B}$. We rename C_N as $C_{\mathcal{B},N}$, and T_N as $T_{\mathcal{B},N}$, and these constants satisfy the requirements of (3).

(4) : The estimate in (2.3.1.4) holds for any $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, and hence, upon choosing a continuous homomorphic section τ to the S-idelic norm, it holds on $\tau(\mathbf{R}_{>0}) \cdot E^1_{F,S}$, where $E^1_{F,S}$ is the S-idelic fundamental domain from (A.2.4.4), for the continuous left-translation action of the S-units $\mathcal{O}^\times_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

Since $\mathcal{B}$ is a fixed bounded subset in $\mathcal{S}(\mathbf{A}_{F,S})$, there exists, by Proposition 2.1.4.1(1), a bounded subset $\mathcal{B}_\infty$ in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, and a bounded subset $\mathcal{B}_f \subset \mathcal{S}(\prod_{v\in S_f} F_v^\times)$, contained within a finite dimensional $\mathbf{C}$-linear subspace of $\mathcal{S}(\prod_{v\in S_f} F_v^\times)$ (since $\mathcal{S}(\prod_{v\in S_f} F_v^\times)$ carries, by Proposition 2.1.2.4 and Proposition C.2.5.1, the finest locally convex topology) linearly generated by finitely many $f_1,\dots,f_m \in \mathcal{S}(\prod_{v\in S_f} F_v^\times)$, such that

$$\mathcal{B} \subset \mathrm{cch}(\mathcal{B}_\infty \otimes \mathcal{B}_f)$$

the convex circled hull of the subset of $\mathcal{S}(\mathbf{A}_{F,S})$ of those elements of the form $f_\infty \otimes f_0$, as f_∞ varies over $\mathcal{B}_\infty$ and f_0 varies over $\mathcal{B}_f$.

The elements of $\mathcal{B}_f$ are finite $\mathbf{C}$-linear combinations of the basis elements $\{f_1,\dots,f_m\}_{m\geq 0}$ whose complex coefficients have bounded absolute value by a positive real number $A>0$ which depends only on $\mathcal{B}_f$.

We may choose $m\geq 0$ large enough so as to arrange each f_k, $1\leq k\leq m$, to be of the form $f_k = \bigotimes_{v\in S_f} f_{k,v}$, for $f_{k,v}$ the indicator function of a compact open subset of F_v.

The subset $I\subset \mathcal{O}_{F,S}^\times$ from (2.3.1.4) depends on $\mathcal{B}$ only through the compact open subsets that serve as the supports of the $f_{k,v}$, for each $v\in S_f$, and $1\leq k\leq m$.

We obtain from (2.3.1.4) that $\Phi_{F,S}(f)$ satisfies the estimate, for $g\in \tau(\mathbf{R}_{>0})\cdot E^1_{F,S}$:

$$|\Phi_{F,S}(f)(g)| \leq mA\sum_{\xi\in I} |f_\infty(\xi g_\infty)| \tag{2.3.1.10}$$

where g_∞ is the image of g into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ under the continuous homomorphic projection $\mathrm{GL}_1(\mathbf{A}_{F,S})\to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and as f varies over the elements of $\mathcal{B}_\infty\otimes\mathcal{B}_f$.

For any integer $N\geq 1$, we let $T_{\mathcal{B},N}>0$ be the real number provided by (4), and define $t_{\mathcal{B},N} := \max\{T_{\mathcal{B},N}, 1/T_{\mathcal{B},N}\}$.

The same argument provided in the proof of (5) of Theorem 2.2.3.4, shows that, over the compact subset $\tau([1/t_{\mathcal{B},N}, t_{\mathcal{B},N}])\cdot E^1_{F,S}$, formation of the map $f_\infty\mapsto \sum_{\xi\in I}|f_\infty(\xi g_\infty)|$ is sequentially continuous on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$.

Since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a Fréchet space (see Proposition 2.1.1.6), it is metrizable, and hence a sequential topological space.

The map $f_\infty\mapsto \sum_{\xi\in I}|f_\infty(\xi g_\infty)|$ is therefore continuous. Since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is Montel by Proposition 2.1.1.6 (being a nuclear Fréchet space), it satisfies the Heine-Borel property, and hence the closure in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ of the bounded subset $\mathcal{B}_\infty$, itself bounded by [**NB**, Thm.6.1.5], is compact.

The continuous map $f_\infty\mapsto \sum_{\xi\in I}|f_\infty(\xi g_\infty)|$ on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is therefore bounded on $\mathcal{B}_\infty$, and hence there exists a positive real number $C_{\mathcal{B}}>0$ such that, over the compact subset $\tau([1/t_{\mathcal{B},N}, t_{\mathcal{B},N}])\cdot E^1_{F,S}$, the estimate:

$$|\Phi_{F,S}(f)(g)| \leq mAC_{\mathcal{B}} \tag{2.3.1.11}$$

holds for all $f\in\mathcal{B}_\infty\otimes\mathcal{B}_f$. It therefore holds over the convex circled hull of $\mathcal{B}_\infty\otimes\mathcal{B}_f$, and hence over $\mathcal{B}$.

Upon renaming the positive real number $\max\{C_{\mathcal{B},N}, mAC_{\mathcal{B}}\}$, for $C_{\mathcal{B},N}>0$ the constant provided by (4), as $C_{\mathcal{B},N}$, we obtain that all the assertions in (4) hold for $T_{\mathcal{B},N}=1$, and using the new constant $C_{\mathcal{B},N}$. □

2.3.1.2. *Homogeneous S-adelic θ-functions.* Throughout the remainder of this susbection, we choose a continuous homomorphic section to the S-idelic norm:

$$\tau : \mathbf{R}_{>0}\to \mathrm{GL}_1(\mathbf{A}_{F,S})$$

and retain the notation $\mu_{S,1}$, $\overline{\mu}_{S,1}$ for the resulting Haar measures on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and $J^1_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathcal{O}^\times_{F,S}$.

We recall that $\overline{\mu}_{S,1}$ does not depend on τ but only on the volume it assigns to the compact Hausdorff abelian group $J^1_{F,S}$.

Definition 2.3.1.3 (ω-homogeneous S-local adelic θ-functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let ω be a continuous character of $J^1_{F,S}$.

We define:

$$\Phi_{F,S,\omega} : \mathcal{S}(\mathbf{A}_{F,S}) \to C(J_{F,S}) \tag{2.3.1.12}$$

to be the linear operator determined by the assignment, for every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$:

$$\Phi_{F,S,\omega}(f)(g) := \int_{J^1_{F,S}} \Phi_{F,S}(f)(gh)\omega(h)\,\mathrm{d}\overline{\mu}_{S,1}(h) \tag{2.3.1.13}$$

as functions of $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$. We refer to $\Phi_{F,S,\omega}$ as the *ω-homogeneous S-local adelic θ-function* of f.

2.3.1.3. *Estimates over $J_{F,S}/J^1_{F,S}$.* It is indeed a consequence of Proposition 2.3.1.4 below, that formation of $\Phi_{F,S,\omega}$ is well-defined, linear and valued in continuous functions on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, $\widetilde{\omega}$-homogeneous under the left-translation action by the closed abelian subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, for $\widetilde{\omega}$ the composition of ω with the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \to J^1_{F,S}$. Equivalently, $\mathcal{O}^\times_{F,S}$-invariant and ω-homogeneous, as continuous functions on $J_{F,S}$, under the continuous left-translation action of $J^1_{F,S}$.

Proposition 2.3.1.4 (Estimates over the S-idele classes - II). *Let F be a number field, S a finite set of places containing all the Archimedean places. Let ω be a continuous character of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathcal{O}^\times_{F,S}$. We denote by $\widetilde{\omega} \in \widehat{\mathrm{GL}_1(\mathbf{A}_{F,S})^1}$ the composition of ω with the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \to J^1_{F,S}$.*

(1) *For every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the integral defining $\Phi_{F,S,\omega}(f)$ in (2.3.1.12) converges absolutely and uniformly over the compact subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S})$. It is a continuous function over $\mathrm{GL}_1(\mathbf{A}_{F,S})$, $\widetilde{\omega}$-homogeneous under the continuous left-translation action by the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.*

(2) *For every integer $N \geq 1$, there exists a constant $C_N > 0$ such that:*

$$|\Phi_{F,S,\omega}(f)(g)| \leq C_N |g|^{-N}_{F,S} \tag{2.3.1.14}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} \to +\infty$.

(3) *Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a bounded subset. There exist real numbers $C_{\mathcal{B},N}, T_{\mathcal{B},N} > 0$ such that (2.3.1.14) holds for with $C_N = C_{\mathcal{B},N}$ and for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} > T_{\mathcal{B},N}$, for all $f \in \mathcal{B}$.*

(4) *For every integer $N \geq 0$, the assertions in (3) all hold with $T_{\mathcal{B},N} = 1$, upon replacing the constant $C_{\mathcal{B},N} > 0$ with a possibly larger, still uniform, constant.*

Proof. Since $J^1_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathcal{O}^\times_{F,S}$ is compact Hausdorff, insofar as it receives a continuous surjection from $C^1_{F,S}$, which is compact Hausdorff by Theorem A.1.5.2(1), and ω is continuous, integration over $\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathcal{O}^\times_{F,S}$ with respect to $\overline{\mu}_{S,1}$ implies that (1) and (2) and (3) are a consequence of Proposition 2.3.1.2, following the argument provided for the global analog of (1) from Lemma 2.2.5.2.

(4) : For each integer $N \geq 1$, this follows from Proposition 2.3.1.2(4) upon replacing the constant $C_{\mathcal{B},N} > 0$ provided by the Proposition, with $\overline{\mu}_{S,1}(J^1_{F,S})C_{\mathcal{B},N}$, after integrating both sides of the estimates asserted in Proposition 2.3.1.2(4) over the compact Hausdorff abelian group $J^1_{F,S}$. □

2.3.2. Compatibility with directional derivatives. In this section we discuss analogs of the global results from Lemma 2.2.4.2 and Lemma 2.2.4.3 on compatibility of θ-function operators with directional derivatives along continuous 1-parameter subgroups.

Lemma 2.3.2.1. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be a continuous 1-parameter subgroup.*

For every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the limit:

$$(\partial_\gamma f)(x) := \lim_{t \to 1} \frac{f(\gamma(t)x) - f(x)}{t - 1} \tag{2.3.2.1}$$

exists for every $x \in \mathbf{A}_{F,S}$, and, as a function of $x \in \mathbf{A}_{F,S}$, it is continuous. Moreover, $\partial_\gamma f$ is itself a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$.

As a linear endomorphism on $\mathcal{S}(\mathbf{A}_{F,S})$, ∂_γ is continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S})$.

PROOF. All the assertions are consequences of Lemma 2.2.4.2 (using the insertion operators $\pi_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_F)$ from (2.1.4.5)). □

PROPOSITION 2.3.2.2 (Existence of directional derivatives over the S-idele classes). *Let F be a number field, S a finite set of places containing all the Archimedean places of F. Let ω be a continuous character of $J^1_{F,S}$.*

Let $\gamma : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be a continuous 1-parameter subgroup.

For every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the limits:

$$\partial_\gamma \Phi_{F,S}(f)(g) := \lim_{t\to 1} \frac{\Phi_{F,S}(f)(\tau(t)g) - \Phi_{F,S}(f)(g)}{t-1} \tag{2.3.2.2}$$

and:

$$\partial_\gamma \Phi_{F,S,\omega}(f)(g) := \lim_{t\to 1} \frac{\Phi_{F,S,\omega}(f)(\tau(t)g) - \Phi_{F,S,\omega}(f)(g)}{t-1} \tag{2.3.2.3}$$

exist for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, and their formation is continuous with respect to g. Moreover, we have:

$$\partial_\gamma \Phi_{F,S}(f) = \Phi_{F,S}(\partial_\gamma f),\ \partial_\gamma \Phi_{F,S,\omega}(f) = \Phi_{F,S,\omega}(\partial_\gamma f) \tag{2.3.2.4}$$

as continuous functions on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, invariant under the continuous left-translation action of $\mathcal{O}^\times_{F,S}$.

REMARK 2.3.2.3. For $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$ we can build the global Schwartz-Bruhat function $f := \pi_S(f_S) \in \mathcal{S}(\mathbf{A}_F)$, and pass to its global adelic θ-function $\Phi_F(f)$. We can also associate to f_S its S-local adelic θ-function $\Phi_{F,S}(f_S)$. The two θ-functions are indeed related. How this goes will be explained throughout the next Chapters up to Chapter 6. Relating $\Phi_{F,S}(f_S)$ and $\Phi_F(f)$ will require use of the Mellin transform, so it is not elementary. In particular, Proposition 2.3.2.2 cannot be deduced from its global analog (see Lemma 2.2.4.3) at this time, so it requires its own argument which is however very similar to the global argument already seen.

PROOF OF PROPOSITION 2.3.2.2. The existence of the limit in (2.3.2.3) follows from the existence of the limit in (2.3.2.2) by Dominated Convergence and compactness of $J^1_{F,S}$, and the fact that $\overline{\mu}_{S,1}$ is a Radon measure on $J^1_{F,S}$, and the identity $\partial_\gamma \Phi_{F,S,\omega}(f) = \Phi_{F,S,\omega}(\partial_\gamma f)$ in (2.3.2.4) follows from the identity $\partial_\gamma \Phi_{F,S}(f) = \Phi_{F,S}(\partial_\gamma f)$. The details of both arguments are similar to those provided in the global situation for Lemma 2.2.4.3 and Theorem 2.2.5.4(3) upon replacing C^1_F with $J^1_{F,S}$, E^1_F with $E_{S,0}$, and $\overline{\mu}_1$ with $\overline{\mu}_{S,1}$. □

2.4. Asymptotic behavior as $|g|_{F,S} \to 0^+$ by lattice point-counting

Let F be a number field, S a finite set of places of F containing all the Archimedean places. In this section we estimate $\Phi_{F,S}(f)(g)$, for $f \in \mathcal{S}(\mathbf{A}_{F,S})$, as a function of $g \in J_{F,S}$ and in the regime $|g|_{F,S} \to 0^+$.

For every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the function $\Phi_{F,S}(f)$ on $J_{F,S}$ is smooth along every continuous 1-parameter subgroup (by Proposition 2.3.2.2. In this section we will be quantifying the unboundedness of $\Phi_{F,S}(f)$ on $J_{F,S}$ as $|g|_{F,S} \to 0^+$, proving an estimate of the following form:

$$\Phi_{F,S}(f) = O(|\log|g|_{F,S}|^{|S|-1})$$

as $|g|_{F,S} \to 0^+$, and discussing the dependencies of the implied constant.

Lattice-point counting methods and Dirichlet's S-unit Theorem are the main ingredients in the calculations that lead to such estimate.

2.4.1. Warm-up: the case of the rational number field. As a warmup, we begin with the special case $F = \mathbf{Q}$ to provide indications as to the origin of the $O(|\log|g|_{F,S}|^{|S|-1})$ bound. The general case is discussed in §2.4.2 (see Theorem 2.4.2.1).

We refer the reader to the discussion preceding [Co, §VII,(4),p.29], where reference to a special case of the result in Proposition 2.4.1.1 below is made.

PROPOSITION 2.4.1.1. *Let S a finite set of places of the rational number field $F = \mathbf{Q}$ containing the unique Archimedean place. Let $f \in \mathcal{S}(\mathbf{A}_{\mathbf{Q},S})$.*

There exists a real number $C_f > 0$ depending on f, such that:

$$|\Phi_{\mathbf{Q},S}(f)(g)| \leq C_f \, |\log|g|_{\mathbf{Q},S}|^{|S|-1} \tag{2.4.1.1}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{\mathbf{Q},S})$ as $|g|_{\mathbf{Q},S} \to 0^+$, and the same holds for $\Phi_{\mathbf{Q},S,\omega}(f)(g)$ for any continuous character ω of $J^1_{\mathbf{Q},S}$.

LEMMA 2.4.1.2. *Let $N, M \geq 1$ be two integers, with $N > 1$, $a_1, \ldots, a_M > 1$ real numbers. Let $\phi : \mathbf{R}_{>0} \to \mathbf{C}$ be function satisfying the following conditions:*

(1) *there exists a real number $C > 0$ such that:*

$$|\phi(t)| \leq C \tag{2.4.1.2}$$

for all $t > 0$.

(2) *There exist real numbers $A, T > 0$ such that $|\phi(t)| \leq At^{-N}$ for all $t \geq T$.*

Then there exist real numbers $T_0, C_\phi > 0$:

$$\sum_{(k_1,\ldots,k_M)\in\mathbf{N}^M} |\phi(a_1^{k_1} \cdots a_M^{k_M} t)| \leq C|\log(t)|^M \tag{2.4.1.3}$$

as $0 < t < T_0$, with $C > 0$ depending on T, M, N and $a_1, \ldots, a_M$.

PROOF. Without loss of generality we may and do assume that we have $1 < a_1 \leq a_2 \leq \ldots \leq a_M$. We fix $t \in (0, T]$.

If any of the k_j in $(k_1, \ldots, k_M) \in \mathbf{N}^M$ is such that

$$k_j \geq -\log_{a_j}(t) + \log_{a_j}(T)$$

then $a_1^{k_1} \cdots a_M^{k_M} t \geq T$, and hence $|\phi(a_1^{k_1} \cdots a_M^{k_M} t)|$ is bounded above by $A(a_1^{k_1} \cdots a_M^{k_M} t)^{-N}$.

We obtain, as an upper bound for the sum over the tuples $(k_1, \ldots, k_M) \in \mathbf{N}^M$ such that $k_j \geq -\log_{a_j}(t) + \log_{a_j}(T)$:

$$\sum_{(k_1,\ldots,k_M)\in\mathbf{N}^M, k_j \geq -\log_{a_j}(t)+\log_{a_j}(T)} |\phi(a_1^{k_1} \cdots a_M^{k_M} t)| \leq A \sum_{(k_1,\ldots,k_M)\in\mathbf{N}^M, k_j \geq -\log_{a_j}(t)+\log_{a_j}(T)} (a_1^{k_1} \cdots a_M^{k_M} t)^{-N} \tag{2.4.1.4}$$

and we have:

$$\sum_{k_j \geq -\log_{a_j}(t)+\log_{a_j}(T)} (a_j^{k_j})^{-N} = \frac{a_j^{N\log_{a_j}(t) - N\log_{a_j}(T)}}{1 - a_j^{-N}} = \frac{(t/T)^N}{1 - a_j^{-N}}. \tag{2.4.1.5}$$

We deduce from (2.4.1.4) and (2.4.1.5) the estimate:

$$\sum_{(k_1,\ldots,k_M)\in\mathbf{N}^M, k_j \geq -\log_{a_j}(t)+\log_{a_j}(T)} |\phi(a_1^{k_1} \cdots a_M^{k_M} t)| \leq A \prod_{1\leq j'\leq M, j'\neq j} \frac{1}{1 - a_{j'}^{-N}} \frac{T^{-N}}{1 - a_j^{-N}} = AT^{-N} \prod_{1\leq j\leq M} \frac{1}{1 - a_j^{-N}} \tag{2.4.1.6}$$

valid for $t \in (0, T]$.

A similar argument shows that each of the sums over the tuples $(k_1, \ldots, k_M) \in \mathbf{N}^M$ such that at least one of the k_j satisfies $k_j \geq -\log_{a_j}(t) + \log_{a_j}(T)$ is bounded as $t \in (0, T]$.

Explicitly, we have, upon denoting by P_m the set of m-tuples of nonnegative integers chosen without repetitions out of the finite set $\{1,\dots,M\}$, for $0 \leq m \leq M$:

(2.4.1.7)
$$\sum_{(k_1,\dots,k_M)\in\mathbf{N}^M, k_{j_r}\geq -\log_{a_{j_r}}(t)+\log_{a_{j_r}}(T),(j_r)_{1\leq r\leq m}\in P_m} |\phi(a_1^{k_1}\cdots a_M^{k_M}t)| \leq At^{-N} \prod_{1\leq j'\leq M, j'\neq j_r, 1\leq r\leq m} \frac{1}{1-a_{j'}^{-N}} \prod_{1\leq r\leq m} \frac{(t/T)^N}{1-a_{j_r}^{-N}}$$

which in turn equals:

$$A \prod_{1\leq j\leq M} \frac{1}{1-a_j^{-N}} t^{(m-1)N}T^{-N} \tag{2.4.1.8}$$

We deduce that we have:

$$\sum_{(k_1,\dots,k_M)\in\mathbf{N}^M} |\phi(a_1^{k_1}\cdots a_M^{k_M}t)| \leq \sum_{(k_1,\dots,k_M)\in\mathbf{N}^M: k_j\leq -\log_{a_j}(t)+\log_{a_j}(T), 1\leq j\leq M} |\phi(a_1^{k_1}\cdots a_M^{k_M}t)| \tag{2.4.1.9}$$

$$+ A \prod_{1\leq j\leq M} \frac{1}{1-a_j^{-N}} \sum_{1\leq m\leq M} \binom{M}{m} t^{(m-1)N}T^{-N} \tag{2.4.1.10}$$

as $t \in (0,T]$. We analyze the sum

$$\sum_{(k_1,\dots,k_M)\in\mathbf{N}^M: k_j\leq -\log_{a_j}(t)+\log_{a_j}(T), 1\leq j\leq M} |\phi(a_1^{k_1}\cdots a_M^{k_M}t)| \tag{2.4.1.11}$$

for $t \in (0,T]$. Since $|\phi(t)| \leq C$ for all $t > 0$, we have:

$$\sum_{(k_1,\dots,k_M)\in\mathbf{N}^M: k_j\leq -\log_{a_j}(t)+\log_{a_j}(T), 1\leq j\leq M} |\phi(a_1^{k_1}\cdots a_M^{k_M}t)| \leq C(1+|\log_{a_1}(t/T)|)^M \tag{2.4.1.12}$$

where we have used $1 < a_1 \leq a_j$ for every $1 \leq j \leq M$. Upon combining (2.4.1.9) with (2.4.1.12), we deduce that (2.4.1.3) holds. □

We can now prove Proposition 2.4.1.1:

Proof of Proposition 2.4.1.1. We call $S_f := S \cap \Sigma_{\mathbf{Q},f}$.

Our aim is to reduce the assertion in (2.4.1.1) of the Proposition to Lemma 2.4.1.2. We first reduce to the case $f \in \mathcal{S}(\mathbf{A}_{\mathbf{Q},S})$ is of the form $f_\infty \otimes \bigotimes_{p\in S_f} 1_{\mathbf{Z}_p}$. Indeed, since the estimate in (2.4.1.1) is stable under finite $\mathbf{C}$-linear combinations of Schwartz-Bruhat functions on $\mathbf{A}_{\mathbf{Q},S}$, at the cost of replacing the implied constant by a larger one, we may and do assume f is of the form $f = f_\infty \otimes \bigotimes_{p\in S_f} 1_{K_p}$ for $K_p \subset \mathbf{Q}_p$ a compact open non-empty subset, for each $p \in S_f$.

Since $\mathbf{Z}_p$ is a compact open subgroup and $\mathbf{Q}_p = \bigcup_{m\geq 0} p^{-m}\mathbf{Z}_p$, there exists a large enough integer $N_p \geq 0$ such that $K_p \subset p^{-N_p}\mathbf{Z}_p$.

Our aim is to estimate the sum

$$\sum_{\xi\in\mathcal{O}_{\mathbf{Q},S}^\times} \left| f_\infty(\xi g_\infty) \prod_{p\in S_f} 1_{K_p}(\xi g_p) \right| \tag{2.4.1.13}$$

where g_∞ is the image of $g \in \mathrm{GL}_1(\mathbf{A}_{\mathbf{Q},S})$ into $\mathrm{GL}_1(\mathbf{A}_{\mathbf{Q},S_\infty})$ and g_p the image of g into $\mathbf{Q}_p$ for each $p \in S_f$, and since $1_{K_p} \leq 1_{p^{-N}\mathbf{Z}_p}$ as real-valued functions on $\mathbf{Q}_p$ then for this purpose it is enough to replace $\mathbf{Q}_p$ with $p^{-N}\mathbf{Z}_p$ at all the finite places $p \in S_f$ such that $K_p \neq \mathbf{Z}_p$ (while we keep $N = 0$ at all finite places $p \in S$ such that $K_p = \mathbf{Z}_p$).

Then for any integer $a \geq N$ we have $p^a p^{-N} \in \mathbf{Z}_p$. We choose a so large that $p^a p^{-N_p} \in \mathbf{Z}_p$ holds for all finite places $p \in S$ (such a exists because S is finite).

We note that $p \in \mathcal{O}^\times_{\mathbf{Q},S}$ since S contains p. The convergent sum in (2.4.1.13) is $\mathcal{O}^\times_{\mathbf{Q},S}$-invariant, i.e. invariant under left-translation by S-units. We operate by left-translation by p^a and reduce to estimating the sum

(2.4.1.14)

$$\sum_{\xi\in\mathcal{O}^\times_{\mathbf{Q},S}}\left|f_\infty(\xi p^a g_\infty)\prod_{v\in S_f}1_{p^{-N_p}\mathbf{Z}_p}(\xi p^a g_p)\right| = \sum_{\xi\in\mathcal{O}^\times_{\mathbf{Q},S}}\left|f_\infty(\xi p^a g_\infty)\prod_{p\in S_f}1_{p^a p^{-N_p}\mathbf{Z}_p}(\xi p^a g_p)\right| \leq \sum_{\xi\in\mathcal{O}^\times_{\mathbf{Q},S}}\left|f_\infty(\xi p^a g_\infty)\prod_{p\in S_f}1_{\mathbf{Z}_p}(\xi g_p)\right|$$

where p^a denotes both the image of the diagonal element $p^a \in \mathcal{O}^\times_{\mathbf{Q},S}$ in $\mathbf{R}^\times$ and in $\mathbf{Q}_\ell^\times$ for every $\ell \in S$ (including $\ell = p$).

We rename the Schwartz-Bruhat function $x \mapsto f_\infty(p^a x)$ as f_∞ and replace K_p with $\mathbf{Z}_p$. We are therefore reduced to the case f is of the form

$$f = f_\infty \otimes \bigotimes_{p\in S_f} 1_{\mathbf{Z}_p}.$$

The reduction is performed at the cost of enlarging the implied constant in the estimate from (2.4.1.1).

It is furthermore enough to estimate the sum

$$\sum_{\xi\in\mathcal{O}^\times_{\mathbf{Q},S}}|f(\xi g)| \tag{2.4.1.15}$$

for f of such special form, on $g = \tau(t)g_0$, for τ a choice of continuous homomorphic section to the S-idelic norm, and with $t \in \mathbf{R}_{>0}$ and $g_0 \in E_{S,0}$, the S-idelic fundamental domain from (A.2.4.5). Under such additional constraint on g, the only $\xi \in \mathcal{O}^\times_{\mathbf{Q},S}$ that contribute to the sum in (2.4.1.15) non-trivially satisfy $|\xi|_p \leq 1$ for every $p \in S_f$, which reads $\xi \in \mathcal{O}^\times_{\mathbf{Q},S} \cap \mathbf{Z}$.

We now note that, upon listing the prime numbers $\{p_1, \ldots, p_{|S|-1}\}$ ordered increasingly, whose non-Archimedean absolute values are contained in S, we have

$$\mathcal{O}^\times_{\mathbf{Q},S} = (\mathbf{Z}[1/(p_1\cdots p_{|S|-1})])^\times$$

and hence $p_1^{-1}, \ldots, p_{|S|-1}^{-1}$ are fundamental units. We define $a_k := p_k$ for $1 \leq k \leq |S|-1$.

For $K_S \subset \mathrm{GL}_1(\mathbf{A}_{\mathbf{Q},S})$ the compact Hausdorff subgroup:

$$K_S = \mu_2 \times \prod_{p\in S_f}\mathbf{Z}_p^\times$$

given as the kernel of the S-logarithm $\mathscr{L}_S$, we use that $f \in \mathcal{S}(\mathbf{A}_{\mathbf{Q}})$ to see that as a function of $t \in \mathbf{R}_{>0}$, the function

$$\phi : g \mapsto \sup_{g_0\in E_{S,0}}\sup_{h\in K_S}|f(\xi h\tau(t)g_0)|$$

on $\mathrm{GL}_1(\mathbf{A}_{\mathbf{Q},S})$ (for τ the chosen continuous homomorphic section to the S-idelic norm, and $E_{S,0}$ the S-idelic fundamental domain from (A.2.4.5)) satisfies

$$\sup_{t\in\mathbf{R}_{>0}}(1+t^N)|\phi(t)| < +\infty$$

for every integer $N \geq 1$.

In light of the inequality, for all $(t, g_0) \in \mathbf{R}_{>0} \times E_{S,0}$:

$$|\Phi_{\mathbf{Q},S}(f)(\tau(t)g_0)| \leq \sum_{\xi\in\mathcal{O}^\times_{\mathbf{Q},S}}|f(\xi\tau(t)g_0)| \leq C\sum_{(k_1,\ldots,k_{|S|-1})\in\mathbf{N}^{|S|-1}}\phi(a_1^{k_1}\cdots a_{|S|-1}^{k_{|S|-1}}t) \tag{2.4.1.16}$$

for a positive real number $C > 0$, we deduce that Lemma 2.4.1.2 applies to ϕ with $M = |S|-1$. The Lemma settles Proposition 2.4.1.1 in the case of S-local adelic θ-functions (i.e. in the non ω-homogeneous case). The

case of ω-homogeneous S-local adelic θ-functions follows upon integrating over the compact Hausdorff abelian group $J^1_{\mathbf{Q},S}$, up to replacing the constant $C > 0$ with $C \cdot \overline{\mu}_{S,1}(J^1_{\mathbf{Q},S})$. □

The preceding result for $F = \mathbf{Q}$ and S the 3-element set containing the real place, as well as the 2-adic and 3-adic places, generalizes an observation made in [**Co**, §VII,p.29] with a similar intent.

More generally and using the S-logarithm $\mathscr{L}_S$ from (A.2.4.7), the effect of left-multiplication by the discrete subgroup of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ generated by the fundamental S-units $\varepsilon_1, \ldots, \varepsilon_{|S|-1}$ is ergodic on all the finite intersections of the S-adelic "hyperplanes":

$$H_v := \{x \in \mathbf{A}_{F,S} : x_v = 0\}$$

for each $v \in S_f$, where we equip the open subset U_v of H_v of those points $x \in H_v$ such that $x_w \neq 0$ for all $w \in S - \{v\}$, with the measure inherited from the product of the multiplicative Tamagawa measures on $\prod_{w \in S-\{v\}} F_w^\times$, upon realizing U_v as an orbit under $\mathrm{GL}_1(\mathbf{A}_{F,S})$ whose stabilizer is $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$.

It is the ergodicity of this action that causes not only the S-local adelic θ-functions $\Phi_{F,S}(f)$ for $f \in \mathcal{S}(\mathbf{A}_{F,S})$, to be unbounded as $|g|_{F,S} \to 0^+$, but also their Mellin transform to have poles on the imaginary line.

For $\chi \in \widehat{J_{F,S}}$ a fixed choice of continuous character, we consider the function of $s \in \mathbf{C}_{>0}$:

$$Z_S(f, \chi|\cdot|^s_{F,S}) := \int_{J_{F,S}} \Phi_{F,S}(f)(g)\chi(g)|g|^s_{F,S}\, \mathrm{d}\overline{\mu}_S(g)$$

where the integral converges absolutely and uniformly for $s \in \mathbf{C}_{>0}$ to a holomorphic function (by the rapid decay of the S-local adelic θ-function $\Phi_{F,S}(f)$, proved in Proposition 2.3.1.2(2)).

The reader familiar with the correspondence between asymptotic expansions as in (2.4.1.1) from Proposition 2.4.1.1 and Proposition 2.3.1.2 (see, for instance, Theorem B.2.1.7 in combination with the Asymptotic Residue Theorem of Theorem B.3.2.1, as well as [**HL**, §2,(2.10),p.385] and [**HL**, §2,Thm.1b,p.385]) will read off from (2.4.1.1) the following piece of information: if the Mellin transform $Z_S(f, \chi|\cdot|^s_{F,S})$ of $\Phi_{F,S}(f)$ extends meromorphically to the complex s-plane (this will be seen using the local functional equations from [**CF**, §2.4,Thm.2.4.1,p.315]), then it may have poles on the imaginary line (and possibly elsewhere too, but not over $\mathbf{C}_{>0}$).

The poles on the imaginary line all have multiplicity bounded above by $|S|$, and this is a direct consequence of Mellin Inversion applied to the estimate from (2.4.1.1).

We will be classifying and studying the poles of the meromorphic functions $Z_S(f, \chi|\cdot|^s_{F,S})$ throughout Chapter 5, and their relation to the Schauder bases that have a fundamental role to play in performing the trace calculation that is the main subject of this work.

It is important to be aware of the relation between such poles, and the action of $\mathcal{O}^\times_{F,S}$ on $\mathbf{A}_{F,S}$ and especially on the "boundary hyperplanes" H_v of which the complement of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ is comprised.

We will be now proving the general version of Proposition 2.4.1.1.

2.4.2. The general case. In the special case of $F = \mathbf{Q}$, direct arguments sufficed to establish the desired estimate on $\Phi_{F,S}(f)(g)$ in the regime as $|g|_{F,S} \to 0^+$.

In the Theorem below, we treat the general case, and show that the implied constants can be arranged to be uniform on bounded subsets within $\mathcal{S}(\mathbf{A}_{F,S})$.

Theorem 2.4.2.1 below will serve the later purpose of showing the operators arising as S-idelic Zeta functions L_S on $\mathcal{S}(\mathbf{A}_{F,S})$ (see (5.1.1.2)) valued in meromorphic functions on the Riemann surface $\Omega(J_{F,S})$, are continuous with respect to finer topologies than that of uniform convergence on the compact subsets away from the poles.

We refer the reader to [**Co**, §VII,(6),p.30], where a similar estimate based on lattice-point counting and a volume calculation, is referred to.

THEOREM 2.4.2.1 (On the asymptotic behavior of the S-local θ-functions as $|g|_{F,S} \to 0^+$). *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a bounded subset.

There exist real numbers $C_{\mathcal{B},1}, C_{\mathcal{B},2} > 0$ such that we have, for every $f \in \mathcal{B}$:

$$|\Phi_{F,S}(f)(g)| \leq C_{\mathcal{B},1}|\log|g|_{F,S}|^{|S|-1} + C_{\mathcal{B},2} \tag{2.4.2.1}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} < 1$.

PROOF. Let $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a bounded subset. By Proposition 2.1.4.1(1), the canonical linear operator $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \bigotimes_{v \in S_f} \mathcal{S}(F_v) \to \mathcal{S}(\mathbf{A}_{F,S})$ is a linear topological isomorphism, and by Proposition C.2.3.13(2), combined with Corollary C.2.3.12 and [**Jar**, Part III,Ch.15,§15.1,p.324], a bounded subset $\mathcal{B}$ in $\mathcal{S}(\mathbf{A}_{F,S})$ is contained in one which is the image of

$$\mathrm{cch}(\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w)$$

the convex circled hull of the subset $\mathcal{B}_\infty \otimes \bigotimes_{w \in S_f} \mathcal{B}_w$, with $\mathcal{B}_w$ bounded in a finite dimensional $\mathbf{C}$-linear subspace of $\mathcal{S}(F_w)$ for each finite place $w \in S$.

Indeed, since, by Lemma 2.1.2.4(1), $\mathcal{S}(F_v)$ carries the finest locally convex topology, $\mathcal{B}_w$ can be assumed to be, for each $w \in S_f$, contained in a finite dimensional subspace of $\mathcal{S}(F_v)$ by Proposition C.2.5.1(4).

There therefore exist finitely many $f_{w,1}, \ldots, f_{w,m_w} \in \mathcal{S}(F_w)$ whose linear span contains $\mathcal{B}_w$. In particular, all the functions in $\mathcal{B}_w$ are supported on a large enough compact open subset $K_w \subset F_w$, and this is so for each $w \in S_f$.

We deduce that there exists a large enough compact open subset $K \subset \prod_{v \in S_f} F_v$ such that every element of $\mathcal{B}$ is a finite absolutely convex $\mathbf{C}$-linear combination (i.e. one whose coefficients have complex Euclidean absolute values that sum to a real number bounded above by 1; see [**Jar**, Part I,Ch.6,§6.1,p.102] for $r = 1$; see also [**NB**, Thm.4.2.11]) of Schwartz-Bruhat functions of the form $f_\infty \otimes \bigotimes_{w \in S_f} f_w$ with each of the f_w supported on the common K whose existence was proved above.

These observations have proceeded as in the proof of Proposition 2.3.1.2(3).

There exists a large enough integer $N \geq 0$ such that $K \subset (1/N)\widehat{\mathcal{O}}_{F,S}$. We may therefore rename $(1/N)\widehat{\mathcal{O}}_{F,S}$ as K.

The left-translation action of $N^{\mathbf{Z}} \subset \prod_{v \in S_f} F_v^\times$ extends to $\mathcal{S}(\mathbf{A}_{F,S})$ through the $\bigotimes_{v \in S_f} \mathcal{S}(F_v)$ tensor factor and the linear topological isomorphism of Proposition 2.1.4.1(1), which is compatible with the left-translation action $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on the source, and the action of each of the factors $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and $\prod_{v \in S_f} F_v^\times$ on the target.

We therefore let $N^{\mathbf{Z}}$ act on $\mathcal{S}(\mathbf{A}_{F,S})$ as $\mathrm{Id}_{\mathcal{S}(\mathbf{A}_{F,S_\infty})} \otimes N$ (where we denote by "N" the action of the element $N \in N^{\mathbf{Z}} \subset \prod_{v \in S_f} F_v$ on $\bigotimes_{v \in S_f} \mathcal{S}(F_v)$).

This action of $N^{\mathbf{Z}}$ on $\mathcal{S}(\mathbf{A}_{F,S})$ is continuous, and therefore sends bounded subsets to bounded subsets. We may rename the pre-image of $\mathcal{B}$ under the action of N^{-1} as $\mathcal{B}$, and assume, without loss of generality, that $K = \widehat{\mathcal{O}}_{F,S}$.

We choose a continuous homomorphic section τ to the S-idelic norm. By connectedness of $\tau(\mathbf{R}_{>0})$, its image is contained in the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$.

For any real number $M > 0$ and any integer $N \geq 2$, there exists a large enough ball $B \subset \mathbf{A}_{F,\infty}$ centered at the origin (the normed structure $\|\cdot\|$ on $\mathbf{A}_{F,S_\infty}$ being any one inducing the Euclidean product topology on $\mathbf{A}_{F,S_\infty}$) such that:

$$\sup_{x \in \mathbf{A}_{F,S_\infty} - B} |f_\infty(x_\infty)| < M\|x_\infty\|^{-N} \tag{2.4.2.2}$$

for every element $f_\infty \in \mathcal{B}_\infty$ (since $\sup_{x \in \mathbf{A}_{F,S_\infty}} |\cdot|$ is a continuous seminorm on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$), upon denoting x_∞ the arbitrary point of $\mathbf{A}_{F,S_\infty}$.

Since $J^1_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathcal{O}^\times_{F,S}$ is compact Hausdorff, the function of (t,g) defined as:

$$\Psi_{F,S,B}(f)(t,g) := \sum_{\xi\in\mathcal{O}^\times_{F,S}} f(\xi\tau(t)g)\cdot(1_{B^c}\otimes 1_{\widehat{\mathcal{O}}_{F,S}})(\xi\tau(t)g)$$

for 1_{B^c} the indicator function of $B^c := \mathbf{A}_{F,S_\infty} - B$, satisfies, in light of (2.4.2.2):

$$|\Psi_{F,S,B}(f)(t,g)| = O(1) \tag{2.4.2.3}$$

for all $(t,g)\in\mathbf{R}_{>0}\times J^1_{F,S}$, with an implied constant that depends on f only through $\mathcal{B}$.

We define $A := \sup_{f_\infty\in\mathcal{B}_\infty}\sup_{\tau(t)g\in B}|f_\infty(\tau(t)g)|$. The supremum defining A exists and it is finite because $\sup_{x\in B}|\cdot|$, is a countinuous seminorm on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $\mathcal{B}_\infty$ is bounded.

We may further replace B with a hypercube containing B, i.e. $Q\subset\mathbf{A}_{F,S_\infty}$ of the form

$$Q = \prod_{v\text{ real}}[-C_v,C_v]\times\prod_{v\text{ complex}}([-C_v,C_v]+i[-C_v,C_v]),$$

for real numbers $C_v>0$ for each Archimedean place v of F.

Up to changing the constants $C_{\mathcal{B},N}$ and $T_{\mathcal{B},N}$, we may further assume $A=1$ without loss of generality.

Up to acting on $\mathbf{A}_{F,S_\infty}$ by left-translation by the elements of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, each of which defines a continuous linear topological automorphism on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, hence sending bounded subsets to bounded subsets, we may and do assume Q is the hypercube:

$$Q = \{x\in\mathbf{A}_{F,S_\infty} : |x|_v\leq 1, v\in S_\infty\}.$$

It follows that it is enough to estimate

$$\sum_{\xi\in\mathcal{O}^\times_{F,S}} u(\xi\tau(t)g)$$

as a function of $(t,g)\in\mathbf{R}_{>0}\times E_{S,0}$, with $u := 1_Q\otimes 1_{\widehat{\mathcal{O}}_{F,S}}$, i.e. the indicator function of $\{x\in\mathbf{A}_{F,S} : |x|_v\leq 1, v\in S\}$, and $E_{S,0}$ the S-idelic fundamental domain from (A.2.4.5) of Definition A.2.4.3.

Since u is real-valued and nonnegative, it is therefore enough to show the following: there exist $C,T>0$ such that:

$$\left|\sum_{\xi\in\mathcal{O}^\times_{F,S}}\int_{E_{S,0}}(1_Q\otimes 1_{\widehat{\mathcal{O}}_{F,S}})(\xi\tau(x)g_0)\,\mathrm{d}\overline{\mu}_{S,1}(g_0)\right| \leq C\,|\log(t)|^{|S|-1} \tag{2.4.2.4}$$

for all $t\in\mathbf{R}_{>0}$ such that $t<T$. This, in turn, holds by Theorem A.4.1.1.

We call $C_{\mathcal{B},1} := C_{\mathcal{B},N}$.

We now call, upon denoting by $T>0$ the constant $T := T_{\mathcal{B},N}$ obtained from the foregoing, $t := \max\{T,1/T\}$, and observe that the proof of Proposition 2.3.1.2(4), implies that for all $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $g\in\tau([1/t,t])\cdot E^1_{F,S}$, for $E^1_{F,S}$ the S-idelic fundamental domain from (A.2.4.4) for the continuous left-translation action of the S-units $\mathcal{O}^\times_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, the estimate:

$$|\Phi_{F,S}(f)(g)|\leq C \tag{2.4.2.5}$$

holds for all $f\in\mathcal{B}$, for a positive real number $C>0$ depending only on $\mathcal{B}$ and $t>0$. Upon defining $C_{\mathcal{B},2} := \max\{C_{\mathcal{B},1}|\log(t)|, C\}$, we obtain that the estimate in (2.4.2.1), for the constant $C_{\mathcal{B},2}$ defined above, and for all $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$ such $|g|_{F,S}<1$. □

2.4.3. Order of growth as $|g|_{F,S}\to 0^+$. We will now collect two consequences of Theorem 2.4.2.1 concerning the asymptotic behavior of the S-local adelic θ-functions as $|g|_{F,S}\to 0^+$, to be used in the subsequent

two Chapters in combination with the local functional equations from [**CF**, Ch.XV,§2.4,Thm.2.4.1,p.315] and of simpler applicability.

PROPOSITION 2.4.3.1. *Let F be a number field, S a finite set of places of F containing all Archimedean places. Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$.*

We have:

(1) *for every $\varepsilon > 0$:*

$$\Phi_{F,S}(f)(g)|g|_{F,S}^{\varepsilon} = O_{\varepsilon}(1) \tag{2.4.3.1}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ as $|g|_{F,S} \to 0^+$, and the same holds for $\Phi_{F,S,\omega}(f)(g)|g|_{F,S}^{\varepsilon}$ for any continuous character ω of $J_{F,S}^1$.

(2) *For $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$ a bounded subset, there exists a real number $T_{\mathcal{B},\varepsilon} > 0$ such that (2.4.3.1) holds with an implied constant that is uniform with respect to f varying over $\mathcal{B}$, and for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} < T_{\mathcal{B},\varepsilon}$. The same holds for $\Phi_{F,S,\omega}(f)(g)|g|_{F,S}^{\varepsilon}$ for any continuous character ω of $J_{F,S}^1$.*

(3) *The assertions in (2) holds with $T_{\mathcal{B},\varepsilon} = 1$, upon replacing the implied constant in (2.4.3.1) with a possibly larger constant depending on $\varepsilon > 0$, and which is still uniform with respect to f varying over the bounded subset $\mathcal{B}$.*

A similar result on the growth of the S-local adelic θ-functions as $|g|_{F,S} \to 0^+$ is present in R. Meyer's [**Me**, §4]. We refer the reader specifically to the beginning of the section, on [**Me**, §4,p.22], where the space $\mathcal{T}_S$ in loc.cit. is characterized. Its elements can be equivalently described as S-local adelic θ-functions. This will follow as a consequence of Theorem 5.4.1.1 from Chapter 4 (i.e. the characterization will be a Theorem, and is not taken as our definition of the analog of Meyer's $\mathcal{T}_S$ throughout this work, which will instead be introduced as a quotient of the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}_{F,S})$; see Definition 5.1.1.1 from §5.1.1).

PROOF OF PROPOSITION 2.4.3.1. (1) : We consider the function:

$$\phi(t,g) := \Phi_{F,S}(f)(\tau(t)g)\omega(g)$$

of $(t,g) \in \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

By Proposition 2.3.2.2, $\phi(\cdot, g)$ is smooth on $\mathbf{R}_{>0}$ for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and the derivative $(t\partial_t)\phi$ is a continuous function of (t,g) on $\mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

Differentiation in the t-variable yields:

$$(x\partial_x)(\phi(x,g)x^{\varepsilon}) = x\left\{\phi'(x,g)x^{\varepsilon} + \varepsilon\phi(x,g)x^{\varepsilon-1}\right\} = (x\partial_x)(\phi)(x,g)x^{\varepsilon} + \varepsilon\phi(x,g)x^{\varepsilon}.$$

By the Fundamental Theorem of Calculus, we have:

$$\phi(t,g)t^{\varepsilon} = -\int_t^{+\infty} (x\partial_x)(\phi(x,g)x^{\varepsilon})\,\frac{\mathrm{d}x}{x}$$

for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and hence

$$|\phi(t,g)t^{\varepsilon}| \leq \int_{\mathbf{R}_{>0}} |(x\partial_x\phi)(x,g)x^{\varepsilon}|\,\frac{\mathrm{d}x}{x} + \varepsilon\int_{\mathbf{R}_{>0}} |\phi(x,g)x^{\varepsilon}|\,\frac{\mathrm{d}x}{x} \tag{2.4.3.2}$$

for all $t \in \mathbf{R}_{>0}$.

By Proposition 2.3.2.2, we have $(t\partial_t\phi)(t,g) = \Phi_{F,S}(\partial_{\gamma_\tau} f)(\tau(t)g)$, for $\gamma_\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ the continuous 1-parameter subgroup that is τ by another name.

Both integrals in (2.4.3.2) are finite, since, by Proposition 2.3.1.4(2), both $(t\partial_t\phi)$ and ϕ are $O_N(t^{-N})$ for every integer $N \geq 1$ as $t \to +\infty$, and by Theorem 2.4.2.1 the singularity of $\phi(t,g)t^{\varepsilon-1}$ as $t \to 0^+$, and of $(t\partial_t)\phi(t,g)t^{\varepsilon-1}$ as $t \to 0^+$, is $\mathrm{d}t$-integrable over $(0,A]$ for every $A > 0$, for all $\varepsilon > 0$.

By Dominated Convergence, formation of both integrals in (2.4.3.2) is continuous as a function of $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and since $\phi(t,g)$ is $\mathcal{O}_{F,S}^\times$-invariant as a function of $(t,g) \in \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S})^1$ with $\mathcal{O}_{F,S}^\times$ acting on the second factor, we may maximize the right side of (2.4.3.2) as a function of g over $E_{S,0}$, which is relatively compact and with non-empty interior, to obtain:

$$|\Phi_{F,S}(f)(g)|g|_{F,S}^\varepsilon| \leq \sup_{(t,g') \in \mathbf{R}_{>0} \times E_{S,0}} |\phi(t,g')|g'|_{F,S}^\varepsilon| < +\infty \tag{2.4.3.3}$$

where the inequality in (2.4.3.3) is valid for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, since $\Phi_{F,S}(f)$ and $|\cdot|_{F,S}$ are both $\mathcal{O}_{F,S}^\times$-invariant functions on $\mathrm{GL}_1(\mathbf{A}_{F,S})$. This shows that (2.4.3.1) holds.

We consider the function:

$$\phi(t) := \int_{J_{F,S}^1} \Phi_{F,S}(f)(\tau(t)g)\omega(g)\,\mathrm{d}\overline{\mu}_{S,1}(g)$$

a function on $\mathbf{R}_{>0}$.

By Proposition 2.3.2.2, ϕ is smooth on $\mathbf{R}_{>0}$.

By the Fundamental Theorem of Calculus, we have:

$$\phi(t)t^\varepsilon = -\int_t^{+\infty} (x\partial_x)(\phi(x)x^\varepsilon)\,\frac{\mathrm{d}x}{x}$$

and hence

$$|\phi(t)t^\varepsilon| \leq \int_{\mathbf{R}_{>0}} |(x\partial_x\phi)(x)x^\varepsilon|\,\frac{\mathrm{d}x}{x} + \varepsilon \int_{\mathbf{R}_{>0}} |\phi(x)x^\varepsilon|\,\frac{\mathrm{d}x}{x} \tag{2.4.3.4}$$

for all $t \in \mathbf{R}_{>0}$.

By Proposition 2.3.2.2, we have $(t\partial_t\phi)(t) = \Phi_{F,S,\omega}(\partial_{\gamma_\tau} f)(\tau(t))$, for $\gamma_\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ the continuous 1-parameter subgroup induced by τ.

Both integrals in (2.4.3.4) are finite, since, by Proposition 2.3.1.4(2), both $(t\partial_t\phi)$ and ϕ are $O_N(t^{-N})$ for every integer $N \geq 1$ as $t \to +\infty$, and the singularity of $\phi(t)t^{\varepsilon-1}$ as $t \to 0^+$ and of $(t\partial_t)\phi(t)t^{\varepsilon-1}$ as $t \to 0^+$, is $\mathrm{d}t$-integrable over $(0,A]$ for every $A > 0$, for all $\varepsilon > 0$, by Theorem 2.4.2.1.

We deduce from (2.4.3.4) that we have:

$$|\phi(t)t^\varepsilon| \leq A$$

for all $t \in \mathbf{R}_{>0}$.

(2) : The uniformity assertion on the implied constant follows from Theorem 2.4.2.1. Indeed, for all $f \in \mathcal{B}$, we can use the constants $C_{\mathcal{B},\varepsilon}$ and $T_{\mathcal{B},\varepsilon}$ to bound the integral of the singularity of $\Phi_{F,S}(f)$ uniformly over $J_{F,S}^{<T_{\mathcal{B},\varepsilon}}$, and formation of the integral $\int_{J_{F,S}^{\geq T_{\mathcal{B},\varepsilon}}} |\Phi_{F,S}(f)(g)|g|_{F,S}^\varepsilon|\,\mathrm{d}\overline{\mu}_S(g)$ is sequentially continuous over $\mathcal{S}(\mathbf{A}_{F,S})$ by Dominated Convergence, and hence continuous by [**NB**, Thm.14.7.3], since $\mathcal{S}(\mathbf{A}_{F,S})$ is a strict LF-space by Proposition 2.1.4.1(2), hence bornological (see [**Scha**, Ch.II,§8,Cor.1,p.62]) and ultrabornological (see [**NB**, Thm.13.2.12]), and the target Hausdorff locally convex topological $\mathbf{C}$-vector space of the linear operator $f \mapsto 1_{J_{F,S}^{\geq T_{\mathcal{B},\varepsilon}}}(g) \cdot \Phi_{F,S}(f)(g)|g|_{F,S}^\varepsilon$ is $L^1(J_{F,S}, \overline{\mu}_S)$, which is webbed by [**NB**, Thm.14.6.2] being a complete normed space, and hence [**NB**, Thm.13.2.12] applies.

In particular, the integral $\int_{J_{F,S}^{\geq T_{\mathcal{B},\varepsilon}}} |\Phi_{F,S}(f)(g)|g|_{F,S}^\varepsilon|\,\mathrm{d}\overline{\mu}_S(g)$ is uniformly bounded as f varies over the bounded subset $\mathcal{B}$: indeed, the function $\alpha : f \mapsto \int_{J_{F,S}^{\geq T_{\mathcal{B},\varepsilon}}} |\Phi_{F,S}(f)(g)|g|_{F,S}^\varepsilon|\,\mathrm{d}\overline{\mu}_S(g)$ is continuous and complex-valued on $\mathcal{S}(\mathbf{A}_{F,S})$ by the foregoing, and since $\mathcal{S}(\mathbf{A}_{F,S})$ is Montel by Proposition 2.1.4.1(2), it has the Heine-Borel property, and hence the closure of $\mathcal{B}$, also bounded by [**Scha**, Ch.I,§5.1,(ii),p.25], is compact. The continuous function α is therefore bounded on $\mathcal{B}$.

(3) : By Theorem 2.4.2.1, the real number $T_{\mathcal{B},\varepsilon}$ used throughout the proof of (2) can be chosen to be $T_{\mathcal{B},\varepsilon} = 1$. □

COROLLARY 2.4.3.2. *Let F be a number field, S a finite set of places of F containing all Archimedean places. Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$.*

For every real number $\varepsilon > 0$, we have:

$$|g|_{F,S}^{\varepsilon} \Phi_{F,S}(f)(g) = o_{\varepsilon}(1) \tag{2.4.3.5}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ as $|g|_{F,S} \to 0^+$, and the same holds for $|g|_{F,S}^{\varepsilon} \Phi_{F,S,\omega}(f)(g)$ for any continuous character $\omega \in \widehat{J^1_{F,S}}$.

PROOF. Let τ be a continuous homomorphic section to the S-idelic norm.

We deduce from (2.4.3.2) that we have:

$$|\Phi_{F,S}(f)(\tau(t)g)t^{\varepsilon/2}| \leq A_{\varepsilon}$$

for all $t \in \mathbf{R}_{>0}$ and all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and hence:

$$|\Phi_{F,S}(f)(\tau(t)g)t^{\varepsilon}| \leq A_{\varepsilon}\, t^{\varepsilon/2} \tag{2.4.3.6}$$

for all $t \in \mathbf{R}_{>0}$, where A_{ε} only depends on f and $\varepsilon > 0$ but not on $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, which implies $\sup_{g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1} |\Phi_{F,S}(\tau(t)g)t^{\varepsilon}| = o_{\varepsilon}(1)$ as $t \to 0^+$. The case of $\Phi_{F,S,\omega}(f)$ is treated in a closely analogous manner, by integration of both sides of (2.4.3.6) over the compact Hausdorff abelian group $J^1_{F,S}$. □

CHAPTER 3

Idelic Zeta Functions

In this Chapter we collect results on how to recover the global idelic Zeta integrals in [**CF**, Ch.XV,§4] as transforms of functions on the idele class group and on the positive reals, as opposed to integrals of Schwartz-Bruhat functions on the adeles restricted to the idele group.

We will further discuss the S-idelic analog of the global idelic Zeta functions from [**CF**, Ch.XV,§4]. Equivalently, we will be discussing in general, the paradigm envisioned by Riemann in his paper "On the Number of Primes Less Than a Given Magnitude" (for which we refer the reader to [**Riemann**]) where he used the Jacobi θ-function to prove the functional equation and existence of the analytic continuation to the Riemann Zeta function.

We will be using the global and S-local adelic θ-functions constructed in the previous Chapter and whose study we initiated therein, to a similar effect. This Chapter will therefore consist in a reformulation of several of the results contained in Tate's dissertation from [**CF**, Ch.XV], with a few complements being discussed.

Subsequently, we will be viewing global and S-local adelic θ-functions as their own function space, i.e. we will be topologizing the $\mathbf{C}$-linear space of all global adelic θ-functions, and all S-local adelic θ-functions, respectively.

The reason for doing this is that certain operators on the spaces of global and S-local adelic θ-functions will enable us to read off non-trivial information about the zeros of Hecke L-functions through trace formulas. The reader familiar with Selberg's classical trace formula and subsequent refinements will know how powerful trace formulas are. They are a pivotal instrument throughout the literature on automorphic representations and the Langlands program.

Trace calculations will be made accessible to us by promoting each of the spaces of adelic θ-functions to Hausdorff locally convex topological $\mathbf{C}$-vector spaces. The trace calculation that is the subject of this work takes place on Sobolev spaces (hence *normed*) but, leading to the main calculation, finer locally convex topologies are also necessary for reasons that will become clear by Chapter 7. They will give us Schauder bases, which will enable us to compute "large pieces" of the trace that interests us.

This is a good time to anticipate for the reader what "local convexity" does for us: it is a condition on bases of 0-neighborhoods in topological vector spaces leading to robust control on several conditions through the machinery of "duality". There is, in other words, good reason to appeal to this side of functional analysis, on which profound and expressive literature is available. Those elements that are relevant to us are discussed in Appendix C.

Topologizing the spaces of S-local and global adelic θ-functions will be the job that the next two Chapters (Chapters 4 and 5 of this work) are tasked with. The linear topologies arising throughout this work will all originate from spaces of Schwartz-Bruhat functions (such as $\mathcal{S}(\mathbf{A}_F)$ and $\mathcal{S}(\mathbf{A}_{F,S})$) and normed completions.

The topological vector spaces of adelic θ-functions will be identified via Mellin transforms with topological vector spaces of meromorphic and holomorphic functions on the Riemann surfaces of continuous quasi-characters of $J_{F,S}$, respectively C_F. The Mellin transform will recover Tate's idelic Zeta integrals in much the same way as the classical Mellin transform of the Jacobi θ-function recovers the Riemann Zeta function, completed with its own Γ-factor.

One of the purposes of Chapter 4 of this work is to describe the image of the Mellin transform as an operator between two Hausdorff locally convex topological $\mathbf{C}$-vector spaces (a space of θ-functions, on the source, and a space of Z-functions, on the target), promoting it to a linear topological isomorphism between a space of θ-functions and a space of meromorphic functions satisfying growth conditions over vertical strips of bounded width and a functional equation, in a similar vein as the Converse Theorems from [**CPS**, Thm.1,p.165] for GL_n.

So far we have only defined the adelic θ-functions, both S-local and global. This is what we have done in Chapter 2. We will be calling the θ-functions the "backward side" of the Mellin transform.

This Chapter, instead, defines the Z-functions. It provides the terminology and basic results to define and study the "forward side" of the Mellin transform, i.e. spaces of meromorphic functions on the Riemann surfaces of idelic and S-idelic quasi-characters.

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3.1. Riemann surfaces of continuous quasi-characters

3.1.1. Generalities. For G a Hausdorff locally compact abelian group, we denote by $\Omega(G)$ the set of all continuous homomorphisms $G \to \mathbf{C}^\times$. These are called *quasi-characters*. We make $\Omega(G)$ into a topological space via the compact-open topology.

If G is modulated, meaning (as in Definition D.2.1.1) that it is equipped with a continuous proper non-trivial homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$ (often called a "module") then the kernel G^1 of the module is compact. There are only two cases:

- $|\cdot|_G$ is surjective, in which case we say that G is $\mathbf{R}_{>0}$-*modulated*.

- $\Gamma := |G|_G \subset \mathbf{R}_{>0}$ is a non-trivial discrete subgroup, hence of the form $\gamma^{\mathbf{Z}}$ for a unique $\gamma \in \mathbf{R}_{>0}$ with $\gamma > 1$, in which case we say that G is *discretely modulated*.

The topological space $\Omega(G)$ is equipped with the structure of a Riemann surface (with possibly uncountably many connected components). Indeed, the natural continuous map

$$\pi : \Omega(G) \to \Omega(G^1) = \widehat{G^1} \tag{3.1.1.1}$$

defined by $c \mapsto c|_{G^1}$ is surjective (characters lift) with discrete target has its open and closed fibres given by:

- ($\mathbf{R}_{>0}$-modulated case)
$$\{\chi|\cdot|_G^s : s \in \mathbf{C}\}$$
for $\chi \in \widehat{G}$. Here we use that $G/G^1 \simeq \mathbf{R}_{>0}$ via $|\cdot|_G$, and it is a torsor under $\Omega(\mathbf{R}_{>0}) = \mathbf{C}$ (identification via $t \mapsto t^s$).
- (discretely modulated case)
$$\left\{\chi|\cdot|_G^s : s \in \mathbf{C}/\frac{2\pi i\mathbf{Z}}{\log(\gamma)}\right\}$$
where we use the isomorphism $G/G^1 \simeq \gamma^{\mathbf{Z}}$ via $|\cdot|_G$ (with $\gamma > 1$), and it is a torsor under $\Omega(\gamma^{\mathbf{Z}}) = \mathbf{C}/\frac{2\pi i\mathbf{Z}}{\log(\gamma)}$ (identification via $\gamma \mapsto \gamma^s$).

We denote by
$$\Omega(G)(c)$$
the connected component of $c \in \Omega(G)$, and if $\omega \in \widehat{G^1}$ is a continuous character then we write $\Omega(G)(\omega)$ for the connected component $\pi^{-1}(\omega)$ (i.e. the connected component of any $c \in \Omega(G)$ such that $c|_{G^1} = \omega$). These connected components do not have a natural unitary base-point, but they have a well-defined continuous surjective "abscissa" function $\sigma : \Omega(G) \to \mathbf{R}$ defined by

$$\sigma(\chi|\cdot|_G^s) := \mathrm{Re}(s) \tag{3.1.1.2}$$

(which is independent of the choice of unitary base-point χ). We refer the reader to [**CF**, Ch.XV,§4.4,p.339] and [**RV**, §7.3,p.271] for a discussion on the connected components of $\Omega(G)$ in the special case of $G = C_F$ (with module homomorphism given by the idelic norm).

Definition 3.1.1.1 (Abscissa of a quasi-character). Let G be a modulated group. For any $c \in \Omega(G)$, we define the *abscissa*, or *exponent* of c to be $\sigma(c)$ defined in (3.1.1.2).

We introduce notation for certain loci in the Riemann surfaces $\Omega(G)$.

Definition 3.1.1.2 (Relative vertical strips). Let G be a modulated group.

(1) For an interval $I \subset \mathbf{R}$ let $\Omega(G)_I$ be the subset of $\Omega(G)$ defined as
$$\Omega(G)_I := \{c \in \Omega(G) : \sigma(c) \in I\} \tag{3.1.1.3}$$
endowed with the subspace topology.
(2) For $I = (a, +\infty)$ we write $\Omega(G)_{>a}$ to denote the open subset $\Omega(G)_I \subset \Omega(G)$.
(3) For any $\omega \in \widehat{G^1}$, we define $\Omega(G)(\omega)_I = \Omega(G)_I \cap \Omega(G)(\omega)$.

For any $c \in \Omega(G)$, we denote by $\widehat{c}$ the quasi-character
$$\widehat{c} = |\cdot|_G\, c^{-1}. \tag{3.1.1.4}$$
The map $c \mapsto \widehat{c}$ is an analytic automorphism of $\Omega(G)$ exchanging the relative strips $\Omega(G)_I$ and $\Omega(G)_{1-I}$ (and so preserving $\Omega(G)_{[0,1]}$). We note that $c \mapsto \widehat{c}$ exchanges the connected components $\Omega(G)(\omega)$ and $\Omega(G)(\overline{\omega})$ for $\omega \in \widehat{G^1}$.

3.1.2. Examples: Riemann surfaces of idelic quasi-characters. Let F be a number field and S be a finite set of places of F containing all the Archimedean places.

Recall that the ring of S-integers $\mathcal{O}_{F,S}$ has unit group $\mathcal{O}_{F,S}^\times$ finitely generated of rank $|S|-1$ (see Theorem A.1.5.2) and the Hausdorff locally compact quotient

$$J_{F,S} := \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times \tag{3.1.2.1}$$

contains $J_{F,S}^1 := \ker(|\cdot|_{F,S})$ (the kernel of the S-idelic norm $|\cdot|_{F,S} := \prod_{v\in S}|\cdot|_v$) as a closed compact (see Theorem A.1.5.2) abelian subgroup. As recalled at the beginning of §3.1.2, $(J_{F,S}, |\cdot|_{F,S})$ is $\mathbf{R}_{>0}$-modulated.

We will be specializing to the cases $G = C_F, J_{F,S}$ and $C_{F,S}$, all equipped with compatible modules given by the idelic norm for C_F and $C_{F,S}$, and the S-idelic norm for $J_{F,S}$ (descended from the idelic norm on $C_{F,S}$). Another important case is that of the modulated groups $F_v^\times$ ($\mathbf{R}_{>0}$-modulated for v Archimedean, discretely modulated for v non-Archimedean), we call their quasi-characters *local*.

For $c \in \Omega(C_F)$, i.e. for c a quasi-character of the idele class group C_F, we denote by c_v the restriction of c to $F_v^\times \subset C_F$ where $F_v^\times$ is embedded as a closed subgroup under the insertion into the v-adic "coordinate" $F_v^\times \to \mathrm{GL}_1(\mathbf{A}_F)$ followed by the quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to C_F$. By [**CF**, Ch.XV,Lemma 3.2.1], we have

$$c = \bigotimes_{v\in\Sigma_F} c_v$$

which means for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$:

$$c(g) = \prod_{v\in\Sigma_F} c_v(g_v)$$

where g_v denotes the projection of g onto the v-indexed "coordinate" (a continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to F_v^\times$).

A similar description as a tensor product of local quasi-characters holds for the quasi-characters of $J_{F,S}$.

We refer the reader to [**CF**, Ch.XV,§2.3] and [**RV**, Ch.5,§5.1,p.182], for a discussion on local, idelic and S-idelic quasi-characters. Moreover we refer the reader to [**CF**, Ch.XV,§4.5,p.342] for a concrete description of quasi-characters of C_F in terms of ideal characters, and especially to [**CF**, Ch.XV,§4.5,p.342] on how to construct examples by a method (explained therein) resting on Dirichlet's S-Unit Theorem, to ensure the existence of solutions to certain systems of linear conditions on the complex parameters of a generic collection of local quasi-characters with the aim of arranging their tensor product (a continuous homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{C}^\times$) to be trivial on $F^\times$.

Throughout the rest of this section we will be making examples: we will describe the Riemann surfaces of local and S-local quasi-characters and recipes to relate them to $\Omega(C_F)$ for later use.

3.1.2.1. *Local quasi-characters.* We begin with the case of a finite place v of F. Let F_v be the resulting non-Archimedean local field. The normalized absolute value $|\cdot|_v$ on $F_v^\times$ promotes $G = F_v^\times$ to a discretely modulated group whose discrete value group is $q_v^{\mathbf{Z}} \subset \mathbf{R}_{>0}$, for $q_v := |\pi_v|_v^{-1}$ for any choice of uniformizer π_v at v. The kernel (which we denoted by G^1 for general modulated groups) of $|\cdot|_v$ is the group of units $\mathcal{O}_{F_v}^\times$ of the complete discrete valuation ring $\mathcal{O}_{F_v}$. This is a compact open subgroup of $F_v^\times$.

If v is Archimedean, then $F_v^{\times,1}$ is μ_2 if v is real, and $\mathrm{U}(1)$ if v is complex. In both cases, Γ_v is isomorphic under $|\cdot|_v$ to $\mathbf{R}_{>0}$ as a Hausdorff locally compact abelian group.

In a manner similar to what was done for $\Omega(G)$ in §3.1.1, any choice of continuous homomorphic section to $|\cdot|_v$ and any choice of continuous quasi-character of $q_v^{\mathbf{Z}}$ induces a bi-holomorphism

$$\Omega(F_v^\times) \simeq \mathbf{C}/\frac{2\pi i\,\mathbf{Z}}{\log(q_v)} \times \widehat{\mathcal{O}_{F_v}^\times}. \tag{3.1.2.2}$$

For v Archimedean, the Archimedean absolute value $|\cdot|_v$ (which, recall, for v complex is the *square* of the complex Euclidean norm) promotes $F_v^\times$ to an $\mathbf{R}_{>0}$-modulated group. The kernel of $|\cdot|_v$ is either a copy of μ_2 (so a finite group of order 2) or U(1), the circle group. We therefore have a bi-holomorphism

$$\Omega(F_v^\times) \simeq \mathbf{C} \times \widehat{\mathcal{O}_{F_v}^\times} \tag{3.1.2.3}$$

induced by a choice of continuous homomorphic section to $|\cdot|_v$, to realize $F_v^\times$ as a product $\mathbf{R}_{>0} \times G^1$, with $G^1 = \mu_2$ if v is real and $G^1 = \mathrm{U}(1)$ if v is complex.

We refer the reader to the discussion on the global holomorphic parametrization of each $\Omega(F_v^\times)$ from [**CF**, Ch.XV,Thm.2.3.1].

3.1.2.2. *Global-local holomorphic submersions.* The injective continuous insertion homomorphisms $j_v : F_v^\times \to C_F$ for each place v of F induce continuous maps

$$\psi_v : \Omega(C_F) \to \Omega(F_v^\times) \tag{3.1.2.4}$$

defined by composition, i.e. $\psi_v(c) := c_v = c \circ j_v$.

We will collect the following basic facts into a Lemma for future reference:

LEMMA 3.1.2.1. *Let F be a number field. For every $v \in \Sigma_F$ the continuous map ψ_v is a local analytic isomorphism.*

3.1.2.3. *Open convering of $\Omega(C_F)$ by analytic covers of the S-local Riemann surfaces.* We now fix a finite set S of places of F containing all of the Archimedean places. As for C_F, the quasi-characters of $J_{F,S}$ are a Riemann surface whose (countably many, since $J_{F,S}^1$ is second-countable) connected components are copies of the complex plane. Fixing a continuous homomorphic section to the S-idelic norm gives a bi-holomorphism

$$\Omega(J_{F,S}) \simeq \mathbf{C} \times \widehat{J_{F,S}^1} \tag{3.1.2.5}$$

We now relate $\Omega(C_F)$ and $\Omega(J_{F,S})$. We define:

$$\Omega_S(C_F) \subset \Omega(C_F) \tag{3.1.2.6}$$

to be the subset of those idelic quasi-characters c such that $c_v|_{\mathcal{O}_{F_v}^\times} = 1$ for every $v \notin S$. We refer to $\Omega_S(C_F)$ as the subset of the idelic quasi-characters that are *unramified* outside S. Clearly $\Omega_S(C_F)$ is open in $\Omega(C_F)$ since the defining condition of $\Omega_S(C_F)$ only concerns the restriction of the idelic quasi-character c to C_F^1, i.e. the image of the point $c \in \Omega(C_F)$ in the discrete parameter space $\widehat{C_F^1}$ over which $\Omega(C_F)$ is fibred, and hence $\Omega_S(C_F)$ is a disjoint union of fibres. It therefore inherits a unique Riemann surface structure promoting the open immersion to a bi-holomorphism onto its image.

The closed continuous homomorphism

$$j_{F,S} : \mathrm{GL}_1(\mathbf{A}_{F,S}) \to \mathrm{GL}_1(\mathbf{A}_F) \tag{3.1.2.7}$$

defined as

$$j_{F,S}(g_S) := (g_v)_{v \in \Sigma_F} \tag{3.1.2.8}$$

where $g_v := (g_S)_v$ for $v \in S$, and $g_v = 1$ otherwise, induces continuous maps:

$$\psi_{F,S} : \Omega_S(C_F) \to \Omega(J_{F,S}). \tag{3.1.2.9}$$

We record the following basic facts about $\psi_{F,S}$ in a lemma for future reference:

LEMMA 3.1.2.2. *Let F be a number field S a finite set of places containing all the Archimedean places. The continuous map $\psi_{F,S}$ of* (3.1.2.9) *is a surjective local analytic isomorphism.*

We note that $\{\Omega_S(C_F)\}_{S\subset\Sigma_F}$ is an open cover of $\Omega(C_F)$ as S varies over all the finite sets of places of F containing all the Archimedean places.

3.2. Global Idelic Zeta Functions

The correspondence between θ-functions and Zeta functions goes back to Riemann's paper [**Riemann**] on the distribution of prime numbers, where he established the functional equation and existence of the analytic continuation for the Riemann Zeta function by exhibiting it as the Mellin transform of the Jacobi θ-function.

Tate's dissertation from [**CF**, Ch.XV] mirrored Riemann's θ-function approach to the study of the analytic properties of abelian L-functions. Therein, however, the focus is mostly on the global Zeta integrals arising as the Fourier-Mellin transforms of Schwartz-Bruhat functions on the adeles.

Chapter 2 of this work initiated the systematic study of their θ-functions. This will be continued throughout the next three Chapters, by studying the linear spaces comprising of those θ-functions, both global and S-local, as topological vector spaces.

The Mellin transform will produce idelic Z-functions from adelic θ-functions, and these too are going to be studied by packaging them into a topological vector space, with the Mellin transform promoted to a linear topological isomorphism.

While the previous Chapter was tasked with defining and studying the first properties of the adelic θ-functions, short of collecting them into a topological vector space, this Chapter is tasked with doing the same for the resulting idelic Z-functions, to be defined and studied herein.

We will be referring to the spaces of adelic θ-functions as the "backward side" of the Mellin transform, and to the spaces of Idelic Z-fuctions as the "forward side" of the Mellin transform.

To paraphrase further, while the previous Chapter was tasked with introducing the main elements comprising the backward side of the Mellin transform, this Chapter is tasked with doing the same for the forward side of the Mellin transform, so that in the subsequent Chapters we can study both sides systematically through the lens of functional and harmonic analysis.

3.2.1. Recollections on Hecke L-functions. Let F be a number field.

We consider the family of idelic quasi-characters

$$c := \chi|\cdot|_F^s$$

with $\chi \in \widehat{C_F}$ a fixed continuous character, and $s \in \mathbf{C}$. Every connected component of $\Omega(C_F)$ has a unitary point, i.e. whose abscissa is $\sigma = 0$. For any place v of F, we thus have

$$c_v = \chi_v|\cdot|_v^s$$

as continuous quasi-characters of $F_v^\times$ (where c_v is the restriction of c to $F_v^\times$).

We denote by S_χ the finite set of finite places of F at which χ is ramified. For any place $v \in \Sigma_F - (S_\chi \cup S_\infty)$, v is non-Archimedean and hence we may choose a uniformizer $\pi_v \in \mathcal{O}_{F_v}$ and evaluate χ_v at π_v. The value $\chi_v(\pi_v)$ is independent of the choice of π_v. We denote it $\chi(\mathfrak{p}_v)$ where $\mathfrak{p}_v \subset \mathcal{O}_F$ is the unique prime ideal corresponding to the finite place v.

Since $|\cdot|_v$ is normalized, there exists a unique $s \in \mathbf{C}/(2\pi i\mathbf{Z}/\log(q_v))$ such that $c_v(\pi_v) = \chi_v(\mathfrak{p}_v)q_v^{-s}$, and so we see that we have

$$1 - \chi(\mathfrak{p}_v)q_v^{-s} = 1 - c_v(\pi_v) \tag{3.2.1.1}$$

as holomorphic functions on the Riemann surface $\Omega(F_v^\times)(1)$, the connected component over $1 \in \widehat{\mathcal{O}_{F_v}^\times}$.

We recall the the following classical fact:

LEMMA 3.2.1.1. *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. The following product is absolutely convergent over $\Omega(C_F)(\chi|_{C_F^1})_{>1}$, uniformly on compact subsets:*

$$\Gamma_F(s,\chi) \prod_{v \in \Sigma_F - (S_\chi \cup S_\infty)} (1 - c_v(\pi_v))^{-1} \tag{3.2.1.2}$$

where:

$$\Gamma_F(s,\chi) := \prod_{v \in S_\infty} L(\chi_v |\cdot|_v^s) \tag{3.2.1.3}$$

with, for v real:

$$L(|\cdot|_v^s) := \pi^{-s/2}\Gamma(s/2),\ L(\mathrm{sgn}|\cdot|_v^s) = \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right) \tag{3.2.1.4}$$

and, for v complex and $\chi_v|\cdot|_v^s = \chi_{n,s}$, for $n \in \mathbf{Z}$, the continuous quasi-character character of $\mathbf{C}^\times$ defined as $(r,\theta) \mapsto r^s e^{in\theta}$

$$L(\chi_v|\cdot|_v^s) := (2\pi)^{-\left(s+\frac{|n|}{2}\right)}\Gamma\left(s + \frac{|n|}{2}\right). \tag{3.2.1.5}$$

The proof is by comparison with the convergence of the Euler product for the Riemann ζ-function over the half-plane of absolute convergence. We refer the reader to [**RV**, Ch.7,Lemma 7.18].

DEFINITION 3.2.1.2. Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. We define the *Hecke L-function* of character χ to be the function $L(s,\chi)$ over $\Omega(C_F)(\chi)_{>1}$ defined by the absolutely and locally uniformly convergent product in (3.2.1.2).

We recall here below the key properties of Hecke L-functions.

PROPOSITION 3.2.1.3 (Regularity and symmetries of Hecke L-functions). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. The function $L(s,\chi)$ defined on $\mathbf{C}_{>1}$ in (3.2.1.2) admits a meromorphic continuation to the complex s-plane, satisfying the following conditions:*

1. *If $\chi|_{C_F^1}$ is non-trivial, $L(s,\chi)$ is entire.*
2. *If $\chi|_{C_F^1}$ is trivial, $L(s,\chi)$ is meromorphic with simple poles at $s = it_0, 1+it_0$, for a unique real number $t_0 \in \mathbf{R}$ such that $\chi = |\cdot|_F^{-it_0}$, whose residues are, respectively, $-\overline{\mu}_1(C_F^1)$ and $|N(\mathfrak{d}_{F/\mathbf{Q}})|^{-1/2}\overline{\mu}_1(C_F^1)$.*
3. *As meromorphic functions on the complex s-plane, $L(s,\chi)$ and $L(s,\overline{\chi})$ are related by the following functional equation:*

$$L(1-s,\overline{\chi}) = \epsilon(s,\chi)L(s,\chi) \tag{3.2.1.6}$$

   where $\epsilon(s,\chi)$ is a nowhere vanishing entire function.
4. *The only zeros of $L(s,\chi)$ over $\mathbf{C}_{>0}$ are contained in the open vertical strip $\mathbf{C}_{(0,1)}$.*

The Proposition is proved upon combining [**RV**, Thm.7.2(iii)] (throughout whose proof the ϵ-factor is seen to be of the form

$$\epsilon(s,\chi) = \prod_{v \in \Sigma_F} \epsilon(\chi_v|\cdot|_v^s)$$

with $\epsilon(\chi_v|\cdot|_v^s)$ computed explicitly in [**RV**, Ch.7,§7.1,p.250,251] for v real, [**RV**, Ch.7,§7.1p.253] for v complex, [**RV**, Ch.7,§7.1,p.258] for v non-Archimedean, with the infinite product having all but finitely many factors equal to 1) with [**RV**, Thm.7.19]. Also, (4) uses the classical zero-free region for $L(s,\chi)$ (see Proposition B.6.1.1) or weaker versions that focus on the right edge of the critical strip.

DEFINITION 3.2.1.4. Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. We define the *finite Hecke L-function* of character χ to be the function over $\Omega(C_F)(\chi|_{C_F^1})_{>1}$ defined by the absolutely and locally

uniformly convergent product

$$L_f(s,\chi) := \prod_{v\in\Sigma_{F,f}-S_\chi} \frac{1}{1-\chi(\mathfrak{p}_v)(N\mathfrak{p}_v)^{-s}}. \tag{3.2.1.7}$$

By Proposition 3.2.1.3, the finite Hecke L-function $L_f(s,\chi)$ extends to a meromorphic function on the complex s-plane, and satisfies the functional equation

$$L_f(s,\chi) = W(s,\chi)L_f(1-s,\overline{\chi}). \tag{3.2.1.8}$$

where

$$W(s,\chi) := \frac{1}{\epsilon(s,\chi)}\frac{\Gamma_F(1-s,\overline{\chi})}{\Gamma_F(s,\chi)} \tag{3.2.1.9}$$

with $\epsilon(s,\chi)$ entire and nowhere-vanishing and $\Gamma_F(1-s,\overline{\chi})$ and $\Gamma_F(s,\chi)$ meromorphic.

By Stirling's Approximation, we have the estimate, for every $0\leq\alpha\leq\beta$ and $0\leq\alpha\leq\sigma\leq\beta$, in light of (3.2.1.4) and and (3.2.1.5):

$$|W(s,\chi)| \ll |t|^{[F:\mathbf{Q}](\frac{1}{2}-\sigma)} \tag{3.2.1.10}$$

as $s=\sigma+it$ satisfies $|t|\to+\infty$, where the implied constant depends on α,β. We refer the reader to [**IK**, Ch.5, §A.4, (5.114)] for the estimate from (3.2.1.10), we further refer the reader to §B.1.

On the definition and properties of ϵ-factors, which, for abelian L-functions, we will be reviewing throughout this Chapter, we further refer the reader to Tate's [**Tate**, §3, (3.6), p.17], to Deligne's [**Del**, §4.11], and to Langlands' [**Langlands**], with the caveat of differing normalizations in the latter two references. We further refer the reader to §14.1.1 of this work.

3.2.2. Global Idelic Zeta Functions. For $a,b\in\mathbf{R}_{>0}$ with $a\leq b$, we adopt the standard notation

$$\mathrm{GL}_1(\mathbf{A}_F)^{[a,b]} := \{g\in\mathrm{GL}_1(\mathbf{A}_F) : |g|_F\in[a,b]\}.$$

The following Theorem is partly contained within [**CF**, Ch.XV,Thm.4.4.1].

THEOREM 3.2.2.1 (Idelic Mellin transform of global adelic θ-functions). *We let $f\in\mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function and $c\in\Omega(C_F)$. Then:*

(1) *the integral*

$$Z(f,c) := \int_{\mathrm{GL}_1(\mathbf{A}_F)} f(g)c(g)\,\mathrm{d}\mu(g) \tag{3.2.2.1}$$

converges absolutely over $\Omega(C_F)_{>1}$, uniformly on compact subsets as a function of $c\in\Omega(C_F)_{>1}$ and it is holomorphic over $\Omega(C_F)_{>1}$.

(a) *It extends to a meromorphic function on $\Omega(C_F)$ with at most simple poles at $c=1$ and $c=|\cdot|_F$, with residues $-\overline{\mu}_1(C_F^1)\,f(0)$, respectively $\overline{\mu}_1(C_F^1)\,\widehat{f}(0)$.*

(b) *It satisfies the functional equation*

$$Z(f,c) = Z(\widehat{f},\widehat{c}). \tag{3.2.2.2}$$

(2) *The integral*

$$L(f,c) := \int_{C_F} \Phi_F(f)(g)c(g)\,\mathrm{d}\overline{\mu}(g) \tag{3.2.2.3}$$

is absolutely convergent over $\Omega(C_F)_{>1}$, uniformly on compact subsets as a function of $c\in\Omega(C_F)_{>1}$ (see Remark 3.2.2.2 for the meaning of such uniformity).

(a) *If $f(0)=\widehat{f}(0)=0$, then the integral defining $L(f,c)$ in (3.2.2.3) is absolutely convergent over $\Omega(C_F)$, uniformly on compact subsets. Under these assumptions, $L(f,c)$ is an entire function.*

(b) *Over all the connected components of* $\Omega(C_F)$ *other than* $\Omega(C_F)(1)$*, the integral defining* $L(f,c)$ *in* (3.2.2.3) *is, with no further assumptions on* f*, absolutely convergent, uniformly on compact subsets, so* $L(f,c)$ *is entire on all the connected components of* $\Omega(C_F)$ *other than* $\Omega(C_F)(1)$ *and it is meromorphic with at most simple poles at* 1 *and* $|\cdot|_F$*, over* $\Omega(C_F)(1)$*.*

(3) *We fix a continuous character* $\omega \in \widehat{C_F^1}$*, equivalently a connected component* $\Omega(C_F)(\omega)$ *of* $\Omega(C_F)$*. Let* $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ *be a continuous homomorphic section to the idelic norm. The integral*

$$\int_{\mathbf{R}_{>0}} \Phi_{F,\omega}(f)(\tau(t))c(\tau(t))\,\frac{\mathrm{d}t}{t} \tag{3.2.2.4}$$

is absolutely convergent over $\Omega(C_F)(\omega)_{>1}$*, uniformly on compact subsets as a function of* $c \in \Omega(C_F)_{>1}$*.*

(a) *If* $\omega = 1$ *and* $f(0) = \widehat{f}(0) = 0$*, then the integral in* (3.2.2.4) *is absolutely convergent over* $\Omega(C_F)(1)$*, uniformly on compact subsets. Under these assumptions, its value is an entire function of* $c \in \Omega(C_F)(1)$*.*

(b) *If* $\omega \neq 1$*, the integral in* (3.2.2.4) *is absolutely convergent on* $\Omega(C_F)(\omega)$*, uniformly on compact subsets, and its value is an entire function of* $c \in \Omega(C_F)(\omega)$*.*

For any $\omega \in \widehat{C_F^1}$*, the following identity holds:*

$$L(f,c) = \int_{\mathbf{R}_{>0}} \Phi_{F,\omega}(f)(\tau(t))c(\tau(t))\,\frac{\mathrm{d}t}{t} \tag{3.2.2.5}$$

as meromorphic functions of $c \in \Omega(C_F)(\omega)$*, with both sides being, in turn, equal to* $Z(f,c)$ *from* (3.2.2.1) *over* $\Omega(C_F)(\omega)$*. In particular, under these assumptions on* f*,* $Z(f,c)$ *is entire over all the connected components of* $\Omega(C_F)$ *parametrized by non-trivial characters of* C_F^1*, meromorphic over* $\Omega(C_F)(1)$ *with at most simple poles at* $1, |\cdot|_F$*, and, under the additional assumption* $f(0) = \widehat{f}(0) = 0$*, it is holomorphic everywhere over* $\Omega(C_F)$*.*

Remark 3.2.2.2. Uniform convergence of the integral in (3.2.2.1) on compacts as a function of c means that the sequence of truncated integrals

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)^{[1/n,n]}} f(g)c(g)\,\mathrm{d}\mu(g)$$

converges uniformly as c varies in any compact subset of $\Omega(C_F)_{>1}$. The same holds with $[1/n,n]$ replaced by any growing sequence of intervals $\{[1/a_n,a_n]\}$ with real numbers $\{a_n\}_{n\geq 0} \subset [1,+\infty)$ such that $\lim_{n\to+\infty} a_n = +\infty$. These truncated integrals are holomorphic in c, by compactness of the domain of integration, so local uniformity of convergence ensures the limiting integral is holomorphic in c.

We briefly digress to discuss continuity of the formation of $Z(f,c)$ for each fixed $c \in \Omega(C_F)_{>1}$, as f varies over $\mathcal{S}(\mathbf{A}_F)$.

Lemma 3.2.2.3 ($Z(\cdot,c)$ is valued in tempered distributions for each $c \in \Omega(C_F)_{>1}$). *Let* F *be a number field. Then for each fixed* $c \in \Omega(C_F)_{>1}$*, formation of* $Z(f,c)$ *is continuous in* f *with respect to the Hausdorff locally convex topology on* $\mathcal{S}(\mathbf{A}_F)$*.*

Proof. We prove that for any Cauchy sequence $\{f_k\}_{k\geq 0}$ with limit f, $Z(f_k,c) \to Z(f,c)$, i.e. $Z(\cdot,c)$ is sequentially continuous as a linear form on $\mathcal{S}(\mathbf{A}_F)$. Although $\mathcal{S}(\mathbf{A}_F)$ is not second-countable, sequential continuity on it still implies continuity because such an implication is a property of continuous linear functionals on bornlogical locally convex spaces [**Scha**, Ch.II, 8.3] and $\mathcal{S}(\mathbf{A}_F)$ is an LF-space (by Lemma 2.1.3.2(1)) so it is also bornological (see [**Scha**, Ch.II, Cor. 2 to 8.2]).

Since $\mathcal{B} := \{f_k\}_{k\geq 0} \cup \{f\}$ is a Cauchy sequence with its limit point, it is a bounded subset in $\mathcal{S}(\mathbf{A}_F)$. By Theorem 2.2.3.4(5), for any integer $N > 1$ there exist real numbers $C_{\mathcal{B},N}, C'_{\mathcal{B},N} > 0$ such that for all $u \in \mathcal{B}$.

$$|\Phi_F(u)(g) - 1_{C_F^{(0,1]}}(g)\cdot|g|_F^{-1}\widehat{u}(0) + 1_{C_F^{(0,1]}}(g)\cdot u(0)| \leq C_{\mathcal{B},N}|g|_F^{-N}\cdot 1_{C_F^{[1,+\infty)}} + C'_{\mathcal{B},N}|g|_F^{N}\cdot 1_{C_F^{(0,1]}}(g)$$

as functions on C_F. By Dominated Convergence, we have $Z(f_k, c) \to Z(f, c)$ as $k \to +\infty$, noting that $u \mapsto u(0)$ and $u \mapsto \widehat{u}(0)$ are also continuous functionals (they are δ_0 and $\widehat{\delta}_0$). □

PROOF OF THEOREM 3.2.2.1. (1) : By Lemma 2.2.2.1 and continuity of Schwartz-Bruhat functions on $\mathbf{A}_F$, along with preservation of $\mathcal{S}(\mathbf{A}_F)$ under the adelic Fourier transform and Fourier inversion on $\mathcal{S}(\mathbf{A}_F)$ (both consequences of [**DE**, Thm.3.5.8] and Dominated Convergence), the complex vector space of Schwartz-Bruhat functions on $\mathbf{A}_F$ is a linear subspace of the complex vector space $\mathfrak{z}$ from [**CF**, Ch.XV,§4.4,p.338] comprised of complex-valued continuous functions on $\mathbf{A}_F$ for which the main global results in Tate's dissertation all hold (see [**CF**, Ch.XV,§4.4]).

To check that Tate's integrability conditions in $\mathfrak{z}_3$ from [**CF**, Ch.XV,§4.4,p.338] (i.e. that for any $\sigma > 1$, $f \cdot |\cdot|_F^\sigma$ and $\widehat{f} \cdot |\cdot|_F^\sigma$ are absolutely μ-integrable over $\mathrm{GL}_1(\mathbf{A}_F)$), we proceed as follows. By Lemma 2.1.3.2(3), it is enough to show this for $f \in \mathcal{S}(\mathbf{A}_F)$ of the special form

$$f = f_\infty \otimes \bigotimes_{v \in \Sigma_{F,f}} 1_{K_v}$$

for $K_v \subset F_v$ a compact open subset for every finite place v of F, with $K_v = \mathcal{O}_{F_v}$ for all but finitely many v. By [**RV**, Prop.5.6(ii), p.186] we have

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)} |f(g)||g|_F^\sigma \, \mathrm{d}\mu(g) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})} |f_\infty(g_\infty)||g_\infty|_{F,S_\infty}^\sigma \, \mathrm{d}\mu_{S_\infty}(g_\infty) \prod_{v \in \Sigma_{F,f}} \int_{K_v} |g_v|_v^\sigma \, \mathrm{d}\mu_v(g_v) \tag{3.2.2.6}$$

where the product on the right side is allowed to diverge. By Proposition 2.4.3.1, the integral over $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ converges since f_∞ is Schwartz-Bruhat over $\mathbf{A}_{F,S_\infty}$. By absolute convergence of the Euler product of the Riemann zeta function, the product of the integrals $\int_{\mathcal{O}_{F_v}} |g_v|_v^\sigma \, \mathrm{d}\mu_v(g_v)$ for all finite places v of F such that $K_v = \mathcal{O}_{F_v}$ converges and so the integral in (3.2.2.6) converges too. The same reasoning applies to $\widehat{f}$ in place of f.

In particular, the integral in (3.2.2.1) defining $Z(f, c)$ converges absolutely for $c \in \Omega(C_F)_{>1}$, and uniformly on the compact subsets of $\Omega(C_F)_{>1}$, therefore defining a holomorphic function of c there. By [**CF**, Ch.XV,Thm.4.4.1] (see also [**RV**, Thm.7.16]), the function $Z(f, c)$ of $c \in \Omega(C_F)_{>1}$ extends to a meromorphic function on the Riemann surface $\Omega(C_F)$ with at most simple poles at $c = 1$ and $c = |\cdot|_F$, whose residues are, respectively, $-\overline{\mu}_1(C_F^1)\, f(0)$ and $\overline{\mu}_1(C_F^1)\, \widehat{f}(0)$.

The meromorphic function $Z(f, c)$ of $c \in \Omega(C_F)$ further satisfies the functional equation in (3.2.2.2).

(2) : The first convergence assertion (without assuming $f(0) = \widehat{f}(0) = 0$) follows from the first two parts of Theorem 2.2.3.4 noting $|\cdot|_F^{\sigma-1}$ has, for $\sigma > 1$, an $\overline{\mu}$-integrable singularity as $|g|_F \to 0$.

By Theorem 2.2.3.4(3), since $|c| = |\cdot|_F^{\sigma(c)}$, upon choosing an integer $N > 0$, there exists a constant $C_N > 0$ depending on N and f but not on c, such that

$$|c(g)\Phi_F(f)(g)| \leq C_N \, e^{-N\,|\log|g|_F| + \sigma(c)\log|g|_F} \tag{3.2.2.7}$$

and since $\sigma(c)\log|g|_F \leq |\sigma(c)|\,|\log|g|_F|$, we have:

$$|c(g)\Phi_F(f)(g)| \leq C_N \, e^{-N\,|\log|g|_F| + \sigma(c)\log|g|_F} \leq C_N \, e^{(|\sigma(c)| - N)\,|\log|g|_F|} \tag{3.2.2.8}$$

being the exponential function $\exp : \mathbf{R} \to \mathbf{R}_{>0}$ strictly increasing.

We choose a bounded interval $I = [a, b] \subset \mathbf{R}$ and an integer $N > \max\{|a|, |b|\}$, and consider the resulting constant $C_N > 0$ from Theorem 2.2.3.4(3).

Upon calling $\overline{\mu}'$ the unique Haar measure on

$$C_F/C_F^1 \overset{|\cdot|_F}{\simeq} \mathbf{R}_{>0}$$

prescribed C_F/C_F^1 by Theorem D.1.2.1 by the choices of the Haar measure $\overline{\mu}$ on C_F and the Haar measure $\overline{\mu}_1$ induced by μ_1 on $\mathrm{GL}_1(\mathbf{A}_F)^1$ constructed in Definition A.3.2.1 and the counting measure on $F^\times$, we have $\overline{\mu}' = \mathrm{d}t/t$.

We have, by (3.2.2.7), the bound

(3.2.2.9)
$$\int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} |c(\tau(t))\Phi_F(f)(\tau(t)g_0)| \, \mathrm{d}\overline{\mu}_1(g_0) \right) \frac{\mathrm{d}t}{t} \leq C_N \overline{\mu}_1(C_F^1) \int_{\mathbf{R}_{>0}} e^{(|\sigma(c)|-N)\,|\log(t)|} \frac{\mathrm{d}t}{t} = 2\, C_N \overline{\mu}_1(C_F^1) \int_0^{+\infty} e^{(|\sigma(c)|-N)x} \, \mathrm{d}x.$$

Since $N > |\sigma(c)|$ for every $c \in \Omega(C_F)_{[a,b]}$, the leftmost integral in (3.2.2.9) converges for such c since the rightmost integral does.

The integral in the leftmost side of (3.2.2.9) reads

$$\int_{C_F/C_F^1} \left(\int_{C_F^1} |c(\tau(t))\Phi_F(f)(\tau(t)g_0)| \, \mathrm{d}\overline{\mu}_1(g_0) \right) \mathrm{d}\overline{\mu}'(g) \tag{3.2.2.10}$$

which therefore converges.

By Lemma D.1.2.3 (see also [**DE**, Cor.1.5.4]), we conclude that the integral in (3.2.2.3) converges absolutely and uniformly over $\Omega(C_F)_{[a,b]}$. The point is that the integrals over $1/n \leq |g|_F \leq n$ in C_F/C_F^1 converge uniformly with respect to $c \in \Omega(C_F)_{[a,b]}$ as $n \to +\infty$ (by an analog of the inequality in (3.2.2.9) for such integrals and observing that for $b > 0$, $\int_M^{+\infty} e^{-bx} \, \mathrm{d}x \to 0$ as $M \to +\infty$).

Since the interval $[a, b]$ is an arbitrary compact interval in the real line, we deduce that the integral in (3.2.2.3) converges absolutely and uniformly on the compact subsets of $\Omega(C_F)$ (in fact, on the closed vertical strips of $\Omega(C_F)$ of bounded width and non-empty interior), and the function $L(f, c)$ of $c \in \Omega(C_F)$ it defines is holomorphic over $\Omega(C_F)$. By Theorem D.1.2.1 (see also [**DE**, Thm.1.5.3]), which applies by Lemma D.1.2.3 (see also [**DE**, Cor.1.5.4]), the following identity holds pointwise on $\Omega(C_F)$:

$$\int_{\mathbf{R}_{>0}} \Phi_{F,\omega}(f)(\tau(t))c(\tau(t)) \frac{\mathrm{d}t}{t} = \int_{C_F} \Phi_F(g)\, c(g) \, \mathrm{d}\overline{\mu}(g)$$

which gives (3.2.2.5).

Since the integral in (3.2.2.1) defining $Z(f, c)$ is absolutely convergent over $\Omega(C_F)_{>1}$, then by Theorem D.1.2.1 again we have the indentity, pointwise over $\Omega(C_F)_{>1}$:

$$\int_{C_F} \Phi_F(f)(g)c(g) \, \mathrm{d}\overline{\mu}(g) = \int_{C_F} \sum_{\xi \in F^\times} f(\xi g)c(g) \, \mathrm{d}\overline{\mu}(g) = \int_{\mathrm{GL}_1(\mathbf{A}_F)} f(g)c(g) \, \mathrm{d}\mu(g). \tag{3.2.2.11}$$

This gives $L(f, c) = Z(f, c)$ over $\Omega(C_F)_{>1}$, and hence, by the Identity Theorem, $L(f, c) = Z(f, c)$ holds everywhere over $\Omega(C_F)$, which establishes all the remaining assertions. □

We explain Theorem 3.2.2.1(2) on an example that connects to the classical theory of Dirichlet series:

Example 3.2.2.4. Let $F = \mathbf{Q}$ and $f := f_\infty \otimes \bigotimes_p f_p$, with $f_2 := \widehat{1_{\mathbf{Z}_2^\times}}$ and $f_p = 1_{\mathbf{Z}_p}$ for every prime $p \neq 2$, where f_∞ is the Gaussian function on $\mathbf{R}$, i.e. $f_\infty(x) := e^{-\pi x^2}$.

We let $c \in \Omega(C_{\mathbf{Q}})(1)$, i.e. the connected component of the trivial quasi-character, be parametrized as $|\cdot|_{\mathbf{Q}}^s$, for $s \in \mathbf{C}$.

For $\mathrm{Re}(s) > 1$, we compute:

$$Z(f, c) = Z(f, |\cdot|_{\mathbf{Q}}^s) = \int_{\mathbf{R}^\times} f_\infty(x)|x|^s \frac{\mathrm{d}x}{|x|} \int_{\mathbf{Q}_2^\times} \widehat{1_{\mathbf{Z}_2^\times}} |x|_2^s \, \mathrm{d}\mu_2(x) \prod_{p>2} \int_{\mathbf{Q}_p^\times} 1_{\mathbf{Z}_p}(x)|x|_p^s \, \mathrm{d}\mu_p(x)$$

to be, up to a positive multiplicative constant, $\pi^{-s/2}\Gamma(s/2)(1 - 2^{1-s}) \prod_p \frac{1}{1-p^{-s}} = (1 - 2^{1-s})\pi^{-s/2}\Gamma(s/2)\zeta(s)$, since, recall, $\mu_p = \frac{p}{p-1} \frac{\mathrm{d}x}{|x|_p}$, the multiplicative local Tamagawa measure from Definition A.3.2.2.

The reader will recognize a phenomenon of resolution of the singularity at $s = 0$ for $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ similar to that for the Dirichlet η-function, which, recall, is the "alternating zeta function" defined as

$$\eta(s) := \sum_{m\geq 1} \frac{(-1)^{m-1}}{m^s}$$

which, over the half-plane of absolute convergence $\mathrm{Re}(s) > 1$, agrees with $(1 - 2^{s-1})\zeta(s)$. The Dirichlet series defining $\eta(s)$ converges conditionally over the half-plane $\mathbf{C}_{>0}$, and absolutely over the half-plane $\mathbf{C}_{>1}$.

See, for example, [**T1**, Ch.IX,§9.13]; see also [**H**], where a proof of the functional equation for $\xi(s)$ is provided using the Dirichlet η-function.

The local Mellin transform of the Fourier transform of $1_{\mathbf{Z}_2^\times}$ resolves the singularity at $s = 1$ of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$, in light of the calculation from Example A.3.1.6, in a similar manner as the factor $(1 - 2^{s-1})$ does for $\eta(s)$.

We observe that indeed $\widehat{f}(0) = 0$, which leads to $Z(f, |\cdot|_{\mathbf{Q}}^s)$ having no pole at $|\cdot|_{\mathbf{Q}}$.

We may think of the idelic Zeta functions $Z(f, c)$ for $f \in \mathcal{S}(\mathbf{A}_F)$ satisfying $f(0) = \widehat{f}(0) = 0$, as idelic analogs of the Dirichlet η-function, and we may think of Theorem 3.2.2.1(2) as the corresponding analog of the conditional convergence over $\mathbf{C}_{>0}$ of the Dirichlet series defining $\eta(s)$.

We introduce the following terminology:

Definition 3.2.2.5 (Global Idelic Z-functions). Let F be a number field, $f \in \mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function.

We refer to the meromorphic function $Z(f, c)$ of $c \in \Omega(C_F)$ defined in (3.2.2.2) of Theorem 3.2.2.1, as the *Global Idelic Z-function* of f.

We will often write $Z(f, c)$ even though the arguments under discussion will concern the global adelic θ-function $\Phi_F(f)$ only, i.e. will depend on f only through its θ-function.

We will start referring to Z as the *Mellin transform* of f, and of $\Phi_F(f)$, alike.

3.3. S-Local Idelic Zeta Functions

Throughout this subsection, we fix a number field F and a finite set S of places of F containing all of the Archimedean places.

3.3.1. S-local Idelic Z-functions. Recall that the local Fourier transforms on each $\mathcal{S}(F_v)$, normalized using the standard continuous additive character for each v, which we constructed in Example A.3.1.8 and Example A.3.1.9, induce the S-adelic Fourier transform on $\mathcal{S}(\mathbf{A}_{F,S})$ with respect to a standard additive character of the S-adeles $\mathbf{A}_{F,S}$ and the additive Tamagawa measure ν_S (the Haar measure $\bigotimes_{v\in S}\nu_v$ on $\mathbf{A}_{F,S}$ from Definition A.3.4.1, with ν_v the local additive Tamagawa measure on the additive topological group of the local field F_v from Definition A.3.1.2). This is a continuous automorphism with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S})$. We further refer the reader to Remark A.3.4.2 and to Remark A.3.2.5.

If S is large enough so as to contain all the places of F ramified over $\mathbf{Q}$ then ν_S is also characterized as the unique Haar measure such that the counting measure on $\mathcal{O}_{F,S}$ and the additive Tamagawa measure ν_S on $\mathbf{A}_{F,S}$ induce the unique Haar measure on the compact Hausdorff quotient $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ (as per Theorem D.1.2.1) giving it volume 1.

The additive Tamagawa measure ν_S on $\mathbf{A}_{F,S}$ is self-dual with respect to the standard S-local additive character (see Example A.3.1.11). Fourier Inversion therefore holds with no normalizing constants and can be deduced from the local case by [**RV**, Prop.5.6(iii),p.186], where the index set in [**RV**, Prop.5.6,(5.2),p.186] is finite.

THEOREM 3.3.1.1 (S-local Idelic Mellin transform of S-local adelic θ-functions). *We let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function and $c \in \Omega(J_{F,S})$. Then*

(1) *the integral*

$$Z_S(f,c) := \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\,\mathrm{d}\mu_S(g) \tag{3.3.1.1}$$

converges absolutely over $\Omega(J_{F,S})_{>0}$, uniformly on compact subsets as a function of $c \in \Omega(J_{F,S})_{>0}$ and so is holomorphic in c.

(a) *It satisfies the functional equation*

$$Z_S(f,c) = \rho_S(c)Z_S(\widehat{f},\widehat{c}) \tag{3.3.1.2}$$

over $\Omega(J_{F,S})_{(0,1)}$, where ρ_S is the meromorphic function on $\Omega(J_{F,S})$ defined by

$$\rho_S(c) := \prod_{v\in S} \rho_v(c_v) \tag{3.3.1.3}$$

with c_v the restriction of c along the closed continuous homomorphism $F_v^\times \to \mathrm{GL}_1(\mathbf{A}_{F,S})$, and ρ_v the explicit meromorphic function on $\Omega(F_v^\times)$ defined in [**CF**, Ch.XV,§2.5] *for each $v \in S$.*

(b) *It extends to a meromorphic function on $\Omega(J_{F,S})$ whose poles, if any, are poles of ρ_S and have pole order at most that of ρ_S.*

(2) *The integral*

$$L_S(f,c) := \int_{J_{F,S}} \Phi_{F,S}(f)(g)c(g)\,\mathrm{d}\overline{\mu}_S(g) \tag{3.3.1.4}$$

is absolutely convergent over $\Omega(J_{F,S})_{>0}$ uniformly on compact subsets as a function of $c \in \Omega(J_{F,S})_{>0}$. In particular, $\Lambda_S(f,c)$ is a holomorphic function of $c \in \Omega(J_{F,S})_{>0}$.

(3) *Let $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be a continuous homomorphic section to the S-idelic norm. Upon defining $\omega := c|_{J^1_{F,S}} \in \widehat{J^1_{F,S}}$, the integral*

$$\int_{\mathbf{R}_{>0}} \Phi_{F,S,\omega}(f)(\tau(t))c(\tau(t))\,\frac{\mathrm{d}t}{t} \tag{3.3.1.5}$$

is absolutely convergent over $\Omega(J_{F,S})_{>0}$ uniformly on compact subsets as a function of $c \in \Omega(J_{F,S})_{>0}$. In particular, it is holomorphic as a function of $c \in \Omega(J_{F,S})_{>0}$. Moreover, the following identity holds

$$L_S(f,c) = \int_{\mathbf{R}_{>0}} \Phi_{F,S,\omega}(f)(\tau(t))c(\tau(t))\,\frac{\mathrm{d}t}{t} \tag{3.3.1.6}$$

as holomorphic functions of $c \in \Omega(J_{F,S})_{>0}$, and both sides are, in turn, equal to $Z_S(f,c)$ from (3.3.1.1) *over $\Omega(J_{F,S})_{>0}$.*

REMARK 3.3.1.2. As for the global version of Theorem 3.3.1.1 (see Theorem 3.2.2.1), by "uniform convergence on compact subsets" for the various integrals we mean that the sequence of truncated integrals over $\mathrm{GL}_1(\mathbf{A}_{F,S})^{[1/n,n]}$ converges as $n \to +\infty$, uniformly as c varies over each compact subset of $\Omega(J_{F,S})_{>0}$. Here $\mathrm{GL}_1(\mathbf{A}_{F,S})^{[1/n,n]}$ is the closed subset given by those S-ideles whose S-idelic norm is in $[1/n,n]$.

PROOF OF THEOREM 3.3.1.1. By Proposition 2.1.4.1(1), for each $f \in \mathcal{S}(\mathbf{A}_{F,S})$ there exist $f_1,\dots,f_m \in \mathcal{S}(\mathbf{A}_{F,S})$ of the form $f_k = f_{k,\infty} \otimes \bigotimes_{v\in S_f} f_{k,v}$, with $S_f := S \cap \Sigma_{F,f}$, and $f_{k,\infty} \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_{k,v} \in \mathcal{S}(F_v)$ for each $v \in S_f$, with complex numbers $c_1,\dots,c_m$, such that $f = \sum_{1\le k\le m} c_k f_k$.

Since formation of $Z_S(f,c)$, $L_S(f,c)$ and $\Phi_{F,S,\omega}$ is $\mathbf{C}$-linear with respect to f, we may and do rename any of the f_k as f and reduce to the case $f = f_\infty \otimes \bigotimes_{v\in S_f} f_v$, with $S_f := S \cap \Sigma_{F,f}$, and $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_v \in \mathcal{S}(F_v)$ for each $v \in S_f$.

(1) : By [RV, Prop.5.6(ii)], since the index set $I = S$ is finite and we may therefore choose $S_0 = S$ in loc.cit., we have:

$$\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\,\mathrm{d}\mu_S(g) = \prod_{v\in S}\int_{F_v^\times} f_v(g_v)c_v(g_v)\,\mathrm{d}\mu_v(g_v) = \prod_{v\in S} Z_v(f_v,c_v)$$

where $Z_v(f_v,c_v)$ is the local zeta integral of [CF, Ch.XV,Def.2.4.1], and $c_v = \psi_v(c)$, where $\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^\times)$ is the holomorphic submersion defined as the restriction of the arbitrary continuous S-idelic quasi-character to the closed subgroup $F_v^\times \subset J_{F,S}$ (see (3.1.2.4), Lemma 3.1.2.1, and Lemma 3.1.2.2).

Since S is finite, (1) reduces to the case $S = \Sigma_\infty$, since $Z_v(f_v,c_v)$ is holomorphic for $\sigma(c) > 0$ and its defining integral

$$Z_v(f_v,c_v) := \int_{F_v^\times} f_v(g)c_v(g)\,\mathrm{d}\mu_v(g) \tag{3.3.1.7}$$

for μ_v the multiplicative local Tamagawa measure on $F_v^\times$ (see Definition A.3.2.2) converges absolutely and uniformly over $\Omega(F_v^\times)_{>0}$, for each v non-Archimedean.

We are under the assumption that f_∞ itself is of the form $\bigotimes_{v\in S_\infty} f_v$ for $f_v \in \mathcal{S}(F_v)$ for each Archimedean place v of F.

Then, by the same token as for the case when v is a non-Archimedean place, the condition $\sigma(c) > 0$, renders the singularity of $c_v|\cdot|_v^{-1}$, for each Archimedean place v of F, ν_v-integrable, where ν_v is the additive local Tamagawa measure on F_v. Boundedness of every Schwartz function $f_v \in \mathcal{S}(F_v)$ around the origin, combined with the rapid decay of f_v for all $g \in F_v^\times$ as $|g|_v \to +\infty$, ensures that the integral from (3.3.1.7) defining $Z_v(f_v,c_v)$ for v Archimedean, is absolutely and uniformly convergent over $\Omega(F_v^\times)_{>0}$.

The same argument applies directly to the integral defining $Z_{S_\infty}(f_\infty,c)$ for general $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and for any $c \in \Omega(J_{F,S_\infty})_{>0}$: the condition $\sigma(c) > 0$ ensures that the singularity of $|\cdot|_{F,S_\infty}^{-1}c$ is absolutely ν_{S_∞}-integrable, noting that $|\cdot|_{F,S_\infty}^{-1}\nu_{S_\infty}$ equals, as a Radon measure on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, μ_{S_∞} up to a positive multiplicative constant. Boundedness of f_∞ around the origin in the Euclidean space $\mathbf{A}_{F,S_\infty}$ combined with the rapid decay at infinity of f_∞ on $\mathbf{A}_{F,S_\infty}$, ensures the absolute and uniform convergence of the integral in (3.3.1.1) in the special case $S = S_\infty$.

Now we digress to show a continuity property of the S-local zeta functions as we vary f for fixed c. In practice and throughout the proof of Theorem 3.3.1.1 we will only need this for $S = S_\infty$. We deal with the general case now for later use:

LEMMA 3.3.1.3 (Z_S is valued in tempered distributions). *For each fixed $c \in \Omega(J_{F,S})_{>0}$, the value $Z_S(f,c)$ is continuous in $f \in \mathcal{S}(\mathbf{A}_{F,S})$ with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S})$ for fixed each $c \in \Omega(J_{F,S})_{>0}$.*

PROOF. As in the proof of Lemma 3.2.2.3, since $\mathcal{S}(\mathbf{A}_{F,S})$ is an LF-space (see Proposition 2.1.4.1(2)) it suffices to show the integral in (3.3.1.1) defining $Z_S(f,c)$ is sequentially continuous in f for each $c \in \Omega(J_{F,S_\infty})_{>0}$ (even though $\mathcal{S}(\mathbf{A}_{F,S})$ is *not* second-countable).

To see the sequential continuity, let $\{f_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{A}_{F,S})$ be a Cauchy sequence with limit $f \in \mathcal{S}(\mathbf{A}_{F,S})$. In particular, $\mathcal{B} := \{f_k\}_{k\geq 0} \cup \{f\}$ is a bounded subset of $\mathcal{S}(\mathbf{A}_{F,S})$. By Theorem 3.3.1.1 (resting on Weil's Quotient Integral Formula: Theorem D.1.2.1) we have the identity, as holomorphic functions over $\Omega(J_{F,S})_{>0}$:

$$Z_S(f,c) = \int_{J_{F,S}} \Phi_{F,S}(f)(g)c(g)\,\mathrm{d}\overline{\mu}_S(g).$$

By Theorem 2.4.2.1 and Proposition 2.3.1.2(4), upon choosing an integer $N > 1$, we obtain the following bound pointwise over $J_{F,S}$ for all $u \in \mathcal{B}$:

$$|\Phi_{F,S}(u)(g)| \leq (C_{\mathcal{B},1}|\log|g|_{F,S}|^{|S|-1} + C_{\mathcal{B},2}) \cdot 1_{J_{F,S}^{(0,1]}}(g) + C_{\mathcal{B},N}|g|_{F,S}^{-N} \cdot 1_{J_{F,S}^{\geq 1}}(g)$$

with suitable constants. By Dominated Convergence, it follows that $Z_S(f_k, c) \to Z_S(f, c)$ for each fixed $c \in \Omega(J_{F,S})_{>0}$

(In fact, the mode of convergence is much stronger: it is uniform as c varies over an arbitrary vertical strip of bounded width within each connected component of $\Omega(J_{F,S})_{>0}$. The strongest form of sequential continuity will be identified later.) □

Returning to the proof of part (1) of Theorem 3.3.1.1, the subspace $\bigotimes_{v\in S} \mathcal{S}(F_v) \subset \mathcal{S}(\mathbf{A}_{F,S})$ is dense and so to prove (3.3.1.2) with ρ_S defined in (3.3.1.3) it suffices to check that for each $v \in S$ we have $Z_v(f_v, c_v) = \rho_v(c_v) Z_v(\widehat{f_v}, \widehat{c_v})$, which holds by [**CF**, Ch.XV,Thm.2.4.1]. This proves (a) in part (1).

Since $Z_S(f, c)$ is holomorphic in $c \in \Omega(J_{F,S})_{>0}$, and $Z_S(\widehat{f}, \widehat{c})$ is holomorphic in $c \in \Omega(J_{F,S})_{<1}$, the poles of the analytic continuation of $Z_S(f, \cdot)$, which we shall denote by $Z_S(f, \cdot)$ again, can only occur among the poles of ρ_S. This settles (b) in part (1).

(2) : Absolute and locally uniform convergence over $\Omega(J_{F,S})_{>0}$ of the integral in (3.3.1.5) is a consequence of the estimates in Proposition 2.3.1.2 (as $|g|_{F,S} \to +\infty$) and Proposition 2.4.3.1 (as $|g|_{F,S} \to 0^+$, see also Theorem 2.4.2.1).

(3) : This follows from compactness of $J^1_{F,S}$: the locally uniform convergence over $\Omega(J_{F,S})_{>0}(\omega)$ of the integral in (3.3.1.5) is a consequence of the estimates in Proposition 2.3.1.4 (as $|g|_{F,S} \to +\infty$ and for the ω-homogeneous θ-function) and Proposition 2.4.3.1 (as $|g|_{F,S} \to 0^+$ and for the ω-homogeneous θ-function, see also Theorem 2.4.2.1).

The identity in (3.3.1.6) holds by compactness of $J^1_{F,S}$ and the absolute convergence, for each $c \in \Omega(J_{F,S})_{>0}$, of the integral in (3.3.1.4) (which renders Theorem D.1.2.1 applicable). □

[1]

REMARK 3.3.1.4 (Tensor decomposition of $Z_S(\cdot, c)$ for $c \in \Omega(J_{F,S})$). Let F be a number field, S a finite set of places containing all the Archimedean places, $c \in \Omega(J_{F,S})_{>0}$ a fixed quasi-character. We call c_∞ the restriction of c to the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and for each place v of F we call c_v the restriction of c to the closed subgroup $F_v^\times$.

We have built the linear functional $Z_S(\cdot, c)$ on $\mathcal{S}(\mathbf{A}_{F,S})$, and want to express the sense in which this is a (completed) tensor product of the functionals $Z_v(\cdot, c_v) : \mathcal{S}(F_v) \to \mathbf{C}$ for $v \in S$. Note that these latter functionals are continuous by inspection of their definition as an integral when $c_v \in \Omega(F_v^\times)_{>0}$ (using for Archimedean v that the topology on $\mathcal{S}(F_v)$ can be described using L^1 seminorms) and via the local functional equation when $c_v \in \Omega(F_v^\times)_{<1}$ (using also continuity of the local Fourier transform on $\mathcal{S}(F_v)$). Here we recall that the spaces $\mathcal{S}(F_v)$ for $v \in S_f$ are equipped with their finest locally convex topology (see Proposition 2.1.2.4 and Proposition C.2.5.1).

On the Archimedean factor $\mathbf{A}_{F,S_\infty}$ Schwartz-Bruhat functions f_∞ are *not* generally built as a product of functions arising from the spaces $\mathcal{S}(F_v)$ for Archimedean v, so the purely algebraic tensorial aspects for handling the finite places don't apply at the Archimedean places. Instead, we recall that naturally $\overline{\bigotimes}_{v\in S}\mathcal{S}(F_v) \simeq \mathcal{S}(\mathbf{A}_{F,S})$ (using the completion of the inductive tensor product, where completion is relevant only for the Archimdean places), so it makes sense to form the continuous linear functional

$$\overline{\bigotimes}_{v\in S} Z_v(\cdot, c_v)$$

on $\overline{\bigotimes}_{v\in S}\mathcal{S}(F_v) \simeq \mathcal{S}(\mathbf{A}_{F,S})$. Our claim is that this agrees with $Z_S(\cdot, c)$.

We first introduce the intermediate linear functional

$$T := Z_{S_\infty}(\cdot, c_\infty) \otimes \bigotimes_{v\in S_f} Z_v(\cdot, c_v),$$

[1]Commented out a Remark on sequential continuity and local boundedness that is never cited.

(with the inductive tensor product topology) which is continuous on $\mathcal{S}(\mathbf{A}_{F,S})$ since $Z_{S_\infty}(\cdot, c_\infty)$ is continuous on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ by Lemma 3.3.1.3. The functional T agrees with $Z_S(\cdot, c)$ on elementary tensors in $\mathcal{S}(\mathbf{A}_{F,S}) \simeq \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\iota \mathcal{S}(\prod_{v\in S_f} F_v)$ (an inductive tensor product), which in turn linearly generate the tensor product linear space. Thus, $Z_S(\cdot, c) = Z_{S_\infty}(\cdot, c) \otimes \bigotimes_{v\in S_f} Z_v(\cdot, c_v)$ as linear functionals on $\mathcal{S}(\mathbf{A}_{F,S})$.

By denseness of $\bigotimes_{v\in S_\infty} C_c^\infty(F_v)$ in $C_c^\infty(\mathbf{A}_{F,S_\infty})$ [Tre, Thm. 39.2] and the denseness of $C_c^\infty(\mathbf{R}^n)$ in $\mathcal{S}(\mathbf{R}^n)$, the algebraic tensor product $\bigotimes_{v\in S_\infty} \mathcal{S}(F_v) \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$ is dense (see also the proof of Lemma D.3.6.1). Thus, $Z_{S_\infty}(\cdot, c_\infty)$ is the completed tensor product functional $\overline{\bigotimes}_{v\in S_\infty} Z_v(\cdot, c_v)$ because both are continuous and agree on the dense subspace generated by elementary tensors (treating separately the cases $c \in \Omega(J_{F,S})_{>0}$ and $c \in \Omega(J_{F,S})_{<1}$).

3.3.1.1. *Estimates on the correcting factor ρ_S from the S-local functional equation.* The following result studies the growth conditions of the correcting factor $\rho_S(c)$ defined in (3.3.1.3), in the S-local functional equation (3.3.1.2) from Theorem 3.3.1.1.

Proposition 3.3.1.5 (Asymptotic behavior of the ρ_S-factor). *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

Let $\rho_S \in \mathscr{M}(\Omega(J_{F,S}))$ be the meromorphic function on $\Omega(J_{F,S})$ defined in (3.3.1.3).

(1) *As meromorphic functions on $\Omega(J_{F,S})$, the following identities:*

$$\rho_S(c)\,\rho_S(\widehat{c}) = c(-1), \quad \rho_S(\overline{c}) = c(-1)\overline{\rho_S(c)}. \tag{3.3.1.8}$$

(2) *On the relative vertical line $\{\sigma(c) = 1/2\} \subset \Omega(J_{F,S})$, ρ_S is regular and valued in* $\mathrm{U}(1)$, *i.e. we have:*

$$|\rho_S(\chi|\cdot|_{F,S}^{1/2+i\mathbf{R}})| = 1 \tag{3.3.1.9}$$

for any continuous character $\chi \in \widehat{J_{F,S}}$.

(3) *For any continuous character $\chi \in \widehat{J_{F,S}}$ and any integer $k \geq 0$, the holomorphic derivative:*

$$\frac{\partial^k}{\partial s^k}\rho_S(\chi|\cdot|_{F,S}^s) \tag{3.3.1.10}$$

is holomorphic over $\mathbf{C}_{>0}$, and we have:

$$\frac{\partial^k}{\partial s^k}\rho_S(\chi|\cdot|_{F,S}^s) = O_\sigma(|\tau|^{\sigma-1/2}|\log(|\tau|)|^k) \tag{3.3.1.11}$$

for $s = \sigma + i\tau$, and for each fixed $\sigma > 0$, as $|\tau| \to +\infty$.

Proof. (1) : By design of ρ_S from (3.3.1.3), we have $\rho_S = \prod_{v\in S} \rho_v$, for ρ_v is the explicit meromorphic function on $\Omega(F_v^\times)$ defined in [CF, Ch.XV,§2.5], for each $v \in S$.

By [CF, Ch.XV,§2.4,Lemma 2.4.3], we have, for each $v \in S$:

$$\rho_v(\psi_v(c))\rho_v(\widehat{\psi_v(c)}) = \psi_v(c)(-1),\ \rho_v(\overline{\psi_v(c)}) = \psi_v(c)(-1)\overline{\rho_v(\psi_v(c))} \tag{3.3.1.12}$$

where $\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^\times)$ is the holomorphic submersion given by restricting a continuous S-idelic quasi-character $c \in \Omega(J_{F,S})$ to the closed subgroup $F_v^\times \subset J_{F,S}$. Explicitly, for any continuous character $\chi \in \widehat{J_{F,S}}$, we have $\psi_v(\chi|\cdot|_{F,S}^s) = \chi_v|\cdot|_v^s$, and hence we have:

$$\rho_v(\chi_v|\cdot|_v^s)\rho_v(\chi_v^{-1}|\cdot|_v^{1-s}) = \chi_v(-1),\ \rho_v(\chi_v^{-1}|\cdot|_v^{\overline{s}}) = \chi_v(-1)\overline{\rho_v(\chi_v|\cdot|_v^s)} \tag{3.3.1.13}$$

and this implies that (3.3.1.8) holds, away from the poles of $\rho_S(c)$ and $\rho_S(\widehat{c})$.

(2) : If $c = \chi|\cdot|_{F,S}^{1/2+it}$ for $\chi \in \widehat{J_{F,S}}$ and $t \in \mathbf{R}$, then $\rho_S(\widehat{c}) = \rho_S(\chi^{-1}|\cdot|_{F,S}^{1/2-it}) = \rho_S(\overline{c})$ from (3.3.1.8), and we have $\rho_S(\overline{c}) = c(-1)\overline{\rho_S(c)}$, from which we deduce:

$$\rho_S(\chi|\cdot|_{F,S}^{1/2+it})\overline{\rho_S(\chi|\cdot|_{F,S}^{1/2+it})}\chi(-1) = \chi(-1) \tag{3.3.1.14}$$

and since $\chi(-1) \neq 0$, we deduce from (3.3.1.14) that (3.3.1.9) holds. It follows that ρ_S is meromorphic and with neither zeros nor poles on the vertical line $\{\sigma(c) = 1/2\}$, hence it is regular there.

(3) : We fix a real number $\sigma > 0$. Upon choosing an enumeration for the finitely many places of F contained in S, i.e. $S = \{v_1, \ldots, v_N\}$, we have, as meromorphic functions of $s \in \mathbf{C}$:

$$\frac{\partial^k}{\partial s^k}\rho_S(\chi|\cdot|_{F,S}^s) = \sum_{(i_1,\ldots,i_N)\in\mathbf{N}^N, k=i_1+\ldots+i_N} \binom{k}{i_1,\ldots,i_N} \prod_{1\leq j\leq N} \frac{\partial^{i_j}}{\partial s^{i_j}}\rho_{v_j}(\chi_{v_j}|\cdot|_{v_j}^s) \tag{3.3.1.15}$$

the sum in (3.3.1.15) being finite.

We analyze for every integer $k \geq 0$ and each $v \in S$, the derivative:

$$\frac{\partial^k}{\partial s^k}\rho_v(\chi_v|\cdot|_v^s). \tag{3.3.1.16}$$

If v is non-Archimedean of prime residue characteristic p, associated to the prime ideal $\mathfrak{p}_v \subset \mathcal{O}_F$, locally generated by $\pi_v \in \mathcal{O}_{F_v}$, then if χ_v is unramified, we have, by [**CF**, Ch.XV,§2.5,p.322]:

$$\rho_v(|\cdot|_v^s) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2}\frac{1-N(\mathfrak{p}_v)^{s-1}}{1-N(\mathfrak{p}_v)^{-s}}$$

and if χ_v is ramified, then up to a vertical translation (noting that all assertions in (3) are stable under replacement of s by a vertical translate) we may assume $\chi_v(\pi_v) = 1$, and we have

$$\rho_v(\chi_v|\cdot|_v^s) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p}\mathfrak{f}_{\chi_v})^{s-1/2}\rho_0(\chi_v|\cdot|_v^s)$$

where $\rho_0(\chi_v|\cdot|_v^s) = \rho_0(\chi_v)$ is the root number, independent of s and valued in $\mathrm{U}(1)$, and $\mathfrak{f}_{\chi_v}$ is the conductor of χ_v.

In the unramified case, it is enough to show that, for every integer $k \geq 0$:

$$\partial_s^k N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2},\ \partial_s^k((1-N(\mathfrak{p}_v)^{s-1})(1-N(\mathfrak{p}_v)^{-s})^{-1})$$

are regular and bounded over $\{\mathrm{Re}(s) = \sigma\}$.

For the former, we have $\partial_s^k N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2} = (\log N(\mathfrak{p}_v))^k N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2}$, which is indeed regular and bounded over $\{\mathrm{Re}(s) = \sigma\}$.

For the latter, it is enough to show that both $\partial_s^k(1-N(\mathfrak{p}_v)^{s-1})$ and $\partial_s^k((1-N(\mathfrak{p}_v)^{-s})^{-1})$ are regular and bounded over $\{\mathrm{Re}(s) = \sigma\}$ for every integer $k \geq 0$.

We have:

$$\partial_s^k(1-N(\mathfrak{p}_v)^{s-1}) = -(\log N(\mathfrak{p}_v))^k N(\mathfrak{p}_v)^{s-1}$$

which is indeed regular and bounded over $\{\mathrm{Re}(s) = \sigma\}$ for every integer $k \geq 0$.

We have:

$$\partial_s^k(1-N(\mathfrak{p}_v)^{-s}) = -(-1)^k(\log N(\mathfrak{p}_v))^k N(\mathfrak{p}_v)^{-s}$$

which is, for each integer $k \geq 0$, regular, nowhere vanishing and, on $\{\mathrm{Re}(s) = \sigma\}$, of constant absolute value $(\log N(\mathfrak{p}_v))^k N(\mathfrak{p}_v)^{-\sigma}$.

We now have:

$$0 = \sum_{(i,j)\in\mathbf{N}^2, i+j=k} \binom{k}{i,j}(\partial_s^i(1-N(\mathfrak{p}_v)^{-s}))\cdot(\partial_s^j((1-N(\mathfrak{p}_v)^{-s})^{-1})) \tag{3.3.1.17}$$

$$= \sum_{(i,j)\in\mathbf{N}^2, i+j=k} \binom{k}{i,j}(-1)^{i+1}(\log N(\mathfrak{p}_v))^i N(\mathfrak{p}_v)^{-s}\cdot(\partial_s^j((1-N(\mathfrak{p}_v)^{-s})^{-1})) \tag{3.3.1.18}$$

which, by induction on $k \geq 0$, the base case $k = 0$ asserting that $(1-N(\mathfrak{p}_v)^{-s})^{-1}$ is regular and bounded on $\{\mathrm{Re}(s) = \sigma\}$, which is indeed the case, implies that so is $\partial_s^k((1-N(\mathfrak{p}_v)^{-s})^{-1}$ over $\{\mathrm{Re}(s) = \sigma\}$.

In the ramified case:

$$\partial_s^k \rho_v(\chi_v|\cdot|_v^s) = (\log N(\mathfrak{d}_{F_v/\mathbf{Q}_p}\mathfrak{f}_{\chi_v})^k \rho_0(\chi_v) N(\mathfrak{d}_{F_v/\mathbf{Q}_p}\mathfrak{f}_{\chi_v})^{s-1/2}$$

which is regular and bounded over $\{\mathrm{Re}(s)=\sigma\}$ for any integer $k\geq 0$.

We are reduced to analyzing $\rho_v(\chi_v|\cdot|_v^s)$ for $\{\mathrm{Re}(s)=\sigma\}$, for v Archimedean.

In light of [**CF**, Ch.XV,§2.5,pp.317,319], it is enough to prove that the following meromorphic functions on the complex s-plane obtained from the Γ-function:

$$\frac{\Gamma(s/2)}{\Gamma((1-s)/2)},\ \frac{\Gamma((s+1)/2)}{\Gamma(((1-s)+1)/2)},\ \frac{\Gamma(s+n/2)}{\Gamma(1-s+n/2)} \tag{3.3.1.19}$$

for every integer $n\geq 0$, are, over the vertical line $\{\mathrm{Re}(s)=\sigma\}$, $O_\sigma(|\tau|^{\sigma-1/2}|\log(|\tau|)|^k)$ for any integer $k\geq 0$.

They are all holomorphic over $\mathbf{C}_{>0}$ (see [**CF**, Ch.XV,§2.4,Cor.2.4.1,p.316] and [**CF**, Ch.XV,§2.5,pp.317,319], noting that $\rho_v(c)$ equals the three functions in (3.3.1.19) up to a factor that is entire, nowhere vanishing, and, on the vertical line $\{\mathrm{Re}(s)=\sigma\}$ is bounded, with bounded derivatives of any order. Such factor is, respectively, $\pi^{1/2-s}$, $i\pi^{1/2-s}$, $(-i)^n(2\pi)^{1-2s}$).

Since $\Gamma(z+1)=z\Gamma(z)$ as meromorphic functions on the complex z-plane, it is enough to prove that the holomorphic derivative of order $k\geq 0$ of the following three meromorphic functions on the complex s-plane:

$$\frac{\Gamma(s/2)}{\Gamma((1-s)/2)},\ \frac{\Gamma((s+1)/2)}{\Gamma(((1-s)+1)/2)},\ \frac{\Gamma(s+1/2)}{\Gamma(1-s+1/2)} \tag{3.3.1.20}$$

is, on the vertical line $\{\mathrm{Re}(s)=\sigma\}$, $O_\sigma(|\tau|^{\sigma-1/2}|\log(|\tau|)|^k)$.

We have, for $k=1$:

$$\frac{\partial}{\partial s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{(1/2)\Gamma'(s/2)}{\Gamma((1-s)/2)} - \frac{-(1/2)\Gamma(s/2)\Gamma'((1-s)/2)}{\Gamma((1-s)/2)^2} \tag{3.3.1.21}$$

$$= \frac{1}{2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\left\{\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)}\right\} \tag{3.3.1.22}$$

$$= \frac{1}{2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\left\{\psi(s/2)+\psi((1-s)/2)\right\} \tag{3.3.1.23}$$

(3.3.1.24)

which, over $\{\mathrm{Re}(s)=\sigma\}$, is indeed $O_\sigma(|\tau|^{\sigma-1/2}|\log(|\tau|)|)$.

A similar argument applies to the holomorphic derivative of order $k=1$ of the remaining two functions in (3.3.1.20) over the vertical lines in $\mathbf{C}_{>0}$.

Induction on $k\geq 0$ using Stirling's Approximation for each of the functions $f(s)$ in (3.3.1.20), combined with an argument similar to the calculation in (3.3.1.21) applied to the logarithmic derivative of $\partial_s^{k-1}f(s)$, leads to the bound $\partial_s^k f(\sigma+i\tau)=O_\sigma(|\tau|^{\sigma-1/2}|\log(|\tau|)|^k)$, as $|\tau|\to+\infty$. □

We note that Theorem 3.3.1.5 implies that $\rho_S(c)$ grows at most polynomially over all the vertical lines in $\Omega(J_{F,S})_{>0}$, and on each such line its derivative of order $k\geq 0$ is asymptotically bounded above by $|\tau|^{\sigma-1/2}|\log(|\tau|)|^k$, τ being the ordinate, and as $|\tau|\to+\infty$.

In particular, along the relative vertical line $\sigma(c)=1/2$, $\rho_S(c)$ is bounded (in fact, valued in U(1)), and its derivative of order $k\geq 0$ grows slowly, at most as $|\log(|\tau|)|^k$.

We now let F be a number field, S a finite set of places of F containing all the Archimedean places. We define:

$$L_S(c) := \prod_{v\in S} L_v(c_v) \tag{3.3.1.25}$$

where $L_v(c_v)$ is the local Euler factor at v of the Hecke L-function $L(c)$ of F, and c_v is the image of $c \in \Omega(J_{F,S})$ under the local analytic isomorphism to $\Omega(F_v^\times)$ obtained by restricting c along the continuous injective closed group homomorphism $F_v^\times \subset J_{F,S}$. We refer to $L_S(c)$ as the *S-local Hecke L-function of F*.

REMARK 3.3.1.6 (On partial Euler products). For any choice of connected component of $\Omega(J_{F,S})$, the restriction of $L_S(c)$ to that connected component is a partial Euler product for the Hecke L-function $L(c)$ restricted to the same connected component. Indeed, upon choosing a holomorphic parametrization $\{\chi|\cdot|_{F,S}^s, s \in \mathbf{C}\}$ of $\Omega(J_{F,S})$, equivalently, choosing a set of continuous characters $\widetilde{J^1_{F,S}} := \{\chi \in \widehat{J_{F,S}}\}$ representing the elements of $\widehat{J^1_{F,S}}$, then, for each $\chi \in \widehat{J^1_{F,S}}$ there exists a continuous character $\chi' \in \widehat{C_F}$ such that its restriction to $C_{F,S} \subset C_F$ agrees with the continuous character of $C_{F,S}$ obtained by composition of the continuous quotient homomorphism $C_{F,S} \to J_{F,S}$ followed by χ.

Indeed, upon calling χ_S such composition, this is a continuous character of the open, hence closed (by [**DE**, Lemma 1.1.7(c)]) subgroup $C_{F,S} \subset C_F$, and by [**DE**, Cor.3.6.2], there exists a continuous character $\chi' \in \widehat{C_F}$ extending χ_S from $C_{F,S}$ to C_F. There therefore exists a finite set of places S' of F containing S such that χ' is unramified outside S', and large enough so as to have that $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$ holds.

3.3.1.2. *ϵ_S-factors and the S-local functional equation.* In the Lemma below, we revisit the S-local functional equation from Theorem 3.3.1.1, and rewrite it in terms of ϵ_S-factors.

LEMMA 3.3.1.7 (ϵ_S-factors and the S-local functional equation). *Let F be a number field, S a finite set of places containing all Archimedean places. Let ω be a continuous unitary character of $J^1_{F,S}$.*

(1) *For every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, $Z_S(f,c)$ is Schwartz on every vertical line of $\Omega(J_{F,S})_{>0}$, and for each $\varepsilon > 0$ it is uniformly Schwartz on vertical strips within $\Omega(J_{F,S})_{>\varepsilon}$ of bounded width (in the sense that the vertical decay constants can be chosen uniformly over such strips).*
(2) *The meromorphic function*

$$\frac{Z_S(f,c)}{L_S(c)} \tag{3.3.1.26}$$

on $\Omega(J_{F,S})$ is entire and satisfies the following growth condition for every continuous unitary character χ of $J_{F,S}$:

$$\frac{Z_S(f,\chi|\cdot|_{F,S}^{\sigma+it})}{L_S(\chi|\cdot|_{F,S}^{\sigma+it})} \ll_{N,a,b} |t|^{-\left\{N+\left(\sigma-\frac{1}{2}\right)\left(\frac{r_1}{2}+r_2\right)\right\}} \cdot e^{\frac{\pi t}{2}\left(\frac{r_1}{2}+r_2\right)} \tag{3.3.1.27}$$

for each integer $N \geq 1$ as $|t| \to +\infty$, and uniformly as σ varies over the arbitrary bounded interval $[a,b]$ of $\mathbf{R}_{>0}$.
(3) *There exists an entire function ϵ_S on $\Omega(J_{F,S})$, such that:*
   (a) *As meromorphic functions of $c \in \Omega(J_{F,S})$,*

$$\epsilon_S(c) = \prod_{v \in S} \epsilon_v(c_v) \tag{3.3.1.28}$$

where ϵ_v is the entire function on $\Omega(F_v^\times)$ defined by:
   (i) *(v real) $\epsilon_v(c_v) = 1$ if $c_v \in \Omega(F_v^\times)(1)$, $\epsilon_v(c_v) = -i$ if $c_v \in \Omega(F_v^\times)(\mathrm{sgn})$,*
   (ii) *(v complex) $\epsilon_v(c_v) = (-i)^{|n|}$ if $c_v \in \Omega(F_v^\times)(\chi_n)$ with $\chi_n : z \mapsto e^{in\theta}$ for $z = re^{i\theta}$ with $r > 0$ and $n \in \mathbf{Z}$,*
   (iii) *(v is non-Archimedean of residue characteristic p) on unramified c_v we have*

$$\epsilon_v(|\cdot|_v^s) := N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2} \tag{3.3.1.29}$$

and if $c_v = \chi_v|\cdot|_v^s$ is ramified at v, with χ_v normalized to satisfy $\chi_v(\pi_v) = 1$ for a choice of local uniformizer π_v at v then

$$\epsilon_v(\chi_v|\cdot|_v^s) := N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-1/2}\rho_0(\chi_v|\cdot|_v^s) \tag{3.3.1.30}$$

where $\rho_0(\chi_v|\cdot|_v^s)$ is the Gauss sum $N(\mathfrak{f})^{-1/2}\sum_u \chi_v(u)e^{2\pi i\Lambda_v(u/\pi_v^{m_v})}$, with $\mathfrak{f}$ the conductor of χ_v, $m_v := v(\mathfrak{d}_{F_v/\mathbf{Q}_p}\mathfrak{f})$, and the sum over any set of representatives for the finite coset space $\mathcal{O}_{F_v}^\times/(1+\mathfrak{f})$.

(b) *ϵ_S is entire over $\Omega(J_{F,S})$ and nowhere-vanishing, and we have $|\epsilon_S(c)| = 1$ for $\sigma(c) = 1/2$.*

(c) *The following identity of entire functions over $\Omega(J_{F,S})$:*

$$\frac{Z_S(f,c)}{L_S(c)} = \epsilon_S(c)\frac{L_S(\widehat{f},\widehat{c})}{L_S(\widehat{c})} \tag{3.3.1.31}$$

holds for all Schwartz-Bruhat functions $f \in \mathcal{S}(\mathbf{A}_{F,S})$.

(d) *The meromorphic function ρ_S from* (3.3.1.3) *satisfies*

$$\rho_S(c) = \epsilon_S(c)\frac{L_S(c)}{L_S(\widehat{c})} \tag{3.3.1.32}$$

as meromorphic functions over $\Omega(J_{F,S})$.

Remark 3.3.1.8. For each place $v \in S$, the standard character $\psi_{v,\mathrm{std}} : x \mapsto e^{2\pi i\,\Lambda_v(x)}$ of F_v in Lemma 3.3.1.7 is defined for v non-Archimedean of residue characteristic p in Example A.3.1.6 as $\psi_{p,\mathrm{std}}(\mathrm{Tr}_{F_v/\mathbf{Q}_p}(\cdot))$ with $\psi_{p,\mathrm{std}}$ the standard character on $\mathbf{Q}_p$ from Example A.3.1.6 and $\Lambda_v = \mathrm{Tr}_{F_v/\mathbf{Q}_p}$. For Archimedean v, it is defined in Example A.3.1.5 with Λ_v the respective local trace form (see Remark A.3.2.5). This standard character for each v makes the additive local Tamagawa measure ν_v from Definition A.3.1.2 on F_v be self-dual upon defining the Fourier transform on $\mathcal{S}(F_v)$ using the additive character $\psi_{v,\mathrm{std}}$ on F_v. In effect, ϵ_S depends on a choice of continuous additive character of F_v for each $v \in S$, which we have taken to be the standard additive characters. We further refer the reader to [**CF**, Ch.XV,§2.5].

Proof of Lemma 3.3.1.7. Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be an arbitrary Schwartz-Bruhat function.

(1) : It is enough to check the assertion on each connected component of $\Omega(J_{F,S})_{>0}$. Upon choosing $\chi \in \widehat{J_{F,S}}$, then $Z_S(f,\chi|\cdot|_{F,S}^s)$ is the Mellin transform of the function $\phi(t) := \chi(\tau(t))\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(\tau(t))$ for τ a choice of continuous homomorphic section to the S-idelic norm.

By Corollary 2.4.3.2 combined with Proposition 2.3.1.4(2), for every $\varepsilon > 0$ the function $t^\varepsilon\phi(t)$ is, as a function of $t > 0$, Schwartz-Bruhat over $\mathbf{R}_{>0}$ with (forward) semi-exponential decay (see Definition B.2.1.1(1)). Concretely, this means that for every integer $N \geq 1$ there exist real numbers $T_N > 0$ and $C_N > 0$ such that:

$$|t^\varepsilon\phi(t)| < C_N t^{-N}$$

for all $t > T_N$.

It follows that $Z_S(f,\chi|\cdot|_{F,S}^s) = M(\phi)(s)$ is holomorphic over the half-plane $\mathbf{C}_{>0}$ and Schwartz on vertical strips of bounded width (meaning Schwartz on each vertical line, and with uniform decay constants on all such lines in the strip) upon applying Lemma B.2.1.5 to $t^\varepsilon\phi(t)$ for every $\varepsilon > 0$.

(2) and (3) : The holomorphy of $Z_S(f,c)/L_S(c)$ over $\Omega(J_{F,S})_{>0}$ follows from (1), and over $\Omega(J_{F,S})_{<0}$ from the functional equation of (3.3.1.2) from Theorem 3.3.1.1. Since $Z_S(f,c)$ and $Z_S(\widehat{f},\widehat{c})$ are both holomorphic over $\Omega(J_{F,S})_{(0,1)}$ by Theorem 3.3.1.1(1), then from (3.3.1.2) the function $Z_S(f,\cdot)$ can only have poles located at the poles of the meromorphic function ρ_S defined in (3.3.1.3).

By [**CF**, Ch.XV,Lemma 2.4.2] (and the incipit of the proof of [**CF**, Ch.XV,Thm.2.4.1]), for every $v \in S$ we have $\rho_v(c_v) = \epsilon_v(c_v)L_v(c_v)/L_v(\widehat{c}_v)$, so $\rho_S(c) = \epsilon_S(c)L_S(c)/L_S(\widehat{c})$ with $1/L_S(\widehat{c})$ entire, so $Z_S(f,\cdot)$ can only

have poles at the poles of L_S. We can therefore rewrite the functional equation from (3.3.1.2) as:

$$\frac{Z_S(f,c)}{L_S(c)} = \epsilon_S(c)\frac{Z_S(\widehat{f},\widehat{c})}{L_S(\widehat{c})} \tag{3.3.1.33}$$

as meromorphic functions over $\Omega(J_{F,S})$. Since the left side is holomorphic over $\Omega(J_{F,S})_{>0}$ and the right side is holomorphic over $\Omega(J_{F,S})_{<1}$, we deduce that each side extends to a holomorphic function over $\Omega(J_{F,S})$.

By (3.3.1.33), it is enough to check the Schwartz condition uniformly over the vertical strips of bounded width of $\Omega(J_{F,S})_{>0}$.

For v non-Archimedean, we analyze $Z_v(f_v,c_v)/L_v(c_v)$. For c_v ramified, we have $L_v(c_v)=1$, and for c_v unramified, $1/L_v(c_v)$ is bounded uniformly over the vertical strips of bounded width of $\Omega(F_v^\times)_{>0}$ (away from the edge). Since $Z_v(f_v,c_v)$ is also bounded uniformly on such vertical strips, so is $Z_v(f_v,c_v)/L_v(c_v)$.

Let $[a,b]\subset \mathbf{R}_{>0}$ be an arbitrary non-empty bounded interval. It follows that $Z_S(f,\chi|\cdot|_{F,S}^{\sigma+it})/L_S(\chi|\cdot|_{F,S}^{\sigma+it})$ can be written as:

$$\frac{Z_S(f,\chi|\cdot|_{F,S}^{\sigma+it})}{L_S(\chi|\cdot|_{F,S}^{\sigma+it})} = \epsilon_S(\chi|\cdot|_{F,S}^{\sigma+it})\frac{1}{L_{S_\infty}(\chi|\cdot|_{F,S_\infty}^{\sigma+it})}\cdot\frac{Z_S(f,\chi|\cdot|_{F,S}^{\sigma+it})}{\prod_{v\in S_f}L_v(\chi_v|\cdot|_v^{\sigma+it})} \tag{3.3.1.34}$$

with $S_f := S\cap\Sigma_{F,f}$. The third factor is Schwartz as a function of t, uniformly with respect to $\sigma\in[a,b]$.[2] Since $1/L_{S_\infty}(\chi|\cdot|_{F,S_\infty}^{\sigma+it})$ is a product of vertical translations of Γ-functions, the estimate from (3.3.1.27) holds by Stirling Approximation (see specifically [**IK**, Ch.5,§A.4,(5.114),(5.115),p.151]).

The formula for ϵ_v for each $v\in S$ follows by comparing (3.3.1.2) with (3.3.1.33), as in the proof of [**RV**, Ch.7,§7.1,Thm.7.2,p.246]. □

3.3.2. Poles and Residues of local L-functions. We review here basic facts on the poles and residues of the local Z-functions $Z_v(f,c)$, defined, for $c\in\Omega(F_v^\times)_{>0}$, by the absolutely and uniformly convergent integral:

$$Z_v(f,c) := \int_{F_v^\times} f(g)c(g)\,\mathrm{d}\mu_v(g) \tag{3.3.2.1}$$

for $f\in\mathcal{S}(F_v)$, $c\in\Omega(F_v^\times)$, for v a place of the number field F, and μ_v the local multiplicative Tamagawa measure on $F_v^\times$ (see Definition A.3.2.2).

We refer the reader to [**CF**, Ch.XV,§2.4,Def.2.4.1], and to [**CF**, Ch.XV,§2.4,Thm.2.4.1], for basic properties of the local Z-functions $Z_v(f,c)$, some of which we have recalled in Theorem 3.3.1.1.

We denote by $Z_v(f,c)$ the unique analytic continuation of the holomorphic function defined on $\Omega(F_v^\times)_{>0}$ by the integral in (3.3.2.1), to a meromorphic function over the local Riemann surface $\Omega(F_v^\times)$.

As $f\in\mathcal{S}(F_v)$ varies, the assignment $Z_v(f,c)$ for $c\in\Omega(F_v^\times)$ defines a meromorphic $\mathcal{S}(F_v)'_\beta$-valued function on the Riemann surface $\Omega(F_v^\times)$. We denote the functional on $\mathcal{S}(F_v)$ obtained by varying f in $Z_v(f,c)$ for each fixed $c\in\Omega(F_v^\times)$ by $Z_v(\cdot,c)$ (for each fixed c it is a tempered distribution on F_v, i.e. it is continuous as f varies over $\mathcal{S}(F_v)$) and if we further vary c then we denote the resulting function $\Omega(F_v^\times)$ valued in $\mathcal{S}(F_v)'_\beta$ by Z_v, a meromorphic distribution-valued function on $\Omega(F_v^\times)$. We will be returning on this perspective later in this Chapter and the next Chapters on this work.

The theory of vector-valued holomorphic and meromorphic functions (which is discussed in Appendix C and specifically in §C.5.2 and §C.5.3) will be used occasionally throughout this work when we want to think about certain residue operators and coefficients of principal parts, as themselves operators on a space of vector valued meromorphic functions. This perspective has indeed appeared in relevant literature before: we have in mind Sarnak's paper on the equidistribution of long horocycles (see [**Sarnak**]) where certain residue operators arise of meromorphic vector-valued Eisenstein series. We will return to Sarnak's paper in Chapters 7 and 8 of this work when we explain the connection of our work with a number of elements of Deninger's program.

[2]Need to give a reference for a proof of this "uniform Schwartz" assertion.

3.3.2.1. *Poles and residues of the local Z-functions at the Archimedean places.* Recall from §3.1.1 on generalities concerning quasi-characters of modulated groups, that for G a modulated group (i.e. a Hausdorff locally compact abelian group equipped with a continuous proper homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$, which we assume to be non-trivial and call *module*) the set of continuous quasi-characters (i.e. continuous homomorphisms $G \to \mathbf{C}^\times$), denoted $\Omega(G)$, is a Riemann surface whose connected components are cylinders, if the image of $|\cdot|_G$ is a discrete subgroup of $\mathbf{R}_{>0}$, complex planes, if $|\cdot|_G$ is surjective.

We have denoted in §3.1.1 by $\Omega(G)(c)$ the connected component of $\Omega(G)$ that contains the quasi-character c. If $\omega \in \widehat{G^1}$, with G^1 the compact kernel of $|\cdot|_G$, then we denote by $\Omega(G)(\omega)$ the connected component parametrized by ω (when we view $\Omega(G)$ as fibred over $\widehat{G^1}$ by restricting quasi-characters of G to characters of G^1). In the former case we call G *discretely modulated* and in the latter we term it $\mathbf{R}_{>0}$*-modulated*. The local fields F_v for v a non-Archimedean place of the number field F are all discretely modulated by the v-adic absolute value $|\cdot|_v$. The Archimedean local fields F_v are $\mathbf{R}_{>0}$-modulated.

We recall here the following result on local Z-functions:

LEMMA 3.3.2.1 (Poles and Residues of the local Z-functions - Archimedean places). *Let F be a number field, v an Archimedean place.*

(1) *The poles of the meromorphic function Z_v on $\Omega(F_v^\times)$ valued in tempered distributions are simple and located at the following quasi-characters:*
   (1) *For v real and $c \in \Omega(F_v^\times)(1)$:*

$$c_{0,m}(x) = |x|_v^{-m}, m \in 2\mathbf{N} \tag{3.3.2.2}$$

   and for v real and $c \in \Omega(F_v^\times)(\mathrm{sgn})$:

$$c_{1,m}(x) = x^{-1}|x|_v^{-m}, m \in 2\mathbf{N} \tag{3.3.2.3}$$

   (2) *If v is complex and, fixing $r \in \mathbf{Z}$, $r \geq 0$, upon calling $\chi_r(z) = \chi_r(Re^{i\theta}) = e^{ir\theta}$, then for $c \in \Omega(F_v^\times)(\chi_r)$:*

$$c_{r,m}(z) = \overline{z}^{-r}|z|_v^{-m},\ m \in \mathbf{N} \tag{3.3.2.4}$$

   and for $c \in \Omega(F_v^\times)(\chi_{-r})$:

$$c_{-r,m}(z) = z^{-r}|z|_v^{-m},\ m \in \mathbf{N}. \tag{3.3.2.5}$$

(2) *The residues of $Z_v(\cdot, c)$ are:*
   (1) *If v is real:*

$$\operatorname*{Res}_{c=c_{a,m}} Z_v(\cdot, c) = A_{v,a,m} D^{a+m}\delta_0,\ a \in \{0,1\}, m \in 2\mathbf{N} \tag{3.3.2.6}$$

   where $D : \mathcal{S}(F_v) \to \mathcal{S}(F_v)$ is the differential operator $\frac{1}{2\pi i}\partial_x$ on $\mathcal{S}(\mathbf{R})$, where $A_{v,a,m} \in \mathbf{C}^\times$ is a constant depending only on the quasi-character, given by the following formula:

$$A_{v,0,m} = \frac{2^{m+1}\pi^m(-1)^{m/2}}{m!},\ A_{v,1,m} = -i\,\frac{2^{m+2}\pi^{m+1}(-1)^{m/2}}{(m+1)!}. \tag{3.3.2.7}$$

   (2) *If v is complex, for $r \geq 0$, and:*

$$\operatorname*{Res}_{c=c_{r,m}} Z_v(\cdot, c) = A_{v,r,m}\overline{D}^{r+m} D^m \delta_0 \tag{3.3.2.8}$$

   and:

$$\operatorname*{Res}_{c=c_{-r,m}} Z_v(\cdot, c) = A_{v,r,m} D^{r+m}\overline{D}^m \delta_0 \tag{3.3.2.9}$$

for the differential operators $D := \frac{1}{2\pi i}\partial_z$ *and* $\overline{D} := \frac{1}{2\pi i}\partial_{\overline{z}}$ *on* $\mathcal{S}(\mathbf{C})$ *for* v *complex, where* $A \in \mathbf{C}^\times$ *is a constant depending only on the quasi-character, given by the following formula:*

$$A_{v,r,m} = (-i)^{|r|}(2\pi)^{1+|r|+2m}\frac{(-1)^m}{\Gamma(1+|r|+m)m!}. \tag{3.3.2.10}$$

PROOF. For any place v of F and $c \in \Omega(F_v^\times)$, we have, for every $f \in \mathcal{S}(F_v)$:

$$Z_v(f,c) = \rho_v(c)Z_v(\widehat{f},\widehat{c}) \tag{3.3.2.11}$$

by [CF, Ch.XV,Thm.2.4.1]. The function $Z_v(\widehat{f},\widehat{c})$ is holomorphic over $\Omega(F_v^\times)_{<1}$ and $Z_v(f,c)$ is holomorphic over $\Omega(F_v^\times)_{>0}$. The functional equation in (3.3.2.11) implies that $Z_v(f,c)$ extends to a meromorphic function over the entire $\Omega(F_v^\times)$, with poles located at the poles of $\rho_v(c)$, of order bounded above by the orders of pole of $\rho_v(c)$, respectively.

Since (3.3.2.11) holds for all $f \in \mathcal{S}(F_v)$ and both sides of the identity are meromorphic functions valued in tempered distributions on F_v, the poles of Z_v as a meromorphic vector-valued function are exactly the poles of ρ_v as a meromorphic complex-valued function. This means that even though there exist Schwartz-Bruhat functions on F_v such that $Z_v(f,c_v)$ has fewer poles that ρ_v, the poles of Z_v as an *operator-valued* meromorphic function (i.e. valued in continuous functionals on $\mathcal{S}(F_v)$) are exactly those of ρ_v because of Lemma C.5.3.3, which computes its residues *as functionals on* $\mathcal{S}(F_v)$ (concretely, for Z_v to have a pole at c it is enough that for *some* $f \in \mathcal{S}(F_v)$, the classical meromorphic function $Z_v(f,\cdot)$ has a pole at c. This existence assertion is proved in Tate's thesis: the "Gaussian" functions constructed in [CF, Ch.XV,§2.5,p.316] are examples).

(1) : If v is Archimedean, we distinguish the two cases when v is real, and v is complex.

For v real, $\Omega(F_v^\times)$ has two connected components, $\Omega(F_v^\times)(1)$ and $\Omega(F_v^\times)(\mathrm{sgn})$.

In the former case, $\rho_v(c)$ is computed in [CF, Ch.XV,§2.5], and it is, upon writing $c = |\cdot|_v^s$ for a unique complex number $s \in \mathbf{C}$:

$$\rho_v(c) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s). \tag{3.3.2.12}$$

The Γ-function has simple poles at $s = -\mathbf{N}$, and the cosine function $\cos\left(\frac{\pi s}{2}\right)$ has simple zeros at the odd integers. The odd negative integers are therefore removable singularities of $\rho_v(|\cdot|_v^s)$, whose only poles are simple and located at $c = |\cdot|_v^{-m}$ for $m \in 2\mathbf{N}$.

For $c \in \Omega(F_v^\times)(\mathrm{sgn})$, we have $c = \mathrm{sgn}\cdot|\cdot|_v^s$ for a unique complex number $s \in \mathbf{C}$:

$$\rho_v(c) = i\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)} = -i\,2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s). \tag{3.3.2.13}$$

The Γ-function has simple poles at $-\mathbf{N}$, and the sine function $\sin\left(\frac{\pi s}{2}\right)$ has simple zeros at the even integers, and so the function $\rho_v(c)$ restricted to $\Omega(F_v^\times)(\mathrm{sgn})$ has simple poles located exactly at the quasi-characters $c = \mathrm{sgn}|\cdot|_v^{-m-1}$, which, evaluated on $x \in \mathbf{R} \simeq F_v$, reads $c(x) = x^{-1}|x|_v^{-m}$, for $m \in 2\mathbf{N}$.

If v is complex, then $\Omega(F_v^\times)$ is fibred over the discrete dual group of $\mathrm{U}(1)$, which is isomorphic to $\mathbf{Z}$ upon choosing the standard unitary character $\chi_1 : F_v^\times \to \mathrm{U}(1)$ defined as $\chi_1(z) = \chi_1(Re^{i\theta}) := e^{i\theta}$.

We fix $r \in \mathbf{Z}$. Note that we have $\chi_r = (\chi_1)^r$. For $c \in \Omega(F_v^\times)(\chi_r)$, we have:

$$\rho_v(c) = (-i)^{|r|}\frac{(2\pi)^{1-s}\Gamma\left(s+\frac{|r|}{2}\right)}{(2\pi)^s\Gamma\left((1-s)+\frac{|r|}{2}\right)}. \tag{3.3.2.14}$$

For $r \geq 0$, the function $\Gamma(s+r/2)$ has simple poles located at the points $-r/2-\mathbf{N}$, and the function $1/\Gamma((1-s)+r/2)$ is entire, with simple zeros located at the points $1+r/2+\mathbf{N}$. The intersection of these two sets comprises of points $s \in \mathbf{C}$ of the form $-r/2-m = 1+r/2-m'$ for some integer $m' \in \mathbf{N}$, and it is empty.

The poles of $\rho_v(c)$ are therefore simple and located at the quasi-characters $c = \chi_r|\cdot|_v^{-r/2-m}$ for $m \in \mathbf{N}$, which evaluated on $z \in \mathbf{C} \simeq F_v$ read $c(z) = \overline{z}^{-r}|z|_v^{-m}$.

For $r \leq 0$, the function $\Gamma(s - r/2)$ has simple poles located at the points $r/2 - \mathbf{N}$, and the function $1/\Gamma((1-s) - r/2)$ is entire, with simple zeros located at the points $1 - r/2 + \mathbf{N}$. The intersection of these two sets is empty, and so the poles of $\rho_v(c)$ are located at the quasi-characters $c = \chi_r|\cdot|_v^{r/2-m}$ for $m \in \mathbf{N}$, which, evaluated at $z \in \mathbf{C} \simeq F_v$, read $c(z) = z^r|z|_v^{-m}$.

(2) : For v real, the residue of $\rho_v(c)$ at $c_{0,m}$ for $m \in 2\mathbf{N}$ is:

$$\frac{2^{m+1}\pi^m(-1)^{m/2}}{m!} \tag{3.3.2.15}$$

and at $c_{1,m}$ for $m \in 2\mathbf{N}$, it is:

$$-i\,\frac{2^{m+2}\pi^{m+1}(-1)^{m/2}}{(m+1)!}. \tag{3.3.2.16}$$

For v complex, the residue of $\rho_v(c)$ at $c_{r,m}$ for $r \in \mathbf{Z}$, $m \in \mathbf{N}$, is:

$$(-i)^{|r|}(2\pi)^{1+|r|+2m}\frac{(-1)^m}{\Gamma(1+|r|+m)m!}. \tag{3.3.2.17}$$

For any v Archimedean the residue satisfies:

$$\mathrm{Res}_c Z_v(f, c') = \mathrm{Res}_c \rho_v(c') Z_v(\widehat{f}, \widehat{c}) \tag{3.3.2.18}$$

as continuous linear operators on $\mathcal{S}(F_v)$.

We compute $Z_v(\widehat{f}, \widehat{c})$ in terms of the differential operators D^k for v real and $k \in \mathbf{N}$, and $D^a\overline{D}^b$ for v complex and $a, b \in \mathbf{N}$, when c is a pole of Z_v.

If v is real and $c = c_{0,m}$ for $m \in 2\mathbf{N}$, we have

$$Z_v(\widehat{f}, \widehat{c_{0,m}}) = \int_{F_v^\times} \widehat{f}(g)|g|_v^{1+m}\,\mathrm{d}\mu_v(g)$$

and upon choosing an isomorphism of topological fields $F_v \simeq \mathbf{R}$, we have:

(3.3.2.19)

$$\int_{F_v^\times} \widehat{f}(g)|g|_v^{1+m}\,\mathrm{d}\mu_v(g) = \int_{\mathbf{R}} \widehat{f}(x)|x|^m\,\mathrm{d}x = \frac{1}{(2\pi i)^m}\int_{\mathbf{R}} (2\pi i x)^m \widehat{f}(x)\,\mathrm{d}x = \frac{1}{(2\pi i)^m}(\partial_x^m f)(0) = (D^m\delta_0)(f)$$

where we used [Fol, Thm.8.22,(e)] and [Fol, Thm.8.26].

If $c = c_{1,m}$ for $m \in 2\mathbf{N}$, we have

$$Z_v(\widehat{f}, \widehat{c_{1,m}}) = \int_{F_v^\times} \widehat{f}(g)c_{1,m}(g)\,\mathrm{d}\mu_v(g)$$

and upon choosing an isomorphism of topological fields $F_v \simeq \mathbf{R}$, we have:

(3.3.2.20)

$$\int_{F_v^\times} \widehat{f}(g)\widehat{c_{1,m}}(g)\,\mathrm{d}\mu_v(g) = \int_{\mathbf{R}} \widehat{f}(x)x|x|^m\,\mathrm{d}x = \frac{1}{(2\pi i)^{m+1}}\int_{\mathbf{R}} (2\pi i x)^{m+1}\widehat{f}(x)\,\mathrm{d}x = \frac{1}{(2\pi i)^{m+1}}(\partial_x^{m+1} f)(0) = (D^{m+1}\delta_0)(f).$$

We deduce that (3.3.2.6) holds.

If v is complex, then we have, for $r, m \in \mathbf{Z}$, $r \geq 0$ and $m \leq 0$, and for $c = c_{r,m}$:

$$Z_v(\widehat{f}, \widehat{c}) = \int_{F_v^\times} \widehat{f}(g)\widehat{c_{r,m}}(g)\,\mathrm{d}\mu_v(g)$$

and upon choosing an isomorphism of topological fields $F_v \simeq \mathbf{C}$, we have:

(3.3.2.21)
$$\int_{F_v^\times} \widehat{f}(g)\widehat{c_{r,m}}(g)\,\mathrm{d}\mu_v(g) = \int_{\mathbf{C}} \widehat{f}(z)\overline{z}^r|z|_v^{1+m}\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|_v} = \int_{\mathbf{C}} \widehat{f}(z)\overline{z}^{r+m}z^m\,|\mathrm{d}z\wedge\mathrm{d}\overline{z}| = \frac{1}{(2\pi i)^{2m+r}}(\partial_{\overline{z}}^{r+m}\partial_z^m\delta_0)(f)$$

which reads $(\overline{D}^{r+m}D^m\delta_0)(f)$.

For $c = c_{-r,m}$ we have:

(3.3.2.22)
$$\int_{F_v^\times} \widehat{f}(g)\widehat{c_{-r,m}}(g)\,\mathrm{d}\mu_v(g) = \int_{\mathbf{C}} \widehat{f}(z)z^r|z|_v^{1+m}\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|_v} = \int_{\mathbf{C}} \widehat{f}(z)z^{r+m}\overline{z}^m\,|\mathrm{d}z\wedge\mathrm{d}\overline{z}| = \frac{1}{(2\pi i)^{2m+r}}(\partial_z^{r+m}\partial_{\overline{z}}^m\delta_0)(f)$$

which reads $(D^{r+m}\overline{D}^m\delta_0)(f)$. □

3.3.2.2. *Poles and residues of the local Z-functions at the non-Archimedean places.* The non-Archimedean analog of Lemma 3.3.2.1 is the following:

LEMMA 3.3.2.2 (Poles and Residues of the local Z-functions - non-Archimedean places). *Let F be a number field, v a non-Archimedean place. The meromorphic function Z_v on $\Omega(F_v^\times)$ valued in tempered distributions on F_v has one simple pole at the trivial quasi-character $c = 1$. Its residue is:*

$$\mathrm{Res}_{c=1}\, Z_v(f,c) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-\frac{1}{2}}\,\frac{1-q_v^{-1}}{\log(q_v)}\,\widehat{f}(0). \tag{3.3.2.23}$$

PROOF. Indeed, for any $f \in \mathcal{S}(F_v)$, we have:

$$Z_v(f,c) = \rho_v(c)Z_v(\widehat{f},\widehat{c})$$

by [CF, Ch.XV,§2.4,Thm.2.4.1,p.315] and the only poles of $Z_v(f,c)$ are located among the poles of $\rho_v(c)$, which is explicitly given by the following formula (see [CF, Ch.XV,§2.5,p.322]) upon writing $q_v = N(\mathfrak{p}_v)$ for $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ corresponding to the finite place v of F, and for $\mathfrak{d}_{F_v/\mathbf{Q}_p}$ the local discriminant, with p the residue characteristic of F_v:

$$\rho_v(|\cdot|_v^s) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{s-it_v-\frac{1}{2}}\,\frac{1-q_v^{s-it_v-1}}{1-q_v^{-s-it_v}},\quad \rho_v(\chi|\cdot|_v^s) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p}\mathfrak{f}_\chi)^{s-\frac{1}{2}}\rho_0(\chi) \tag{3.3.2.24}$$

where $\mathfrak{f}_\chi$ is the conductor of a ramified unitary character χ of $F_v^\times$ and $\rho_0(\chi)$ is the root number, a complex number of absolute value 1, independent of $s \in \mathbf{C}$ modulo $2\pi i\,\mathbf{Z}/\log(q_v)$. The only poles of $\rho_v(c)$ therefore occur over the connected component of $\Omega(F_v^\times)$ fibred over the trivial character of $\mathcal{O}_{F_v}^\times$, and within such connected component, parametrized by $|\cdot|_v^s$ with s unique modulo $2\pi i\,\mathbf{Z}/\log(q_v)$, the pole occurs at the image of the origin in the complex s-plane into the cylinder $\mathbf{C}/(2\pi i\,\mathbf{Z}/\log(q_v))$. The pole is simple, because the zeros of $1-q_v^{-s-it_v}$ in the complex s-plane are simple, and its residue is:

$$\mathrm{Res}_{c=1}\, Z_v(f,c) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-\frac{1}{2}}\,\frac{1-q_v^{-1}}{\log(q_v)}Z_v(\widehat{f},|\cdot|_v) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-\frac{1}{2}}\,\frac{1-q_v^{-1}}{\log(q_v)}\int_{F_v^\times}\widehat{f}(g)|g|_v\,\mathrm{d}\mu_v(g) \tag{3.3.2.25}$$

and, given that $\nu_v = |\cdot|_v\mu_v$ as Radon measures over $F_v^\times$, which has ν_v-measure zero complement in F_v, the rightmost side of (3.3.2.25) indeed is the rightmost side of (3.3.2.23). □

3.4. Eigendistributions and Weil's Identity

In this section we discuss what we refer to, throughout this work, as *Weil's Identity*. It is a formula to compute the meromorphic function $Z(f,\chi|\cdot|_F^s)$ from Theorem 3.2.2.1 in terms of the finite Hecke L-function $L_f(s,\chi)$, for any Schwartz-Bruhat function f on $\mathbf{A}_F$.

This was observed by Weil in his paper on eigendistributions (see [**Weil-a**, §3]). The study of the global Idelic Z-functions defined in Definition 3.2.2.5 through the lens of functional analysis, was initiated by Weil in his paper [**Weil-a**] on adelic eigendistributions.

Therein, Weil gave a direct construction and a characterization (proving their uniqueness up to normalization, in a similar vein as for uniqueness of the Haar measure on a Hausdorff locally compact abelian group) of those adelic tempered distributions, i.e. elements of the strong dual $\mathcal{S}(\mathbf{A}_F)'_\beta$, satisfying a homogeneity condition with weight a continuous idelic quasi-character, under the adjoint of the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}(\mathbf{A}_F)$.

They turn out to be equal to the continuous functionals:

$$\frac{Z(\cdot, c)}{L(c)}$$

on $\mathcal{S}(\mathbf{A}_F)$ for each $c \in \Omega(C_F)$, and they vary holomorphically with respect to the continuous quasi-character $c \in \Omega(C_F)$. They define a holomorphic family of eigendistributions, equivalently an $\mathcal{S}(\mathbf{A}_F)'_\beta$-valued holomorphic function on $\Omega(C_F)$: its value at each $c \in \Omega(C_F)$ is homogeneous of weight c under the adjoint of the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}(\mathbf{A}_F)$.

The Theory of holomorphic and meromorphic functions valued in a quasi-complete Hausdorff topological $\mathbf{C}$-vector space, is developed in §C.5. The reader is referred to §C.5 (and specifically to Theorem C.5.2.6), and the references contained therein.

Eigendistributions will be reprised throughout the next Chapter, where we will be studying the Hausdorff locally convex topological $\mathbf{C}$-vector spaces of Adelic (both global and S-local) θ-functions, as well as the resulting Hausdorff locally convex topological $\mathbf{C}$-vector spaces of meromorphic functions on $\Omega(C_F)$ under the Mellin transform Z. We refer the reader to §5.5 of Chapter 4.

In this section, we record Weil's main results from [**Weil-a**, §1,3], and provide a few complements, for later reference.

3.4.1. Weil's Identity: the unramified case.

We begin by recalling the following:

NOTATION 3.4.1.1. We recall that for $c \in \Omega(C_F)$, i.e. for c a continuous quasi-character of C_F, we denote by c_v the restriction of c to $F_v^\times \subset C_F$, where $F_v^\times$ is embedded as a closed subgroup under the insertion into the v-adic "coordinate" $F_v^\times \to \mathrm{GL}_1(\mathbf{A}_F)$ followed by the quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_F) \to C_F$. By [**CF**, Ch.XV,§3.2,Lemma 3.2.1,p.324] we have $\bigotimes_{v \in \Sigma_F} c_v = c$.

Recall that we say c is *unitary* if it is valued in $\mathrm{U}(1) \subset \mathbf{C}^\times$. Any other unitary quasi-character c' of C_F with the same restriction to C_F^1, i.e. within the same connected component of $\Omega(C_F)$, is related to c by the formula

$$c' = c|\cdot|_F^{ix_0}$$

for a unique real number $x_0 \in \mathbf{R}$, and if c' is possibly non-unitary then there exists a unique complex number s such that $c' = c|\cdot|_F^s$.

We refer the reader to §3.1.1 for generalities on quasi-characters.

We will reserve the letter χ to denote an arbitrary continuous character of C_F. We denote by S_χ the finite set of non-Archimedean places of F at which χ is ramified.

The following Theorem relates the global Idelic Zeta functions from (3.2.2.1) of Theorem 3.2.2.1 to the finite Hecke L-function $L_f(s, \chi)$ from Definition 3.2.1.4 up to finitely many Euler factors.

We will be generalizing it by the end of this Chapter using Weil's Eigendistributions (see Theorem 3.4.3.3). The special case discussed in Theorem 3.4.1.1 below is already of interest and does not require the reader to have familiarity with distributions. We record it here separately for later use.

THEOREM 3.4.1.1 (Weil's Identity - I). *Let F be a number field. Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function. Assume f is of the form $f = f_\infty \otimes \bigotimes_{v\in\Sigma_{F,f}} f_v$, where $f_v \in \mathcal{S}(F_v)$ is of the form $f_v = 1_{K_v}$ for $K_v \subset F_v$ a compact open subset, with $K_v = \mathcal{O}_{F_v}$ for all but finitely many finite places v.*

Then the following product is convergent absolutely and uniformly on compact subsets of $\Omega(C_F)_{>0}$ to a holomorphic function of $c \in \Omega(C_F)_{>0}$:

$$\widetilde{Z}(f,c) := \widetilde{Z}_{S_\infty}(f_\infty, c_\infty) \prod_{v\in\Sigma_{F,f}} \widetilde{Z}_v(f_v, c_v) \tag{3.4.1.1}$$

where

$$\widetilde{Z}_{S_\infty}(f_\infty, c_\infty) := \int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})} f_\infty(g_\infty)\, c_\infty(g_\infty)\, \mathrm{d}\mu_{S_\infty}(g_\infty) \tag{3.4.1.2}$$

where $c_\infty := \bigotimes_{v\in S_\infty} c_v$, and for each finite place v of F

$$\widetilde{Z}_v(f_v, c_v) := \int_{F_v^\times} (f_v(g_v) - f_v(\pi_v^{-1} g_v))\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) \tag{3.4.1.3}$$

if c is unramified at v, where π_v denotes a choice of uniformizer for F_v and

$$\widetilde{Z}_v(f_v, c_v) := \int_{F_v^\times} f_v(g_v) c_v(g_v)\, \mathrm{d}\mu_v(g_v) = Z_v(f_v, c_v) \tag{3.4.1.4}$$

if c is ramified at v.

Moreover, for a fixed choice of arbitrary continuous character $\chi \in \widehat{C_F}$ and S any finite set of places of F containing $S_\infty \cup S_\chi$ such that $f_v = 1_{\mathcal{O}_{F_v}}$ for all finite $v \notin S$, then as holomorphic functions over $\mathbf{C}_{>0}$ we have

$$\widetilde{Z}(f, \chi|\cdot|_F^s) = Z_{S_0}(f_{S_0}, \chi_{S_0}|\cdot|_{F,S_0}^s) \prod_{v\in S - S_0} \widetilde{Z}_v(f_v, \chi_v|\cdot|_v^s) \tag{3.4.1.5}$$

where $S_0 = S_\infty \cup S_\chi \subset S$, $f_{S_0} := f_\infty \otimes \bigotimes_{v\in S_\chi} f_v$, and χ_{S_0} is the unique continuous character of J_{F,S_0} obtained from χ by descent by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S_0}$-invariance of the restriction of χ to C_{F,S_0}. Moreover, as holomorphic functions over $\mathbf{C}_{>0}$, we have

$$Z(f, \chi|\cdot|_F^s) = Z_{S_0}(f_{S_0}, \chi_{S_0}|\cdot|_{F,S_0}^s) \prod_{v\in S - S_0} \widetilde{Z}_v(f_v, \chi_v|\cdot|_v^s) L_{\Sigma_F - S_0}(s, \chi) \tag{3.4.1.6}$$

where $L_{\Sigma_F - S_0}(s,\chi) := \prod_{v\in S_\chi}(1 - \chi(\mathfrak{p}_v) N(\mathfrak{p}_v)^{-s}) L_f(s,\chi)$.

Recall that $Z_S(f,c)$ is the S-idelic Zeta function of the Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, which we defined in Definition 5.1.1.1 and studied throughout §5.1.1.

PROOF. To see this, we observe that the first factor integral defining $\widetilde{Z}_{S_\infty}(f_\infty, c_\infty)$ from (3.4.1.2) is absolutely convergent, because, upon calling B a compact neighborhood of $0 \in \mathbf{A}_{F,S_\infty} \simeq \prod_{v\in S_\infty} F_v$ of the form $\prod_{v\in S_\infty} Z_v$, for $B_v \subset F_v$ a compact neighborhood of zero in F_v, then the integral over $B \cap \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ of $f_\infty\, c_\infty \prod_{v\in S_\infty} |\cdot|_v^{-1}$ with respect to the Lebesgue measure on the Euclidean space $\mathbf{A}_{F,S_\infty}$ is absolutely convergent so long as $\sigma(c_v) > 0$ for every Archimedean place v of F, since such convergence is ensured by the convergence of $\int_{B_v} |\cdot|_v^{\sigma(c_v)-1}\, \mathrm{d}x_v$ for $\sigma(c_v) > 0$ and by boundedness over B of f_∞. Since $\sigma(c) > 0$, then $\sigma(c_v) > 0$ for every Archimedean place v of F, and so the desired convergence holds, locally uniformly with respect to c_v for each such v.

The Schwartz condition on f_∞ then implies the absolute convergence of the integral over the complement of B in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$. Both convergent integrals converge locally uniformly over $\Omega(C_F)_{>0}$ as functions of c through $c_\infty = \bigotimes_{v\in S_\infty} c_v$.

We note that in the special case c is unramified outside S_∞, then c_∞ is trivial on the global units, and we have $\widetilde{Z}_{S_\infty}(f_\infty, c_\infty) = Z_{S_\infty}(f_\infty, c_\infty)$, the S_∞-idelic Z-function of f_∞ over the connected component of c_∞ in $\Omega(J_{F,S_\infty})$.

We call S'_f the finite set of finite places of F such that $f_v \neq 1_{\mathcal{O}_v}$. We fix $c \in \Omega(C_F)$, and we compute, for every finite place $v \notin (S_f \cup S_c)$, where S_c is the finite set of non-Archimedean places at which c is ramified:

$$\begin{aligned}\int_{F_v^\times} (f_v(g_v) - f_v(\pi_v^{-1} g_v)) c_v(g_v)\, \mathrm{d}\mu_v(g_v) &= \int_{F_v^\times} (1_{\mathcal{O}_v} - 1_{\pi_v \mathcal{O}_v})(g_v) c_v(g_v)\, \mathrm{d}\mu_v(g_v) \\ &= \int_{F_v^\times} 1_{\mathcal{O}_v^\times}(g_v) c_v(g_v)\, \mathrm{d}\mu_v(g_v) \\ &= \int_{\mathcal{O}_v^\times} \mathrm{d}\mu_v(g_v) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2}.\end{aligned}$$

We refer the reader to Definition A.3.2.2 for the normalization defining the local multiplicative Tamagawa measures on the local fields $F_v^\times$. Note that in particular $\mu_v(\mathcal{O}_{F_v}^\times) = 1$ for all but finitely many finite places v of F.

For every $v \in S'_f$ at which c is unramified, we have

$$\begin{aligned}\int_{F_v^\times} (f_v(g_v) - f_v(\pi_v^{-1} g_v))\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) &= \int_{F_v^\times} f_v(g_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) - \int_{F_v^\times} f_v(\pi_v^{-1} g_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) \\ &= \int_{F_v^\times} f_v(g_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) - \int_{F_v^\times} f_v(g_v)\, c_v(\pi_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) \\ &= (1 - c_v(\pi_v)) \int_{F_v^\times} f_v(g_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v) \\ &= (1 - c_v(\pi_v))\, Z(f_v, c_v)\end{aligned}$$

where the integral $\int_{F_v^\times} f_v(g_v)\, c_v(g_v)\, \mathrm{d}\mu_v(g_v)$ converges, for each $v \in S'_f$ at which c is unramified, absolutely and uniformly over $\Omega(F_v^\times)_{>0}$, where $\Omega(F_v^\times)$ is the Riemann surface of the continuous quasi-characters of the non-Archimedean local field F_v, and $\Omega(F_v^\times)_{>0} \subset \Omega(F_v^\times)$ is the open subset of those quasi-characters of exponent $\sigma(c_v) > 0$.

It follows that the product defining $\widetilde{Z}(f, c)$ in (3.4.1.1) is finite and its factors are holomorphic over $\Omega(C_F)_{>0}$, since each is either $\widetilde{Z}_{S_\infty}(f_\infty, c_\infty)$ (which is holomorphic as a function of $c \in \Omega(C_F)_{>0}$ since it factors through the monomial function $\otimes_{v \in S_\infty} c_v$ of $c \in \Omega(C_F)_{>0}$) or of the form $Z(f_v, c_v)$ or $(1 - c_v(\pi_v))Z(f_v, c_v)$ for finite v (both holomorphic over $\Omega(F_v^\times)_{>0}$ by [**CF**, Ch.XV,Lemma 2.4.1]). As a function of c, $\widetilde{Z}(f, c)$ is therefore holomorphic over $\Omega(C_F)_{>0}$ because it is a finite product of functions that are each holomorphic over $\Omega(C_F)_{>0}$ by Lemma 3.1.2.1.

We now observe that, for $c = \chi|\cdot|_F^s$ with χ a continuous character of C_F, upon parametrizing $\Omega(C_F)(\chi|_{C_F^1})_{>1}$ by means of the global holomorphic chart

$$c = \chi\, |\cdot|_F^s \tag{3.4.1.7}$$

with $\mathrm{Re}(s) > 1$, then we have that the identities of holomorphic functions in (3.4.1.5) and (3.4.1.6) hold over $\Omega(C_F)(\chi|_{C_F^1})_{>1}$.[3]

Since $\widetilde{Z}(f, \chi|\cdot|_F^s)$ is holomorphic over $\Omega(C_F)(\chi|_{C_F^1})_{>0}$, and since both $L_f(s, \chi)$, and hence also $L_{\Sigma_F - S_0}(s, \chi)$, and $Z(f, \chi|\cdot|_F^s)$ extend to meromorphic functions over $\Omega(C_F)(\chi|_{C_F^1})_{>0}$, where $L_f(s, \chi)$ is regarded as a meromorphic function on $\Omega(C_F)(\chi|_{C_F^1})_{>0}$ by means of the parametrization of $\Omega(C_F)(\chi|_{C_F^1})_{>0}$ from (3.4.1.7) with $\mathrm{Re}(s) > 0$, the identities in (3.4.1.5) and (3.4.1.6) hold over $\Omega(C_F)(\chi|_{C_F^1})_{>0}$ by the Identity Theorem. □

3.4.2. Applications of the unramified case of Weil's Identity.

[3] Why do such identities hold? Must say more about how this follows.

3.4.2.1. *Convergence on the relative* 1*-line.* We record here below an application of Weil's Identity from Theorem 3.4.1.1: an observation that, while not used throughout this work, we find opportune to mention here. It concerns the validity of the identity in (3.2.2.1) from Theorem 3.2.2.1, which is taken as the definition of $Z(f,c)$ over $\Omega(C_F)_{>1}$, over the relative 1-line, i.e. the discretely-parametrized family of vertical lines:

$$\{\sigma(c)=1, c\in\Omega(C_F)\}\subset\Omega(C_F)$$

over the discrete parameter space $\widehat{C_F^1}$.

PROPOSITION 3.4.2.1 (Convergence on the relative 1-line). *Let F be a number field. For any real number $A>0$, we define the subset:*

$$L_A:=\{\sigma(c)=1, c\in\Omega(C_F), c|_{C_F^1}\neq 1\}\cup\{|\cdot|_F^{1+ix}, x\in\mathbf{R}-(-A,A)\}\subset\Omega(C_F).$$

For any Schwartz-Bruhat function $f\in\mathcal{S}(\mathbf{A}_F)$, as a function of $c\in L_A$ the Lebesgue integral

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)} f(g)c(g)\,\mathrm{d}\mu(g) \tag{3.4.2.1}$$

is finite and varies continuously in such c. As a function of c, it equals the restriction of $Z(f,c)$ to L_A.

PROOF. The complex measure $c\mu$ on $\mathrm{GL}_1(\mathbf{A}_F)$ is Radon, and formation of the integral in (3.4.2.1) is $\mathbf{C}$-linear with respect to $f\in\mathcal{S}(\mathbf{A}_F)$. By Proposition 2.1.3.2(3), it is therefore sufficient to prove the assertions of the Proposition in the special case f is of the form $f=f_\infty\otimes\bigotimes_{v\in\Sigma_{F,f}}f_v$, with $f_\infty\in\mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_v\in\mathcal{S}(F_v)$ for every finite place v of F, and with $f_v=1_{\mathcal{O}_{F_v}}$ for all but finitely many finite places of F

By [**RV**, Ch.5,§5.1,Prop.5.6(i),p.186], the integral from (3.4.2.1) equals the possibly divergent product:

$$\int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})} f_\infty(g_\infty)c_\infty(g_\infty)\,\mathrm{d}\mu_\infty(g_\infty)\prod_{v\in\Sigma_{F,f}}\int_{F_v^\times} f_v(g_v)c_v(g_v)\,\mathrm{d}\mu_v(g_v). \tag{3.4.2.2}$$

We prove that product in (3.4.2.2) converges, for any $A>0$, pointwise over L_A and uniformly over its compact subsets. If so, then its limit function is therefore continuous over L_A, and hence the same holds for the integral from (3.4.2.1).

The convergence assertions on the product in (3.4.2.2) can be checked on each connected component of $\Omega(C_F)$, respectively. We let $\omega\in\widehat{C_F^1}$ be arbitrary, and choose a base point $\chi\in\widehat{C_F}$ for the connected component parametrized by $\omega=\chi|_{C_F^1}\in\widehat{C_F^1}$. We parametrize $\Omega(C_F)(\omega)$ holomorphically as $\{\chi|\cdot|_F^s, s\in\mathbf{C}\}$.

Let S_χ be the finite set of finite places of F at which χ is ramified, and let S_0 be the finite set of finite places of F such that $f_v\neq 1_{\mathcal{O}_{F_v}}$.

We call $S:=S_\infty\cup S_0\cup S_\chi$. By Weil's Identity (see (3.4.1.6) from Theorem 3.4.1.1), it is enough to show that the countable net of meromorphic functions over the complex s-plane, defined by the partial Euler products:

$$S'\mapsto\prod_{v\in S'}L_v(\chi_v|\cdot|_v^s)$$

for the Hecke L-function $L(s,\chi)$, as S' varies over all finite sets of finite places of F such that $S'\cap S=\varnothing$, converges pointwise to $L_f(s,\chi)\prod_{v\in S_0}L_v(\chi_v|\cdot|_v^s)^{-1}$ over $1+i\mathbf{R}$ if $\chi|_{C_F^1}\neq 1$ (uniformly on its compact subsets), and $1+i(\mathbf{R}-(-A-x_0,A-x_0))$ if $\chi=|\cdot|_F^{ix_0}$ (uniformly on its compact subsets).

This is indeed the case by Corollary B.7.2.5 (a consequence of Perron's Formula; see also [**T2**, Ch.III,§3.15,p.66] for the special case $F=\mathbf{Q}$ and $\chi=1$). □

3.4.2.2. *Estimates on adelic θ-functions, uniformly on Schwartz-Bruhat-bounded subsets.* We record here the following result as another application of Theorem 3.4.1.1, presenting a different method to establish some

of the estimates from Theorem 2.2.5.4 on the decay properties of the functions $\Phi_F(f)(g)$ over C_F as $|g|_F \to 0^+$ and as $|g|_F \to +\infty$, for f a Schwartz-Bruhat function on $\mathbf{A}_F$.

The Theorem below was already established in Theorem 2.2.5.4. The proof we provide has one important advantage though. It gives a concrete interpretation to the constants C_N that appear in the estimates stated in Theorem 2.2.5.4, revealing why they can be maximized over bounded subsets in $\mathcal{S}(\mathbf{A}_F)$. The existence of maxima comes down to $\mathcal{S}(\mathbf{A}_F)$ satisfying the Heine-Borel property. Already through the proof of the existence of uniform versions of the constants C_N from Theorem 2.2.5.4 (uniform as f varies over a fixed bounded subset $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_F)$) the Heine-Borel property was used to the same effect: maximizing certain auxiliary continuous functions (seminorms) arising in the process of building the uniform constants $C_{\mathcal{B},N}$ from parts (5) and (6) of Theorem 2.2.5.4.

The idelic Mellin transform in the proof below reveals the nature of those constants, and why they can be rendered uniform on bounded subsets of $\mathcal{S}(\mathbf{A}_F)$.

THEOREM 3.4.2.2 (Estimates over the Idele classes - III). *Let F be a number field, $f \in \mathcal{S}(\mathbf{A}_F)$ a Schwartz-Bruhat function. Let $\omega \in \widehat{C_F^1}$ be a fixed continuous character. The continuous function $\Phi_{F,\omega}(f)$ on C_F satisfies the following conditions:*

(1) *For every integer $N \geq 0$, there exists a real number $C_N > 0$ depending on N and f, such that for every $g \in C_F$, we have*

$$|\Phi_{F,\omega}(f)(g)| \leq C_N |g|_F^{-N} \tag{3.4.2.3}$$

as $g \in C_F$ is such that $|g|_F \to +\infty$.

(2) *For every integer $N \geq 0$, there exists a real number $C_N > 0$ depending on N and f, such that for every $g \in C_F$, we have*

$$|\Phi_{F,\omega}(f)(g) - (|g|_F^{-1}\, \widehat{f}(0) - \, f(0))\, \delta_\omega(g)| \leq C_N\, |g|_F^N \tag{3.4.2.4}$$

as $g \in C_F$ is such that $|g|_F \to 0^+$, where

$$\delta_\omega : C_F \to \mathbf{C} \tag{3.4.2.5}$$

is the bounded function on C_F defined as:

$$\delta_\omega(g) := 1_{\{x \in C_F : |x|_F \leq 1\}}(g),\ g \in C_F \tag{3.4.2.6}$$

if $\omega = 1$, and identically zero otherwise.

Moreover, the constants C_N from (3.4.2.3) *and* (3.4.2.4) *can be made to be uniform with respect to f varying over a fixed arbitrary bounded subset $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_F)$.*

REMARK 3.4.2.3 (On the nature of the uniform constants $C_{\mathcal{B},N}$ from Theorem 2.2.5.4). The main points of the argument that follows, are two. Let $f \in \mathcal{S}(\mathbf{A}_F)$ be of the form $\pi_S(f_S)$ for some $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$. Recall that $\pi_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_F)$ is the transition operator from (2.1.4.5) defined by sending f_S to $f_S \otimes 1_K$ with $K = \prod_{v \notin S} \mathcal{O}_{F_v}$.

- Use Weil's Identity in the unramified case (i.e. (3.4.1.6) from Theorem 3.4.1.1) to deduce properties of $Z(f, \chi|\cdot|_F^s)$ from properties of $Z_S(f_S, \chi_S|\cdot|_{F,S}^s)$ combined with well known properties of the finite Hecke L-function itself $L_f(s,\chi)$ (these are shown in Appendix B). We know much about the growth properties of $L_f(s,\chi)$ uniformly on vertical strips of bounded width of the complex s-plane, and π_S is turned into multiplication by $L_f(s,\chi)$ with finitely many Euler factors "removed" (all the Archimedean Euler factors, and all the S-indexed non-Archimedean Euler factors too). We refer the reader to §B.1 and specifically to Proposition B.1.1.3.
- Exhibit the constants appearing in the estimates from Theorem 2.2.3.4 (see (2.2.3.4) and (2.2.3.5)) as L^1-norms of $Z(f,c)$ on certain vertical lines over $\Omega(C_F)$ of prescribed abscissa (dependent on N).

The second point illustrates to the reader why one can indeed successfully make those constants uniform on bounded subsets of $\mathcal{S}(\mathbf{A}_F)$. This, we already did in Theorem 2.2.5.4 obtaining the constant $C_{\mathcal{B},N}$, through direct arguments via geometry of numbers.

Therein, we used the following important fact about $\mathcal{S}(\mathbf{A}_F)$: it satisfies the Heine-Borel property, i.e. closed bounded subsets are compact. It is a Montel space by Lemma 2.1.3.2, and hence it satisfies this property, just as the Schwartz space of any Euclidean space (in the special case of Euclidean spaces, this is a consequence of Ascoli-Arzelà). We used the Heine-Borel property to maximize certain continuous functions on bounded subsets in $\mathcal{S}(\mathbf{A}_F)$ on our way to build the uniform constants $C_{\mathcal{B},N}$. Upon exhibiting such constants C_N in the statement of Theorem 2.2.5.4 as L^1-norms of $Z(f,c)$ on vertical lines of $\Omega(C_F)_{>1}$, then the fact that one can render them uniform on bounded subsets is not only reproved, but explained: such L^1-norms induce continuous seminorms on $\mathcal{S}(\mathbf{A}_F)$ and by the Heine-Borel property they attain maxima on bounded subsets.

We write, for $c \in \Omega(C_F)$ unramified outside a finite set S of places of F containing all the Archimedean places, c_S the unique quasi-character of $J_{F,S}$ obtained from c by restricting it to $C_{F,S}$ and descending it by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance (where $\widehat{\mathcal{O}}^\times_{\Sigma_F-S} = \prod_{v \notin S} \mathcal{O}^\times_{F_v}$).

Proof of Theorem 3.4.2.2. By Theorem 3.2.2.1, $Z(f,c)$ has at most two poles, at the quasi-characters $c = 1$ and $c = |\cdot|_F$. Both of them are located in the connected component of $\Omega(C_F)$ parametrized by $\omega = 1 \in \widehat{C_F^1}$, i.e. in the fibre of π from (3.1.1.1) over $\{1\}$ in $\widehat{C_F^1}$.

We now restrict $Z(f,c)$ to $\Omega(C_F)(1)$, the connected component of the trivial character, and obtain by Theorem 3.2.2.1(3), upon parametrizing $\Omega(C_F)(1)$ by $s \mapsto |\cdot|_F^s$ and choosing a continuous homomorphic section τ to the idelic norm $|\cdot|_F$:

$$Z(f, |\cdot|_F^s) = \int_{\mathbf{R}_{>0}} \Phi_{F,\omega=1}(f)(\tau(t)) t^s \, \frac{\mathrm{d}t}{t} = M(\Phi_{F,\omega=1}(f) \circ \tau)(s)$$

adhering to the notation $M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \, \frac{\mathrm{d}t}{t}$ for the Mellin transform (see (B.2.1.2)).

By Theorem 3.2.2.1(3), we know that, for $\phi := \Phi_{F,\omega=1}(f) \circ \tau$, $M(\phi)(s)$ converges absolutely and uniformly on the compact subsets of $\mathbf{C}_{>1}$.

By Theorem 3.2.2.1(1), the Mellin transform $M(\phi)(s)$ extends to a meromorphic function over the entire complex s-plane, with simple poles at most at $s = 0, 1$, whose residues are $-\overline{\mu}_1(C_F) f(0)$ and $\overline{\mu}_1(C_F)\widehat{f}(0)$, respectively.

We now use Weil's Identity. Let S be a finite set of places of F containing all the Archimedean places. We fix $f \in \mathcal{S}(\mathbf{A}_F)$ of the form $f = f_\infty \otimes \bigotimes_{v \in \Sigma_{F,f}} f_v$ with $f_v = 1_{K_v}$ for $K_v \subset F_v$ a compact open subset for every finite place v, and $K_v = \mathcal{O}_{F_v}$ for every $v \notin S$. This we can arrange once such f is chosen, up to enlarging S. We denote by f_S the Schwartz-Bruhat function in $\mathcal{S}(\mathbf{A}_{F,S})$ defined as $f_S := f_\infty \otimes \bigotimes_{v \in S_f} 1_{K_v}$. For simplicity, we choose S containing S_∞ and minimal with respect to the above property (i.e. that $K_v \neq \mathcal{O}_{F_v}$ should occur for $v \in S$).

As a consequence of (3.4.1.6) from Theorem 3.4.1.1 (applied to a large enough finite set of places S of F containing all the Archimedean places), we have, as meromorphic functions on the complex s-plane, away from the zeros of $\prod_{v \in S_f}(1 - q_v^{-s})$ and the poles of $\zeta_F(s)$ and $\widetilde{Z}_v(1_{K_v}, |\cdot|_v^s)$ (if any):

$$Z(f, |\cdot|_F^s) = Z_S(f_S, |\cdot|_{F,S}^s) \prod_{v \in S_f} (1 - q_v^{-s}) \zeta_F(s) \prod_{v \in S_f} \widetilde{Z}_v(1_{K_v}, |\cdot|_v^s) \tag{3.4.2.7}$$

where $\zeta_F(s)$ is the Dedekind zeta function of F, $q_v := N(\mathfrak{p}_v)$ and $\mathfrak{p}_v$ the prime ideal of $\mathcal{O}_F$ corresponding to the finite place v, and $\widetilde{Z}_v$ was defined in (3.4.1.3). It is the meromorphic function $\frac{Z_v(1_{K_v}, |\cdot|_v^s)}{1 - q_v^{-s}}$ (which turns out to be entire).

The idea now is to show that, in light of the identity from (3.4.2.7), $Z(f, |\cdot|_F^s)$ satisfies the assumptions of the Asymptotic Residue Theorem (see Theorem B.3.2.1). This will lead, through Mellin Inversion, to all the estimates stated in the Theorem, and to new information on the constants C_N.

We now apply the Asymptotic Residue Theorem (see Theorem B.3.2.1) to the meromorphic function

$$Z(f, |\cdot|_F^s)t^{-s} = M(\Phi_{F,\omega=1}(f) \circ \tau)(s)t^{-s}$$

of $s \in \mathbf{C} - \{0, 1\}$.

In order to apply the Asymptotic Residue Theorem over a vertical strip of bounded width $\mathbf{C}_I$, for $I \subset \mathbf{R}$ a connected open non-empty interval, it is sufficient to show the following (compare with the statement of Theorem B.3.2.1). We must show that $Z(f, |\cdot|_F^s)$ can be written as a product

$$\Phi(s)u(s)$$

over a slightly larger open vertical strip $\mathbf{C}_J$ of bounded width containing $\mathbf{C}_I$, with Φ holomorphic over $\mathbf{C}_J$ and u meromorphic on $\mathbf{C}_J$ with *slow growth near its poles* (for this concept, the reader is referred to Definition B.3.1.1). The function Φ must further satisfy the assumption that Φ is Schwartz uniformly on any open vertical strip of bounded width contained within $\mathbf{C}_J$.

Indeed, $\prod_{v \in S_f}(1 - q_v^{-s})$ is bounded uniformly on vertical strips of bounded width, together with all of its derivatives, and its zeros resolve the singularities of $Z_S(f_S, |\cdot|_{F,S}^s)$ located on the imaginary line of the complex s-plane, with the sole possible exception of a single point, i.e. $s = 0$. The meromorphic function $\zeta_F(s)$ grows polynomially uniformly over the vertical strips of bounded width of the complex s-plane (see Proposition B.1.1.3), and it has simple poles at $s = 0$ and $s = 1$. The functions $\widetilde{Z}_v(1_{K_v}, |\cdot|_v^s)$ are, for each $v \in S_f$, entire (by (3.4.1.3) combined with the local functional equation, see [**CF**, Ch.XV,Thm.2.4.1] and [**CF**, Ch.XV,§2.5]) and bounded uniformly on vertical strips (by vertical periodicity) together with all of their derivatives.

Finally, $Z_S(f_S, |\cdot|_{F,S}^s)$ is Schwartz uniformly over the vertical strips of bounded width of the complex half-plane $\mathbf{C}_{>0}$ by the Uniform Riemann-Lebesgue Lemma of Proposition B.2.2.3, which applies as a consequence of Theorem 2.3.1.4 and Proposition 2.3.2.2. The S-local functional equation (see (3.3.1.3) from Theorem 3.3.1.1) together with the estimates on ρ_S and its derivatives from Proposition 3.3.1.5) then exhibit $Z_S(f_S, |\cdot|_{F,S}^s)$ as being indeed of the form $\Phi(s)u(s)$ with Φ Schwartz uniformly on the vertical strips of bounded width of the complex half-plane $\mathbf{C}_{>-N}$ for each fixed $N > 0$, and u meromorphic with slow growth near its poles therein.

We give the idea of how to see this. Once $N > 0$ is fixed, we want to apply the Asymptotic Residue Theorem to $Z(f, |\cdot|_F^s)$ over a rectangle contained in $\mathbf{C}_{>-N-\varepsilon}$ for $\varepsilon > 0$ small enough and fixed (whose rightmost edge has abscissa $\sigma > 1$ and horizontal edges are let to tend to $\pm\infty$ vertically). Over the half-plane $\mathbf{C}_{>-N-\varepsilon}$, $Z_S(f_S, |\cdot|_{F,S}^s)$ has its poles within the discrete subset of the zeros of $\prod_{v \in S_f}(1 - q_v^{-s})$ (with multiplicity bounded above by their respective order of vanishing) and within the finite subset of the poles of the Archimedean local factors of the Hecke L-function $L(s, \chi)$. In particular, we define the function

$$\Phi(s) := Z_S(f_S, |\cdot|_{F,S}^s) \prod_{v \in S_f}(1 - q_v^{-s})p_N(s)\zeta_F(s)s(s-1)$$

with p_N any entire polynomial function on $\mathbf{C}$ whose zeros are located at the poles of $Z_S(f_S, |\cdot|_{F,S}^s)$ counted with multiplicity. The meromorphic function Φ is entire, and Schwartz uniformly on the open non-empty vertical strips of bounded width contained within $\mathbf{C}_{>-N-\varepsilon}$ (to see this, we use Lemma D.3.1.1 and the Uniform Riemann-Lebesgue Lemma of Proposition B.2.2.3, which informs us of the rapid decay of $Z_S(f_S, |\cdot|_{F,S}^s)$ over $\mathbf{C}_{>0}$) and since $u_N(s) := 1/(p_N(s)s(s-1))$ is meromorphic and has slow growth near its poles, then indeed $Z(f, |\cdot|_{F,S}) = \Phi(s)u_N(s)$ by (3.4.2.7), as claimed, and the Asymptotic Residue Theorem applies to $Z(f, |\cdot|_{F,S}^s)$ over $\mathbf{C}_{>-N-\varepsilon}$.

The identity from (3.4.2.7) then implies that the Asymptotic Residue Theorem applies to $Z(f, |\cdot|_F^s)$. We obtain, for every fixed real number $\sigma > 1$, and every integer $N > 0$:

(3.4.2.8)
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M(\Phi_{F,\omega=1}(f)\circ\tau)(s)t^{-s}\,\mathrm{d}s = t^{-1}\overline{\mu}_1(C_F)\widehat{f}(0) - \overline{\mu}_1(C_F)f(0) + \frac{1}{2\pi i}\int_{-N-i\infty}^{-N+i\infty} M(\Phi_{F,\omega=1}(f)\circ\tau)(s)t^{-s}\,\mathrm{d}s.$$

The leftmost side of (3.4.2.8) reads $\Phi_{F,\omega=1}(f)(\tau(t))$ by Fourier-Mellin inversion. We therefore deduce from (3.4.2.8) that the following identity holds:

$$\Phi_{F,\omega=1}(f)(\tau(t)) - t^{-1}\overline{\mu}_1(C_F)\widehat{f}(0) + \overline{\mu}_1(C_F)f(0) = \frac{1}{2\pi i}\int_{-N-i\infty}^{-N+i\infty} M(\Phi_{F,\omega=1}(f)\circ\tau)(s)t^{-s}\,\mathrm{d}s. \tag{3.4.2.9}$$

The following estimate therefore holds:

$$|\Phi_{F,\omega=1}(f)(\tau(t)) - t^{-1}\overline{\mu}_1(C_F)\widehat{f}(0) + \overline{\mu}_1(C_F)f(0)| \leq \frac{t^N}{2\pi}\|M(\Phi_{F,\omega=1}(f)\circ\tau)(-N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)} \tag{3.4.2.10}$$

as $t \to 0^+$ (the only regime in which the bound is useful). Note that the estimate in (3.4.2.10) does not depend on σ.

We now apply the Asymptotic Residue Theorem (see Theorem B.3.2.1) one more time, to obtain, for every integer $N > 1$ and for every real number $\sigma \in (1, N)$:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M(\Phi_{F,\omega=1}(f)\circ\tau)(s)t^{-s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{N-i\infty}^{N+i\infty} M(\Phi_{F,\omega=1}(f)\circ\tau)(s)t^{-s}\,\mathrm{d}s \tag{3.4.2.11}$$

since $Z(f, |\cdot|_F^s)$ has no poles in the vertical strip $\mathbf{C}_{(\sigma-\varepsilon,N+\varepsilon)}$ for all small enough $\varepsilon > 0$ and hence the same can be said about $Z(f, |\cdot|_F^s)t^{-s}$.

The leftmost side of (3.4.2.11) is $\Phi_{F,\omega=1}(f)(\tau(t))$ by Mellin inversion (see Theorem B.2.1.7 and Proposition B.2.1.8), and the rightmost side is bounded above by $\frac{t^{-N}}{2\pi}\|M(\Phi_{F,\omega=1}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)}$, leading to the estimate:

$$|\Phi_{F,\omega=1}(f)(\tau(t))| \leq \frac{t^{-N}}{2\pi}\|M(\Phi_{F,\omega=1}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)} \tag{3.4.2.12}$$

as $t \to +\infty$. Once again we observe that the estimate in (3.4.2.12) does not depend on σ.

For $\omega \neq 1$, the meromorphic function $Z(f, c)$ restricted to $\Omega(C_F)(\omega)$ has no poles, and the same argument as in the foregoing shows that the Asymptotic Residue Theorem applies to it, leading to the estimates, for each integer $N > 1$:

$$|\Phi_{F,\omega}(f)(\tau(t))| \leq \frac{t^{-N}}{2\pi}\|M(\Phi_{F,\omega}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)} \tag{3.4.2.13}$$

as $t \to +\infty$, and, for every integer $N > 0$:

$$|\Phi_{F,\omega}(f)(\tau(t))| \leq \frac{t^{N}}{2\pi}\|M(\Phi_{F,\omega}(f)\circ\tau)(-N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)} \tag{3.4.2.14}$$

as $t \to 0^+$. We established the Theorem for $N > 1$, so we are done. □

Remark 3.4.2.4. Using compatibility of the global θ-function operator Φ_F with directional derivatives (see Lemma 2.2.4.3) in combination with the Uniform Riemann-Lebesgue Lemma of Theorem D.3.4.2, it is indeed possible to prove the "non ω-homogeneous" version of the estimates in Theorem 3.4.2.2 proved in Theorem 2.2.3.4.

This reads, for any fixed integer $N \geq 0$, for any $f \in \mathcal{S}(\mathbf{A}_F)$:

$$|\Phi_{F,\omega}(f)(g)| \leq C_N|g|_F^{-N} \tag{3.4.2.15}$$

as $|g|_F \to +\infty$ for some real number $C_N > 0$, and

$$|\Phi_F(f)(g) - (|g|_F^{-1}\,\widehat{f}(0) - \,f(0))\,\delta(g)| \leq C_N\,|g|_F^N \tag{3.4.2.16}$$

for some real number $C_N > 0$ and as $|g|_F \to 0^+$, where

$$\delta : C_F \to \mathbf{C}$$

is the bounded function on C_F defined in (3.4.2.6). We note that the Fourier transform of the bounded function δ in (3.4.2.6) is supported on the connected component of the trivial character in $\widehat{C_F}$ (namely the space of characters c trivial on C_F^1).

In other words, through Weil's Identity which we discussed in Theorem 3.4.1.1, we can read off the estimates from Theorem 2.2.3.4 from the forward side of the Mellin transform, through properties of Hecke L-functions (analytic continuation, functional equation, polynomial growth of $L_f(s,\chi)$ uniformly on vertical strips of bounded width, all of which are discussed in Appendix B).

We sketch the method here, to shed light on the nature of the uniform constants $C_{\mathcal{B},N}$ from Theorem 2.2.5.4. We use the functional equation from (3.2.2.2), giving the identity

$$M(\Phi_{F,\omega}(f)\circ\tau)(-N+ix) = M(\Phi_{F,\overline{\omega}}(\widehat{f})\circ\tau)(1+N-ix)$$

as functions of $x \in \mathbf{R}$. We now assume $N > 1$, so that the estimates in (3.4.2.10) and (3.4.2.12), respectively (3.4.2.13) and (3.4.2.14), all hold at the same time with this fixed choice of N.

The functions of ω given by

$$\|M(\Phi_{F,\omega}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R},\mathrm{d}x)},\ \|M(\Phi_{F,\overline{\omega}}(f)\circ\tau)(1+N-ix)\|_{L^1(\mathbf{R},\mathrm{d}x)}$$

are integrable with respect to the Plancherel measure to the Haar measure $\overline{\mu}_1$ on C_F^1. Note that since C_F^1 is compact Hausdorff, the Plancherel measure on $\widehat{C_F^1}$ of the measure $\overline{\mu}_1$ on C_F^1 (see [**DE**, Thm.3.4.8] for the definition of the Plancherel measure on the dual of a Hausdorff locally compact abelian group equipped with a choice of Haar measure) is a nonzero multiple of the counting measure.

We observe that it implies (3.4.2.15) and (3.4.2.16). Indeed, from (3.4.2.10), (3.4.2.12), (3.4.2.13) and (3.4.2.14), and using the Hausdorff-Young inequality applied to the Fourier transform on C_F^1, we obtain the estimates

$$|\Phi_F(f)(g)| \leq \frac{|g|_F^N}{2\pi}\|M(\Phi_{F,\overline{\omega}}(\widehat{f})\circ\tau)(1+N-ix)\|_{L^1(\mathbf{R}\times\widehat{C_F^1},\mathrm{d}x\otimes\mathrm{d}\widehat{\overline{\mu}}_1)} \tag{3.4.2.17}$$

as $|g|_F < 1$, and

$$|\Phi_F(f)(g) - (|g|_F^{-1}\,\widehat{f}(0) - f(0))\,\delta(g)| \leq \frac{|g|_F^{-N}}{2\pi}\|M(\Phi_{F,\omega}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R}\times\widehat{C_F^1},\mathrm{d}x\otimes\mathrm{d}\widehat{\overline{\mu}}_1)}. \tag{3.4.2.18}$$

as $|g|_F > 1$.

We now, at last, recognize the constants

$$C_N(f) := \frac{1}{2\pi}\|M(\Phi_{F,\overline{\omega}}(\widehat{f})\circ\tau)(1+N-ix)\|_{L^1(\mathbf{R}\times\widehat{C_F^1},\mathrm{d}x\otimes\mathrm{d}\widehat{\overline{\mu}}_1)},\ C'_N(f) := \frac{1}{2\pi}\|M(\Phi_{F,\omega}(f)\circ\tau)(N+ix)\|_{L^1(\mathbf{R}\times\widehat{C_F^1},\mathrm{d}x\otimes\mathrm{d}\widehat{\overline{\mu}}_1)}$$

as serving the same role of the constants C_N from Theorem 2.2.5.4. They depend on f (and on N, as they should), but they do so continuously in f. This will be shown much later throughout this treatise, and the proof of this fact will *not* use Theorem 2.2.5.4 and the uniform constants $C_{\mathcal{B},N}$ as an input!

Proving continuity of $C_N(f)$ and $C'_N(f)$ on the forward side of the Mellin transform only, is, in other words, indeed possible. All the work we did in Theorem 2.2.3.4 and Theorem 2.2.5.4 however has to be shifted elsewhere, then. It is indeed shifted into Theorem 2.4.2.1 and Proposition 2.3.1.2 (the S-local counterparts to Theorem 2.2.3.4 and Theorem 2.2.5.4), which will lead to a proof of continuity of the S-idelic Zeta

function operators Z_S taking values in $\mathcal{S}_{(0,+\infty)}(J_{F,S})$ (the nuclear strict LF-space of I-exponentially decaying Schwartz-Bruhat functions on $J_{F,S}$, with $I = (0, +\infty)$, see Definition D.3.3.3 in Appendix D).

The remaining ingredient is then the Convexity Bound on the finite Hecke L-function (see Proposition B.1.1.3), combined with Weil's Identity in the unramified case (Theorem 3.4.1.1) and in the general case, to be proved in Theorem 3.4.3.3 by the end of this Chapter (using eigendistributions).

This Remark wants to point out to the reader how the uniform constants $C_{\mathcal{B},N}$ from the estimates from Theorem 2.2.3.4 come to life, i.e. where they come from and how they can be thought of. They arise as maxima over bounded subsets in $\mathcal{S}(\mathbf{A}_F)$, of continuous seminorms on $\mathcal{S}(\mathbf{A}_F)$. This is the punchline of this Remark. The fact that such maxima exist is due to an important structural property of $\mathcal{S}(\mathbf{A}_F)$: it is a Montel space. This means, concretely, that a version of Ascoli-Arzelà holds within it. Specifically, bounded subsets are relatively compact (equivalently precompact, since $\mathcal{S}(\mathbf{A}_F)$ is complete). This is why continuous functions (hence in particular the continuous functions $C_N(f)$ and $C'_N(f)$), can be maximized over the bounded subsets of $\mathcal{S}(\mathbf{A}_F)$. Ultimately, however, some work with concrete estimates and lattice point-counting techniques has to be done: indeed, Theorem 2.4.2.1 and Proposition 2.3.1.2 are, for all practical purposes, unavoidable.

Proving the continuity aspects of the formation of $Z_S(f, c)$, even for $c \in \Omega(J_{F,S})_{>0}$, requires the bounds proved therein as inputs. Throughout the arguments in the proofs of Theorem 2.4.2.1 and Proposition 2.3.1.2 and those bounds, we find that once again the Montel property of the S-local Schwartz-Bruhat spaces $\mathcal{S}(\mathbf{A}_{F,S})$ is used therein, to maximize certain continuous seminorms on bounded subsets in the process of maximizing constants dependent on the Schwartz-Bruhat function, and obtain the $C_{\mathcal{B},1}$ and $C_{\mathcal{B},2}$ in Theorem 2.4.2.1, and the $C_{\mathcal{B},N}$ in Proposition 2.3.1.2.

This gives a big-picture idea to the reader of how the elements of this work fit together: a detailed study of the S-local θ-functions and Zeta functions is required, because proving assertions on their global analogs is then done by reduction to the S-local case, where more flexibility is available and arguments are concrete and manageable. This is what the first nine Chapters of this work are for.

"Global-to-S-local" is the motto to keep in mind, entering the next Chapter and up to the very main trace calculation. The "global-to-S-local" general rule of thumb will underlie the trace calculation itself.

3.4.3. Eigendistributions and Weil's Identity: the general case. We will need an extension of the identity in (3.4.1.6) to the case χ is not unramified outside S. To this end we will be using the completed Hecke L-function $L(s, \chi)$ as opposed to the finite Hecke L-function $L_f(s, \chi)$ with finitely many Euler factors removed, i.e. $L_{\Sigma_F - S}(\chi, s)$, and we will be using results of Weil on local and global eigendistributions from [**Weil-a**, §1,3], whose main elements we briefly recall below, to be reprised in the next Chapter.

Definition 3.4.3.1. Let F be a number field, v a place of F, F_v the completion of F at v.

(1) Let $c \in \Omega(F_v^\times)$ be a quasi-character. We say $T \in \mathcal{S}(F_v)'_\beta$ is an *eigendistribution of weight* c if and only if:

$$\rho'_v(g, T) = c(g)T \tag{3.4.3.1}$$

for all $g \in F_v^\times$, where ρ'_v is the dual of the continuous right-translation action $\rho_v : F_v^\times \times \mathcal{S}(F_v) \to \mathcal{S}(F_v)$; i.e., $(\rho'_v(g, T))(f) = T(f \circ m_g)$ where m_g is g-multiplication on F_v. We denote by $\mathcal{S}(F_v)'_\beta(c)$ the closed subspace of $\mathcal{S}(F_v)'_\beta$ of eigendistributions on F_v of weight c.

(2) For $c \in \Omega(C_F)$ an idelic quasi-character, say $T \in \mathcal{S}(\mathbf{A}_F)'_\beta$ is a *global eigendistribution of weight* c if

$$\rho'_F(g, T) = c(g)T \tag{3.4.3.2}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$, where ρ'_F is the dual of the continuous right-translation action $\rho_F : \mathrm{GL}_1(\mathbf{A}_F) \times \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$.

We denote by $\mathcal{S}(\mathbf{A}_F)'_\beta(c)$ the closed subspace of $\mathcal{S}(\mathbf{A}_F)'_\beta$ of eigendistributions on $\mathbf{A}_F$ of weight c. We refer to the elements of $\mathcal{S}(\mathbf{A}_F)'_\beta(c)$ as *global eigendistributions of weight* c.

Definition 3.4.3.1(2) will be recalled again in Definition 5.5.1.1.

3.4.3.1. *Standard Gaussians local and global Schwartz-Bruhat functions.* We begin with the following definition:

DEFINITION 3.4.3.2 (Standard Gaussians). Let F be a number field, $\omega \in \widehat{C_F^1}$ a continuous character (labeling a connected component of $\Omega(C_F)$).

We define Schwartz-Bruhat functions $E_{\omega_v,v}$ on F_v, for each place v of F.

(1) for v real Archimedean:
   (a) if $\omega_v = 1$

$$E_{1,v}(x) := e^{-\pi x^2}. \tag{3.4.3.3}$$

   (b) if $\omega_v = \mathrm{sgn}$, the sign character

$$E_{\mathrm{sgn},v}(x) := xe^{-\pi x^2}. \tag{3.4.3.4}$$

(2) For v complex Archimedean, if $\omega_v = \omega_{t,m} : re^{i\theta} \mapsto r^t e^{im\theta}$
   (a) for $m \geq 0$

$$E_{m,v}(z) := \frac{1}{2\pi} z^m e^{-2\pi z\overline{z}}. \tag{3.4.3.5}$$

   (b) For $m < 0$

$$E_{m,v}(z) := \frac{1}{2\pi} \overline{z}^{-m} e^{-2\pi z\overline{z}} \tag{3.4.3.6}$$

(3) For v non-Archimedean:

$$E_v(x) := 1_{\mathcal{O}_{F_v}}. \tag{3.4.3.7}$$

For S_ω the finite set of finite places of F at which ω is ramified, we further define for $v \in S_\omega$:

$$E_{\omega_v,v}(x) := \omega_v^{-1} 1_{\mathcal{O}_{F_v}^\times}. \tag{3.4.3.8}$$

For $c \in \Omega(C_F)$ a fixed quasi-character we denote by $E_{c,v}$, abbreviating to E_v if no confusion can arise, the local Schwartz-Bruhat function defined above associated to the character $c|_{C_F^1}$ of C_F^1. We define the global Schwartz-Bruhat function:

$$E_c := \bigotimes_{v \in \Sigma_F} E_{c_v,v}. \tag{3.4.3.9}$$

We abbreviate the notation for E_c to E when no confusion can arise.

We refer to the local Schwartz-Bruhat functions $E_{c_v,v}$, for v Archimedean, respectively non-Archimedean, and to the global Schwartz-Bruhat function E_c, as *standard Gaussian* Schwartz-Bruhat functions.

The completed Hecke L-function $c \mapsto L(c)$ is defined, upon choosing a parametrization $\chi|\cdot|_F^s$ of $\Omega(C_F)$ for $\chi \in \widehat{C_F}$ a continuous character, to be $Z(E_\chi, s)$. This function near c depends only on the connected component of c in $\Omega(C_F)$, which is to say on restriction of c to C_F^1.

3.4.3.2. *Weil's Identity: the general case.* Weil classified the local and global eigendistributions. We provide the statements below for later reference:

THEOREM 3.4.3.3 (Weil's Identity - II). *Let F be a number field, v a place of F. We equip $\mathrm{GL}_1(\mathbf{A}_F)$ and each $F_v^\times$ with the global, respectively local, multiplicative Tamagawa measures μ, μ_v.*

(1) *For every $c \in \Omega(F_v^\times)$, we have $\dim_{\mathbf{C}} \mathcal{S}(F_v)'(c) = 1$. There exists a unique generator $\Delta_v'(c)$ of $\mathcal{S}(F_v)'(c)$ determined by the condition:*

(1) *If v is non-Archimedean and c is unramified at v:*

$$\langle \Delta'_v(c), E_v \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)} = 1. \tag{3.4.3.10}$$

(2) *If v is non-Archimedean and c is ramified at v:*

$$\langle \Delta'_v(c), c^{-1} 1_{\mathcal{O}_{F_v}^\times} \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)} = 1. \tag{3.4.3.11}$$

(3) *If v is Archimedean:*

$$\langle \Delta'_v(c), E_{c_v, v} \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)} = 1. \tag{3.4.3.12}$$

For any place v of F, the function $c \mapsto \Delta'_v(c)$ is a holomorphic function $\Omega(F_v^\times) \to \mathcal{S}(F_v)'_\beta$. The following identity holds, as meromorphic $\mathcal{S}(F_v)'_\beta$-valued functions on $\Omega(F_v^\times)$:

$$Z_v(\cdot, c) = L_v(c) \Delta'_v(c) \tag{3.4.3.13}$$

where L_v is the local factor at v of the completed Hecke L-function.

(2) *For every $c \in \Omega(C_F)$, we have $\dim_{\mathbf{C}} \mathcal{S}(\mathbf{A}_F)'(c) = 1$. The following tempered distribution in $\mathcal{S}(\mathbf{A}_F)'$*

$$\Delta'(c) := \bigotimes_{v \in \Sigma_F} \Delta'_v(c) \tag{3.4.3.14}$$

is a global eigendistribution of weight c (see Remark 3.4.3.4 for the meaning of the right side as a continuous linear functional on $\mathcal{S}(\mathbf{A}_F)$). The function $c \mapsto \Delta'(c)$ is a holomorphic function $\Omega(C_F) \to \mathcal{S}(\mathbf{A}_F)'_\beta$. The following identity holds, as meromorphic $\mathcal{S}(\mathbf{A}_F)'_\beta$-valued functions on $\Omega(C_F)$:

$$Z(\cdot, c) = L(c) \Delta'(c) \tag{3.4.3.15}$$

where $c \mapsto L(c)$ is the completed Hecke L-function.

Remark 3.4.3.4. For the right side of (3.4.3.14) to act continuously on $\mathcal{S}(\mathbf{A}_F)$, we use that $\mathcal{S}(\mathbf{A}_F) = (\overline{\otimes}_{v|\infty} \mathcal{S}(F_v)) \otimes (\otimes_{v \nmid \infty} \mathcal{S}(F_v))$ as a locally convex inductive limit in which all tensor products are given the inductive topology and $\overline{\otimes}$ denotes completion for $\otimes_{v|\infty} \mathcal{S}(F_v)$. The topology on the latter (even prior to completion) coincides with both the projective and injective topologies on that tensor product since $\mathcal{S}(F_v)$ for archimedean v is Fréchet: Corollary C.2.3.12 gives equality with the projective topology, and nuclearity of the $\mathcal{S}(F_v)$'s for such v gives equality of the projective and injective tensor product topologies.

Proof. We refer the reader to [**Weil-a**, §1,3] for (1) and (2) with the exception of the holomorphy aspects, which, in turn, follow from Theorem C.5.2.6 and Theorem C.5.2.4, since, for every $f \in \mathcal{S}(F_v)$, the function $c \mapsto \langle \Delta'_v(c), f \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)}$ is a holomorphic function of $c \in \Omega(F_v^\times)$, and, for every $f \in \mathcal{S}(\mathbf{A}_F)$, the function $c \mapsto \langle \Delta'_v(c), f \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)}$ is a holomorphic function of $c \in \Omega(C_F)$. We note that Theorem C.5.2.6 and Theorem C.5.2.4 apply since $\mathcal{S}(F_v)'_\beta$ is complete, hence quasi-complete.

We further refer the reader to [**Kudla**, pp.109-130] and specifically to [**Kudla**, Lemma 3.2,3.3] for the structural results of local eigendistributions. Especially to [**Kudla**, Thm.3.4] for their existence and uniqueness up to nonzero scalar multiplication, i.e. the one-dimensionality of $\mathcal{S}(F_v)'_\beta(c)$ for any continuous quasi-character c of $F_v^\times$. This proves (1).

We refer the reader to [**Kudla**, Lemma 4.1], for the case of global eigendistributions, i.e. c-homogeneous continuous functionals in $\mathcal{S}(\mathbf{A}_F)'$, for $c \in \Omega(C_F)$, and particularly to [**Kudla**, Thm.4.2] for the existence and uniqueness of global eigendistributions up to nonzero scalar multiplication, i.e. the one-dimensionality of $\mathcal{S}(\mathbf{A}_F)'(c)$ for any continuous idelic quasi-character. Similarly, the function $c \mapsto \Delta'(c)$ is a holomorphic $\mathcal{S}(\mathbf{A}_F)'_\beta$-valued function, by Theorem C.5.2.6 and Theorem C.5.2.4, both of which apply since $\mathcal{S}(\mathbf{A}_F)'_\beta$ is complete, hence quasi-complete (see Proposition 2.1.3.2). □

CHAPTER 4

Periodic Schauder Bases on modulated groups

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4.1. Schauder bases

In this section we briefly discuss basic facts about Schauder bases in Hausdorff locally convex topological $\mathbf{C}$-vector spaces, providing references to the literature and recording some foundational results for later use.

4.1.1. Basic concepts and properties. Finite dimensional **C**-vector spaces admit finite basic frames (i.e. **C**-linear bases), and several invariants of linear endomorphisms of a finite dimensional **C**-vector spaces, while intrinsic, can be computed upon making a choice of basic coordinate frame, i.e. a **C**-linear isomorphism to $\mathbf{C}^N$ for some integer $N \geq 0$.

The theory of complex Hilbert spaces is another source of example where bases provide a robust tool to make calculations. If the Hilbert space E is infinite dimensional, not all linear endomorphsms of E can be understood in terms of their effect on a basis of E, and not all types of bases of E offer robust control over meaningful classes of linear endomorphisms of E.

The Hilbertian normed topology on E makes orthonormal frames particularly well suited to understand important classes of *continuous* **C**-linear endomorphsims of E.

We refer the reader to §C.3, and particularly to §C.3.6, for a discussion of Hilbert-Schmidt and nuclear (equivalently, by [**Kub**, Thm.6.1], trace class) operators, and more generally on the Schatten-von Neumann classes $S_p(E)$ for every real number $p \geq 1$ (nuclear operators being the class $S_1(E)$, and Hilbert-Schmidt operators being the class $S_2(E)$).

See also Lemma C.3.4.1, for more on how these classes of operators can be understood from their effects on an orthonormal frame for E (i.e. an orthonormal basis).

Even in Hilbert spaces, orthonormal frames are not the only type of bases available, and capable of offering control over classes of continuous operators. Riesz bases are images of orthonormal frames under linear topological automorphisms. In particular, they are not necessarily orthonormal. Orthonormal bases and more generally Riesz bases are examples of Schauder bases in Hilbert spaces.

Weaker types of bases are also available on Hilbert, and, more generally, Banach spaces: Markushevich bases. We will be discussing a few in the later Chapters of this work, and solely in the special cases of interest. They are often obtained as images of a Schauder basis under a continuous injective linear operator with dense range, whose target is a Banach space originating as the normed completion of a locally convex source. These conditions are not sufficient for such image to be a Markushevich basis, but often times such is the origin of Markushevich bases in Banach spaces. We refer the reader to [**Jar**, Part II,Ch.14,§14.1,p.289].

We further refer to [**BJL**, Ch.IV,§7] for criteria to detect "completeness" of a basic system of holomorphic functions on non-empty domains of the complex plane with respect to the topology of uniform convergence on compact subsets, criteria to which we shall return throughout this Chapter.

An example to bear in mind is that of the exponential system on $C^\infty(\mathbf{T})$, where $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ is the 1-dimensional real torus, which we equip with the volume-1 Haar measure, and $C^\infty(\mathbf{T})$ is topologized with the Fréchet topology on uniform convergence on functions and their derivatives of arbitrary order.

Thus topologized, $C^\infty(\mathbf{T})$ is a nuclear Fréchet space (see [**Tre**, Part III,Ch.51,Thm.51.3,p.528] combined with [**Tre**, Part III,Ch.51,Thm.51.5,p.529]).

We refer to the complete metrizable topology on $C^\infty(\mathbf{T})$ as the *smooth topology*.

The 1-periodic smooth functions:

$$e_m(x) := e^{2\pi i m x},\ x \in \mathbf{R}, m \in \mathbf{Z}$$

uniquely determine smooth functions on $\mathbf{T}$, which we also denote by e_m, and the countable collection $\{e_m\}_{m\in\mathbf{Z}}$ is an orthonormal basis for the L^2-norm completion of $C^\infty(\mathbf{T})$, i.e. $L^2(\mathbf{T})$.

The continuous dual $C^\infty(\mathbf{T})'$ contains the continuous functionals $a_m : C^\infty(\mathbf{T}) \to \mathbf{C}$ obtained by sending a smooth 1-periodic function on $\mathbf{R}$, equivalently a smooth function on $\mathbf{T}$, to its mth Fourier coefficient, for $m \in \mathbf{Z}$.

Every element $\phi \in C^\infty(\mathbf{T})$ has a unique Fourier series expansion:

$$\phi(x) = \sum_{m\in\mathbf{Z}} a_m(\phi) e^{2\pi i m x} \tag{4.1.1.1}$$

where the series converges with respect to the smooth topology on $C^\infty(\mathbf{T})$ (a consequence of [**Fol-2**, Ch.2,§2.3,Thm.2.6,p.41]).

For any $\phi \in L^2(\mathbf{T})$, the Fourier series expansion $\phi(x) = \sum_{m\in\mathbf{Z}} a_m(\phi)e^{2\pi imx}$ still converges, pointwise almost everywhere over $\mathbf{T}$, and with respect to the L^2-norm topology (this latter is a consequence of [**Fol-2**, Ch.3,§3.3,Thm.3.5,p.78]. The convergence almost everywhere is a Theorem of Carlson, for which we refer the reader to [**Carleson**, Thm.(c)]).

Both in $C^\infty(\mathbf{T})$ and in $L^2(\mathbf{T})$, the datum

$$\{(e_m, a_m)\}_{m\in\mathbf{Z}} \in E \times E'$$

with $E = C^\infty(\mathbf{T}), L^2(\mathbf{T})$, is an example of a Schauder basis.

The defining property of a Schauder basis is extrapolated from this example: it is the existence of a series expansion as in (4.1.1.1), convergent with respect to the intrinsic Hausdorff locally convex topology on E, together with continuity of the coefficient functionals (without this latter condition, the resulting basis is termed a *topological basis*. We refer the reader to [**Jar**, Part II,Ch.14,§14.2,p.292]).

We further refer the reader to [**Jar**, Part II,Ch.14,§14.2,p.292], for the definition of Schauder basis in a Hausdorff locally convex topological $\mathbf{C}$-vector space E, which we record here below for ease of access:

DEFINITION 4.1.1.1. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $\{(x_m, f_m)\}_{m\geq 0}$ a bi-orthogonal sequence in $E \times E^*$, i.e. $f_m(x_n) = \delta_{m,n}$ for all $m, n \geq 0$.

(1) $\{x_m\}_{m\geq 0}$ is a *topological basis for* E, if and only if for every $u \in E$, the series:

$$\sum_{m\geq 0} f_m(u)x_m \tag{4.1.1.2}$$

converges in E, and its limit is u.

(2) Assume $\{x_m\}_{m\geq 0}$ is a topological basis for E. We say $\{x_m\}_{m\geq 0}$ is a Schauder basis, if and only if we have $\{f_m\}_{m\geq 0} \subset E'$.

We refer to the sequence $\{f_m\}_{m\geq 0} \subset E^*$ as the sequence of *coefficient functionals* of $\{x_m\}_{m\geq 0}$.

The Schauder basis $\{(e_m, a_m)\}_{m\in\mathbf{Z}}$ in $C^\infty(\mathbf{T})$ is termed the *exponential system*, and it is the aim of this Chapter to generalize it to the case of $\mathbf{R}_{>0}$-modulated abelian groups G (see Definition D.2.1.1), recording the growth properties of the basis elements with respect to the generic continuous seminorm on the Schwartz-Bruhat space of certain compact quotients of G by closed monogenic discrete subgroups.

To clarify what is meant by "growth properties" of the basis elements, we consider the continuous seminorm:

$$p_N(\phi) := \sup_{x\in[0,1)} |\partial_x^N \phi(x)|$$

for $\phi \in C^\infty(\mathbf{T})$, composed with the quotient homomorphism $\mathbf{R} \to \mathbf{T}$, and $N \geq 0$ an integer.

Then

$$p_N(e_m) = (2\pi|m|)^N$$

which, as a function of $m \in \mathbf{Z}$, grows polynomially.

Examples of normed completion E of $C^\infty(\mathbf{T})$ such that the exponential system maps to a sequence in E which is *not* a Schauder basis, even though the coefficient functionals $\{a_m\}_{m\geq 0}$ remain continuous functionals on E, are $E = C(\mathbf{T})$, equipped with the uniform norm, and $E = L^1(\mathbf{T})$, both Banach spaces.

The exponential system remains a Schauder basis in $L^p(\mathbf{T})$ for all real numbers $1 < p < \infty$ (see [**Katzn**, Ch.II,§1.5,Theorem,p.59]), and the convergence of (4.1.1.1) is not only with respect to the normed topology, but almost everywhere (a generalization of Carleson's result; see [**Hunt**]).

These examples amount to the well known phenomenon that continuous, respectively absolutely integrable, functions on $\mathbf{T}$ do not generally have a uniformly convergent Fourier series, respectively convergent with

respect to the L^1-norm; see [**Katzn**, Ch.II,§1.3,p.57], resting on the Uniform Boundedness Principle, for the former, and Kolmogorov's famous example (for which we refer the reader to [**Kol**]), for the latter.

Much as finite basic frames on finite dimensional $\mathbf{C}$-linear spaces, and orthonormal or Riesz frames on complex Hilbert spaces, are useful to understand classes of operators and compute some of their invariants, to Schauder bases goes the same role, on more general Hausdorff locally convex topological $\mathbf{C}$-vector spaces E.

Depending on robustness of properties of the topology on E, Schauder bases in E will enjoy stronger stability properties, such as absolute summability of their basic series expansions (as for (4.1.1.1) with respect to the smooth topology on $C^\infty(\mathbf{T})$, which promotes the exponential system to an *absolute* Schauder basis).

These will be recalled throughout this Chapter and the subsequent ones, as the need arises.

We will be using Schauder bases on countable strict inductive limits of nuclear Fréchet spaces, and with the specific aim of recognizing certain nuclear operators, and computing their nuclear traces.

It is the content of [**Lit-II**, §4,Cor.2], that Schauder bases can be used to compute nuclear traces of nuclear operators, under reasonable assumptions on the Hausdorff locally convex topology on E (i.e. E has the approximation property and admits a Schauder basis; see [**Lit-I**]).

4.1.2. On Absolute Schauder Bases and duality. With a view towards the following discussion, we refer the reader to [**Jar**, Part II,Ch.14,§14.6,p.305] for the concept of *unconditionally summable* sequences within Hausdorff locally convex topological $\mathbf{C}$-vector spaces.

For a sequence $\{x_m\}_{m\geq 0} \subset E$ in a Hausdorff locally convex topological $\mathbf{C}$-vector space E, $\{x_m\}_{m\geq 0}$ is unconditionally summable if and only if for any countable collection of mutually disjoint finite subsets $\{\Sigma_k\}_{k\geq 0}$ with $\Sigma_k \subset \mathbf{N}$ for all $k \geq 0$, and $\bigcup_{k\geq 0} \Sigma_k = \mathbf{N}$, we have that the sequence $\{s_k\}_{k\geq 0}$, with $s_k := \sum_{m\in\Sigma_k} x_m$, converges in E.

Equivalently, for any bijection $\sigma : \mathbf{N} \to \mathbf{N}$, the sequence $\{\sum_{0\leq m\leq N} x_{\sigma(m)}\}_{N\geq 0}$ converges in E.

We further refer the reader to [**Jar**, Part II,Ch.14,§14.7,p.314] for the concept of *absolute* Schauder basis, which will be the most discussed throughout this Chapter, and for the following properties of Schauder bases: equicontinuous (see [**Jar**, Part II,Ch.14,§14.2,p.296]), unconditional (see [**Jar**, Part II,Ch.14,§14.7,p.309], boundedly complete, respectively shrinking (see [**Jar**, Part II,Ch.14,§14.5,p.302]), regular (see [**Kal**, §1,Def.1.1]) and normalized (see [**Kal**, §1,Def.1.2]), all of which we recall below for ease of access.

Definition 4.1.2.1 (Properties of Schauder bases). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $\{x_m\}_{m\geq 0} \subset E$ a Schauder basis, $\{f_m\}_{m\geq 0} \subset E'$ its sequence of coefficient functionals. We say $\{x_m\}_{m\geq 0}$ is:

(1) *equicontinuous*, if and only if the basic projections $P_k := \sum_{0\leq m\leq k} f_m \cdot x_m$ are such that the sequence $\{P_k\}_{k\geq 0} \subset \mathscr{L}(E)$ is equicontinuous.
(2) *unconditional*, if and only if for every $u \in E$, the sequence $\{f_m(u)x_m\}_{m\geq 0}$ is unconditionally summable.
(3) *absolute*, if and only if for any continuous seminorm q on E, there exists a continuous seminorm q' on E satisfying:

$$\sum_{m\geq 0} q(f_m(u)x_m) \leq q'(u). \tag{4.1.2.1}$$

(4) *boundedly complete*, if and only if for every sequence $\{\alpha_m\}_{m\geq 0} \subset \mathbf{C}$ such that the sequence $\{\sum_{0\leq m\leq k} \alpha_m x_m\}_{k\geq 0} \subset E$ is bounded, then $\sum_{m\geq 0} \alpha_m x_m$ converges in E.
(5) *shrinking*, if and only if $\{f_m\}_{m\geq 0} \subset E'$ is a Schauder basis for E'_β.
(6) *regular*, if and only if there exists a 0-neighborhood U such that $x_m \notin U$ for all integers $m \geq 0$. It is *normalized*, if and only if it is regular and bounded.

Remark 4.1.2.2. Note that in (4.1.2.1), we have $\sum_{m\geq 0} q(f_m(u)x_m) = \sum_{m\geq 0} |f_m(u)|q(x_m)$ by absolute homogeneity of q. We further note that even if the Schauder basis $\{x_m\}_{m\geq 0} \subset E$ is equicontinuous, this

does not make the sequence of coefficient functionals $\{f_m\}_{m\geq 0} \subset E'$ an equicontinuous subset; see [**Jar**, Part II,Ch.14,§14.3,No.7,Remarks,p.299]. We further refer the reader to [**Kal**, §2,Prop.2.1,p.92], where it is shown that if $\{f_m\}_{m\geq 0}$ is equicontinuous (as a subset of E', i.e. without necessarily being itself a Schauder basis for E' with respect to any polar topology), then $\{x_m\}_{m\geq 0}$ is regular. The converse also holds if E carries the Mackey topology and $\tau(E', E)$ is sequentially complete.

REMARK 4.1.2.3 (On regular Schauder bases and continuous norms). Let E be a Fréchet space. Then by [**Kal**, §2,Thm.2.3,p.93], if E has a regular Schauder basis, then E has a continuous norm, and admits no complemented closed subspace linearly and topologically isomorphic to $\prod_{m\in\mathbf{N}} \mathbf{C}$.

We will be using this observation to run a few sanity checks, as we construct Schauder bases on certain nuclear Fréchet spaces arising as subquotients of the strict LF-spaces $\mathcal{S}_I(G)$ of I-exponentially decaying Schwartz-Bruhat functions on a second-countable Hausdorff locally compact abelian group G.

4.1.2.1. *Schauder bases in the presence of nuclearity.* Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. We let $\{x_m\}_{m\geq 0}$ be an absolute Schauder basis, and write $\{f_m\}_{m\geq 0} \subset E'_\beta$ for its sequence coefficient functionals. In particular, $\{(x_m, f_m)\}_{m\geq 0}$ is a bi-orthogonal sequence in $E \times E'$.

Under assumptions on E, $\{f_m\}_{m\geq 0} \subset E'_\beta$ is itself a Schauder basis, i.e. $\{x_m\}_{m\geq 0}$ is shrinking.

Let $\iota_E : E \to E'' = (E'_\beta)'$ be the evaluation map sending u to the continuous functional $\iota_E(u) := \langle u, \cdot\rangle_{E,E'}$ (see (C.4.1.1) and Lemma C.4.1.1(1)).

Lemma 4.1.2.4 below provides a criterion to detect Schauder bases in E'_β provided $\{x_m\}_{m\geq 0}$ is an absolute Schauder basis.

LEMMA 4.1.2.4 (On absolute and unconditional Schauder bases). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. Assume E is complete, barrelled and reflexive.*

Let $\{x_m\}_{m\geq 0} \subset E$ be an absolute Schauder basis, and $\{f_m\}_{m\geq 0}$ its sequence of coefficient functionals. Then $\{x_m\}_{m\geq 0}$ is shrinking and boundedly complete, and $\{f_m\}_{m\geq 0}$ is an unconditional, boundedly complete Schauder basis for E'_β.

If E is, in addition, nuclear, then $\{f_m\}_{m\geq 0}$ too is shrinking.

PROOF. For any continuous seminorm q on E, there exists a continuous seminorm q' on E such that, for any $u \in E$:

$$\sum_{m\geq 0} |f_m(u)|q(x_m) \leq q'(u). \tag{4.1.2.2}$$

The evaluation map ι_E is a linear topological isomorphism, and (4.1.2.2) then reads:

$$\sum_{m\geq 0} |f_m(\iota_E^{-1}\iota_E(u))|((\iota_E)_* q(\iota_E(x_m)) \leq (\iota_E)_*(q')(\iota_E(u)) \tag{4.1.2.3}$$

where for any continuous seminorm q on E, we define $(\iota_E)_* q$ to be the continuous seminorm on E'' assigned by $\varphi \mapsto ((\iota_E)_* q)(\varphi) = q(\iota_E^{-1}(\varphi))$ for any $\varphi \in E''$. All continuous seminorms on E'' are obtained in this manner, being ι_E a linear topological isomorphism.

We may therefore rewrite (4.1.2.3) as:

$$\sum_{m\geq 0} |(f_m \circ \iota_E^{-1})(\varphi)|q(\iota_E(x_m)) \leq q'(\varphi) \tag{4.1.2.4}$$

for all $\varphi \in E''$, for q, q' as in the foregoing, but as continuous seminorms on E''.

To see that indeed this makes sense and the sum in (4.1.2.4) converges, we observe that the inequality in (4.1.2.4) holds for all the partial sums.

Note that, for any $\varphi \in E''$ and for $u := \iota_E^{-1}(\varphi)$, we have:

$$(f_m \circ \iota_E^{-1})(\iota_E(u)) = \langle u, f_m\rangle_{E,E'} = \langle \iota_E(u), f_m\rangle_{E'',E'} = \langle \varphi, f_m\rangle_{E'',E'}. \tag{4.1.2.5}$$

In particular, we have $(f_m \circ \iota_E^{-1})(\varphi) = \langle \varphi, f_m\rangle_{E'',E'}$, which we denote $\varphi(f_m)$, for each integer $m \geq 0$. By (4.1.2.4) and since E'' is complete by reflexivity, since E is complete, we deduce that the sum $\sum_{m\geq 0} \varphi(f_m)\iota_E(x_m)$ converges in E'' (i.e. with respect to the strong topology on the continuous dual of the strong dual E'_β; see [**Jar**, Part II,Ch.11,§11.2]), and we have $\varphi = \sum_{m\geq 0} \varphi(f_m)\iota_E(x_m)$.

This implies that $\{f_m\}_{m\geq 0}$ is a Schauder basis in E'_β (alternatively, by [**Jar**, Part II,Ch.14,§14.5,Thm.1(a),p.302], $\{x_m\}_{m\geq 0}$ is shrinking, since E is reflexive).

By (4.1.2.4), the sum $\sum_{m\geq 0} \varphi(f_m)\iota_E(x_m)$ is unconditionally convergent, and hence the Schauder basis $\{f_m\}_{m\geq 0}$ is unconditional.

If E is nuclear, then it is Montel, since it is barrelled, complete and reflexive (see [**Tre**, Part III,Ch.50,Cor.1,p.520]). By [**Jar**, Part II,Ch.11,§11.5,Prop.4(f),p.230], E'_β is also reflexive, and hence by [**Jar**, Part II,Ch.14,§14.5,Thm.1(a),p.302] the Schauder basis $\{f_m\}_{m\geq 0}$ is shrinking. □

We record here below the following Theorem of Dynin-Mityagin, providing a criterion to detect absolute Schauder bases. Often times, we will be showing the absoluteness of the Schauder basis under consideration directly, to describe finer information, such as growth conditions on its elements with respect to the generic continuous seminorm.

We refer the reader to [**Jar**, Part III,Ch.21,§21.10,Thm.1,p.510], and to [**McAr**, §9,Thm.9.1,p.900] and [**DyM**].

Remark 4.1.2.5 (Cautionary remark on the concept of absolute Schauder basis). In [**McAr**, §9,p.900], a Schauder basis $\{(x_m, f_m)\}_{m\geq 0}$ in a Hausdorff locally convex topological **C**-vector space E, is termed *absolute* if and only if for every continuous seminorm p on E, the following series:

$$\sum_{m\geq 0} p(f_m(u)x_m)$$

converges for every $u \in E$.

If E is a Fréchet space, this concept of absolute basis, which, a priori, differs from that defined in [**Jar**, Part II,Ch.14,§14.7,(*),p.314], which is the one we adopt throughout this work, agrees with it, by the Closed Graph Theorem (see [**Jar**, Part I,Ch.5,Cor.3,p.94]), since absolute summability of the sequence $\{f_m(u)x_m\}_{m\geq 0} \subset E$ for each $u \in E$ implies closedness of the induced linear operator $E \to \ell^1\{E\}$, and since E is Fréchet, so is $\ell^1\{E\}$, the space of absolute Cauchy sequences in E (see [**Jar**, Part II,Ch.15,§15.6]. Note that for $E = \mathbf{C}$, we obtain $\ell^1\{E\} = \ell^1$, the Banach space of absolutely summable sequences of complex numbers).

Similarly, if E is a strict LF-space, it is strictly webbed (see [**Jar**, Part I,Ch.5,§5.6,Thm.2,p.97]), and so [**Jar**, Part I,Ch.5,§5.4,Thm.1,p.92] applies, so long as we show that $\ell^1\{E\}$ is webbed.

This is indeed the case, if E is a strict inductive limit of a sequence of nuclear Fréchet spaces. To see this, we proceed as follows. Let $\{E_m\}_{m\geq 0}$ be a defining sequence of Fréchet spaces.

By [**Jar**, Part II,Ch.15,§15.8,Cor.6,p.341], we have a linear topological isomorphism $\ell^1 \mathbin{\widehat{\otimes}_\pi} E \to \ell^1\{E\}$. Since ℓ^1 is a Banach space, then by [**Jar**, Part II,Ch.15,§15.5,Cor.4(a),p.334], the continuous linear map $\varinjlim_{m\geq 0} \ell^1 \otimes_\pi E_m \to \ell^1 \otimes_\pi E$ is a linear topological isomorphism.

Corollary C.2.3.12 applies to $\ell^1 \otimes_\iota E_m \to \ell^1 \otimes_\pi E_m$, and hence the continuous linear operator $\ell^1 \otimes_\iota E_m \to \ell^1 \otimes_\pi E_m$ is a linear topological isomorphism for each integer $m \geq 0$.

We refer the reader to the discussion following (C.2.3.22) and to Proposition C.2.3.13 of §C.2.3, for more on inductive tensor products.

By Proposition C.2.3.13(2), $\ell^1 \overline{\otimes} E$ is therefore the strict inductive limit of a countable sequence of Fréchet spaces (using nuclearity of the members of the defining sequence for E, to deduce strictness from [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348]). In particular, $\ell^1 \overline{\otimes} E$ is webbed and strictly webbed.

The foregoing further shows that $\ell^1 \overline{\otimes} E \to \ell^1 \widehat{\otimes}_\pi E$ is a linear topological isomorphism, and we deduce that $\ell^1\{E\}$ is webbed and strictly webbed.

In particular, if E is either a nuclear Fréchet space, or a strict inductive limit of nuclear Fréchet spaces, the concept of absolute Schauder basis in E is unambiguous: the one implicitly used in [**McAr**, §9] is, in these special cases, equivalent to the one defined in [**Jar**, Part II,Ch.14,§14.7,(*),p.314].

THEOREM 4.1.2.6 (Dynin-Mityagin). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space.*

(1) *If E is nuclear, then every equicontinuous Schauder basis in E is absolute.*
(2) *Assume E is a nuclear Fréchet space. Then every Schauder basis is absolute.*
(3) *Assume E is a strict inductive limit of a sequence of nuclear Fréchet spaces. Then every Schauder basis is absolute.*

PROOF. (1) : This is [**Jar**, Part III,Ch.21,§21.10,Thm.1,p.510].

(2) : By [**Jar**, Part II,Ch.14,§14.3,Thm.4,p.296], every Schauder basis in E is equicontinuous, and hence by (1), it is absolute.

(3) : By [**Jar**, Part II,Ch.14,§14.3,Thm.4,p.296], since strict countable inductive limits of Fréchet spaces are complete, bornological and ultrabornological (see [**Jar**, Part II,Ch.13,§13.1,Cor.4,p.273] and [**Jar**, Part II,Ch.13,§13.1,Cor.5,p.273]) and strictly webbed (see [**Jar**, Part I,Ch.5,§5.6,Prop.1,p.96] and [**Jar**, Part I,Ch.5,§5.6,Thm.2,p.97]), every Schauder basis (noting that every Schauder basis is, in particular, a weak Schauder basis), is also an equicontinuous basis. By (1), it is an absolute basis; see also [**Jar**, Part II,Ch.14,§14.3,Cor.3,p.296], noting that strict LF-spaces are barrelled (see [**Scha**, Ch.II,§7.2,Cor.2,p.61]), and hence the Corollary applies. □

A consequence of Theorem 4.1.2.6 is that every nuclear Fréchet space E with a Schauder basis is linearly and topologically isomorphic to a Köthe sequence space (see [**Jar**, Part III,Ch.21,§21.6,p.496]). It does not however describe the corresponding Köthe set, which depends specifically on growth conditions satisfied by the basis elements with respect to the generic continuous seminorm on E.

In the special cases of interest to this work, which we discuss in this Chapter for the later purpose of identifying copies of the Schauder bases constructed herein, into subquotients of spaces of S-local and global adelic θ-functions, it is these growth conditions that we are required to describe, which happen to exhibit the resulting sequence spaces as being linearly and topologically isomorphic to $\mathfrak{s}$, the nuclear Fréchet space of rapidly decreasing sequences (see [**Tre**, Part III,Ch.51,p.527]).

4.2. Bases of exponentially decaying Schwartz-Bruhat functions: the case $G = \mathbf{R}_{>0}$

In this section we exhibit the existence of absolute Schauder bases on subquotients of the Hausdorff locally convex space $\mathcal{S}_{\mathrm{E}}(G)$ of exponentially decaying Schwartz-Bruhat functions on an $\mathbf{R}_{>0}$-modulated group G (see Definition D.2.1.1, Proposition D.3.2.5 and Theorem D.3.3.7) related from averaging operators over the action of a discrete subgroup of the form $\gamma^{\mathbf{Z}} \subset G$.

In the final subsection of this Chapter, §4.6.3, we will be propagating the existence of these Schauder bases, to subquotients of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}_I(G)$ of I-exponentially decaying Schwartz-Bruhat functions, with I an arbitrary non-empty open interval of the real line, with a view towards applications in the later Chapters of this work.

4.2.1. The exponential system on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$. We begin with a first construction: a special case of the periodization operator from (D.1.7.2), associated to a Hausdorff locally compact abelian group G, and

a closed subgroup $k \subset G$. We refer the reader to Theorem D.1.7.1, for more on periodization over closed subgroups of a Hausdorff locally compact abelian groups, and for a distributional version of the Poisson Summation Formula.

In this subsection, we discuss the special case $G = \mathbf{R}_{>0}$ and $k = \gamma^{\mathbf{Z}}$, for some $\gamma \in \mathbf{R}_{>0}$, $\gamma > 1$. Under these assumptions, we prove the main properties of the periodization operator over $\gamma^{\mathbf{Z}}$-translates in $\mathbf{R}_{>0}$, for Schwartz-Bruhat functions on $\mathbf{R}_{>0}$, directly.

We will then discuss the case of G a second-countable $\mathbf{R}_{>0}$-modulated group with module $|\cdot|_G$, and $k = \gamma^{\mathbf{Z}}$, for $\gamma \in G$ with $|\gamma|_G > 1$.

Definition 4.2.1.1 (Monogenic periodization over $\mathbf{R}_{>0}$). Let $\gamma \in \mathbf{R}_{>0}$ be a positive real number, such that $\gamma > 1$.

We define the operator

$$\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) = C^\infty(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \tag{4.2.1.1}$$

assigned by:

$$\Phi_\gamma(\phi)(x) := \sum_{k \in \mathbf{Z}} \phi(\gamma^k x) \tag{4.2.1.2}$$

as a function of $x \in \mathbf{R}_{>0}$. We refer to the linear operator Φ_γ as the $\gamma^{\mathbf{Z}}$-*periodization operator*.

We now study the operators Φ_γ in a manner that is similar to our study of the θ-function operators $\Phi_{F,S}$ and Φ_F from Chapters 2 and 3.

We first prove the following well known:

Lemma 4.2.1.2 (Exponential system in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$). *Let $\gamma \in \mathbf{R}_{>0}$ be a real number, $\gamma > 1$.*
The countable set of functions:

$$\left\{ e_m := t^{\frac{2\pi i m}{\log(\gamma)}} \right\}_{m \in \mathbf{Z}} \tag{4.2.1.3}$$

satisfies the following properties.

1. *It is an absolute Schauder basis for $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$.*
2. *Its bi-orthogonal sequence of coefficient functionals is an unconditional Schauder basis for $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$, and it is the countable set of continuous functionals $\{a_m(\cdot)\}_{m \in \mathbf{Z}}$, where, for any $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$:*

$$a_m(\phi) := \frac{1}{\log(\gamma)} \int_{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}} \phi(g)\overline{\chi_m}(g)\, \mu_\gamma(g) \tag{4.2.1.4}$$

   is the mth Fourier coefficient functional, and where μ_γ is the unique Haar measure determined, as per Theorem D.1.2.1, by the choices of $\frac{\mathrm{d}t}{t}$ on $\mathbf{R}_{>0}$, and the counting measure on $\gamma^{\mathbf{Z}} \subset \mathbf{R}_{>0}$, and where χ_m is the unique continuous character of $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ extending to the $\gamma^{\mathbf{Z}}$-invariant continuous character of $\mathbf{R}_{>0}$ defined by $t \mapsto t^{\frac{2\pi i m}{\log(\gamma)}}$.
3. *For each integer $m \in \mathbf{Z}$ and every $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, the following formula:*

$$a_m(\phi) := \frac{1}{\log(\gamma)} \int_1^\gamma \phi(t) t^{\frac{-2\pi i m}{\log(\gamma)}}\, \frac{\mathrm{d}t}{t} \tag{4.2.1.5}$$

   holds, where $\phi(t)$ is the smooth function of $t \in \mathbf{R}_{>0}$ obtained by composition of ϕ with the smooth quotient homomorphism $\mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$.

Proof. We note that indeed $a_m(e_m) = 1$ and $a_m(e_k) = 0$ for all integers $k \neq m$. In particular, the pair $(\{e_m\}_{m \in \mathbf{Z}}, \{a_m(\cdot)\}_{m \in \mathbf{Z}})$ is a bi-orthogonal sequence with respect to the duality $\langle \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}), \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})' \rangle$.

This implies that $\{e_m\}_{m \in \mathbf{Z}}$ is a topologically free set by [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289]. Let $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ be an arbitrary smooth function.

We denote by $t\partial_t$ the differential operator obtained as the directional derivative along the continuous, hence smooth, 1-parameter group given by the quotient homomorphism $\mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$.

Then the sums:

$$\sum_{m\in\mathbf{Z}:|m|\leq M} a_m(\phi)(t\partial_t)^k e_m \tag{4.2.1.6}$$

for each integer $k \geq 0$ and each integer $M \geq 0$, converge as $M \to +\infty$ uniformly, since we have:

$$\left|\sum_{m\in\mathbf{Z}:|m|\leq M} a_m(\phi)(t\partial_t)^k e_m\right| \leq \left(\frac{2\pi}{\log(\gamma)}\right)^k \sum_{m\in\mathbf{Z}:|m|\leq M} |a_m(\phi)|m^k \tag{4.2.1.7}$$

and the coefficients $a_m(\phi)$ are, for each such ϕ, a sequence of rapid decay, i.e. an element of $\mathcal{S}(\mathbf{Z}) = \mathfrak{s}$.

To see this, we observe that they are, as a function on $\frac{2\pi i\mathbf{Z}}{\log(\gamma)} \simeq \widehat{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}}$, the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \to \mathcal{S}\left(\widehat{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}}\right) \simeq \mathcal{S}(\frac{2\pi i\mathbf{Z}}{\log(\gamma)})$$

of ϕ.

Indeed, for each integer $m \in \mathbf{Z}$, upon calling $\chi_m := e_m$ as a continuous character $\chi_m \in \widehat{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}}$, we have

$$a_m(\phi) = \frac{1}{\log(\gamma)} \int_{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}} \phi(g)\overline{\chi_m}(g)\, \mathrm{d}\mu_\gamma(g) = \mathcal{F}(\phi)(\chi_m)$$

and since as $m \in \mathbf{Z}$ varies over all the integers, $m \mapsto \chi_m$ is an isomorphism of discrete finitely generated abelian groups $\mathbf{Z} \to \widehat{\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}}$, we deduce that the composition of $\mathcal{F}(\phi)$ with it, is a Schwartz-Bruhat function on $\mathbf{Z}$, hence a sequence of rapid decay, i.e. an element of $\mathfrak{s}$.

This proves that the series in (4.2.1.6) converges absolutely for each $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, and hence that the sequence of finite rank operators $\{\sum_{m\in\mathbf{Z}:|m|\leq M} a_m(\cdot)e_m\}_{M\geq 0}$ converge absolutely with respect to the topology of simple convergence. Since $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is a nuclear Fréchet space, the argument in Remark 4.1.2.5 implies that the sequence $\{e_m\}_{m\in\mathbf{Z}}$ is an absolute Schauder basis in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$. Alternatively, the convergence of (4.2.1.6) for every $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ implies that $\{e_m\}_{m\in\mathbf{Z}}$ is a Schauder basis for $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, and since $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is a nuclear Fréchet space, then by the Dynin-Mityagin Theorem (see Theorem 4.1.2.6) it is an absolute Schauder basis.

By reflexivity of $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, Lemma 4.1.2.4 applies, giving that the sequence $\{a_m(\cdot)\}_{m\in\mathbf{Z}}$ of coefficient functionals is an unconditional Schauder basis for the strong dual $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$. □

4.2.1.1. *Recollections on the two strong topologies on orthogonals and the bounded lifting property.* We recall the following:

Lemma 4.2.1.3 (Homomorphic weak and strong adjoints to quotient maps). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, and $M \subset E$ a closed subspace.*

The linear bijection:

$$(E/M)' \to M^\perp \tag{4.2.1.8}$$

where the orthogonal $M^\perp$ is formed with respect to the duality $\langle E, E'\rangle$, is a linear topological isomorphism, upon endowing the source with the weak topology, and the target with the subspace topology of the weak topology inherited from E'_σ.

(1) *If E is a nuclear Fréchet space, then* (4.2.1.8) *is a linear topological isomorphism, upon endowing the source with the strong topology, and the target with the subspace topology of the strong topology inherited from E'_β.*

(2) *If E is a Hilbert space, then* (4.2.1.8) *is a linear topological isomorphism, upon endowing the source with the strong topology, and the target with the subspace topology of the strong topology inherited from E'_β.*

PROOF. The linear operator $(E/M)' \to M^\perp$ is defined by sending a continuous functional T to the composition of T by the quotient $E \to E/M$. By [**NB**, Thm.8.12.1(a)], it is a linear bijection.

By [**NB**, Thm.8.13.2], which applies by [**NB**, Thm.8.11.3(c)], the linear operator $(E/M)' \to E'$ is a linear topological embedding with respect to the weak topology on source and target. By [**NB**, Thm.8.12.1(a)], its range is $M^\perp$, and hence the linear operator from (4.2.1.8) is a linear topological isomorphism upon endowing source and target with the weak topologies.

(1) : By [**Apiola**, §4,Thm.4.3], nuclear Fréchet spaces are separated quotients of Köthe Schwartz spaces, and by [**MV-2**, Thm.7], they are quasi-normable.

By [**Jar**, Part II,Ch.10,§10.7,Cor.3,p.215] and [**Tre**, Part III,Ch.50,Prop.50.2(50.12),p.519], nuclear Fréchet spaces are therefore Fréchet-Schwartz spaces, and by [**Val-b**, Thm.2.4(2)], for any closed subspace $M \subset E$ of the nuclear Fréchet space E, the quotient map $E \to E/M$ lifts bounded subsets.

Let $H \subset (E/M)''$ be an equicontinuous subset. Since E/M is reflexive, being nuclear Fréchet (by [**Scha**, Ch.III,§7,Thm.7.4,p.103]), hence Montel by [**Tre**, Part III,Ch.50,Cor.3,p.520], hence reflexive, then by [**Scha**, Ch.III,§4.1,Corollary,p.83] H is bounded with respect to the strong topology, and hence its pre-image under the evaluation linear topological isomorphism $\iota_{E/M} : E/M \to (E/M)''$ is bounded. There therefore exists a bounded subset $B \subset M$ whose image under the quotient map $E \to E/M$ is $\iota_{E/M}^{-1}(H)$. Under the linear topological isomorphism $\iota_E : E \to E''$, $\iota_E(B)$ is therefore a bounded subset of E'', hence equicontinuous by [**Scha**, Ch.III,Thm.4.2,p.83] combined with [**Scha**, Ch.III,§3.4,Corollary,p.82]. Its image under the continuous linear operator $E'' \to (E/M)''$, which is the continuous adjoint of the continuous adjoint $(E/M)'_\beta \to E'_\beta$, is H.

By [**NB**, Thm.8.13.4], applied to $X = (E/M)'_\beta$, $Y = E'_\beta$, $A : X \to Y$ the continuous adjoint of the quotient map $E \to E/M$, by weak relative openness of A, which was shown in the foregoing, and since we have shown that every equicontinuous subset of X' is the image under A' of some equicontinuous subset of Y', we deduce that A is relatively open. Its range is $M^\perp$, and hence (4.2.1.8) is a linear topological isomorphism, upon topologizing the source with the strong topology, and the target with the subspace topology of the strong topology.

(2) : By [**Val-b**, Thm.2.4(1)] (or by direct construction), for any closed subspace $M \subset E$ of the Hilbert space E, the quotient map $E \to E/M$ lifts bounded subsets. By total reflexivity of Hilbert spaces (i.e. E and all of its separated quotients are reflexive) and again applying [**Scha**, Ch.III,§4.1,Corollary,p.83], [**Scha**, Ch.III,Thm.4.2,p.83] and [**Scha**, Ch.III,§3.4,Corollary,p.82], the same argument provided for (1) shows that [**NB**, Thm.8.13.4] applies, and (4.2.1.8) is a linear topological isomorphism, upon topologizing the source with the strong topology, and the target with the subspace topology of the strong topology. □

REMARK 4.2.1.4. We note that the same assertion as in the Lemma does not hold for the weak topology replaced by the strong topology, unless additional assumptions on E and the quotient E/M are in force.

Assume E is a totally reflexive Fréchet space (i.e. every separated quotient of E is also reflexive; see [**Gro-e**, §II.3,Def.2,p.99]), and $M \subset E$ is a closed subspace.

The continuous adjoint $(E/M)'_\sigma \to E'_\sigma$ is relatively open by Lemma 4.2.1.3.

Assume that $(E/M)'_\beta \to E'_\beta$ is a linear topological embedding. Then by reflexivity, [**NB**, Thm.8.13.4] applies to A the continuous adjoint of the linear operator from (4.2.1.8), and to $X = (E/M)'_\beta$ and $Y = M^\perp$ equipped with the subspace topology of the strong topology.

It follows that the quotient map $E \to E/M$ lifts equicontinuous subsets (with respect to the $\beta(E'', E')$-topology, using reflexivity of E).

By [**Scha**, Ch.III,§4,4.1,Corollary,p.83], any equicontinuous subset of E/M and E is strongly bounded (each identified with the strong dual of $(E/M)'_\beta$, respectively E'_β, by reflexivity; see also [**NB**, Thm.11.3.5] and

[**NB**, Thm.11.3.6]), and since E is barrelled by [**NB**, Thm.11.7.2] (see also [**Scha**, Ch.II,§7,Thm.7.1,p.60]), the converse also holds by [**Scha**, Ch.III,§4,Thm.4.2,p.83].

It follows that the quotient map $E \to E/M$ would have to lift bounded subsets.

By [**Val-b**, Thm.2.4(2)], E is either a Fréchet-Schwartz space, a Banach space, or a product of a Banach space by ω.

Nuclear Fréchet spaces are totally reflexive and Fréchet-Schwartz. We further refer the reader to [**FlM**, §5.5,Corollary,p.110], for several sufficient conditions for a homomorphism between Hausdorff locally convex topological $\mathbf{C}$-vector spaces, to have homomorphic strong adjoint.

We refer the reader to Definition C.1.5.9 and subsequent discussion for information on the bounded lifting property for continuous linear maps between Hausdorff locally convex topological $\mathbf{C}$-vector spaces.

4.2.1.2. *Monogenic periodization on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ and absolute bases of point-evaluation functionals.* The following Lemma studies the periodization operator $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ and its kernel.

We recall that the Mellin transform $M : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C})$, where $A\mathcal{S}(\mathbf{C})$ is the *Schwartz-Bergman space* defined in Definition D.3.1.3 and studied in §D.3.1, the space of entire functions that are Schwartz uniformly over the non-empty open vertical strips of bounded width of the complex s-plane, topologized with the Hausdorff locally convex topology defined in Lemma D.3.1.2, promoting it to a separable nuclear Fréchet space, is a linear topological isomorphism; see Proposition D.3.2.5.

The Mellin transform is defined, on each $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, as:

$$M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \, \frac{\mathrm{d}t}{t}$$

as a function of $s \in \mathbf{C}$, and it is defined and studied in §B.2.1.

LEMMA 4.2.1.5 (On the kernel of the monogenic periodization operator Φ_γ). *Let $\gamma \in \mathbf{R}_{>0}$ be a positive real number, such that $\gamma > 1$. Let $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ of exponential decay. Then:*

(1) *$\Phi_\gamma(f)$ is a Schwartz-Bruhat function on the compact Hausdorff abelian group $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$.*
(2) *The linear operator Φ_γ from (4.2.1.1) is continuous, surjective and open.*
(3) *$\ker(\Phi_\gamma)$ is the closure in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as functions of x and as ϕ varies over all the elements of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$. In particular, the continuous functionals $\phi \mapsto \int_{\mathbf{R}_{>0}} \phi(t) t^{-\frac{2\pi i m}{\log(\gamma)}} \frac{\mathrm{d}t}{t}$ are an unconditional Schauder basis for the orthogonal of $\ker(\Phi_\gamma)$ with respect to the duality $\langle \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}), \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})' \rangle$, equipped with the strong topology on the dual $(\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/\ker(\Phi_\gamma))'$, equivalently with the subspace topology of the strong topology on the dual $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'$.*

REMARK 4.2.1.6. Note that for $M \subset \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ a closed subspace, the orthogonal $M^\perp$ of M with respect to the duality $\langle \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}), \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})' \rangle$ is linearly isomorphic to the continuous dual of the quotient $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/M$ (see [**NB**, Thm.8.12.1(a)]). There are therefore two "strong topologies" on $M^\perp$: the topology induced by the strong topology of $(\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/M)'_\beta$, and the topology induced by the subspace topology of the strong topology of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta$. These two topologies are equivalent, because $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is a nuclear Fréchet space (see Lemma D.3.2.3(1) applied with $I = \mathbf{R}$), and therefore it has the bounded lifting property, i.e. for every closed subspace $M \subset \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, every bounded subset of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/M$ is contained in the image of a bounded subset of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ under the quotient map $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/M$.

We refer the reader to Lemma 4.2.1.3 and to [**FlM**, §5.5,Corollary,part (1),p.110].

The key to this fact is quasi-normability of Fréchet-Schwartz spaces, of which nuclear Fréchet spaces are an example. We further refer the reader to Valdivia's characterization of those Fréchet spaces that possess the bounded lifting property (see [**Val-b**, Thm.2.4(2)]).

In particular, there is no ambiguity, throughout the statement of Lemma 4.2.1.5, as to what strong topology orthogonals of closed subspaces of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ are equipped with.

Proof of Lemma 4.2.1.5. (1) and (2) : The exponential decay of f ensures that formation of the derivative $(t\partial_t)^k f$ and of the $\partial_\alpha^k \Phi_\gamma(f)$, where $\alpha : \mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ is the continuous 1-parameter group defined by the quotient homomorphism, commute, i.e. $\partial_\alpha^k \Phi_\gamma(f) = \Phi_\gamma((t\partial_t)^k f)$ as continuous functions on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$. We will be showing this towards the end of the argument.

It follows that $\Phi_\gamma(f)$ is a smooth function on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, and hence Schwartz-Bruhat on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, since $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ is a compact Hausdorff abelian group.

Let $\{f_m\}_{m\geq 0}$ be a Cauchy sequence in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ converging to $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$. Then $\{f_m\}_{m\geq 0}$ is bounded in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, and $(t\partial_t)^k f_m \to (t\partial_t)^k f$ as $m \to +\infty$ uniformly over $\mathbf{R}_{>0}$ for each integer $k \geq 0$.

For every integer $N > 0$, there therefore exists $T > 0$ and an integer $m_T > 0$ depending on N, such that for all integers $m \geq m_T$, we have, for some real number $C_N > 0$:

$$|(t\partial_t)^k f_m - (t\partial_t)^k f| < C_N e^{-N|\log(t)|}$$

for all $t \in \mathbf{R}_{>0} - (1/T, T)$.

For $t > T$, $\gamma^k t \geq T$ if and only if $\gamma^k \in t^{-1} \cdot [T, +\infty)$. There exists an integer $k_T(t)$ such that this is the case if and only if $k \geq k_T(t)$. We note that we have $k_T(t) \geq -\log_\gamma(t) + \log_\gamma(T)$, and we have, for every $t > T$:

$$\left|\Phi_\gamma((t\partial_t)^k f_m) - \Phi_\gamma((t\partial_t)^k f)\right| \leq C_N \sum_{k\geq k_T(t)} |\gamma^k t|^{-N} \tag{4.2.1.9}$$

where the series on the right side of (4.2.1.9) converges geometrically.

Explicitly, we have:

$$\sum_{k\geq k_T(t)} |\gamma^k t|^{-N} = t^{-N} \sum_{k_T(t)\leq k<0} \gamma^{-k} + t^{-N} \sum_{k\geq 0} \gamma^{-k} \tag{4.2.1.10}$$

$$= t^{-N} \sum_{k_T(t)\leq k<0} \gamma^{-k} + \frac{t^{-N}}{1-\gamma^{-1}} \tag{4.2.1.11}$$

$$= -t^{-N} + t^{-N} \frac{1-\gamma^{-k_T(t)+1}}{1-\gamma} + \frac{t^{-N}}{1-\gamma^{-1}} \tag{4.2.1.12}$$

$$\leq -t^{-N} + t^{-N} \frac{\gamma t/T - 1}{\gamma - 1} + \frac{t^{-N}}{1-\gamma^{-1}}. \tag{4.2.1.13}$$

Upon coosing $N > 1$ from the outset, then for any $\varepsilon > 0$, we may choose $T_1 > T$ as large as we please so as to arrange to have:

$$\sup_{t>T_1} C_N \sum_{k\geq k_T(t)} |\gamma^k t|^{-N} < \varepsilon/3. \tag{4.2.1.14}$$

The case $t \in (0, 1/T)$ is treated similarly, and we obtain that for some large enough $T' > T$ depending on $N > 1$ and on ε, then over $(0, 1/T'] \cup [T', +\infty)$ we have

$$\left|\Phi_\gamma((t\partial_t)^k f_m) - \Phi_\gamma((t\partial_t)^k f)\right| < (2/3)\varepsilon$$

for all $m \geq m_T$.

Over $(1/T', T')$, the convergence of $(t\partial_t)^k f_m$ to $(t\partial_t)^k f$ is uniform, and for every $\varepsilon_0 > 0$, there exists an integer $m_0 = m_0(\varepsilon_0) \geq m_{T'}$ such that for all $m \geq m_0$ we have:

$$\sup_{t\in(1/T',T')} |(t\partial_t)^k f_m - (t\partial_t)^k f| < \varepsilon_0. \tag{4.2.1.15}$$

Since the subgroup $\gamma^{\mathbf{Z}} \subset \mathbf{R}_{>0}$ is discrete, by the assumption $\gamma \neq 1$, only finitely many integers $k \in \mathbf{Z}$ are such that $\gamma^k(1/T', T') \cap (1/T', T')$ is non-empty. Let $M_{T'}$ be their number.

We deduce that we have:

$$|\Phi_\gamma((t\partial_t)^k f_m)(x) - \Phi_\gamma((t\partial_t)^k f)(x)| < M_{T'}\varepsilon_0 \tag{4.2.1.16}$$

for all $x \in (1/T', T')$.

We choose $\varepsilon_0 := \varepsilon/(3M_{T'})$ and obtain that for all $m \geq m_0$, we have $\|\Phi_\gamma((t\partial_t)^k f_m) - \Phi_\gamma((t\partial_t)^k f)\|_u < \varepsilon$, and hence $\Phi_\gamma((t\partial_t)^k f_m) \to \Phi_\gamma((t\partial_t)^k f)$ uniformly.

Since this is the case for every integer $k \geq 0$, we deduce that we have $\Phi_\gamma(f_m) \to \Phi_\gamma(f)$ in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ (whose topology is equivalent to that of $\mathcal{D}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, the $\mathbf{C}$-linear space of smooth functions on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ topologized with the nuclear Fréchet topology of uniform convergence of functions and their derivatives of arbitrary order), so long as we show that $\Phi_\gamma(f)$ is smooth, and that the commutation property $(t\partial_t)\Phi_\gamma(f) = \Phi_\gamma(t\partial_t f)$ holds.

If so, then Φ_γ is sequentially continuous by the foregoing, as an operator $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$. It is therefore continuous since both source and target are metrizable.

We now return to the proof of smoothness of $\Phi_\gamma(f)$, and we then show the commutation property $(t\partial_t)\Phi_\gamma(f) = \Phi_\gamma(t\partial_t f)$ for any $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$.

We let $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ be arbitrary, and let $\{t_m\}_{m\geq 0} \subset \mathbf{R}_{>0}$ be an arbitrary sequence such that $\lim_{m\to+\infty} t_m = 1$. We define

$$f_m(x) := \frac{f(t_m^{-1}x) - f(x)}{t_m - 1} \tag{4.2.1.17}$$

itself a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ with exponential decay, for every integer $m \geq 0$. The sequence $\{f_m\}_{m\geq 0}$ converges to $(t\partial_t)f$ uniformly over $\mathbf{R}_{>0}$ (indeed, every $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is, in particular, a Schwartz-Bruhat function on $\mathbf{R}_{>0}$, and hence it is uniformly differentiable over $\mathbf{R}_{>0}$).

We proved that we have $\lim_{m\to+\infty} \Phi_\gamma(f_m) = \Phi_\gamma((t\partial_t)f)$ pointwise, and uniformly on the compact subsets of $\mathbf{R}_{>0}$.

Since $\{t_m\}_{m\geq 0}$ is arbitrary, this shows that $\Phi_\gamma(f)$ is smooth, and that indeed $(t\partial_t)\Phi_\gamma(f) = \Phi_\gamma((t\partial_t)f)$ holds, as continuous functions over $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$.

We now let $u \in \mathcal{D}(\mathbf{R}_{>0})$ be a smooth function, compactly supported within an open neighborhood with compact closure of $[\gamma^N, \gamma^{N+1}]$ for a choice of integer $N \geq 0$ large enough so as to have $\gamma^N > 3$.

We assume u is real-valued, nonnegative, and $u = 1$ on $[\gamma^N, \gamma^{N+1}]$ and $u = 0$ away from $[\gamma^N - \varepsilon, \gamma^{N+1} + \varepsilon]$ for a choice of $\varepsilon > 0$ so small so as to arrange $[\gamma^N - \varepsilon, \gamma^{N+1} + \varepsilon] \cap (0, 2) = \varnothing$.

We call $u_0 := \Phi_\gamma(u)$ as a smooth function of $t \in \mathbf{R}_{>0}$, and define, for each $\phi \in \mathcal{D}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, $\phi_u := (\phi \circ \pi) \cdot (u/u_0)$, for $\pi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ the quotient homomorphism.

Since u_0 does not vanish on $[\gamma^N - \varepsilon, \gamma^{N+1} + \varepsilon]$, the definition of ϕ_u makes sense, providing a smooth compactly supported function on $\mathbf{R}_{>0}$ such that $\Phi_\gamma(\phi_u) = \phi$ as smooth functions on $\mathbf{R}_{>0}$, invariant under the action of $\gamma^{\mathbf{Z}}$.

This proves that Φ_γ is surjective. By [**NB**, Thm.14.4.5(b)], it is open.

(3) : The kernel $\ker(\Phi_\gamma) \subset \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, a closed subspace by (2), contains the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$.

The Mellin transform sends its closure K_γ onto the closed subspace $(1 - \gamma^{-s})A\mathcal{S}(\mathbf{C})$ (by the Asymptotic Residue Theorem, Theorem B.3.2.1, from which it follows that it is sequentially closed, hence closed, being $A\mathcal{S}(\mathbf{C})$ a Fréchet space by Lemma D.3.1.2).

It follows that the orthogonal to $(1 - \gamma^{-s})A\mathcal{S}(\mathbf{C})$ with respect to the duality $\langle A\mathcal{S}(\mathbf{C}), A\mathcal{S}(\mathbf{C})'\rangle$, equipped with the strong topology on the continuous dual of the quotient $A\mathcal{S}(\mathbf{C})/(1 - \gamma^{-s})A\mathcal{S}(\mathbf{C})$, equivalently (by Lemma 4.2.1.3(1), which applies since $A\mathcal{S}(\mathbf{C})$ is a nuclear Fréchet space by Proposition D.3.2.5 and Lemma

D.3.2.3(1)) with the subspace topology of the strong topology on $A\mathcal{S}(\mathbf{C})'$, is linearly and topologically generated by the continuous point-evaluation functionals at the points of the complex s-plane $\left\{\frac{2\pi i m}{\log(\gamma)}, m \in \mathbf{Z}\right\}$.

To see this, we proceed as follows. The linear inclusion $A\mathcal{S}(\mathbf{C}) \to \mathscr{O}(\mathbf{C})$ of the Schwartz-Bergman space $A\mathcal{S}(\mathbf{C})$ into the linear space of holomorphic functions on the complex s-plane, is continuous, upon endowing the target with the complete metrizable topology of uniform convergence on compact subsets. It is injective with dense range. Since both source and target are Montel, hence reflexive, then by [**NB**, Thm.8.10.6(e)], the continuous adjoint to the injective continuous linear operator $A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C}) \to \mathscr{O}(\mathbf{C})/(1-\gamma^{-s})\mathscr{O}(\mathbf{C})$ with dense range, is itself injective and with weakly dense range.

Since such range is a linear subspace, hence, in particular, convex, it is dense by [**Jar**, Part I,Ch.8,§8.2,Prop.5,p.149] (see also [**Scha**, Ch.IV,§3,Thm.3.1,p.130]. We use reflexivity of $E := A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C})$ to identify the topologies $\sigma(E', E)$ and $\sigma(E', (E'_\beta)') = \sigma(E'_\beta, (E'_\beta)'))$.

We now observe that the strong dual of $\mathscr{O}(\mathbf{C})/(1-\gamma^{-s})\mathscr{O}(\mathbf{C})$ is the linear space $\bigoplus_{z\in\mathbf{C}:1-\gamma^{-z}=0}\mathbf{C}\delta_z$, for δ_z the continuous point-evaluation functional on $\mathscr{O}(\mathbf{C})$ at $z \in \mathbf{C}$, equipped with the finest locally convex topology.

In turn, to see this, we proceed as follows.

Since the complex s-plane $\mathbf{C}$ is Stein, then, upon denoting by $\mathscr{J} \subset \mathscr{O}$ the ideal sheaf globally generated by $1-\gamma^{-s}$, and by $\check{\mathrm{H}}(\mathbf{C}, \cdot)$ Čech cohomology (see [**Gdm**, Ch.I,§5.1,5.2] and [**Gdm**, Ch.I,§5.2,Thm.5.2.4,p.209] and its Corollary [**Gdm**, Ch.I,§5.2,Corollaire,p.209]), we have

$$\check{\mathrm{H}}^1(\mathbf{C}, \mathscr{O}/\mathscr{J}) = \mathrm{H}^1(\mathbf{C}, \mathscr{O}/\mathscr{J}) = 0$$

and therefore upon choosing a locally finite countable open cover $\{B_m\}_{m\geq 0}$ of the complex s-plane by open balls each of which contains at most one of the zeros z of $1-\gamma^{-s}$, then, upon denoting by $B_{\underline{k}}$ the intersection $\bigcap_{m\in\underline{k}} B_m$ for $\underline{k}$ varying over all non-empty finite subsets of the nonnegative integer, we deduce that $\mathrm{H}^0(B_{\underline{k}}, \mathscr{O}/\mathscr{J})$ is trivial if $\underline{k}$ contains at least two elements, and it is either trivial or 1-dimensional if $\underline{k}$ contains one element exactly.

To see this, we let $z \in \mathbf{C}$ be a zero of $1-\gamma^{-s}$ contained in the open ball $B := B_m$ only. We observe that we have $\mathscr{O}(B)/(1-\gamma^{-s})\mathscr{O}(B) = \mathscr{O}(B)/(s-z)\mathscr{O}(B)$ linearly and topologically, since $1-\gamma^{-s}$ has only simple zeros.

Since $\mathscr{O}(B)$ is $\mathbf{C}[s]$-flat as an algebra over $\mathbf{C}[s]$, and $\mathbf{C}[s]$ is dense in $\mathscr{O}(B)$ by Runge's Theorem (see [**Cnw-b**, Ch.VIII,Cor.1.15,p.200]), then $\mathbf{C}[s]/(s-z) = \mathbf{C}$ is mapped densely into $\mathscr{O}(B)/(s-z)\mathscr{O}(B)$. Since $\mathbf{C}[s]/(s-z)$ is 1-dimensional, it carries the subspace topology inherited from $\mathscr{O}(B)/(s-z)\mathscr{O}(B)$, which is the Euclidean topology, and by completeness and denseness, the map $\mathbf{C}[s]/(s-z) \to \mathscr{O}(B)/(s-z)\mathscr{O}(B)$ is also surjective.

This computes

$$\mathscr{O}(\mathbf{C})/(1-\gamma^{-s})\mathscr{O}(\mathbf{C}) = \check{\mathrm{H}}^0(\mathbf{C}, \mathscr{O}/\mathscr{J})$$

to be $\prod_{z:1-\gamma^{-z}=0}\mathbf{C}$. The quotient topology $\mathscr{O}(\mathbf{C})/(1-\gamma^{-s})\mathscr{O}(\mathbf{C})$ matches the product topology. Indeed, since any compact subset $K \subset \mathbf{C}$ is contained in the union of at most finitely many of the B_m, then the linear bijection $\mathscr{O}(\mathbf{C})/(1-\gamma^{-s})\mathscr{O}(\mathbf{C}) \to \prod_{z:1-\gamma^{-z}=0}\mathbf{C}$ is continuous. Being source and target Fréchet spaces, it is a linear topological isomorphism by [**NB**, Thm.14.4.5(b)].

We have shown that the strong dual of $A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C})$, equivalently of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/K_\gamma$, is linearly and topologically generated by the continuous point-evaluation functionals at the points of the complex s-plane $\left\{\frac{2\pi i m}{\log(\gamma)}, m \in \mathbf{Z}\right\}$, equivalently their images under the continuous adjoint of the Mellin transform.

Such functionals are, each respectively, the image of the continuous linear functionals sending $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ to its mth Fourier coefficient, and hence, being the latter set of functionals a Schauder basis for

$\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$ by Lemma 4.2.1.2, the continuous linear map:

$$\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta \to (\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/K_\gamma)'_\beta$$

is injective with dense range.

By [**NB**, Thm.8.10.6(e)] and reflexivity, the continuous linear operator $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/K_\gamma \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is injective. However, by (2), it is also surjective, and hence it is a continuous linear bijection.

Being source and target Fréchet spaces, it is a linear topological isomorphism by [**NB**, Thm.14.4.5(b)]. □

We recall that an equicontinuous Schauder basis may not be an equicontinuous set; see [**Jar**, Part II,Ch.14,§14.3,No.7,Remarks,p.299].

We now prove the following:

COROLLARY 4.2.1.7. *Let $\gamma \in \mathbf{R}_{>0}$ be a real number, $\gamma > 1$. The unconditional Schauder basis $\{a_m(\cdot)\}_{m\in\mathbf{Z}}$ from* (4.2.1.4) *for $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$ is absolute, and, as a subset of $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$, it is bounded and equicontinuous in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$.*

PROOF. Since $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is a nuclear Fréchet space, its strong dual is barrelled (see [**Bou-TVS**, Ch.IV,§3,No.4,Corollary,IV.24], which applies since nuclear Fréchet spaces are Montel by [**Tre**, Part III,Ch.50,Cor.3,p.520], hence reflexive), and hence [**Jar**, Part II,Ch.14,§14.3,p.296] applies, giving that the Schauder basis $\{a_m(\cdot)\}_{m\in\mathbf{Z}}$ for $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$ is equicontinuous. By the Dynin-Mityagin Theorem (see Theorem 4.1.2.6(1)), since $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$ is itself nuclear by [**Tre**, Part III,Ch.50,Prop.50.6,p.523], $\{a_m(\cdot)\}_{m\in\mathbf{Z}}$ is an absolute Schauder basis.

By Lemma 4.2.1.5(2), the continuous linear operator $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is surjective and open. By [**FlM**, §5.5,Corollary,part (1),p.110], which applies since $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is nuclear Fréchet, and hence it has the Heine-Borel property (see [**Tre**, Part III,Ch.50,Cor.3,p.520] and [**Tre**, Part III,Ch.50,Prop.50.2,(50.12),p.519]), $\Phi'_\gamma : \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta \to \mathcal{S}(\mathbf{R}_{>0})'_\beta$ is a linear topological embedding (see also Lemma 4.2.1.3).

We call $B := \{a_m(\cdot)\}_{m\in\mathbf{Z}} \subset \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$ the set of continuous functionals from (4.2.1.4). To see that B is bounded in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$, it is enough to show that its image under Φ'_γ is bounded in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta$.

To this end, we let $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ be arbitrary. By Theorem D.1.2.1, the image of $a_m(\cdot)$ under Φ'_γ is the unique continuous functional on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ whose image under the adjoint of the Mellin transform M is the point-evaluation functional on $A\mathcal{S}(\mathbf{C})$ supported at the point $z_m := \frac{2\pi i m}{\log(\gamma)}$, i.e.

$$\Phi'_\gamma(a_m(\cdot))(\phi) = \int_{\mathbf{R}_{>0}} \phi(t) t^{\frac{2\pi i m}{\log(\gamma)}} \frac{\mathrm{d}t}{t} = M(\phi)(z_m).$$

For any $\sigma > 0$, by the Asymptotic Residue Theorem (see Theorem B.3.2.1) we obtain, being $M(\phi)$ Schwartz uniformly over the vertical strips of bounded width of the complex s-plane:

$$M(\phi)(z_m) = -\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{M(\phi)(z)}{z - z_m}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{M(\phi)(z)}{z - z_m}\,\mathrm{d}z. \tag{4.2.1.18}$$

We deduce the estimate:

$$|M(\phi)(z_m)| \leq \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} \frac{1}{\sigma^2 + |\tau - \frac{2\pi m}{\log(\gamma)}|^2}\,\mathrm{d}\tau\right)^{1/2} \left\{\|M(\phi)\|_{L^2(-\sigma+i\mathbf{R},\mathrm{d}\tau)} + \|M(\phi)\|_{L^2(\sigma+i\mathbf{R},\mathrm{d}\tau)}\right\} \tag{4.2.1.19}$$

and since the right side of (4.2.1.19) is independent of $m \in \mathbf{Z}$, we deduce that $\Phi'_\gamma(B)$ is simply bounded in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta$. Since $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is a Fréchet space by Lemma D.3.1.2, hence barrelled, then by [**Scha**, Ch.III,§4.2,Thm.p.83] we deduce that $\Phi'_\gamma(B)$ is equicontinuous (and hence so is B in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})'_\beta$). By [**Scha**, Ch.III,§4.1,Corollary,p.83], it is bounded in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta$, which is what we wanted to show. □

REMARK 4.2.1.8 (On the unboundedness of the exponential system). Note that throughout the proof of Lemma 4.2.1.5, we have shown that the canonical continuous linear map:

$$\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$$

is a linear topological isomorphism. We will be reprising this in general in Corollary 4.5.3.1.

We have shown that the absolute Schauder basis $\{e_m\}_{m\in\mathbf{Z}}$ given by the exponential system on $E := \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}), \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ respectively, is shrinking, i.e. its sequence of continuous coefficient functionals $\{a_m(\cdot)\}_{m\in\mathbf{Z}} \subset E'$ is itself a Schauder basis. It is also strongly bounded and an equicontinuous subset of E'.

By [**Kal**, §2,Prop.2.1,p.92], $\{e_m\}_{m\in\mathbf{Z}}$ is a regular Schauder basis (see Definition 4.1.2.1(6)), and by [**Kal**, §5,Thm.5.6,p.103], it cannot be bounded, otherwise it would be normalized (see Definition 4.1.2.1(6)), and hence normal, in the sense of [**Kal**, §1,Def.1.3,p.91]. However, since E is a nuclear Fréchet space, hence Montel, this cannot be the case, given [**Kal**, §5,Thm.5.6,p.103].

Indeed, let $N \geq 0$ be an integer, and consider the continuous seminorm p_N on $\mathcal{S}(\mathbf{R}_{>0})$ defined as

$$p_N(\phi) := \sup_{t\in\mathbf{R}_{>0}} |(t\partial_t)^N \phi(t)|.$$

Then $p_N(e_m) = \left(\frac{2\pi|m|}{\log(\gamma)}\right)^N$. p_N induces a continuous seminorm on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ which, evaluated on e_m, grows polynomially with respect to $|m|$ as $|m| \to +\infty$, showing that $\{e_m\}_{m\in\mathbf{Z}}$ is not bounded in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, and hence in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$.

4.2.2. Schauder bases on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ with a support condition. Recall that for any compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, we denote by $\mathcal{D}_K(\mathbf{R}_{>0}) \subset \mathcal{D}(\mathbf{R}_{>0})$ the linear subspace of those smooth compactly supported functions on $\mathbf{R}_{>0}$ whose support is contained in K.

By [**Sch-a**, Thm.II,p.66] and [**Scha**, Ch.II,§6,6.4,p.58], the topology induced on $\mathcal{D}_K(\mathbf{R}_{>0})$ by the locally convex inductive limit topology on $\mathcal{D}(\mathbf{R}_{>0})$ is the topology of uniform convergence on functions and their derivatives of any order, i.e. the continuous linear inclusion $\mathcal{D}_K(\mathbf{R}_{>0}) \subset \mathcal{D}(\mathbf{R}_{>0})$ is a linear topological embedding.

We let $\gamma \in \mathbf{R}_{>0}$ be a real number satisfying $\gamma > 1$.

We will now show that the exponential system in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, which, so far, we have shown can be lifted to a sequence in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, can in fact be lifted to a sequence of smooth compactly supported functions all contained in $\mathcal{D}_K(\mathbf{R}_{>0})$, for a large enough compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, depending only on γ.

This means, in particular, that all basis elements comprising the exponential system in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ can be lifted to smooth compactly supported functions on $\mathbf{R}_{>0}$ whose support is contained within the *same* compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, so long as K satisfies a largeness condition dependent on γ.

The argument rests on one of the forms of the Paley-Wiener Theorem, and specifically on [**Ru-2**, Ch.19,Thm.19.3,p.375].

In this form, recall that the Paley-Wiener asserts the following:

THEOREM 4.2.2.1 (Paley-Wiener Theorem on compactly supported Fourier transforms). *Let $A > 0$ be a real number, $f \in \mathscr{H}(\mathbf{C})$ an entire function.*

The following conditions on f are equivalent:

(1) *There exists $F \in L^2((-A, A), \lambda)$, for λ the normalized Lebesgue measure, such that:*

$$f(z) = \int_{-A}^{A} F(x)e^{ixz}\,\mathrm{d}x \tag{4.2.2.1}$$

as entire functions of $z \in \mathbf{C}$.

(2) *There exists a real number $C > 0$, such that for every $z \in \mathbf{C}$, we have $|f(z)| \leq Ce^{A|z|}$, and the restriction of f to the real line is in $L^2(\mathbf{R}, \lambda)$.*

We refer the reader to [**Ru-2**, Ch.19,Thm.19.3,p.375]. Theorem 4.2.2.1 can be read through the lens of the smooth change of variables $x = -i\log(t)$: part (1) asserts the existence of $F \in L^2((1/A, A), \mathrm{d}t/t)$ such that:

$$f(s) = \int_{1/A}^{A} F(t)t^s \frac{\mathrm{d}t}{t}$$

as entire functions of $s \in \mathbf{C}$.

Part (2) asserts the existence of a real number $C > 0$ such that $|f(s)| \leq CA^{|\sigma|}$ for all $s \in \mathbf{C}$, with $\sigma := \mathrm{Re}(s)$, and that the restriction of f to the imaginary line is in $L^2(i\mathbf{R}, \lambda)$.

PROPOSITION 4.2.2.2. *Let $\gamma \in \mathbf{R}_{>0}$ be a real number, $\gamma > 1$.*

(1) *If K has non-empty interior and the quotient $\mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ maps K surjectively onto $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, then the induced continuous linear map:*

$$\Phi_\gamma : \mathcal{D}_K(\mathbf{R}_{>0})/(\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}) \tag{4.2.2.2}$$

is a linear topological isomorphism. The assumption on K is satisfied by the compact subsets of the form $K = b \cdot [1/a, a]$ for $a, b > 0$ real numbers, $a > 1$ with $2\log(a) > \log(\gamma)$.

(2) *Under the assumption on K from (1), assume that we have $K = b \cdot [1/a, a]$ for $a, b > 0$ real numbers, $a > 1$. Let ϕ be an element in the intersection $\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})$. Let $A_0 > 0$ be a real number such that $K \subset [1/A_0, A_0]$, and define $A := |-\log(\gamma) + \log(A_0)|$, and $K_A := [e^{-A}, e^A] \subset \mathbf{R}_{>0}$.*

There exists $\psi \in \mathcal{D}_{K_A}(\mathbf{R}_{>0})$ such that $\psi(x) - \psi(\gamma x) = \phi(x)$ as smooth compactly supported functions on $\mathbf{R}_{>0}$, the identity holding in $\mathcal{D}_{K_A \cup \gamma^{-1}K_A}(\mathbf{R}_{>0})$.

REMARK 4.2.2.3. Note that the largeness condition on K from (1) is satisfied, for instance, if K contains an interval of the form $\gamma^N[1 - \varepsilon, \gamma + \varepsilon]$ for some integer $N \geq 1$ and a real number $0 < \varepsilon < 1$.

We begin with the following:

LEMMA 4.2.2.4. *For any compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, the linear subspace $\mathcal{D}_K(\mathbf{R}_{>0}) \subset \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is closed, and the subspace topology agrees with the topology on $\mathcal{D}_K(\mathbf{R}_{>0})$ of uniform convergence of functions and their derivatives of arbitrary order.*

PROOF. The subspace topology on $\mathcal{D}_K(\mathbf{R}_{>0})$ is stronger than the Fréchet topology of uniform convergence on functions and their derivatives of arbitrary order (because it is defined by a larger fundamental family of seminorms; see Definition D.3.3.1).

With respect to the subspace topology, $\mathcal{D}_K(\mathbf{R}_{>0})$ is a closed subspace, since such topology preserves the support condition upon formation of limits in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ of Cauchy sequences of elements in $\mathcal{D}_K(\mathbf{R}_{>0})$, and hence $\mathcal{D}_K(\mathbf{R}_{>0})$ is sequentially closed in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, hence closed, being $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ a Fréchet space by Lemma D.3.1.2.

Since $\mathcal{D}_K(\mathbf{R}_{>0})$ is a Fréchet space with respect to the topology of uniform convergence on functions and their derivatives of arbitrary order, the two topologies are equivalent by the Open Mapping Theorem (see [**NB**, Thm.14.4.5(b)]). □

PROOF OF PROPOSITION 4.2.2.2. (1) : Under the given assumptions on K, the continuous linear operator $\mathcal{D}_K(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is surjective, hence open by the Open Mapping Theorem (see [**NB**, Thm.14.4.5(b)]), being source and target Fréchet spaces. Since it induces an injective linear operator from the Hausdorff quotient $\mathcal{D}_K(\mathbf{R}_{>0})/(\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0}))$, itself a Fréchet space, to $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, the linear operator Φ_γ from (4.2.2.2) is therefore a linear topological isomorphism.

We show that if $K = b \cdot [1/a, a]$ for $a, b \in \mathbf{R}_{>0}$, with $a > 1$, then K satisfies the surjectivity condition in (2), i.e. Φ_γ from (4.2.2.2) is surjective in this case.

Indeed, let $u_0 \in \mathcal{D}_K(\mathbf{R}_{>0})$ be a real valued nonnegative function whose support contains $b' \cdot [1, \gamma]$ in its interior for some $b' > 0$. u_0 exists because $2\log(a) > \log(\gamma)$, and so $[1, \gamma] \subset ab \cdot (1/a, a)$, and it is enough to choose $b' := 1/a$.

The compact subset $b' \cdot [1, \gamma]$ maps surjectively onto $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ under the quotient homomorphism $\pi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, since $\gamma^{\mathbf{Z}} \cdot (b' \cdot [1, \gamma]) = b' \cdot (\bigcup_{m \in \mathbf{Z}} \gamma^m \cdot [1, \gamma]) = b' \cdot \mathbf{R}_{>0} = \mathbf{R}_{>0}$. Then $u_1 := \Phi_\gamma(u_0)$ is real valued, nonnegative, nowhere vanishing over $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$. As a function of $x \in \mathbf{R}_{>0}$, it is smooth and $\gamma^{\mathbf{Z}}$-invariant, and it is nowhere vanishing over $b' \cdot [1, \gamma]$, and hence over $\gamma^{\mathbf{Z}} \cdot (b' \cdot [1, \gamma]) = \mathbf{R}_{>0}$.

Let $\phi \in \mathcal{D}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$. Then $\widetilde{\phi} := u_0 \cdot \frac{\phi \circ \pi}{u_1}$ is smooth, compactly supported with support contained within the support of u_0, which is in turn contained in $b \cdot [1/a, a]$, and we have

$$\Phi_\gamma(\widetilde{\phi}) = \Phi_\gamma(u_0) \cdot \frac{\phi \circ \pi}{u_1} = \Phi_\gamma(u_0) \cdot \frac{\phi \circ \pi}{\Phi_\gamma(u_0)} = \phi \circ \pi$$

since $\frac{\phi \circ \pi}{u_1}$ is $\gamma^{\mathbf{Z}}$-invariant. As a function on $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, this is ϕ, which proves that Φ_γ is surjective in the case $K = b \cdot [1/a, a]$ with $2\log(a) > \log(\gamma)$.

We further refer the reader to the argument in [**Bou-II**, Ch.VII,§2,no.2,Remark 1,INT VII.30].

(2) : We let $K \subset \mathbf{R}_{>0}$ be a compact subset of the form $K = b \cdot [1/a, a]$ for $a, b > 0$ real numbers, $a > 1$. If the assertion holds for $K' = [1/a, a]$, then it holds for K. Indeed, $\phi \in \ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})$, then the same holds for $\phi(bx)$ as a function of $x \in \mathbf{R}$, and if ψ_0 satisfies conditions (a) and (b) for $\phi(bx)$, then $K_0 \subset [e^{-A}, e^A]$ with $A := |-\log(\gamma) + \log(a)|$. Then upon choosing $A_0 := \max\{ab, a/b\}$, we have $b \cdot [1/a, a] \subset [1/A_0, A_0]$, and $\psi(b^{-1}x)$ is supported over a compact subset of $[e^{-A'}, e^{A'}]$ for $A' := |-\log(\gamma) + \log(A_0)|$.

It is therefore enough to treat the case $b = 1$, and that the assertion holds in the special case $K = [1/a, a]$ for any real number $a > 1$ satisfying the largeness condition for (2). In particular, a satisfies $2\log(a) \geq \log(\gamma)$.

Let $\phi \in \ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})$ be the arbitrary element. Then $M(\phi)(s)$ is an entire function on the complex s-plane, Schwartz uniformly over the open vertical strips of bounded width, and, by the Paley-Wiener Theorem (see [**Ru-2**, Ch.19,Thm.19.3,p.375] which applies since we have in particular $\phi \in L^2((1/a, a), \mathrm{d}t/t)$), it satisfies the estimate:

$$M(\phi)(s) = O(a^{|\mathrm{Re}(s)|}) \tag{4.2.2.3}$$

as $|\mathrm{Re}(s)| \to +\infty$. The function:

$$u(s) := \frac{1}{1 - \gamma^{-s}} M(\phi)(s) \tag{4.2.2.4}$$

is meromorphic, with poles located at the countable discrete subset of $i\mathbf{R}$ comprised by the points

$$\left\{ z_m := \frac{2\pi i m}{\log(\gamma)}, m \in \mathbf{Z} \right\}.$$

Since the function $1 - \gamma^{-s}$ is periodic, so is its reciprocal as a meromorphic function on the complex s-plane, and hence it has slow growth near its poles in the sense of Definition B.3.1.1. The Asymptotic Residue Theorem therefore applies (see Theorem B.3.2.1). Since $\phi \in \ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})$, then, by Proposition 4.2.1.5, u has only removable singularities. The Asymptotic Residue Theorem therefore gives, for any $\sigma > 0$ and any integer $N \geq 0$:

$$-\frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} \frac{z^N u(z)}{z - s} \, \mathrm{d}z + \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{z^N u(z)}{z - s} \, \mathrm{d}z = \mathrm{Res}_{z=s} \frac{z^N u(z)}{z - s} = s^N u(s) \tag{4.2.2.5}$$

where u is redefined as its unique extension across each removable singularity $s = z_m$ for $m \in \mathbf{Z}$. Since u is Schwartz on every vertical line of nonzero abscissa, the two line integrals in (4.2.2.5) are bounded uniformly with respect to s varying over the vertical strip of bounded width $(-\sigma + \varepsilon, \sigma - \varepsilon)$, upon choosing σ large

enough, and $\varepsilon > 0$ such that $\sigma - \varepsilon > 0$. It follows that u is also entire and Schwartz uniformly on the open vertical strips of bounded width. Moreover, since $M(\phi)$ satisfies (4.2.2.3), then u satisfies:

$$u(s) = O(e^{|\mathrm{Re}(s)|\cdot|\log(\gamma)-\log(a)|}) \tag{4.2.2.6}$$

as $|\mathrm{Re}(s)| \to +\infty$.

Another application of the Paley-Wiener Theorem gives that the inverse Fourier transform $\psi \in \mathcal{S}(\mathbf{R}_{>0})$, is, in fact, compactly supported over $K_0 := [e^{-|-\log(\gamma)+\log(a)|}, e^{|-\log(\gamma)+\log(a)|}]$ (see the proof of [**Ru-2**, Ch.19,Thm.19.3,p.375], which shows that under the stated conditions, $\psi(e^x)$ must vanish, as a function of $x \in \mathbf{R}$, over the complement of the interval $[-A, A]$, for $A := |-\log(\gamma)+\log(a)|$).

We have, $\phi(x) = \psi(x) - \psi(\gamma x)$ as smooth functions of $x \in \mathbf{R}_{>0}$, since

$$M(\phi)(s) = (1-\gamma^{-s})u(s) = (1-\gamma^{-s})M(\psi)(s).$$

Moreover, if $\gamma > a$, K_0 is contained within $[a/\gamma, \gamma/a]$. If $a > \gamma$, then K_0 is contained within $[\gamma/a, a/\gamma]$. □

As a Corollary of Lemma 4.2.1.5 and Proposition 4.2.2.2, we obtain the following:

PROPOSITION 4.2.2.5. *Let $\gamma \in \mathbf{R}_{>0}$ be a real number, $\gamma > 1$.*
Then:

(1) *the continuous functionals $\delta_{z_m} : \phi \mapsto \int_{\mathbf{R}_{>0}} \phi(t) t^{z_m} \frac{\mathrm{d}t}{t}$, for $z_m := -\frac{2\pi i m}{\log(\gamma)}$ with $m \in \mathbf{Z}$, are an absolute Schauder basis for the orthogonal of $\ker(\Phi_\gamma)$ with respect to the duality $\langle \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}), \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})' \rangle$, equipped with the strong topology on the dual $(\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/\ker(\Phi_\gamma))'$, equivalently with the subspace topology inherited from the strong topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta$.*
(2) *The exponential system in $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ can be lifted to a sequence of smooth compactly supported functions on $\mathbf{R}_{>0}$, whose support can be chosen to be contained in $\mathcal{D}_K(\mathbf{R}_{>0})$ for $K = b \cdot [1/a, a]$ and $a, b \in \mathbf{R}_{>0}$ satisfying $2\log(a) > \log(\gamma)$, and its image in the quotient Fréchet space $\mathcal{D}_K(\mathbf{R}_{>0})/(\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0}))$ is an absolute Schauder basis. Under this assumption on a, the image of the functionals δ_{z_m} is an absolute Schauder basis for the strong dual of $\mathcal{D}_K(\mathbf{R}_{>0})/(\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0}))$.*

PROOF. (1) is Lemma 4.2.1.5(3), and by Proposition 4.2.2.2(2) the assertion on the compact support of the bi-orthogonal sequence to $\{\delta_{z_m}\}_{m\in\mathbf{Z}}$ holds for $K = b \cdot [1/a, a]$. The assertion on the image of $\{\delta_{z_m}\}_{m\in\mathbf{Z}}$ under the continuous adjoint to the continuous linear map:

$$\mathcal{D}_K(\mathbf{R}_{>0})/(\ker(\Phi_\gamma) \cap \mathcal{D}_K(\mathbf{R}_{>0})) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/\ker(\Phi_\gamma) \tag{4.2.2.7}$$

follows upon observing that (4.2.2.7) is a linear topological isomorphism by Proposition 4.2.2.2(2), and hence so is its adjoint, and linear topological isomorphisms preserve the defining conditions of an absolute Schauder basis. □

REMARK 4.2.2.6. We note that the proofs of Lemma 4.2.1.5 and Proposition 4.2.2.2 give the existence of continuous linear sections to the $\gamma^{\mathbf{Z}}$-periodization operators $\Phi_\gamma : \mathcal{S}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ and $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, associated to a choice of smooth compactly supported function u_1 identically equal to 1 over a compact interval of the form $[1, \gamma] \subset \mathbf{R}_{>0}$, and which vanishes outside of a small open neighborhood of $[1, \gamma]$.

Indeed, such section is defined as, upon denoting by $\pi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$ the smooth quotient homomorphism, $\eta(\phi) := \frac{u_1}{\Phi_\gamma(u_1)} \cdot (\phi \circ \pi)$. Compactness of the support of u_1, and hence of $\frac{u_1}{\Phi_\gamma(u_1)}$, ensures the sequential continuity of η, and since both $\mathcal{S}(\mathbf{R}_{>0})$ and $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ are metrizable, η is continuous.

We note that since $\mathbf{R}_{>0}/\gamma^{\mathbf{Z}}$, $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is linearly and topologically isomorphic to the nuclear Fréchet space $\mathfrak{s}$ of rapidly decreasing sequences. Hermite functions, moreover, provide a Schauder basis for $\mathcal{S}(\mathbf{R}_{>0})$ identifying, linearly and topologically, $\mathcal{S}(\mathbf{R}_{>0})$ with the nuclear Fréchet space of rapidly decreasing sequences $\mathfrak{s}$ itself (see [**Simon**, §1,Thm.1,p.140]).

The kernel of Φ_γ in $\mathcal{S}(\mathbf{R}_{>0})$ is a copy of $\mathcal{S}(\mathbf{R}_{>0})$ itself, and so, up to linear topological isomorphism, $\mathcal{S}(\mathbf{R}_{>0})$ is realized as being isomorphic to a split linear topological extension of $\mathfrak{s}$ by $\mathfrak{s}$.

This is, of course, well known, since we have a linear topological isomorphism $\mathfrak{s} \times \mathfrak{s} \simeq \mathfrak{s}$.

4.3. Preparations on isotypic projectors

In this section, we study the continuous action of a compact Hausdorff abelian group k on a Hausdorff locally convex topological $\mathbf{C}$-vector space E, and discuss assumptions on E and on the action of k on E, so that E splits, linearly and topologically as a continuous $\mathbf{C}$-linear representation of k, into a (completed, under suitable completeness conditions on E) topological direct sum of generalized eigenspaces for the continuous action of k on E.

We will be first discussing the special case of E a separable Hilbert space and $k := G^1$ the compact Hausdorff closed subgroup of elements of module-1 in an $\mathbf{R}_{>0}$-modulated group G. We recall from Definition D.2.1.1 that we term $\mathbf{R}_{>0}$-*modulated* those Hausdorff locally compact abelian groups G equipped with a continuous proper surjective homomorphism $|\cdot|_G$ to $\mathbf{R}_{>0}$, i.e. $G^1 := \ker(|\cdot|_G)$.

We further refer the reader to §D.3.3 for more on modulated groups, and to Theorem D.3.3.7 for the structure of their Schwartz-Bruhat spaces (of I-exponential decaying Schwartz-Bruhat functions).

Situation 4.3.0.1 (Isotypic projectors - preparatory notation). Let F be a number field, S a finite set of places of F containing all the Archimedean places.

(1) We denote by G either the Hausdorff locally compact abelian group C_F, or $J_{F,S}$.
(2) We denote ν the Haar measure $\overline{\mu}$ on C_F, respectively $\overline{\mu}_S$ on $J_{F,S}$.
(3) We write G^1 for the closed subgroups C_F^1 of C_F, respectively $J_{F,S}^1$ of $J_{F,S}$, and we denote by ν_1 the Haar measure $\overline{\mu}_1$ on C_F^1, respectively $\overline{\mu}_{S,1}$ on $J_{F,S}^1$, obtained upon choosing a continuous homomorphic section τ to the idelic norm, respectively to the S-idelic norm.

In practice, G will be often assumed to satisfy the conditions from Situation 4.3.0.1.

Throughout this section and this Chapter, unless otherwise stated, it will be assumed to be a second-countable $\mathbf{R}_{>0}$-modulated group with closed subgroup of module-1 elements G^1. We will denote by $|\cdot|_G$ the module of G, i.e. the continuous proper surjective homomorphism to $\mathbf{R}_{>0}$ promoting G to an $\mathbf{R}_{>0}$-modulated group.

4.3.1. Strong continuity and unitarity. We let G be a second-countable $\mathbf{R}_{>0}$-modulated group with module $|\cdot|_G$. We denote by G^1 the kernel of $|\cdot|_G$. We let ν a choice of Haar measure on G. We further choose a Haar measure ν_1 on the compact Hausdorff abelian group G^1.

In this subsection we record some results on G^1-isotypes within the left-regular representation

$$R(G) := L^2(G, \nu)$$

of G.

By this we mean, for each continuous character ω of G^1, the closed subspace

$$R(G)(\omega) := \{\phi \in R(G) : \rho_G(g, \phi) = \omega(g)\phi \text{ for all } g \in G^1\}$$

where we denote by ρ_G the left-translation action by G^1 on $R(G)$ induced by that of G.

The closed subspaces $R(G)\omega)$ are simpler to study since they are separable Hilbert spaces, being $R(G)$ a separable (since G is second-countable; see [**Fol**, Prop.6.7]) Hilbert space. We will be recalling some important facts about them, for later use in the early arguments of this Chapter, throughout this subsection.

We will later introduce a locally convex analog of the spaces $R(G)(\omega)$, which will require more work to describe.

4.3.1.1. *Left-regular representation of G on $L^2(G,\nu)$.* Before we arrive at the main construction, we need information on the left-regular representation of G on $L^2(G,\nu)$.

DEFINITION 4.3.1.1. Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$, closed subgroup of module-1 elements G^1. Let ν be a choice of Haar measure on G, and ν_1 a choice of Haar measure on G^1.

We define the Hilbert space

$$R(G) := L^2(G,\nu) \tag{4.3.1.1}$$

to be the space of ν-square-integrable complex-valued functions on G with respect to the Haar measure ν.

We write

$$\langle \phi_1, \phi_2 \rangle_{R(G)} := \int_G \phi_1(g)\,\overline{\phi_2(g)}\,\mathrm{d}\nu(g) \tag{4.3.1.2}$$

for the Hermitian inner product on $R(G)$, and $\|\cdot\|_{R(G)}$ for its L^2-norm $\langle\cdot,\cdot\rangle_{R(G)}^{1/2}$.

REMARK 4.3.1.2. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

Recall that the idelic "norm" $|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$ is defined as the product $\prod_v |\cdot|_v$ of all v-adic absolute values on each F_v; see (A.1.3.2). $|\cdot|_F$ is not a norm and it is only referred to as such, by convention. Rather, it is only a continuous homomorphism of Hausdorff locally compact abelian topological groups. Moreoever, recall that by Artin's product formula, we have $|F^\times|_F = 1$. We will denote by $\mathrm{GL}_1(\mathbf{A}_F)^1$ the closed subgroup of "norm-one" elements in $\mathrm{GL}_1(\mathbf{A}_F)$, i.e. the kernel of $|\cdot|_F$. Its topology is the subspace topology. Similar assertions hold for the S-idelic norm $|\cdot|_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and its class group $J_{F,S} := \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$.

The reader may assume that the $\mathbf{R}_{>0}$-modulated group G satisfies the assumptions from Situation 4.3.0.1, i.e. it is either C_F or $J_{F,S}$.

As for $G = C_F, J_{F,S}$, a general $\mathbf{R}_{>0}$-modulated group admits continuous homomorphic sections to the module $|\cdot|_G$; see Remark D.2.1.2.

We recall the following facts about the Haar measure ν on G.

LEMMA 4.3.1.3. *The measure ν on G is regular, Radon and σ-finite.*

PROOF. Since ν is Radon as a Borel measure on G, as any Haar measure on a locally compact Hausdorff abelian group, ν is finite on compact subsets (see [**DE**, Thm.1.3.5]). It is therefore regular Radon by [**Fol**, Thm.7.8]. Since G is second-countable, then it is σ-compact, and hence ν is σ-finite. □

REMARK 4.3.1.4. One important consequence of Lemma 4.3.1.3 is that $C_c(G)$, the complex vector space of complex valued continuous compactly supported functions on G, is contained in $L^2(G,\nu) = R(G)$ and it is dense in the L^2-norm topology. This is [**Fol**, Prop.7.9] or [**DE**, Prop.1.3.3], applied to $X = G$, μ in either Theorem is the Haar measure ν, and $p = 2$.

We note that $R(G)$ carries an action of G by left-translation on functions through its continuous left-translation action on itself.

It is defined as follows: on continuous compactly supported functions, we define

$$\rho_G : G \times C_c(G) \to C_c(G) \tag{4.3.1.3}$$

by the assignment, for any $\phi \in C_c(G)$, and any $(g,x) \in G \times G$

$$\rho_G(g,\phi)(x) := \phi(g^{-1}x) \tag{4.3.1.4}$$

We will recall below that ρ_G extends uniquely to $R(G)$ as a continuous representation of G.

LEMMA 4.3.1.5 (Basic properties of the left-regular representation $R(G)$). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let τ be a choice of continuous homomorphic section to the module $|\cdot|_G$. The Hilbert space $R(G)$ satisfies the following conditions:*

(1) *$R(G)$ is separable.*
(2) *ρ_G from (4.3.1.4) uniquely extends to a continuous unitary representation of G on $R(G)$. For every $\phi_1, \phi_2 \in R(G)$ and every $g \in G$, we have*

$$\langle \rho_G(g, \phi_1), \rho_G(g, \phi_2) \rangle_{R(G)} = \langle \phi_1, \phi_2 \rangle_{R(G)}. \tag{4.3.1.5}$$

(3) *Upon defining the assignment*

$$U_{G,\tau} : \mathbf{R}_{>0} \times R(G) \to R(G) \tag{4.3.1.6}$$

as $U_{G,\tau}(t)(\phi) := \rho_G(\tau(t), \phi)$ for every $\phi \in R(G)$, then $U_{G,\tau}$ is a strongly continuous one-parameter group of unitary operators on $R(G)$.

On the concept of continuity for ρ_G being used, we devote a few words.

REMARK 4.3.1.6 (Strong continuity of the left-translation action of G on $R(G)$). We note that continuity of ρ_G from (4.3.1.4) is often referred to as *strong* continuity, usually in the context of one-parameter groups or semi-groups of unitary operators. It means that the map

$$\rho_G : G \to \mathrm{GL}(R(G)) \tag{4.3.1.7}$$

sending g to $\rho_G(g, \cdot)$ is a continuous map with respect to the topology on the Hausdorff locally compact abelian group G on the source, and the subspace topology of the strong operator topology on the target, $\mathrm{GL}(R(G))$ being the group of invertible continuous $\mathbf{C}$-linear endomorphisms of the Hilbert space $R(G)$.

That $\mathrm{GL}(R(G))$ is indeed a group, is a consequence of the Open Mapping Theorem (see [**Cnw-a**, Ch.III,Thm.12.1]).

The strong operator topology on $\mathscr{B}(R(G))$, the $\mathbf{C}$-vector space of bounded $\mathbf{C}$-linear endomorphisms of $R(G)$, is the Hausdorff locally convex topology defined by the separating family of seminorms $\{q_\phi\}_{\phi \in R(G)}$ with $q_\phi(T) := \|T(\phi)\|_{R(G)}$.

The group $\mathrm{GL}(R(G))$ is not a topological group with respect to the strong operator topology, if $R(G)$ is infinite dimensional (equivalently, if G is not a finite abelian group).

The Lemma however says that ρ_G factors through $\mathrm{U}(R(G))$, the unitary group of $R(G)$. The subgroup $\mathrm{U}(R(G))$ of $\mathrm{GL}(R(G))$ is a topological group with respect to the strong operator topology (because all subsets of $\mathrm{U}(R(G))$ are strongly bounded, hence equicontinuous by the Uniform Boundedness Principle (see [**Scha**, Ch.III,§4,Thm.4.2,p.83]; see also [**Cnw-a**, Ch.IX,§1,Prop.1.3(d),p.256]). It is Hausdorff, complete (by another application of the Uniform Boundedness Principle) but not locally compact, unless $R(G)$ is finite dimensional, equivalently if G is a finite abelian group. It is second-countable, metrizable (see [**EU**, Lemma 1.6]) and hence paracompact.

We further note that by the Spectral Theorem for normal operators (see [**Cnw-a**, Ch.IX,§2,Thm.2.2,p.263]), $\mathrm{U}(R(G))$ is connected and path-connected, since for each $A \in \mathrm{U}(R(G))$, there exists a bounded, self-adjoint endomorphism $\phi(A) \in \mathscr{B}(R(G))$, for $\phi(z) := \theta$, $z = e^{i\theta}$ for $\theta \in [0, 2\pi)$, a bounded Borel-measurable function $\phi : \mathrm{U}(1) \to [0, 2\pi) \subset \mathbf{R}$, and since the spectrum $\sigma(A)$ is contained in $\mathrm{U}(1)$ (see [**Cnw-a**, Ch.IX,§2,Thm.2.3,p.264]), and the strongly continuous (by [**Cnw-a**, Ch.X,§5,Thm.5.1,p.327]) path $\gamma(t) := \exp(it\phi(A))$ connects $A = \gamma(1)$ to the identity $\mathrm{Id} = \gamma(0)$.

Note that indeed $\phi(A)$ is self-adjoint, since the image of ϕ is contained in $[0, +\infty)$, and hence, by [**Cnw-a**, Ch.IX,§2,(2.4),p.264] and [**Cnw-a**, Ch.IX,§1,Lemma 1.9,p.257] (noting that orthogonal projectors are positive operators), [**Cnw-a**, Ch.II,§2,Prop.2.12,p.33] applies.

Strong continuity of ρ_G amounts to the homomorphism:

$$\rho_G : G \to \mathrm{U}(R(G)) \tag{4.3.1.8}$$

being a continuous homomorphism of Hausdorff topological groups.

We refer the reader to [**Cnw-a**, Ch.IX,Def.1.2], and to [**Cnw-a**, Ch.X,Def.5.3]. In Lemma 4.3.1.5(3), continuous unitary actions of $\mathbf{R}_{>0}$ on $R(G)$ are studied, induced by the action ρ_G of G. Continuity of $\mathbf{R}_{>0}$ is intended with respect to the strong operator topology.

We refer to [**HP**, Ch.II,§2.16,p.51] for more on the strong operator topology, and to [**Cnw-a**, Ch.X,Thm.5.4] and to [**HP**, Ch.X] for comparisons among various notions of topologies on linear operators, for one-parameter unitary groups. Note that the strong operator topology on $\mathrm{U}(R(G))$ is equivalent to the weak operator topology (see [**EU**, Lemma 1.5]), while the norm-topology is generally strictly finer (see [**EU**, pp.50-51]). We further note that the strong operator topology on $\mathrm{U}(R(G))$ is equivalent to the topology of pre-compact convergence, by Uniform Boundedness and [**Scha**, Ch.III,§4,Thm.4.5,p.85]. In particular, $\mathrm{U}(R(G))$ is contractible, by [**AS**, Appendix 2,Prop.A2.1].

We further refer the reader to §C.3.3 and §C.3.4 in Appendix C, for more on the strong operator topology on separable Hilbert spaces and its properties.

Proof of Lemma 4.3.1.5. (1) : Since the space G is second-countable, hence separable (see Remark A.5.2.9, and, for the special case in which G satisfies the assumptions from Situation 4.3.0.1, see Proposition A.2.3.10), and the measure ν is Radon and σ-finite by Lemma 4.3.1.3, $R(G)$ is a separable metric space by [**Coh**, Prop.3.4.5], hence separable as a Hilbert space.

(2) : For any $g \in G$, we have

$$\int_G \phi_1(g^{-1}x)\,\overline{\phi_2(g^{-1}x)}\,\mathrm{d}\nu(x) = \int_G \phi_1(x)\,\overline{\phi_2(x)}\,\mathrm{d}\nu(gx) \tag{4.3.1.9}$$

$$= \int_G \phi_1(x)\,\overline{\phi_2(x)}\,\mathrm{d}\nu(x) \tag{4.3.1.10}$$

$$= \langle \phi_1, \phi_2 \rangle_{R(G)} \tag{4.3.1.11}$$

where we used $\mathrm{d}\nu(gx) = \mathrm{d}\nu(x)$, which is left invariance of the Haar measure ν, along with [**Bog**, Thm.3.6.1] applied to $\nu(g^{-1}\,\cdot) = \nu$.

This proves (4.3.1.5). This proves in particular that $\|\rho_G(g,\phi)\|_{R(G)} = \|\phi\|_{R(G)}$ and continuity with respect to the strong operator topology on $\mathrm{GL}(R(G))$ follows from the definition. Note that in particular the operator $\rho_G(g,\cdot)$ has bounded operator norm (by 1) for all $g \in G$.

Continuity follows from [**DE**, Lemma 6.1] too. Each $\rho_G(g,\cdot)$ on $C_c(G)$ therefore extends uniquely to $R(G)$ for every $g \in G$.

Stability of the operators $\rho_G(g,\cdot)$ under compositions in g on $C_c(G)$ propagates the same property to all elements of G, providing the existence of the desired extension to a strongly continuous $\mathbf{C}$-linear action on $R(G)$. Uniqueness is ensured by denseness of $C_c(G)$ within $R(G)$, which holds by Lemma 4.3.1.3 (see Remark 4.3.1.4).

(3) : Since the section τ to the module $|\cdot|_G$ is continuous, composing ρ_G with $(\tau, \mathrm{Id}_{R(G)})$ gives a continuous action $U_{G,\tau} : \mathbf{R}_{>0} \times R(G) \to R(G)$. We need only check that it is unitary.

This follows from the fact that $|\tau(t)| = |t|_{\mathbf{R}}$, where $|t|_{\mathbf{R}}$ is the Euclidean absolute value on $\mathbf{R}_{>0}$, and so unitarity of $U_{G,\tau}$ follows from (4.3.1.5). □

4.3.2. Orthogonal decomposition into G^1-isotypes. We continue to assume that the Hausdorff locally compact abelian group G is a second-countable $\mathbf{R}_{>0}$-modulated group, whose module (a given proper surjective continuous homomorphism onto $\mathbf{R}_{>0}$) we denote by $|\cdot|_G$, and we write G^1 for the compact Hausdorff closed subgroup of module-1 elements, i.e. $G^1 := \ker(|\cdot|_G)$.

We further refer the reader to §D.2.1, for more on modulated groups.

We will be applying the contents of this subsection to the case G satisfies the assumptions from Situation 4.3.0.1, and hence is, in particular, either C_F, or $J_{F,S}$, equipped with the module given by the idelic norm $|\cdot|_F$, respectively the S-idelic norm $|\cdot|_{F,S}$.

That, under these assumptions, G^1 is compact Hausdorff (and hence G is $\mathbf{R}_{>0}$-modulated) is a consequence of Theorem A.1.4.3(1).

For any continuous character ω of G^1, we have a continuous linear map:

$$\Pi_{G,\omega}(\phi)(x) := \int_{G^1} \overline{\omega(g)}\phi(g^{-1}x)\,\mathrm{d}\nu_1(g) = \int_{G^1} \omega(g)\phi(gx)\,\mathrm{d}\nu_1(g) \tag{4.3.2.1}$$

defined for ν-almost all $x \in G$, where ν_1 is, recall, a choice of Haar measure on G^1.

In the special case G satisfies the assumptions from Situation 4.3.0.1, then ν is the Tamagawa measure $\overline{\mu}$ on C_F, respectively $\overline{\mu}_S$ on $J_{F,S}$, i.e. the Haar measures from Definition **??** on G, and ν_1 is uniquely determined as per Theorem D.1.2.1 by requiring compatibility with ν and $\mathrm{d}t/t$ on $\mathbf{R}_{>0} \simeq G/G^1$ (explicitly, uniquely determined by requiring that, upon choosing a continuous homomorphic section τ to the module $|\cdot|_G$, i.e. to the idelic, respectively S-idelic, norm, then ν satisfies $\nu = \nu_1 \otimes \tau_*(\mathrm{d}t/t)$, for $\mathrm{d}t/t$ the canonical Haar measure on $\mathbf{R}_{>0}$. See Definition D.2.1.1(2), Remark D.2.1.3 and Remark D.2.1.4).

We choose a continuous homomorphic section τ to the module $|\cdot|_G$.

The operator $\Pi_{G,\omega}$ is defined on $R(G) := L^2(G,\nu)$ and its composition with the pull-back along τ is valued in $L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t)$, where we define the action of G on $L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t)$ as:

$$\overline{\rho}_{G,\omega} : G \times L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \to L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \tag{4.3.2.2}$$

on $\phi \in C_c(\mathbf{R}_{>0})$, by:

$$\overline{\rho}_{G,\omega}(g,\phi)(x) := \omega(g)\phi(|g|_G^{-1}x) \tag{4.3.2.3}$$

and extending $\overline{\rho}_{G,\omega}$ to $L^2(\mathbf{R}_{>0}, \mathrm{d}t/t)$ by continuity and denseness of $C_c(\mathbf{R}_{>0})$ in $L^2(\mathbf{R}_{>0}, \mathrm{d}t/t)$.

The composition of $\Pi_{G,\omega}$ with the pull-back along τ, which we denote by $\Pi_{G,\omega,\tau}$, is therefore promoted to a continuous operator:

$$\Pi_{G,\omega,\tau} : L^2(G,\nu) \to L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \tag{4.3.2.4}$$

equivariant with respect to the action of G^1 obtained by restriction of ρ_G to G^1, on the source and the action of G^1 obtained by restriction of the action $\overline{\rho}_{G,\omega}$ to G^1, on the target.

On $\phi \in C_c(G)$, we see that we have

$$\|\Pi_{G,\omega,\tau}(\phi)\|_{L^2_\omega(\mathbf{R}_{>0},\mathrm{d}t/t)} \leq \|\phi\|_{L^2(G,\nu)} \tag{4.3.2.5}$$

and so this remains true on all $\phi \in L^2(\Gamma,\nu)$ by continuity.

In particular, the unique continuous operator:

$$\Pi_\tau : L^2(G,\nu) \to \prod_{\omega \in \widehat{G^1}} L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \tag{4.3.2.6}$$

to the Hausdorff locally convex product factors through a continuous operator valued into the linear subspace

$$\Pi_\tau : L^2(G,\nu) \to \widehat{\bigoplus_{\omega \in \widehat{G^1}}} L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \tag{4.3.2.7}$$

the Hilbert sum of the Hilbert spaces $L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t)$.

This is the linear subspace

$$\{(\phi_\omega)_{\omega \in \widehat{C_F^1}} : \phi_\omega \in L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t) \text{ for all } \omega \in \widehat{C_F^1},\ \sum_{\omega \in \widehat{C_F^1}} \|\phi_\omega\|^2_{L^2(\mathbf{R}_{>0},\mathrm{d}t/t)} < \infty\}$$

for which we refer the reader to [**Cnw-a**, Ch.I,Prop.6.2,p.24].

REMARK 4.3.2.1. Note that the metric topology on the Hilbert sum is finer than the subspace topology inherited from the locally convex product topology. Also note that by compactness of G^1 (which, in the case $G = C_F$, is a consequence of Theorem A.1.4.3(1), and for $G = J_{F,S}$, is a consequence of Theorem A.1.5.2(1), since $J^1_{F,S}$ is a quotient of $C^1_{F,S}$) the dual group $\widehat{G^1}$ is discrete, and since G is second-countable, so is G^1, and hence $\widehat{G^1}$ is countable.

Countability of $\widehat{G^1}$ ensures that [**Cnw-a**, Ch.I,Prop.6.2,p.24] applies.

Note that in the special cases of interest, $G = C_F$ and $G = J_{F,S}$, the dual group of G^1 can be described explicitly enough to infer its countability directly. For instance, in the case $G = C_F$, the description of the connected component of the identity given in [**AT**, Ch.IX,Thm.3], provides a description of $\widehat{C^1_F}$, and we proceed as follows.

An application of [**DE**, Cor.3.6.2] reduces the proof of countability of $\widehat{C^1_F}$ to that of the dual of the connected component $(C^1_F)^0$ of the identity of C^1_F, and of the dual of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. The connected component $(C^1_F)^0$ is isomorphic as a compact Hausdorff topological abelian group, to $(\mathbf{A_Q}/\mathbf{Q})^{r_1+r_2-1} \times (\mathbf{R}/\mathbf{Z})^{r_2}$, by [**AT**, Ch.IX,Thm.3], and its dual group $\widehat{(C^1_F)^0}$ is isomorphic as a discrete abelian group to $\mathbf{Q}^{r_1+r_2-1} \times \mathbf{Z}^{r_2}$, which is countable.

The dual group of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is also countable, since $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ has a countable neighborhood base at the identity comprising of open subgroups of finite index, and hence its dual group injects into a countable product of finite groups and it is therefore countable.

For any finite set S of places of F containing all the Archimedean places, the dual group of $C^1_{F,S}$ is a quotient of the dual group of C^1_F, and it is therefore also countable. The dual group of $J^1_{F,S}$ is a closed subgroup of the dual group of $C^1_{F,S}$, and it is therefore also countable.

LEMMA 4.3.2.2 (Orthogonal decomposition into G^1-isotypes). *The operator* Π_τ *from* (4.3.2.7) *is a surjective isometry.*

PROOF. By a consequence of the Peter-Weyl Theorem (see [**Kn**, Thm.1.12]), we have an isometry of complex Hilbert spaces, compatible with the structures of continuous unitary representations of G^1 on source and target:

$$L^2(G,\nu) \simeq \widehat{\bigoplus}_{i\in I} H_i \tag{4.3.2.8}$$

where I is a set, and each H_i is in turn a Hilbert space, isometric to a completed direct sum of one-dimensional continuous unitary representations of G^1 whose character is ω_i, for a unique continuous character ω_i of G^1 such that $\rho_G(g, \phi_{i,n_i}) = \omega_i(g)\,\phi_{i,n_i}$, and for every pair of indeces $i_1, i_2 \in I$ with $i_1 \neq i_2$, $\omega_1 \neq \omega_2$.

For every nonzero $\phi \in L^2(G,\nu)$ there exists $i \in I$ such that the image ϕ_i of ϕ under the isomorphism of Hilbert spaces from (4.3.2.8), is nonzero. G^1 acts on ϕ_i by multiplication by the continuous unitary character ω_i, since the isomorphism of Hilbert spaces in (4.3.2.8) is compatible with the continuous representation of G^1 structure on source and target.

Each H_i is isometric under $\Pi_{G,\omega_i,\tau}$ to $L^2_{\omega_i}(\mathbf{R}_{>0}, \mathrm{d}t/t)$. We further deduce that the map $i \mapsto \omega_i$ is an injective map $I \to \widehat{G^1}$. On the other hand, for each continuous character $\omega \in \widehat{G^1}$, the closed subspace $H_\omega \subset L^2(G,\nu)$ defined as:

$$H_\omega := \{\phi \in L^2(G,\nu) : \rho_G(g,\phi) = \omega(g)\phi, g \in G^1\} \subset L^2(G,\nu)$$

is such that $H_{\omega_i} = H_i$ for each $i \in I$.

If $\omega \in \widehat{G^1}$ exists which is not in the image of the injective map $I \to \widehat{G^1}$ from the foregoing, then H_ω maps to the trivial subspace of the Hilbert sum $\widehat{\bigoplus}_{i\in I} H_i$, and hence $H_\omega = 0$, since the continuous linear operator from (4.3.2.8) is a surjective isometry.

This gives a contradiction (since $\varphi \otimes \overline{\omega}$ for any nonzero continuous compactly supported function φ on $\mathbf{R}_{>0}$ induces a continuous compactly supported function on G which is a nonzero element of H_ω). We deduce that we have $I = \widehat{G^1}$. □

We introduce the following:

TERMINOLOGY 4.3.2.1 (G^1-isotypes on the regular representation $R(G)$ of an $\mathbf{R}_{>0}$-modulated group). Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. Let ν a choice of Haar measure on G, and $R(G) = L^2(G, \nu)$ its left-regular representation. Let ν_1 be a choice of Haar measure on $G^1 := \ker(|\cdot|_G)$.

Let $\omega \in \widehat{G^1}$ be a continuous character.

We denote by $R(G)(\omega)$ the closed linear subspace of the Hilbert space $R(G)$, of those elements that are ω-homogeneous with respect to the continuous left-translation action by G^1 on $R(G)$ obtained by restriction to G^1 of the continuous left-translation action of G on $R(G)$:

$$R(G)(\omega) := \{\phi \in R(G) : \rho_G(g, \phi) = \omega(g)\phi, \text{ for all } g \in G^1\} \subset R(G).$$

We refer to $R(G)(\omega)$ as the *ω-isotype* of $R(G)$, and to the orthogonal projector $\nu_1(G^1)^{-1}\Pi_{G^1,\omega}$ as the *projection onto the ω-isotype*.

We note that for each $\omega \in \widehat{G^1}$, the closed subspace $R(G)(\omega)$ is isometric to $L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t)$ under the operator $\Pi_{G,\omega,\tau}$ of (4.3.2.7): the operator $\Pi_{G,\omega,\tau}$ factors through an isometry $R(G)(\omega) \to L^2_\omega(\mathbf{R}_{>0}, \mathrm{d}t/t)$, and the kernel of $\Pi_{G,\omega,\tau}$ is linearly and topologically generated within $R(G)$ by the elements of the form $\phi - \overline{\omega}(g)\rho_G(g, \phi)$ as (ϕ, g) varies over $R(G) \times G^1$.

4.3.3. G-invariants in a quasi-complete Hausdorff locally convex topological vector space. Using the results on integration in the sense of Pettis from §C.4), the following Lemma 4.3.3.1 studies formation of continuous invariants and co-invariants under the continuous action of a compact Hausdorff abelian group G, on a quasi-complete Hausdorff locally convex topological $\mathbf{C}$-vector space E.

The Lemma will be used systematically throughout this Chapter in the presence of a continuous left-translation action by a compact Hausdorff abelian group, and we record it here for convenience.

Throughout the remainder of this subsection, G will denote a compact Hausdorff abelian group, as opposed to a Hausdorff locally compact $\mathbf{R}_{>0}$-modulated group.

We recall that a Hausdorff locally convex topological $\mathbf{C}$-vector space E is *quasi-complete* if and only if every *bounded* Cauchy net in E converges in E (see also Remark C.4.3.4).

Quasi-completeness is also termed, in light of the definition, bounded completeness, and a quasi-complete E is also termed *boundedly complete*. We further refer the reader to §C.4.3 for more on quasi-completeness, a condition that, combined with the Bipolar Theorem (see [**Bou-TVS**, Ch.II,§6,No.3,Thm.1,TVS II.44]), ensures that the closed convex circled hull of a precompact subset of E is contained in E (see Lemma C.4.1.2(1)). This leads to E-valued integration, for E quasi-complete, to exist and be well behaved, as discussed in §C.4.3 (see Theorem C.4.3.5).

We refer the reader to §C.2, §C.5.1 and §C.3.3 for classes of examples of quasi-complete Hausdorff locally convex topological $\mathbf{C}$-vector spaces.

LEMMA 4.3.3.1 (Norm isomorphism of continuous representations of compact Hausdorff abelian groups). *Let G be a compact Hausdorff abelian group, E a Hausdorff locally convex topological $\mathbf{C}$-vector space equipped with a continuous $\mathbf{C}$-linear action ρ_G of G by linear topological automorphisms. Let μ_G be a choice of Haar measure on G. Assume that E is quasi-complete and that either one of the following two conditions is satisfied:*

(a) *the action ρ_G is uniformly continuous, i.e. for every $u \in E$ and for every net $\{u_i\}_{i\in I} \subset E$ converging to u, we have $\rho_G(g, u_i) \to \rho_G(g, u)$ uniformly as $g \in G$.*

(b) *E is bornological, and condition* (a) *is satisfied for every $u \in E$ and for every sequence $\{u_m\}_{m\geq 0} \subset E$ converging to $u \in E$.*

Then:

(1) *For every $u \in E$, the $\mathbf{C}$-linear form on E':*

$$\Phi_G(u) : \varphi \mapsto \int_G \langle \rho_G(g,u), \varphi\rangle_{E,E'} \, \mathrm{d}\mu_G(g) \tag{4.3.3.1}$$

is in the image of the canonical continuous linear operator $\iota_E : E \to (E')^*$, i.e. $\Phi_G(u)$ is the evaluation functional on E' on a unique element of E. Equivalently, the linear operator $\Phi_G : E \to (E')^*$ assigned by $u \mapsto \Phi_G(u)$, factors through ι_E.

Its formation is continuous with respect to the intrinsic topology on E, i.e. $\Phi_G : E \to E$ is a continuous linear operator.

(2) *Φ_G surjects onto the closed subspace $\mathrm{H}^0(G,E)$ of G-invariant elements of E, and $\mu_G(G)^{-1}\Phi_G$ is idempotent.*

Upon denoting by $\mathrm{H}_{0,\mathrm{cont}}(G,E)$ the quotient of E by the kernel of Φ_G, then Φ_G factors through a linear topological isomorphism:

$$N_G : \mathrm{H}_{0,\mathrm{cont}}(G,E) \to \mathrm{H}^0(G,E) \tag{4.3.3.2}$$

with respect to the quotient topology on the source and the subspace topology on the target.

(3) *The closed subspace $\ker(\Phi_G) \subset E$ is the closure of the $\mathbf{C}$-linear subspace V_E generated by the elements $\{u - \rho_G(g,u), u \in E, g \in G\}$ as u varies over E.*

Remark 4.3.3.2 (On E-valued integration). The reader may think of the operator Φ_G as an E-valued integral:

$$\Phi_G = \int_G \rho_G(g,\cdot) \, \mathrm{d}\mu_G(g)$$

in the sense of Pettis, for which we refer to Appendix C, and specifically to §C.4.

We highlight for the reader that the theory of E-valued integration is indeed needed for the purposes of this work beyond the case of Banach spaces, for multiple reasons, all related to the following fact.

Let G be a Hausdorff locally compact abelian group, assume G is compactly generated, and let $k \subset G$ be a compact subgroup with compactly generated orthogonal $k^\perp$. Then $\mathcal{S}(k,G)$, the closed subspace of k-invariant Schwartz-Bruhat functions on G, is a nuclear Fréchet space, and hence it is, in particular, never a Banach space, unless G/k is finite.

Vector-valued integrals will often be, throughout this work, valued in $\mathcal{S}(G)$ or quotients of $\mathcal{S}(G)$, or in closed subspaces of $\mathcal{S}(k,G)$ and their quotients. This leads us to the necessity of vector-valued integration beyond the case of the Banach-valued integral of Bochner. We will be using continuity properties of the integral in the sense of Pettis, and the reader familiar with the Banach-valued theory of integration will recognize extensions of the known results on the Bochner integral, which is recovered as a spacial case, in §C.4.

Terminology 4.3.3.1 (Norm isomorphisms). We refer to the linear topological isomorphism N_G from (4.3.3.2) as the *norm isomorphism* of ρ_G on E.

Proof of Lemma 4.3.3.1. (1) : By Theorem C.4.3.5, applied to $X = G$, $\mu = \mu_G$, $f : X \to E$ defined as $f(g) := \rho_G(g,u)$ for each $u \in E$ fixed, we see that f is μ-integrable in the sense of Pettis, which implies that $\Phi_G(u)$, a priori only an element of $(E')^*$, is contained in $\iota_E(E)$. In particular, the assignment $u \mapsto \Phi_G(u)$ is $\iota_E(E)$-valued and linear, and hence it factors through ι_E as a $\mathbf{C}$-linear operator on $E \to (E')^*$. We denote by Φ_G the resulting unique linear operator $\Phi_G : E \to E$ such that its composition with ι_E is $\Phi_G : E \to (E')^*$.

By Lemma C.4.3.12, $\Phi_G : E \to E$ is continuous with respect to the intrinsic topology on E, if condition (a) is satisfied.

If condition (b) is satisfied, instead, then Lemma C.4.3.12 implies that Φ_G is sequentially continuous. By [**NB**, Thm.13.2.7(a)], which applies to $X = Y = E$ since E is bornological, then Φ_G is continuous (since a sequentially continuous linear map $f : X \to Y$ is, in the terminology of [**NB**, Ch.6,§6.4,p.163], locally bounded, i.e. it maps bounded subsets to bounded subsets, by [**NB**, Thm.6.1.4(a)], and hence [**NB**, Thm.13.2.7(a)] applies. We further refer the reader to Remark C.4.3.4, noting that convergent sequences are bounded in any topological **C**-vector space, while Cauchy nets may not be).

(2) : Since $\mathrm{H}^0(G, E)$ is the intersection of the linear subspaces $\ker(\mathrm{Id}_E - \rho_G(g, \cdot))$ as g varies over G, each of which is closed, it is itself a closed subspace of E.

We have, for every $u \in E$ and every continuous functional $\varphi \in E'$:

$$\langle \rho_G(g, \Phi_G(u)), \varphi\rangle_{E,E'} = \langle \Phi_G(u), \rho'_G(g, \varphi)\rangle_{E,E'} \tag{4.3.3.3}$$

where ρ'_G is the adjoint action on the continuous dual E' of E. We have, from (4.3.3.3):

$$\langle \Phi_G(u), \rho'_G(g, \varphi)\rangle_{E,E'} = \int_G \langle \rho_G(g', u), \rho'_G(g, \varphi)\rangle_{E,E'}\, \mathrm{d}\mu_G(g') = \int_G \langle \rho_G(g, \rho_G(g', u)), \varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g') \tag{4.3.3.4}$$

By G-invariance of the Haar measure μ_G, the righmost side of (4.3.3.4) reads:

$$\int_G \langle \rho_G(g'', u), \varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g'') = \langle \Phi_G(u), \varphi\rangle_{E,E'}.$$

We deduce that Φ_G is valued in the closed subspace $\mathrm{H}^0(G, E)$. To see that it surjects onto it, we observe that we have, for any $u \in \mathrm{H}^0(G, E)$, for every $\varphi \in E'$:

$$\Phi_G(u)(\varphi) = \int_G \langle \rho_G(g, u), \varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g) = \mu_G(G)\langle u, \varphi\rangle_{E,E'} \tag{4.3.3.5}$$

which reads $\Phi_G(u) = \mu_G(G)u$, and hence u is in the image of Φ_G.

N_G therefore admits a continuous right-inverse, given by the composition of the linear topological embedding with closed range $\mathrm{H}^0(G, E) \subset E$, followed by multiplication by $1/\mu_G(G)$, and by the quotient $E \to E/\ker(\Phi_G)$. By [**Bou-GT**, Ch.I,§3.5,Prop.9,p.41], Φ_G is open, and hence a linear topological isomorphism, since it is a continuous linear bijection.

(3) : The kernel of Φ_G is annihilated by all continuous linear functionals on E invariant under the adjoint action of G on E', since N_G surjects onto $\mathrm{H}^0(G, E)$, and these are the only continuous functionals on E with this property. Indeed, let $\varphi \in E'$ be such that $\langle u, \varphi\rangle_{E,E'} = 0$ for all $u \in \ker(\Phi_G)$. Then φ is in the image of the linear inclusion $\mathrm{H}_{0,\mathrm{cont}}(G, E)' \to E'$ by (2) and [**NB**, Thm.8.12.1(a)].

We have, for every $u \in E$:

$$\langle \Phi_G(u), \varphi\rangle_{E,E'} = \langle u, \varphi\rangle_{E,E'}$$

since we may write $u = \Phi_G(u) + u_0$ for some $u_0 \in \ker(\Phi_G)$, being, by (4.3.3.5), the continuous linear endomorphism $\mu_G(G)^{-1}\Phi_G$ of E, idempotent, and hence with linearly and topologically complemented range and kernel.

This identifies the image of the linear inclusion $\mathrm{H}_{0,\mathrm{cont}}(G, E)' \to E'$ with the linear subspace of E' of those continuous functionals that are in the range of the continuous adjoint Φ'_G.

A continuous functional $\varphi \in E'$ is in the image of Φ'_G if and only if for every $u \in E$, we have:

$$\langle u, \varphi\rangle_{E,E'} = \langle \Phi_G(u), \varphi\rangle_{E,E'}$$

which implies:

$$\langle \rho_G(g, u), \varphi\rangle_{E,E'} = \langle \Phi_G(\rho_G(g, u)), \varphi\rangle_{E,E'} = \langle \Phi_G(u), \varphi\rangle_{E,E'} = \langle u, \varphi\rangle_{E,E'}$$

which reads:

$$\langle u, \rho'_G(g, \varphi)\rangle_{E,E'} = \langle u, \varphi\rangle_{E,E'}$$

for every $u \in E$, and hence φ is invariant under the adjoint action ρ'_G, since E separates the points of E'. Conversely, if $\varphi \in E'$ is invariant under the adjoint G-action, then we have, for every $u \in \ker(\Phi_G)$:

$$\langle u, \rho'_G(g,\varphi)\rangle_{E,E'} = \langle \rho_G(g,u), \varphi\rangle_{E,E'}$$

for every $g \in G$, and upon integrating over G we obtain:

$$\Phi_G(u)(\varphi) = \int_G \langle \rho_G(g,u), \varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g) = 0 \tag{4.3.3.6}$$

since $u \in \ker(\Phi_G)$. By G-invariance of φ, (4.3.3.6) reads:

$$\mu_G(G)\langle u, \varphi\rangle_{E,E'} = 0 \tag{4.3.3.7}$$

and hence $\langle u, \varphi\rangle_{E,E'} = 0$ for all $u \in \ker(\Phi_G)$. This shows that the orthogonal $\ker(\Phi_G)^\perp$ with respect to the duality $\langle E, E'\rangle$ is the closed linear subspace of E' of all continuous functionals on E invariant under the adjoint G-action, which we shall denote by $\mathrm{H}^0(G, E')$.

We now note that the subspace V_E also satisfies $V_E^\perp = \mathrm{H}^0(G, E')$. Indeed, let $\varphi \in E'$ be such that $\langle \varphi, u - \rho_G(g,u)\rangle_{E,E'} = 0$ for all $(g,u) \in G \times E$. This is equivalent to the vanishing condition:

$$\langle u, \varphi - \rho'_G(g,\varphi)\rangle_{E,E'} = 0$$

for all $(g,u) \in G \times E$, where ρ'_G is the adjoint action of G on E'. Since E separates the points of E', we deduce that we have $\varphi - \rho'_G(g,\varphi) = 0$ for all $g \in G$, which is equivalent to the condition $\varphi \in \mathrm{H}^0(G, E')$.

By [**NB**, Thm.8.9.2], it follows that the $\sigma(E,E')$-closure of V_E is the $\sigma(E,E')$-closure of $\ker(\Phi_G)$. We now use [**Jar**, Part II,Ch.8,§8.2,Prop.5,p.149], to deduce that the weak closure of V_E, respectively of $\ker(\Phi_G)$, agree with their closures with respect to the initial topology on E. □

We record here for later use, a consequence of Lemma 4.3.3.1:

Corollary 4.3.3.3 (Isotypes of the continuous action of a compact Hausdorff abelian group). *Let G be a compact Hausdorff abelian group, E a Hausdorff locally convex topological $\mathbf{C}$-vector space equipped with a continuous $\mathbf{C}$-linear action ρ_G of G by linear topological automorphisms. Let μ_G be a choice of Haar measure on G. Assume that E is quasi-complete and bornological, and that ρ_G is sequentially uniformly continuous.*

Let $\omega \in \widehat{G}$ be a continuous character.

Then:

(1) *the $\mathbf{C}$-linear form on E':*

$$\Phi_{G,\omega}(u) : \varphi \mapsto \int_G \langle \overline{\omega}(g)\rho_G(g,u), \varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g) \tag{4.3.3.8}$$

defines a linear operator $\Phi_{G,\omega} : E \to (E')^*$ which factors uniquely through $\iota_E : E \to (E')^*$, and the resulting unique linear operator $\Phi_{G,\omega} : E \to E$ is continuous.

(2) *$\Phi_{G,\omega}$ surjects onto the closed subspace $\mathrm{H}^0(\omega, E)$ of ω-homogeneous elements, i.e. $\mathrm{H}^0(\omega, E) := \{u \in E : \rho_G(g,u) = \omega(g)u\}$, and $\mu_G(G)^{-1}\Phi_{G,\omega}$ is idempotent.*

Upon denoting by $\mathrm{H}_{0,\mathrm{cont}}(\omega, E)$ the quotient of E by $\ker(\Phi_{G,\omega})$, then $\Phi_{G,\omega}$ factors through a linear topological isomorphism:

$$N_{G,\omega} : \mathrm{H}_{0,\mathrm{cont}}(\omega, E) \to \mathrm{H}^0(\omega, E) \tag{4.3.3.9}$$

with respect to the quotient topology on the source and the subspace topology on the target.

(3) *The closed subspace $\ker(\Phi_{G,\omega}) \subset E$ is the closure of the $\mathbf{C}$-linear subspace $V_{E,\omega}$ generated by the elements $\{u - \overline{\omega}(g)\rho_G(g,u), g \in G\}$ as u varies over E.*

TERMINOLOGY 4.3.3.2 (Locally convex isotypes). Let G be a compact Hausdorff abelian group, E a Hausdorff locally convex topological $\mathbf{C}$-vector space equipped with a continuous $\mathbf{C}$-linear action ρ_G of G on E by linear topological automorphisms.

Assume (ρ_G, E) satisfy either of the assumptions (a) and (b) of Corollary 4.3.3.3.

Let $\omega \in \widehat{G}$ be a continuous character.

We refer to both $\mathrm{H}_{0,\mathrm{cont}}(\omega, E)$ and $\mathrm{H}^0(\omega, E)$ as the *ω-isotype* of E. As the need arises, we will refer to the former, equipped with the quotient topology, as the space of *continuous ω-co-invariants of E*, and to the latter, equipped with the subspace topology, as the space of *ω-homogeneous elements of E*.

PROOF OF COROLLARY 4.3.3.3. We define the continuous $\mathbf{C}$-linear action $\eta_{G,\omega} := \overline{\omega}\rho_G$ of G on E, i.e. for every $g \in G$ and $u \in E$, $\eta_{G,\omega}(g, u) := \overline{\omega}(g)\rho_G(g, u)$.

Since ω, and hence $\overline{\omega} = \omega^{-1}$, is a continuous character of the compact Hausdorff abelian group G, then $\eta_{G,\omega}$ is a continuous $\mathbf{C}$-linear action of G on E, since ρ_G is. For each $g \in G$, $\eta_{G,\omega}(g, \cdot)$ is a linear topological automorphism whose continuous inverse is $\eta_{G,\omega}(g^{-1}, \cdot)$.

(1), (2) and (3) follow from (1), (2) and (3) of Lemma 4.3.3.1 applied to $\eta_{G,\omega}$ in place of ρ_G of the Lemma, upon observing the following on the continuous action $\eta_{G,\omega}$ of G on E.

Since ρ_G is assumed to be sequentially uniformly continuous, and since the image of ω, and hence of $\overline{\omega}$, is the unit circle in the complex plane, then, by absolute homogeneity of any continuous seminorm on E, $\eta_{G,\omega}$ is also uniformly sequentially continuous. Indeed, for any continuous seminorm p on E, and any $u \in E$, and for any sequence $\{u_m\}_{m\geq 0} \subset E$ converging to u in E, we have

$$\sup_{g\in G} p(\eta_{G,\omega}(g, u_m - u)) = \sup_{g\in G} |\overline{\omega}(g)|\, p(\rho_G(g, u_m - u)) = \sup_{g\in G} p(\rho_G(g, u_m - u)) \to 0.$$

Since E is bornological, (1.b) of Lemma 4.3.3.1 applies.

We then observe that if G acts on E through $\eta_{G,\omega}$, then the resulting $\mathrm{H}^0(G, E)$ is the closed subspace of those elements $u \in E$ satisfying, for every $g \in G$:

$$u = \eta_{G,\omega}(g, u)$$

which reads $u = \overline{\omega}(g)\rho_G(g, u)$, equivalently $\rho_G(g, u) = \omega(g)u$. $\mu_G(G)^{-1}\Phi_{G,\omega}$ is idempotent by the same assertion from Lemma 4.3.3.1(2), applied to $\eta_{G,\omega}$ in place of ρ_G.

These observations having been made, Lemma 4.3.3.1 implies all the assertions of the Corollary. □

4.3.3.1. *Hilbertian and locally convex isotypic projectors.* We now show how to recover the orthogonal projectors $\Pi_{G,\omega}$ from (4.3.2.1) and Lemma 4.3.2.2, using Lemma 4.3.3.1 and Corollary 4.3.3.3.

COROLLARY 4.3.3.4 (Hilbertian and locally convex isotypic projectors). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$ and compact Hausdorff closed subgroup of module-1 elements G^1. Let $R(G)$ be the Hilbert space $L^2(G, \nu)$, for μ_G a choice of Haar measure on G. Let ν_1 be a choice of Haar measure on G^1.*

Then, as continuous $\mathbf{C}$-linear endomorphisms of $R(G)$, we have:

$$\Pi_{G^1,\omega} = \Phi_{G^1,\omega} \tag{4.3.3.10}$$

where $\Pi_{G^1,\omega}$ is the continuous linear endomorphism of $R(G)$ defined in (4.3.2.1)*, and $\Phi_{G^1,\omega}$ is the continuous linear endomorphism of $R(G)$ defined in* (4.3.3.17)*.*

Moreover, $\nu_1(G^1)^{-1}\Pi_{G^1,\omega}$, equivalently $\nu_1(G^1)^{-1}\Phi_{G^1,\omega}$, are orthogonal projectors, whose ranges are, for distinct continuous characters ω of G^1, mutually orthogonal.

PROOF. We note that the Hilbert space $E := R(G)$ is complete, hence quasi-complete. All metrizable Hausdorff locally convex topological $\mathbf{C}$-vector spaces are bornological (see [**NB**, Example 13.2.8(b)]), and by (4.3.2.5), $\Pi_{G^1,\omega}$ is a continuous linear operator.

The continuous left-translation action of G on E is unitary (see Lemma 4.3.1.5(2)) and hence uniformly continuous. Its restriction to the compact Hausdorff abelian subgroup G^1, therefore also is.

Lemma 4.3.3.1 therefore applies, and gives $\Phi_{G^1,\omega}$ through its Corollary 4.3.3.3.

The composition $\iota_E \circ \Pi_{G^1,\omega}$ agrees with $\iota_E \circ \Phi_{G^1,\omega}$ by the Riesz Representation Theorem (see [**Fol**, Ch.6,§6.2,Thm.6.15,p.190]). Since $\nu_1(G^1)^{-1}\Phi_{G^1,\omega}$ is idempotent by Corollary 4.3.3.3(2), it is an orthogonal projector by [**Cnw-a**, Ch.II,§3,Prop.3.3(f)], so long as we show that it is a positive operator. The linear direct sum $\bigotimes_{\omega' \in \widehat{G^1}} \mathrm{H}^0(\omega', E)$ of mutually orthogonal closed subspaces of E, is dense in E by the Peter-Weyl Theorem (see [**Kn**, Thm.1.12]). By continuity of the Hermitian inner product on E, and continuity of $\Phi_{G,\omega}$, it is enough to check that $\Phi_{G,\omega}$ is positive on $\mathrm{H}^0(\omega, E)$ (i.e. for $\omega' = \omega$, since $\Phi_{G,\omega}$ annihilates $\mathrm{H}^0(\omega', E)$ for every continuous character ω' of G^1 such that $\omega' \neq \omega$). $\Phi_{G,\omega}$ is positive on $\mathrm{H}^0(\omega, E)$ since it acts as multiplication by $\nu_1(G^1)$. □

Corollary 4.3.3.4 provides an alternative take on Lemma 4.3.2.2, noting that a form of the Peter-Weyl Theorem is used in the proof of both, respectively, to infer than the linear subspace linearly generated by the matrix coefficients is dense.

EXAMPLE 4.3.3.5 (Finite dimensional representations of compact abelian groups). Let G be a second-countable compact abelian group, and E a finite dimensional $\mathbf{C}$-vector space, equipped with the Euclidean topology, i.e. the unique, up to equivalence, linear topology promoting E to a Hausdorff locally convex topological $\mathbf{C}$-vector space.

Let $\rho_G : G \times E \to E$ be a continuous $\mathbf{C}$-linear action of G by linear topological automorphisms, equivalently linear automorphisms.

There exists a compact subgroup $k \subset G$ with finitely generated orthogonal $k^\perp \subset \widehat{G}$, such that ρ_G restricts to k to the trivial action of k on E, i.e. ρ_G factors trough G/k, which is a compact abelian Lie group, and the resulting action of G/k is faithful, i.e. ρ_G induces an injective continuous homomorphism $\rho : G/k \to \mathrm{GL}(E)$. Indeed, the kernel k of the continuous homomorphism $\rho : G \to \mathrm{GL}(E)$ qualifies, since $\mathrm{GL}(E)$ is itself a Lie group, and hence so is the image of $\rho(G)$ since a closed subgroup of a Lie group is an embedded Lie subgroup, since $\rho(G)$ is compact, hence closed in $\mathrm{GL}(E)$.

We may replace G with G/k and rename G/k as G.

Then G is a compact abelian Lie group, and it acts faithfully and semisimply on E. We choose a Haar measure μ_G on G.

The operators $\mu_G(G)^{-1}\Phi_{G,\omega}$, as ω varies over the continuous characters of G, are the projectors onto the ω-isotypic component $E(\omega)$ of E. Since E is finite dimensional, only finitely many continuous characters $\omega \in \widehat{G}$ exist such that $E(\omega) \neq 0$.

EXAMPLE 4.3.3.6 (Finite dimensional representations of finite groups). Let G be a finite group. Then upon choosing μ_G to be any Haar measure on G, there exists a unique positive real number $r > 0$ such that μ_G is $r\nu_G$, for ν_G the counting measure on G (see [**DE**, Prop.1.4.4(c)]).

Let $E := \mathbf{C}[G]$, the finite dimensional regular representation of G, equipped with the Euclidean topology.

The linear operator $\mu_G(G)^{-1}\Phi_{G,\omega}$ does not depend on the choice of Haar measure μ_G, and we may therefore assume $r = 1$. The linear operator $\mu_G(G)^{-1}\Phi_{G,\omega}$ is, therefore, the isotypic projector $u \mapsto \frac{1}{|G|}\sum_{g \in G} \overline{\omega}(g) g.v$ where $g.v$ is the effect of the action of G on $\mathbf{C}[G]$ by left-translation on itself.

4.3.3.2. *Averaging over relatively compact locally closed subsets of a Hausdorff locally compact abelian group.* We conclude this subsection by recording here below and integrability and continuity criterion for the

vector-valued continuous functions obtained from the action of a Hausdorff locally compact abelian group G, acting continuously on a quasi-complete Hausdorff locally convex topological $\mathbf{C}$-vector space E, restricted to the compact subsets of G.

LEMMA 4.3.3.7 (Integrability and continuity of actions of relatively compact subsets). *Let G be a Hausdorff locally compact abelian group, $X \subset G$ a relatively compact locally closed subset with non-empty interior. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, equipped with a continuous $\mathbf{C}$-linear action ρ_G by linear topological automorphisms. Let μ_G be a choice of Haar measure on G.*

Assume E is quasi-complete, and that either of the following two conditions on ρ_G and E is satisfied:

(a) *the action ρ_G is continuous, uniformly on the compact subsets of G, i.e. for every $u \in E$ and for every net $\{u_i\}_{i\in I} \subset E$ converging to u, we have, for every compact subset $K \subset G$, $\rho_G(g, u_i) \to \rho_G(g, u)$ uniformly as $g \in K$.*

(b) *E is bornological, and condition* (a) *is satisfied for every $u \in E$ and for every sequence $\{u_m\}_{m\geq 0} \subset E$ converging to $u \in E$.*

Then:

(1) *For every $u \in E$, the $\mathbf{C}$-linear form on E':*

$$\Phi_X(u) : \varphi \mapsto \int_X \langle \rho_G(g, u), \varphi\rangle_{E,E'} \, \mathrm{d}\mu_G(g) \tag{4.3.3.11}$$

is in the image of the canonical continuous linear operator $\iota_E : E \to (E')^*$, i.e. $\Phi_X(u)$ is the evaluation functional on E' on a unique element of E. Equivalently, the linear operator $\Phi_X : E \to (E')^*$ assigned by $u \mapsto \Phi_X(u)$, factors through ι_E.

Its formation is continuous with respect to the intrinsic topology on E, i.e. $\Phi_X : E \to E$ is a continuous linear operator.

(2) *Φ_X surjects onto the closed subspace $\mathrm{H}^0(X, E)$ of elements u of E satisfying the equations $\rho_G(g, u) = u$ for all $g \in X$, and $\mu_G(X)^{-1}\Phi_X$ is idempotent.*

(3) *The $\sigma(E, E')$-closure of the closed subspace $\ker(\Phi_X) \subset E$ is the $\sigma(E, E')$-closure of the $\mathbf{C}$-linear subspace V_E generated by the elements $\{u - \rho_G(g, u), u \in E, g \in X\}$ as u varies over E.*

PROOF. (1) : By Theorem C.4.3.5, applied to X, $\mu = \mu_G|_X$ (the restriction of the Radon measure μ_G to X, itself a nonzero Radon measure, since X is relatively compact and its interior is non-empty; see [**Bou-I**, Ch.III,§2,No.1,INT III.23] for the case X is open, and [**Bou-I**, Ch IV,§5,No.7,Def.4,INT IV.74] for the general case X is a locally compact subspace of G. In particular, if X is locally closed), $f : X \to E$ defined as $f(g) := \rho_G(g, u)$ for each $u \in E$ fixed, we deduce that f is μ-integrable in the sense of Pettis, which implies that $\Phi_X(u)$, a priori only an element of $(E')^*$, is contained in $\iota_E(E)$. In particular, the assignment $u \mapsto \Phi_X(u)$ is $\iota_E(E)$-valued and linear, and hence it factors through ι_E as a $\mathbf{C}$-linear operator on $E \to (E')^*$. We denote by Φ_X the resulting unique linear operator $\Phi_X : E \to E$ such that its composition with ι_E is $\Phi_X : E \to (E')^*$.

By Lemma C.4.3.12, $\Phi_X : E \to E$ is continuous with respect to the intrinsic topology on E, if condition (a) is satisfied.

If condition (b) is satisfied, instead, then we proceed as in the proof of (1) from Lemma 4.3.3.1: by Lemma C.4.3.12, Φ_X is sequentially continuous. By [**NB**, Thm.13.2.7(a)], which applies to $X = Y = E$ (notation as in loc.cit.) since E is bornological, then Φ_X is continuous.

(2) : Since $\mathrm{H}^0(X, E)$ is the intersection of the linear subspaces $\ker(\mathrm{Id}_E - \rho_G(g, \cdot))$ as g varies over X, each of which is closed, it is itself a closed subspace of E.

We have, for every $u \in E$, for every continuous functional $\varphi \in E'$, and for every $g \in G$:

$$\langle \rho_G(g, \Phi_X(u)), \varphi\rangle_{E,E'} = \langle \Phi_X(u), \rho'_G(g, \varphi)\rangle_{E,E'} \tag{4.3.3.12}$$

where ρ'_G is the adjoint action on the continuous dual E' of E. We have, from (4.3.3.12):

(4.3.3.13)
$$\langle \Phi_X(u), \rho'_G(g,\varphi)\rangle_{E,E'} = \int_X \langle \rho_G(g',u), \rho'_G(g,\varphi)\rangle_{E,E'}\, \mathrm{d}\mu_G(g') = \int_X \langle \rho_G(g,\rho_G(g',u)),\varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g').$$

By G-invariance of the Haar measure μ_G, the righmost side of (4.3.3.13) reads:

$$\int_X \langle \rho_G(g'',u),\varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g'') = \langle \Phi_X(u),\varphi\rangle_{E,E'}.$$

We deduce that Φ_X is valued in the closed subspace $\mathrm{H}^0(X,E)$. To see that it surjects onto it, we proceed as in the proof of (2) of Lemma 4.3.3.1, and we observe that we have, for any $u \in \mathrm{H}^0(X,E)$, for every $\varphi \in E'$:

$$\Phi_G(u)(\varphi) = \int_X \langle \rho_G(g,u),\varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g) = \mu_G(X)\langle u,\varphi\rangle_{E,E'} \tag{4.3.3.14}$$

which reads $\Phi_X(u) = \mu_G(X)u$, and hence u is in the image of Φ_X.

(3) : The kernel of Φ_X is annihilated by all continuous linear functionals on E invariant under the adjoint action of the elements of G contained in X, on E'. Indeed, let $\varphi \in E'$ be such that $\langle u,\varphi\rangle_{E,E'} = 0$ for all $u \in \ker(\Phi_X)$.

We have, for every $u \in E$:

$$\langle \Phi_X(u),\varphi\rangle_{E,E'} = \langle u,\varphi\rangle_{E,E'}$$

since we may write $u = \Phi_X(u) + u_0$ for some $u_0 \in \ker(\Phi_X)$, being, by (4.3.3.14), the continuous linear endomorphism $\mu_G(X)^{-1}\Phi_X$ of E, idempotent, and hence with linearly and topologically complemented range and kernel.

A continuous functional $\varphi \in E'$ is in the image of Φ'_X if and only if for every $u \in E$, we have:

$$\langle u,\varphi\rangle_{E,E'} = \langle \Phi_X(u),\varphi\rangle_{E,E'}$$

which implies, for every $g \in X$:

$$\langle \rho_G(g,u),\varphi\rangle_{E,E'} = \langle \Phi_X(\rho_G(g,u)),\varphi\rangle_{E,E'} = \langle \Phi_X(u),\varphi\rangle_{E,E'} = \langle u,\varphi\rangle_{E,E'}$$

which reads:

$$\langle u,\rho'_G(g,\varphi)\rangle_{E,E'} = \langle u,\varphi\rangle_{E,E'}$$

for every $u \in E$, and hence φ is invariant under the adjoint action ρ'_G of every element of G contained in X, since E separates the points of E'. Conversely, if $\varphi \in E'$ is invariant under the adjoint G-action of the elements of X, then we have, for every $u \in \ker(\Phi_X)$:

$$\langle u,\varphi\rangle_{E,E'} = \langle u,\rho'_G(g,\varphi)\rangle_{E,E'} = \langle \rho_G(g,u),\varphi\rangle_{E,E'}$$

for every $g \in X$, and upon integrating over X we obtain:

$$\mu_G(X)\langle u,\varphi\rangle_{E,E'} = \Phi_X(u)(\varphi) = \int_X \langle \rho_G(g,u),\varphi\rangle_{E,E'}\, \mathrm{d}\mu_G(g) = 0 \tag{4.3.3.15}$$

since $u \in \ker(\Phi_X)$, which further reads:

$$\mu_G(X)\langle u,\varphi\rangle_{E,E'} = 0 \tag{4.3.3.16}$$

and hence $\langle u,\varphi\rangle_{E,E'} = 0$ for all $u \in \ker(\Phi_X)$. This shows that the orthogonal $\ker(\Phi_X)^\perp$ with respect to the duality $\langle E,E'\rangle$ is the closed linear subspace of E' of all continuous functionals on E invariant under the adjoint G-action of the elements of X, which we shall denote by $\mathrm{H}^0(X,E')$.

We now note that the subspace V_E also satisfies $V_E^\perp = \mathrm{H}^0(X,E')$. Indeed, let $\varphi \in E'$ be such that $\langle u - \rho_G(g,u),\varphi\rangle_{E,E'} = 0$ for all $(g,u) \in X \times E$. This is equivalent to the vanishing condition:

$$\langle u,\varphi - \rho'_G(g,\varphi)\rangle_{E,E'} = 0$$

for all $(g,u) \in X \times E$, where ρ'_G is the adjoint action of G on E'. Since E separates the points of E', we deduce that we have $\varphi - \rho'_G(g,\varphi) = 0$ for all $g \in X$, which is equivalent to the condition $\varphi \in \mathrm{H}^0(X, E')$.

By [NB, Thm.8.9.2], it follows that the $\sigma(E,E')$-closure of V_E is the $\sigma(E,E')$-closure of $\ker(\Phi_X)$. □

The following Corollary is in analogy with Corollary 4.3.3.3:

COROLLARY 4.3.3.8 (Isotypes of the continuous action of a relatively compact subset). *Let G be a Hausdorff locally compact abelian group, E a Hausdorff locally convex topological $\mathbf{C}$-vector space equipped with a continuous $\mathbf{C}$-linear action ρ_G of G by linear topological automorphisms.*

Let $X \subset G$ be a relatively compact subset with non-empty interior.

Let μ_G be a choice of Haar measure on G. Assume that E is quasi-complete and bornological, and that ρ_G is sequentially continuous uniformly on the compact subsets of G.

Let $\omega \in \widehat{G}$ be a continuous character.

Then:

(1) *the $\mathbf{C}$-linear form on E':*

$$\Phi_{X,\omega}(u) : \varphi \mapsto \int_X \langle \overline{\omega}(g)\rho_G(g,u), \varphi\rangle_{E,E'} \, \mathrm{d}\mu_G(g) \tag{4.3.3.17}$$

defines a linear operator $\Phi_{X,\omega} : E \to (E')^*$ which factors uniquely through $\iota_E : E \to (E')^*$, and the resulting unique linear operator $\Phi_{X,\omega} : E \to E$ is continuous.

(2) *$\Phi_{X,\omega}$ surjects onto the closed subspace $\mathrm{H}^0(\omega|_X, E)$ of those elements $u \in E$ satisfying the equations $\rho_G(g,u) = \omega(g)u$ for all $g \in X$, and $\mu_G(X)^{-1}\Phi_{X,\omega}$ is idempotent.*

(3) *The $\sigma(E,E')$-closure of the closed subspace $\ker(\Phi_{X,\omega}) \subset E$ is the $\sigma(E,E')$-closure of the $\mathbf{C}$-linear subspace $V_{E,\omega}$ generated by the elements $\{u - \overline{\omega}(g)\rho_G(g,u), g \in X\}$ as u varies over E.*

PROOF. The Corollary follows from the argument from the proof of Corollary 4.3.3.3, resting, in the role of Lemma 4.3.3.1 from therein, on Lemma 4.3.3.7. □

We refer the reader to Proposition D.1.9.2, which proves that for G a second-countable, compact Hausdorff abelian group, the continuous left-translation action of G on $\mathcal{S}(G)$, the Schwartz-Bruhat space of G (for which we refer the reader to §2.1) is uniformly continuous (hence sequentially uniformly continuous). Since $\mathcal{S}(G)$ is bornological, sequential uniform continuity already suffices, and this provides a source of examples of Hausdorff locally convex topological $\mathbf{C}$-vector spaces equipped with continuous G-module structures, to which Lemma 4.3.3.1, Lemma 4.3.3.7 and their corollaries, apply.

4.4. Exponential system on monogenic quotients of $\mathbf{R}_{>0}$-modulated groups

The reader may use Lemma 4.2.1.5 and Proposition 4.2.2.7 from the previous section, as guidance through the following more general analogs, which cover the case of an arbitrary $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group G equipped with an element $\gamma \in G$ with $|\gamma|_G > 1$ (therefore) generating a free closed subgroup $\gamma^{\mathbf{Z}}$.

This generality is needed for the intended applications. The arguments for the respective results below, closely follow those we have just provided for the special case $G = \mathbf{R}_{>0}$, and are, for the most part, only notationally more involved.

4.4.1. Complements on the Mellin transform on $\mathcal{S}_I(G)$. We make a brief interlude on the Mellin transform on the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}_I(G)$ of I-exponentially decaying Schwartz-Bruhat functions on G, a Hausdorff locally compact second-countable $\mathbf{R}_{>0}$-modulated abelian group.

In this subsection, we show how to reconcile all of the various instances of the Mellin transform arising throughout this work, comparing them through changes of variables and across linear topological isomorphisms.

Our main references will be Proposition D.3.2.5, Theorem D.3.3.7 and Theorem D.3.3.13.

We let G be an $\mathbf{R}_{>0}$-modulated group.

Recall from Definition D.2.1.1 and the subsequent discussion, that this amounts to the datum of a Hausdorff locally compact abelian group G, equipped with a *module*, i.e. a continuous homomorphism:

$$|\cdot|_G : G \to \mathbf{R}_{>0}$$

that is assumed to be proper (hence the closed subgroup $G^1 := \ker(|\cdot|_G)$ is compact Hausdorff) and surjective (hence the nomenclature $\mathbf{R}_{>0}$-*modulated*).

We defined in Definition D.3.3.1, the Hausdorff locally compact topological $\mathbf{C}$-vector space $\mathcal{S}_I(G)$ of I-exponentially decaying Schwartz-Bruhat functions o G, with $I \subset \mathbf{R}$ a non-empty connected interval of the real line.

Recall that this is the linear subspace of the $\mathbf{C}$-vector space of all continuous functions ϕ on G, such that $|\cdot|_G^a \phi \in \mathcal{S}(G)$ for every $a \in I$. The topology on $\mathcal{S}_I(G)$ is the projective topology induced by the linear operators $T_a : \mathcal{S}_I(G) \to \mathcal{S}(G)$, defined as $T_a(\phi) := |\cdot|_G^a \phi$ and by the Hausdorff locally convex topology on $\mathcal{S}(G)$.

We choose a continuous homomorphic section τ to the module $|\cdot|_G$ (whose existence is proved in Remark D.2.1.2), we assume G is second-countable, and we choose a Haar measure μ_1 on G such that μ is the push-forward, under the isomorphism of Hausdorff locally compact abelian groups $\mathbf{R}_{>0} \times G^1 \to G$ induced by the choice of τ, of the Radon product $\frac{\mathrm{d}t}{t} \otimes \mu_1$.

We now refer the reader to Theorem D.3.3.7, where it is shown that $\mathcal{S}_I(G)$ is, thus topologized, a nuclear strict LF-space, linearly and topologically isomorphic to

$$\mathcal{S}_I(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(G^1)$$

where the tensor product is inductive (see the discussion following (C.2.3.22)).

In other words, to understand $\mathcal{S}_I(G)$, it is enough to understand the case $G = \mathbf{R}_{>0}$ (which is studied in §B.2.1, §D.3.2 through the lens of the Mellin transform, and Lemma D.3.2.3), and the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(G^1)$ of Schwartz-Bruhat functions on a compact Hausdorff abelian group, together with the inductive topology on tensor products (for which we refer the reader to the discussion following (C.2.3.22), and to Proposition C.2.3.13 for basic permanence properties. We further refer the reader to [**Gro-b**, Ch.II,§3,no.1,p.73] in Grothendieck's AMS memoir).

Much as in the special case $G = \mathbf{R}_{>0}$ and I open, where the Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$ is defined by $M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \frac{\mathrm{d}t}{t}$, and valued in $A\mathcal{S}(\mathbf{C}_I)$, where $\mathbf{C}_I$ is the open non-empty connected vertical strip within the complex s-plane $\mathbf{C}$, whose intersection with the real line is the base interval of abscissas I, the Mellin transform is defined on $\mathcal{S}_I(G)$ too, by the following assignment:

$$M(\phi)(c) := \int_G \phi(g) c(g) \, \mathrm{d}\mu(g) \tag{4.4.1.1}$$

for μ a choice of Haar measure on G (which can be selected to be canonical, as per Definition D.2.1.1(2); see also Remark D.2.1.3. We will not be making this assumption: we will be allowing G^1 to have nonzero volume different than 1).

The Mellin transform is a function of continuous quasi-characters c of G, the set of which can be given the structure of a Riemann surface $\Omega(G)$, fibred over the discrete set $\widehat{G^1}$. We will be assuming G is second-countable, in which case $\widehat{G^1}$ is a countable discrete set.

If $\phi \in \mathcal{S}_I(G)$, for $I \subset \mathbf{R}$ an open connected interval, then $M(\phi)$ is holomorphic over the relative vertical strip $\Omega(G)_I := \{c \in \Omega(G) : \sigma(c) \in I\}$, where $\sigma(c)$ is the unique real number such that $c = \chi |\cdot|_G^{\sigma(c)+it}$, for some $t \in \mathbf{R}$, and for a choice of continuous character $\chi \in \widehat{G}$ such that $\chi|_{G^1} = c|_{G^1}$.

By Theorem D.3.3.13, the Mellin transform induces a linear topological isomorphism:

$$M : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I)\,\overline{\otimes}\,\mathcal{S}(G^1) \tag{4.4.1.2}$$

obtained by composing the linear topological isomorphism $\mathcal{S}_I(G) \to \mathcal{S}_I(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(G^1)$ induced by a choice of continuous homomorphic section τ to the module $|\cdot|_G$ (whose existence is discussed in Remark D.2.1.2), with $M\,\overline{\otimes}\,\check{\mathcal{F}}_{\mathcal{G}}$, where $\check{\mathcal{F}}_{\mathcal{G}}$ is the composition of the Fourier transform on $\mathcal{S}(G^1)$ composed by inversion (because the Fourier transform is defined as $\mathcal{F}_{G^1}(\psi)(\chi) = \int_{G^1} \psi(g)\overline{\chi}(g)\,\mathrm{d}\mu_1(g)$, therefore necessitating inversion to restore the variation with respect to χ as opposed to $\overline{\chi}$).

Explicitly, we have, for $(t, g) \in \mathbf{R}_{>0} \times G^1$:

$$M(\phi)(c(\tau(t)g)) = \int_{\mathbf{R}_{>0}} c(\tau(t)) \int_{G^1} \phi(\tau(\cdot)g)c(g)\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t} = M_{\mathbf{R}_{>0}}(\mathcal{F}_{G^1}(\phi(\tau(t)g))(\overline{c}|_{G^1}))(c \circ \tau)$$

where $M_{\mathbf{R}_{>0}}$ denotes the Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$, and where $\mathcal{F}_{G^1}$ denotes the Fourier transform on $\mathcal{S}(G^1)$, the element $\phi(\tau(t)g)$ being viewed as a function of $g \in G^1$ for each $t \in \mathbf{R}_{>0}$.

In the special case $\phi = \psi \otimes \phi_0$, with $\psi \in \mathcal{S}_I(\mathbf{R}_{>0})$ and $\phi_0 \in \mathcal{S}(G^1)$, then we indeed obtain

$$M(\phi)(c) = M_{\mathbf{R}_{>0}}(\psi)(c \circ \tau)\mathcal{F}_{G^1}(\phi_0)(\overline{c}|_{G^1}).$$

We let $\gamma \in G$ be an element such that $|\gamma|_G > 1$, and discuss an interlude on the action of the closed subgroup $\gamma^{\mathbf{Z}} \subset G$ on $\mathcal{S}_I(G)(\chi|_{G^1})$, for $\chi \in \widehat{G}$ a continuous character, seen through the lens of the Mellin transform $M : \mathcal{S}_I(G)(\chi|_{G^1}) \to A\mathcal{S}(\mathbf{C})$ from Proposition D.3.2.5 and Theorem D.3.3.13 (applied to $I = \mathbf{R}$).

Discussion 4.4.1.1 (On the Mellin transform on $\mathcal{S}_I(G)$). We let I be a non-empty open connected interval of the real line.

We let G be an $\mathbf{R}_{>0}$-modulated group, and τ be a choice of continuous homomorphic section to the module $|\cdot|_G$.

We let $\gamma \in G$ be an element such that $|\gamma|_G > 1$. In particular, $\gamma \notin G^1$.

We choose μ_1 any Haar measure on G^1, and denote by μ the Haar measure on G matching under push-forward along the isomorphism of Hausdorff locally compact abelian groups $G \simeq G^1 \times \mathbf{R}_{>0}$ induced by the choice of τ, the Radon measure $\mu_1 \otimes \frac{\mathrm{d}t}{t}$.

We fix a continuous character $\chi \in \widehat{G}$ and denote by

$$\Phi_{G,\chi} : \mathcal{S}_I(G) \to \mathcal{S}_I(G)(\chi|_{G^1})$$

the continuous surjective linear operator $u \mapsto \Phi_{G,\chi}(u)(x) := \int_{G^1} \chi(g)u(gx)\,\mathrm{d}\mu_1(g)$, where, recall, $\mathcal{S}_I(G)(\chi|_{G^1})$ is the quotient of $\mathcal{S}_I(G)$ by the kernel of $\Phi_{G,\chi}$, and $\Phi_{G,\chi}$ is continuous surjective and open, making the quotient Hausdorff and complete since so is $\mathcal{S}_I(G)$, and linearly and topologically isomorphic to $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_I(G))$, the closed subspace of $\mathcal{S}_I(G)$ of the $\chi|_{G^1}$-homogeneous I-exponentially decaying Schwartz-Bruhat functions; see Proposition D.3.3.12 (where $\Phi_{G,\chi}$ is linearly and topologically isomorphic, under multiplication by $\mu_1(G^1)$, to $\Phi^I_{G,\omega}$, for $\omega = \chi|_{G^1}$).

We consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_I(G) & \xrightarrow{M(\cdot)(\chi|\cdot|_G^s)} & A\mathcal{S}(\mathbf{C}_I) \\ {\scriptstyle \Phi_{G,\chi}}\downarrow & & \downarrow{\scriptstyle T_{-ix_0}} \\ \mathcal{S}_I(G)(\chi|_{G^1}) & \xrightarrow[M_{\mathbf{R}_{>0}} \circ \tau]{} & A\mathcal{S}(\mathbf{C}_I) \end{array} \tag{4.4.1.3}$$

where M is the Mellin transform on $\mathcal{S}_I(G)$, hence assigned by:

$$M(\phi)(\chi|\cdot|_G^s) := \int_G \phi(g)\chi(g)|g|_G^s\,\mathrm{d}\mu(g). \tag{4.4.1.4}$$

Upon defining $x_0 \in \mathbf{R}$ to be the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$, the rightmost vertical arrow is the effect of the vertical translation by $-ix_0$, i.e. $T_{-ix_0} : s \mapsto s - ix_0$, promoting the diagram in (4.4.1.3) to a commutative diagram of continuous linear operators.

Recall that the bottom horizontal arrow is defined as:

$$M_{\mathbf{R}_{>0}}(\phi)(s + ix_0) = \int_{\mathbf{R}_{>0}} \phi(t) t^{s+ix_0} \frac{\mathrm{d}t}{t}$$

and it is the Mellin transform on the $\mathbf{R}_{>0}$-modulated group $\mathbf{R}_{>0}$ itself, evaluated on the I-exponentially decaying Schwartz-Bruhat functions obtained as $\phi(t) = \Phi_{G,\chi}(\psi)(\tau(t))$, for $\psi \in \mathcal{S}_I(G)$.

The left-translation action by $\pi := \gamma^{-1} \in \gamma^{\mathbf{Z}}$ has the following effect on the diagram in (4.4.1.3):

(1) It sends $\phi \in \mathcal{S}_I(G)$ to $\phi(\pi^{-1}x)$ as a function of $x \in G$.
(2) The operator M is compatible with the effect described in (1) on the source, and with the multiplication operator on $A\mathcal{S}(\mathbf{C}_I)$ by $\chi(\pi)q^{-s}$, where $q := |\pi|_G^{-1}$.
(3) We let $u \in G$ be the element $\tau(q)\pi \in G^1$. Upon calling $\omega \in \widehat{C_F^1}$ the continuous character $\omega := \chi|_{C_F^1}$, we have that Φ is compatible with the effect described in (1) on the source, and with the multiplication operator by $\omega(u)$ on the target, composed with the left-translation action by $q^{-1} \in \mathbf{R}_{>0}$ on the function of $t \in \mathbf{R}_{>0}$ given by $\phi(\tau(t))$ for $\phi \in \mathcal{S}_I(G)(\chi|_{G^1})$. In particular, the left-translation action by $\pi^{\mathbf{Z}} = \gamma^{\mathbf{Z}}$ descends to a continuous action on $\mathcal{S}_I(G)(\chi|_{G^1})$: the effect of its generator π is the one that has just been described.
(4) The operator $M_{\mathbf{R}_{>0}} \circ \tau$ is compatible with the effect described in (3) of the action of $(q^{-1}, u) \in \mathbf{R}_{>0} \times G^1$ (i.e. of π through the isomorphism of Hausdorff locally compact abelian groups $G \simeq \mathbf{R}_{>0} \times G^1$ induced by the choice of τ) on the source, and the effect of the multiplication operator by $q^{-s}\omega(u)$ on the target.
(5) The rightmost vertical arrow is compatible with the multiplication operator by $\chi(\pi)q^{-s}$ on the source, and with the multiplication operator by $\omega(u)q^{-s}$ on the target, because

$$\chi(\pi)q^{-s+ix_0} = \chi(\tau(q)\pi)\chi(\tau(q)^{-1})q^{-s+ix_0} = \omega(u)q^{-s+ix_0-ix_0} = \omega(u)q^{-s}.$$

We will be denoting by $A\mathcal{S}(\mathbf{C}_I)(\chi)$, respectively $A\mathcal{S}(\mathbf{C}_I)(\omega)$, the top, respectively bottom copy of $A\mathcal{S}(\mathbf{C}_I)$ in the diagram from (4.4.1.3).

We further note that upon denoting by $\mathbf{C}(\omega)$ the 1-dimensional complex representation of G^1 through its character ω, then the Mellin transform M is a linear topological isomorphism $\mathcal{S}_I(G)(\omega) \to A\mathcal{S}(\mathbf{C}_I) \mathbin{\widehat{\otimes}}_\pi \mathbf{C}(\omega) = A\mathcal{S}(\mathbf{C}_I)(\omega)$, as per Theorem D.3.3.7.

In practice, we will be choosing γ such that $|\gamma|_G > 1$, of the form $\gamma = \pi^{-1}$, with π a global or S-local idele, or idele class, corresponding to a choice of uniformizer at a non-Archimedean place of the number field F. The $\mathbf{R}_{>0}$-modulated group will either be C_F or $J_{F,S}$, and the module will either be the idelic norm $|\cdot|_F$, or the S-idelic norm $|\cdot|_{F,S}$.

4.4.2. The exponential system on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$. Reprising Discussion 4.4.1.1, we now use analogs of the periodization operators Φ_γ on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, and absolute Schauder bases on the quotient $\mathcal{S}(G/\gamma^{\mathbf{Z}})$, as opposed to $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, provided by analogs of Lemma 4.2.1.5 and Proposition 4.2.2.7, to exhibit absolute Schauder bases on subquotients of $\mathcal{S}_{\mathrm{E}}(G)$, analogous to that in (4.2.2.7), for G a second-countable $\mathbf{R}_{>0}$-modulated group (which will be $J_{F,S}$ in the applications) and incorporating the effect of multiplication by a continuous character of G^1.

Definition 4.4.2.1 ($\gamma^{\mathbf{Z}}$-periodization operators). Let G be an $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group. Let μ_1 be a Haar measure on G^1.

Let $\gamma \in G$ be an element such that $|\gamma|_G > 1$. Let $\chi \in \widehat{G}$ be a continuous character.

We define the operators:

$$\Phi_\gamma, \Phi_{\gamma,\chi} : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}}) = C^\infty(G/\gamma^{\mathbf{Z}}) \tag{4.4.2.1}$$

assigned by:

$$\Phi_\gamma(\phi)(x) := \sum_{k\in\mathbf{Z}} \phi(\gamma^k x),\ \Phi_{\gamma,\chi}(\phi)(x) := \sum_{k\in\mathbf{Z}} \int_{G^1} \chi(g)\phi(\gamma^k g x)\,\mathrm{d}\mu_1(g) \tag{4.4.2.2}$$

as a functions of $x \in G$ invariant under $\gamma^{\mathbf{Z}}$, respectively invariant under $\gamma^{\mathbf{Z}}$ and $\chi|_{G^1}$-homogeneous under the continuous left-translation action of G^1 on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

We refer to the operators Φ_γ, respectively $\Phi_{\gamma,\chi}$, as the *$\gamma^{\mathbf{Z}}$-periodization operator*, respectively the *$\chi|_{G^1}$-homogeneous $\gamma^{\mathbf{Z}}$-periodization operator*, on $\mathcal{S}_{\mathrm{E}}(G)$.

Remark 4.4.2.2. In practice, we will be choosing a continuous homomorphic section τ to the module $|\cdot|_G$, and μ the unique Haar measure on G determined as per Theorem D.1.2.1 by the choices $\frac{\mathrm{d}t}{t}$ and μ_1 on $\mathbf{R}_{>0}$, respectively G^1, and by the isomorphism of Hausdorff locally compact abelian groups $G \simeq \mathbf{R}_{>0} \times G^1$ induced by the choice of τ.

We note that $\Phi_{\gamma,\chi}$ does not depend on the choice of τ, but only on the choice of Haar measure μ_1 on G^1, which amounts to choosing the volume of G^1, since $|\cdot|_G$ is proper.

We will be adopting the assumptions and notation from the following:

Situation 4.4.2.3. Let G be an $\mathbf{R}_{>0}$-modulated group. Assume G is second-countable.

(1) Let τ be a choice of continuous homomorphic section to the module $|\cdot|_G$.
(2) Let μ_1 be a choice of Haar measure on G^1, and μ the unique Haar measure on G matching the Radon product $\frac{\mathrm{d}t}{t} \otimes \mu_1$ under push-forward along the isomorphism $G \simeq \mathbf{R}_{>0} \times G^1$ of Hausdorff locally compact abelian groups induced by the choice of τ.
(3) Let $\gamma \in G$ be a real number such $|\gamma|_G > 1$.
(4) We choose, for each continuous character $\omega \in \widehat{G^1}$, a continuous character $\chi \in \widehat{G}$ such that $\chi|_{G^1}$. We denote by $\widetilde{G^1}$ the subset of $\widehat{G}$ given by the chosen continuous characters χ.
(5) We call, for each $\chi \in \widetilde{G^1}$, $x_0(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0(\chi)}$ as smooth functions of $t \in \mathbf{R}_{>0}$ and $t_0(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\gamma) = |\gamma|_G^{it_0(\chi)}$, and such that $t_0(\chi) \in [0, 2\pi/\log|\gamma|_G)$.

Remark 4.4.2.4. The choice of $\widetilde{G^1}$ amounts to choosing a holomorphic parametrization for the Riemann surface $\Omega(G)$:

$$\chi|\cdot|_G^s,\ s \in \mathbf{C}. \tag{4.4.2.3}$$

As s varies over $i\mathbf{R}$, (4.4.2.3) provides a smooth parametrization for the topological space $\widehat{G}$, fibred over the discrete set $\widehat{G^1}$ (which, if we assume that G, and hence G^1, is second-countable, is a countable discrete set).

Since G is $\mathbf{R}_{>0}$-modulated, it is isomorphic, as a Hausdorff locally compact abelian group, to $\mathbf{R}_{>0} \times G^1$, with G^1 compact Hausdorff, under the choice τ of continuous homomorphic section to the module $|\cdot|_G$. The dual group of $\mathbf{R}_{>0}$ is smoothly isomorphic to $\mathbf{R}$, and the continuous quasi-characters of $\mathbf{R}_{>0}$ are of the form $t \mapsto t^s$, with $\widehat{\mathbf{R}_{>0}}$ embedding as $i\mathbf{R}$ within the complex s-plane of all continuous quasi-characters.

Note that such section τ exists by Remark D.2.1.2.

4.4.2.1. *The exponential system on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.* We let G be an $\mathbf{R}_{>0}$-modulated group. We fix an element $\gamma \in G$ such that $|\gamma|_G > 1$.

We begin with the following description of the orthogonal $(\gamma^{\mathbf{Z}})^\perp \subset \widehat{G}$, i.e. the subgroup:

$$(\gamma^{\mathbf{Z}})^\perp = \{\chi \in \widehat{G} : \chi(\gamma^m) = 1,\ m \in \mathbf{Z}\}.$$

This is a discrete subgroup of $\widehat{G}$, isomorphic to the dual group of the compact group $G/\gamma^{\mathbf{Z}}$ (see Lemma D.1.4.6).

LEMMA 4.4.2.5 (Orthogonal of a monogenic discrete group). *Let G be an $\mathbf{R}_{>0}$-modulated group. Let $\gamma \in G$ be such that $|\gamma|_G > 1$.*

We choose, for each continuous character $\omega \in \widehat{G^1}$, a continuous character $\chi \in \widehat{G}$ such that $\chi|_{G^1} = \omega$, and denote by $\widetilde{G^1} \subset \widehat{G}$ the set of the chosen representatives of the elements of $\widehat{G^1}$.

We have, as discrete subsets of $\widehat{G}$:

$$(\gamma^{\mathbf{Z}})^{\perp} = \{\overline{\chi}|\cdot|_G^{-\frac{2\pi i m}{\log|\gamma|_G}+it_0(\chi)}, \chi \in \widetilde{G^1}, m \in \mathbf{Z}\} \tag{4.4.2.4}$$

where $t_0(\chi) \in \mathbf{R}$ is, for each $\chi \in \widetilde{G^1}$, the unique real number such that $\chi(\gamma) = |\gamma|_G^{it_0(\chi)}$, and $t_0(\chi) \in [0, 2\pi/\log|\gamma|_G)$.

PROOF. We note that the subset of $\widehat{G}$ defined by the right side of (4.4.2.4) is contained in $(\gamma^{\mathbf{Z}})^{\perp}$. Let now $\chi_0 \in \widehat{G}$ be an element of $(\gamma^{\mathbf{Z}})^{\perp}$. This means that for every $m \in \mathbf{Z}$, we have $\chi_0(\gamma^m) = 1$. For the unique $\chi \in \widetilde{G^1}$ such that $\chi_0|_{G^1} = \chi|_{G^1}$, there exists a unique $s \in \mathbf{C}$ such that $\chi_0 = \chi|\cdot|_G^s$, and since χ_0 is unitary, then $s \in i\mathbf{R}$. There therefore exists $t_1 \in \mathbf{R}$ such that $\chi|\cdot|_G^{it_1} \in (\gamma^{\mathbf{Z}})^{\perp}$, and since the set of all such t_1's is a torsor under $\mathbf{Z}/(\frac{2\pi}{\log|\gamma|_G}\mathbf{Z})$, there exists a unique $t_1 \in [-2\pi/\log|\gamma|_G, 0)$ satisfying the same property. We denote by $t_1(\chi)$ this preferred t_1, and we deduce that every other element of $(\gamma^{\mathbf{Z}})^{\perp}$ is of the form $\chi|\cdot|_G^{it_1(\chi)+\frac{2\pi i m}{\log|\gamma|_G}}$ for some integer $m \in \mathbf{Z}$.

We define $t_0(\chi) := -t_1(\chi)$, and deduce that we have $\chi(\gamma) = |\gamma|_G^{it_0(\chi)}$, and $t_0(\chi) \in [0, 2\pi/\log|\gamma|_G)$. Since $(\gamma^{\mathbf{Z}})^{\perp}$ is stable under complex conjugation, we deduce that (4.4.2.4) holds.

The subspace topology on $(\gamma^{\mathbf{Z}})^{\perp}$ is the discrete topology by Lemma D.1.4.6, since $G/\gamma^{\mathbf{Z}}$ is compact Hausdorff. The topology on the subset $A := \{\chi \in \widetilde{G^1} : \overline{\chi}|\cdot|_G^{-\frac{2\pi i m}{\log|\gamma|_G}+it_0(\chi)}\}$ is the quotient topology induced by the surjective map $\widetilde{G^1} \times \mathbf{Z} \to A$ defined by $(\chi, m) \mapsto \overline{\chi}|\cdot|_G^{-\frac{2\pi i m}{\log|\gamma|_G}+it_0(\chi)}$, with $\widetilde{G^1}$ topologized with the quotient topology induced from the discrete topology on G^1 by the map $\omega \mapsto \chi$ (χ being the chosen lift in $\widetilde{G^1}$ of $\omega \in \widehat{G^1}$). $\widetilde{G^1}$ is therefore a discrete subset of $\widehat{G}$, and A carries the discrete topology since the fibres of the map $\widetilde{G^1} \times \mathbf{Z} \to A$ are single points. □

We prove the following analog of Lemma 4.2.1.2. It will be our source of absolute Schauder bases on subquotients of $\mathcal{S}_{\mathrm{E}}(G)$, for G a second-countable $\mathbf{R}_{>0}$-modulated group.

LEMMA 4.4.2.6 (Exponential system on $\mathbf{R}_{>0}$-modulated groups). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the conditions and choices from Situation 4.4.2.3. In particular, we assume that G is second-countable.*

(1) *Any enumeration of the countable set of smooth functions:*

$$\left\{e_{\chi,m}(t,g) := \overline{\omega}(g)t^{\frac{-2\pi i m}{\log(|\gamma|_G)}+it_0(\chi)-ix_0(\chi)} = \overline{\chi}|\cdot|_G^{-\frac{2\pi i m}{\log(|\gamma|_G)}+it_0(\chi)}\right\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}} = (\gamma^{\mathbf{Z}})^{\perp} \subset \widehat{G} \tag{4.4.2.5}$$

on G, is an absolute Schauder basis for $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

Its dual basis is the countable set of continuous functionals $\{a_{\chi,m}(\cdot)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$, where, for any $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$:

$$a_{\chi,m}(\phi) := \frac{1}{\log(|\gamma|_G)\mu_1(G^1)}\int_{G/\gamma^{\mathbf{Z}}} \phi(g)\chi(g)|g|_G^{\frac{2\pi i m}{\log(\gamma)}-it_0(\chi)}\,\mathrm{d}\overline{\mu}_{\gamma}(g) \tag{4.4.2.6}$$

is the mth Fourier coefficient functional, and where $\overline{\mu}_{\gamma}$ is the unique Haar measure on G/γ determined, as per Theorem D.1.2.1, by the choices of μ on G and the counting measure on $\gamma^{\mathbf{Z}} \subset G$.

(2) *Let $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$ be the arbitrary element.*

(a) *The countable set* $\{a_{\chi,m}(\phi)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$ *uniquely determines a Schwartz-Bruhat function* $v_\phi \in \mathcal{S}((\gamma^{\mathbf{Z}})^\perp)$ *such that, for* $\xi \in (\gamma^{\mathbf{Z}})^\perp$ *the unique character corresponding to the pair* $(\chi, m) \in \widetilde{G^1} \times \mathbf{Z}$*, we have*

$$v_\phi(\xi) = a_{\chi,m}(\phi). \tag{4.4.2.7}$$

(b) *Let* q *be any continuous seminorm on* $\mathcal{S}(G/\gamma^{\mathbf{Z}})$*. Then, for each* $\chi \in \widetilde{G^1}$*, we have* $\{q(e_{\chi,m})\}_{m\in\mathbf{Z}}$ *is an element of* $\mathcal{O}_M(\mathbf{Z})$*, i.e. a complex-valued function* $u : \mathbf{Z} \to \mathbf{C}$ *such that for any Schwartz-Bruhat function* $v \in \mathcal{S}(\mathbf{Z})$*, we have* $uv \in \mathcal{S}(\mathbf{Z})$*.*

Remark 4.4.2.7. We note that the Haar measure on $G/\gamma^{\mathbf{Z}}$ given by $A^{-1}\overline{\mu}_\gamma$, for $A := \mu_1(G^1)$, agrees with the unique Haar measure obtained from the canonical measure on the $\mathbf{R}_{>0}$-modulated group G, and the volume-1 measure on G^1, as per Theorem D.1.2.1. We further refer the reader to Definition D.2.1.1 for the normalization required for a Haar measure on an $\mathbf{R}_{>0}$-modulated group to be the canonical measure. In particular, the Haar measure on $G/\gamma^{\mathbf{Z}}$ obtained as $B^{-1}\overline{\mu}_\gamma$ for $B := \log(|\gamma|_G)\mu_1(G^1)$, is the unique volume-1 Haar measure on the compact Hausdorff abelian group $G/\gamma^{\mathbf{Z}}$.

Remark 4.4.2.8. The notation $\mathcal{O}_M(\Gamma)$, for Γ a Hausdorff locally compact abelian group, is borrowed from [**Sch-a**, Ch.VII,§5,pp.243-244] and denotes the multiplier space of $\mathcal{S}(\Gamma)$, i.e. the $\mathbf{C}$-vector space of those continuous functions $u : \Gamma \to \mathbf{C}$ such that $u \cdot \mathcal{S}(\Gamma) \subset \mathcal{S}(\Gamma)$, topologized by transport of structure from $\mathcal{S}(\Gamma)$.

$\mathcal{O}_M(\Gamma)$ will only play a role here, to enable us to deduce that certain products, such as uv in Lemma 4.4.2.6(2), are sequences of rapid decay (see also [**Tre**, Part III,Ch.51,p.527]).

In the special case $G = \mathbf{R}_{>0}$, we choose $\chi = 1$, in which case $(\gamma^{\mathbf{Z}})^\perp$ is the subgroup of $\widehat{G}$ of those characters of the form $t \mapsto t^{\frac{2\pi i m}{\log(\gamma)}}$ (noting that we have $|\gamma|_G = \gamma$ in this case).

The sequence $a_{\chi,m}(\phi)$ is the sequence of Fourier coefficients of ϕ, and since $\phi \in \mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$, it is a sequence of rapid decay, i.e. an element of $\mathcal{S}(\frac{2\pi i\mathbf{Z}}{\log(\gamma)}) \simeq \mathcal{S}((\gamma^{\mathbf{Z}})^\perp)$.

A fundamental family of continuous seminorms $\{q_k\}_{k\geq 0}$ on $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$ is obtained from the family of seminorms $\widetilde{q}_k := \|(t\partial_t)^k(\cdot)\|_u$ on $\mathcal{S}(\mathbf{R}_{>0})$ following the definition of quotient seminorms from [**NB**, Thm.5.11.2]. In particular

$$q_k(|\cdot|_{\mathbf{R}}^{-\frac{2\pi i m}{\log(\gamma)}}) \leq \widetilde{q}_k(|\cdot|_{\mathbf{R}}^{-\frac{2\pi i m}{\log(\gamma)}}) = \left(\frac{2\pi|m|}{\log(\gamma)}\right)^k \widetilde{q}_0(|\cdot|_{\mathbf{R}}^{-\frac{2\pi i m}{\log(\gamma)}}) = \left(\frac{2\pi|m|}{\log(\gamma)}\right)^k$$

which illustrates the meaning of Lemma 4.4.2.6(2.b): the sequence $q_k(e_{\chi,m})$, for each $k \geq 0$ (i.e. for each member of a fundamental family of continuous seminorms, and hence, up to absolute constants, for any continuous seminorm) is a sequence of polynomial growth, and hence a Schwartz multiplier for $\mathcal{S}(\mathbf{Z})$.

Lemma 4.4.2.6(2.b) will be used to prove the convergence of the sequence of basis projections in several situations at the end of Chapter 6 of this work, enabling us to prove that certain continuous linear injective operators are actually linear topological isomorphisms, compatibly with C_F-actions on source and target.

Remark 4.4.2.9. We note that for each pair $(\chi, m) \in \widetilde{G^1} \times \mathbf{Z}$, there exists a unique continuous character $\xi \in (\gamma^{\mathbf{Z}})^\perp$ such that $\xi = e_{\chi,m}$. A more general and stronger form of (2.b) is the assertion that $\widehat{G} \subset \mathcal{O}_M(G)$ holds. In particular, since $(\gamma^{\mathbf{Z}})^\perp \subset \widehat{G}$, we have that every $e_{\chi,m}$ is a Schwartz multiplier, and hence in particular a smooth function on G such that multiplication by any Schwartz-Bruhat function on G by it, is again a Schwartz-Bruhat function.

In particular, for any continuous seminorm q on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$, for each $\chi \in \widetilde{G^1}$, there exists a polynomial $p \in \mathbf{C}[X]$ such that $q(e_{\chi,m}) \ll_q p(m)$ as the integer $m \in \mathbf{Z}$ satisfies $|m| \to +\infty$.

Proof. (1) and (2) : We note that indeed $a_{\chi,m}(e_{\chi,m}) = 1$ and $a_{\chi,m}(e_{\chi',m'}) = 0$ for all pairs $(\chi', m') \neq (\chi, m)$. We are using the compactness of $G/\gamma^{\mathbf{Z}}$ to deduce that the smooth functions $e_{\chi,m}$ are all $\overline{\mu}_\gamma$-absolutely integrable over $G/\gamma^{\mathbf{Z}}$.

In particular, the pair

$$(\{e_{\chi,m}\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}, \{a_{\chi,m}(\cdot)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}})$$

is a bi-orthogonal sequence with respect to the duality $\langle \mathcal{S}(G/\gamma^{\mathbf{Z}}), \mathcal{S}(G/\gamma^{\mathbf{Z}})'\rangle$.

This implies that $\{e_{\chi,m}\}_{m\in\mathbf{Z}}$ is a topologically free set by [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289]. Let $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$ be an arbitrary smooth function.

We denote by $t\partial_t$ the derivation operator along the continuous 1-parameter group of $G/\gamma^{\mathbf{Z}}$ induced by the choice of $\tau : \mathbf{R}_{>0} \to G$, the fixed continuous section to the module homomorphism $|\cdot|_G$, composed with the continuous quotient homomorphism $G \to G/\gamma^{\mathbf{Z}}$.

Since $\gamma^{\mathbf{Z}} \subset G$ is discrete and co-compact, there exists a smooth compactly supported function $\psi \in \mathcal{D}(G)$ such that $\sum_{m\in\mathbf{Z}} \psi(\gamma^m x) = \phi(x)$ as smooth functions of $x \in G$, invariant under the action of $\gamma^{\mathbf{Z}}$ by left-translation.

Since G is second-countable and compactly generated (since G^1 is compact), there exists a sequence of compact subgroups $\{k_r\}_{r\geq 0}$, with $k_r \subset G$ such that $k_r^{\perp} \subset \widehat{G}$ is open and compactly generated for each integer $r \geq 0$, such that $\bigcup_{r\geq 0} k_r^{\perp} = \widehat{G}$ (see Lemma D.1.4.4), and such that G/k_r is an abelian Lie group.

It follows that there exists a large enough $r \geq 0$ such that ψ is k_r-invariant.

We consider the sum:

$$\sum_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}:\chi\in F_j, |m|\leq M} a_{\chi,m}(\phi)e_{\chi,m} \tag{4.4.2.8}$$

for each integer $M \geq 0$ and each finite subset $F_j \subset \widetilde{G^1}$ indexed by an integer $j \geq 0$, such that $\bigcup_{j\geq 0} F_j = \widetilde{G^1}$.

We now note that the assignment:

$$(\chi, m) \mapsto a_{\chi,m}(\phi) \tag{4.4.2.9}$$

uniquely determines a continuous function on the discrete group $(\gamma^{\mathbf{Z}})^{\perp} \simeq \widehat{G/\gamma^{\mathbf{Z}}}$ (by [**DE**, Prop.3.1.5(a)] and [**DE**, Prop.3.6.1(a)], since $G/\gamma^{\mathbf{Z}}$ is compact Hausdorff) satisfying the following property: for $\mathcal{F} : \mathcal{S}(G/\gamma^{\mathbf{Z}}) \to \mathcal{S}\left(\widehat{G/\gamma^{\mathbf{Z}}}\right)$ the Fourier transform, defined as

$$\phi \mapsto \mathcal{F}(\phi)(\eta) := \frac{1}{\log(|\gamma|_G)\mu_1(G^1)} \int_{G/\gamma^{\mathbf{Z}}} \phi(g)\overline{\eta}(g)\, \mathrm{d}\overline{\mu}_{\gamma}(g)$$

we have:

$$\mathcal{F}(\phi)(\overline{\eta}) = a_{\chi,m}(\phi) \tag{4.4.2.10}$$

for $\eta = \overline{\chi}|\cdot|_G^{\frac{2\pi i m}{\log(|\gamma|_G)}+it_0(\chi)}$, noting that all $\eta \in (\gamma^{\mathbf{Z}})^{\perp}$ are described uniquely in this manner, by Lemma 4.4.2.5.

We further note that $\mathcal{F}(\phi)(\overline{\eta})$ is supported over $k_r^{\perp} \cap (\gamma^{\mathbf{Z}})^{\perp} \subset \widehat{G}$. In particular, $a_{\chi,m}(\phi) = 0$ for all $(\chi, m) \in \widetilde{G^1} \times \mathbf{Z}$ satisfying $\chi \in k_r^{\perp}$.

In order to show that the sum in (4.4.2.8) converges absolutely (which we are required to do, in order to verify the defining properties of an absolute Schauder basis; see the discussion preceding [**Jar**, Part II,Ch.14,§14.2,Prop.3,p.298] and the discussion at the beginning of [**Jar**, Part II,Ch.14,§14.6,p.305], as well as the discussion following [**Jar**, Part II,Ch.14,§14.7,Cor.6,p.314]) as $M \to +\infty$ and $j \to +\infty$, it is enough to show that the following assertion holds.

Let G_0 be a compact abelian Lie group (which, in our situation, is chosen to be $G/(\gamma^{\mathbf{Z}} \cdot k_r)$, where we may choose $k_r \subset G^1$ for each integer $r \geq 0$ large enough), μ_0 the volume-1 Haar measure on G_0, $\phi \in \mathcal{S}(G_0) = \mathcal{D}(G_0)$.

Then the sum:

$$\sum_{\xi\in\widehat{G_0}} \mathcal{F}(\phi)(\xi)\overline{\xi} \tag{4.4.2.11}$$

converges uniformly over G_0 along with all of its directional derivatives along any smooth 1-parameter subgroup $\alpha : \mathbf{R}_{>0} \to G_0$.

Recall that for G_0 a compact abelian Lie group, this mode of convergence is, by definition, the convergence with respect to the Hausdorff locally convex topology on $\mathcal{D}(G_0)$. The assertion is therefore saying that the sum of smooth functions on G_0 from (4.4.2.11), converges in $\mathcal{D}(G_0)$.

To clarify the meaning of the convergence assertion further, we note that we regard $\overline{\xi}$ in (4.4.2.11) as a smooth function on G_0 (since it is a continuous character, hence smooth), and the collection $\{\mathcal{F}(\phi)(\xi)\}_{\xi\in\widehat{G_0}}$ as a collection of complex coefficients that multiply each $\overline{\xi}$ obtaining the smooth function $g \mapsto \mathcal{F}(\phi)(\xi)\overline{\xi}(g)$ of $g \in G_0$. The sum in (4.4.2.11) is the limit of finite sums of such smooth functions on G_0, and the assertion to be proven is that it converges uniformly, together with all of its directional derivatives, i.e. in $\mathcal{D}(G_0)$.

We will now establish this. Indeed, we have $\partial_\alpha\overline{\xi} = p_\alpha(\overline{\xi})\overline{\xi}$, where $p_\alpha(\overline{\xi}) = (\partial_\alpha\overline{\xi})(e)$ for $e \in G_0$ the neutral element.

Multiplication by $p_\alpha(\overline{\xi})$ on $\mathcal{S}(\widehat{G_0})$ is a linear topological automorphism (it is multiplication by a polynomial function of $\xi \in \mathbf{Z}^A \times \mathbf{T}^B \times F$ for $A, B \in \mathbf{N}$ and F a finite group) and $\mathcal{F}(\phi)(\xi)$ is a function of $\xi \in \mathbf{Z}^A \times \mathbf{T}^B \times F$ of rapid decay.

In particular, $p_\alpha(\overline{\xi})\mathcal{F}(\phi)(\xi)$ is an element of $\mathcal{S}(\widehat{G_0})$, and as such, it is absolutely integrable with respect to the Plancherel measure on $\widehat{G_0}$ of the Haar measure $(\mu_1(G^1)\log|\gamma|_G)^{-1}\overline{\mu}_\gamma$. This, is a multiple of the counting measure on the discrete subgroup $(\gamma^{\mathbf{Z}})^\perp \subset \widehat{G_0}$, by [**DE**, Prop.1.4.4].

We now note that we have, for each finite subset $V \subset \widehat{G_0}$:

$$\sup_{g\in G_0}\left|\sum_{\xi\in V} p_\alpha(\overline{\xi})\mathcal{F}(\phi)(\xi)\overline{\xi}(g)\right| \leq \sum_{\xi\in V}|p_\alpha(\overline{\xi})\mathcal{F}(\phi)(\xi)| \tag{4.4.2.12}$$

and by absolute integrability of the function $p_\alpha(\overline{\xi})\mathcal{F}(\phi)(\xi)$ on $\widehat{G_0}$, we deduce that the rightmost side of (4.4.2.12) is bounded above by the convergent sum $\sum_{\xi\in\widehat{G_0}}|p_\alpha(\overline{\xi})\mathcal{F}(\phi)(\xi)|$, which proves that the finite sums:

$$\sum_{\xi\in V}\mathcal{F}(\phi)(\xi)(\partial_\alpha\overline{\xi})$$

converge absolutely and uniformly over G_0 as smooth functions on G_0. This proves the convergence assertion for the sum in (4.4.2.11) in $\mathcal{D}(G_0)$.

Note that $(\gamma^{\mathbf{Z}})^\perp$ contains $(\gamma^{\mathbf{Z}} \cdot k_r)^\perp$ for every $r \geq 0$, which is isomorphic to the dual group of $G/(\gamma^{\mathbf{Z}} \cdot k_r)$ by [**DE**, Prop.3.6.1(a)], and so indeed proving that for ϕ a k_r-invariant element of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ (under the left-translation action on the right-, equivalently left-cosets), the sum (indexed by the sole elements of $(\gamma^{\mathbf{Z}})^\perp$ contained in $(\gamma^{\mathbf{Z}} \cdot k_r)^\perp$) from (4.4.2.11), converges, for $G_0 = G/(\gamma^{\mathbf{Z}} \cdot k_r)$, suffices.

For each fixed $\chi \in \widehat{G^1}$, however, there exists a large enough $r \geq 0$ such that every $\xi \in (\gamma^{\mathbf{Z}})^\perp$ such that $\xi|_{G^1} = \chi|_{G^1}$, is already contained in $(\gamma^{\mathbf{Z}} \cdot k_r)^\perp$, and so the restriction of p_α to the subset of such characters $\xi \in (\gamma^{\mathbf{Z}})^\perp$ such that $\xi|_{G^1} = \chi|_{G^1}$ has absolute value that depends on ξ through the unique integer $m \in \mathbf{Z}$ such that $\xi = \chi|\cdot|_G^{\frac{2\pi i m}{\log(|\gamma|_G)} - it_0(\chi)}$, and, as a function of $|m|$, grows polynomially.

We conclude that (2) holds, and that the sum on the right side of (4.4.2.8) is absolutely convergent as $j \to +\infty$ and $M \to +\infty$ in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

This therefore proves that the finite rank operators

$$\sum_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}:\chi\in F_j,|m|\leq M} a_{\chi,m}(\cdot)e_{\chi,m}$$

converge absolutely (see the discussion preceding [**Jar**, Part II,Ch.14,§14.2,Prop.3,p.298] and the discussion at the beginning of [**Jar**, Part II,Ch.14,§14.6,p.305]) with respect to the topology of simple convergence, as $j \to +\infty$ and $M \to +\infty$. The limit of such sums is the identity on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$, since it is so pointwise.

It follows that $\{e_{\chi,m}, a_{\chi,m}(\cdot))\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$ is not only a bi-orthogonal sequence with respect to the duality $\langle \mathcal{S}(G/\gamma^{\mathbf{Z}}), \mathcal{S}(G/\gamma^{\mathbf{Z}})'\rangle$. It is also a topological basis, and hence a Schauder basis by [**Jar**, Part II,Ch.14,§14.2,Prop.3,p.292], and an absolute Schauder basis, either by the foregoing, which shows that the mode of convergence of the sum in (4.4.2.8), as $j, M \in \mathbf{N}$ tend to grow, is absolute (in the sense of [**Jar**, Part II,Ch.14,§14.7,(*),p.314]) for every $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$, or by Theorem 4.1.2.6(3), since $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is nuclear and $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is a strict inductive limit of a sequence of nuclear Fréchet spaces by Lemma D.1.5.6(2) and second-countability of G.

By Lemma 4.1.2.4, which applies since $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is Montel, and hence barrelled and reflexive, by Theorem D.1.5.7(2), the countable set of coefficient functionals $\{a_{\chi,m}(\cdot)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$ is a Schauder basis (with respect to any choice of enumeration of the elements of $\widetilde{G^1}$). Since $\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta$ is itself nuclear by Theorem D.1.5.7(1), then, with respect to any such enumeration of the countably many elements of $\widetilde{G^1}$, the resulting sequence $\{a_{\chi,m}(\cdot)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$ is an equicontinuous basis by [**Jar**, Part II,Ch.14,§14.3,Cor.3,p.296], and hence, by Theorem 4.1.2.6(1), it is an absolute Schauder basis. □

4.5. Periodic Schauder Bases in the absence of multiplicity

In this section, we exhibit absolute Schauder bases for the quotients of $\mathcal{S}_{\mathrm{E}}(G)$ by the kernel of each of the monogenic periodization operators Φ_γ on $\mathcal{S}_{\mathrm{E}}(G)$ introduced in Definition 4.4.2.1.

We recall from the definition that we defined:

$$\Phi_\gamma(f)(x) := \sum_{k\in\mathbf{Z}} f(\gamma^k x)$$

and:

$$\Phi_{\gamma,\chi}(f)(x) := \sum_{k\in\mathbf{Z}} \int_{G^1} \chi(g) f(\gamma^k g x)\, \mathrm{d}\mu_1(g)$$

for each $f \in \mathcal{S}_{\mathrm{E}}(G)$. We have, however, yet to show that indeed these assignments are well posed, i.e. that the sums converge, uniformly over the compact Hausdorff abelian group $G/\gamma^{\mathbf{Z}}$, and with respect to the Hausdorff locally convex topology on $G/\gamma^{\mathbf{Z}}$.

This, we will be doing throughout the following subsection, along with establishing properties of the operators Φ_γ and $\Phi_{\gamma,\chi}$.

4.5.1. Periodic Schauder Bases: the monogenic case. Let G be an $\mathbf{R}_{>0}$-modulated group, $\gamma \in G$ an element such that $|\gamma|_G > 1$, and $\chi \in \widehat{G}$ a continuous character.

4.5.1.1. *Monogenic periodization on $\mathbf{R}_{>0}$-modulated groups.* In this section, we use the exponential system on $\mathcal{S}_{\mathrm{E}}(G/\gamma^{\mathbf{Z}})$ to source absolute Schauder bases for the ranges of the periodization operators Φ_γ, $\Phi_{\gamma,\chi}$ from Definition 4.4.2.1, for then extending the constructions and results to the case of finitely many elements $\{\gamma_1, \ldots, \gamma_N\} \subset G$ such that $|\gamma_k|_G > 1$ for $1 \leq k \leq N$, in the next subsection.

Proposition 4.5.1.1 (Sourcing Periodic Schauder Bases - I). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the conditions and choices from Situation 4.4.2.3.*

Then:

(1) *The linear map:*

$$\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}}) \tag{4.5.1.1}$$

defined by the assignment:

$$\Phi_\gamma : f \mapsto \sum_{k\in\mathbf{Z}} f(\gamma^k x) \tag{4.5.1.2}$$

as a function of $x \in G$, is continuous, surjective, and sends the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G)$ of exponentially decaying Schwartz-Bruhat functions on G that are $\chi|_{G^1}$-homogeneous with respect to the left-translation action by G^1, onto a closed subspace of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

(2) *$\Phi_{\chi,\gamma}(f)$ is a Schwartz-Bruhat function on the compact Hausdorff abelian group $G/\gamma^{\mathbf{Z}}$.*

(3) *The linear operator $\Phi_{\chi,\gamma}$ from (4.4.2.1) is continuous, relatively open, with closed range.*

(4) *$\ker(\Phi_{\chi,\gamma}) \cap \mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ is the closure in $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as functions of $x \in G$ and as ϕ varies over all the elements of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$.*

(5) *$\ker(\Phi_\gamma)$ is the closure in $\mathcal{S}_{\mathrm{E}}(G)$ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as functions of $x \in G$ and as ϕ varies over all the elements of $\mathcal{S}_{\mathrm{E}}(G)$.*

(6) *The linear map:*

$$P_\gamma : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_{\mathrm{E}}(G) \tag{4.5.1.3}$$

assigned by $P_\gamma(\phi)(x) := \phi(x) - \phi(\gamma x)$ has closed range, and its formation commutes with passage to the G^1-isotypes for the continuous left-translation acton of G^1 on $\mathcal{S}_{\mathrm{E}}(G)$.

Remark 4.5.1.2 (On the non-bornologicity of $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$). Note that Φ_γ is itself open. Indeed, $\mathcal{S}_I(G)$ is linearly and topologically isomorphic to $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ by Theorem D.3.3.7(3), and hence Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]) applies, being $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ itself a strict LF-space, since G is assumed to be second-countable.

We note that Φ_γ extends to a sequentially continuous linear map:

$$\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1) \to \mathcal{S}(G/\gamma^{\mathbf{Z}}).$$

If $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is bornological, which happens if G^1 has compactly generated dual group, i.e. in the case G^1 is an abelian Lie group, then the canonical continuous linear operator $\mathcal{S}_I(G) \to \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is a linear topological isomorphism by Corollary C.2.3.12.

See also Remark 4.5.1.3 below.

Remark 4.5.1.3 (Caveat on tensor products and inductive limits). The phenomenon highlighted in Remark 4.5.1.2 is caused by a general result observed in [**Gro-b**, Ch.I,§1,No.3,Rem.1,p.47]: let E be a Fréchet space, and F a strict LF-space. Assume that E is not normable and that there exists a defining sequence $\{F_m\}_{m\geq 0}$ for F, presenting F as the Hausdorff locally convex inductive limit of the Fréchet spaces F_m. Further assume that the sequence is such that $F_m \subset F_{m+1}$ is a strict containment.

Then $\varinjlim_{m\geq 0}(F_m \mathbin{\widehat{\otimes}}_\pi E) \to F \mathbin{\widehat{\otimes}}_\pi E$ is not a linear topological isomorphism.

In the case of $\mathcal{S}_I(G)$ for G a general modulated group, by Theorem D.3.3.7(3) any choice of continuous homomorphic section τ to the module $|\cdot|_G$ induces a linear topological isomorphism $\mathcal{S}_I(G) \simeq \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$, with $G^1 := \ker(|\cdot|_G)$ a compact Hausdorff abelian closed subgroup.

The Schwartz-Bruhat space $\mathcal{S}(G^1)$ is indeed a strict LF-space, since G is assumed to be second-countable, which forces G^1 to also be second-countable.

If G^1 does not have finitely generated dual group, then a defining sequence $\{F_m\}_{m\geq 0}$ as in the foregoing can be found, i.e. such that the closed inclusions $F_m \subset F_{m+1}$ are strict.

Since $\mathcal{S}_I(\mathbf{R}_{>0})$ is a non-normable Fréchet space, we are led to the conclusion from Remark 4.5.1.2, i.e. the inductive tensor product topology on $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ and the projective tensor product topology on $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ are not equivalent. The canonical continuous linear operator $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1) \to \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is, however, a continuous bijection.

Its linear inverse is sequentially continuous (as an application of Proposition C.2.3.11) and hence source and target have equivalent von-Neumann bornologies.

PROOF. (1) : By Theorem D.3.3.7(3), applied for $I = \mathbf{R}$, the choice τ of continuous homomorphic section to the module homomorphism $|\cdot|_G$ induces by pullback a linear topological isomorphism:

$$\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(G^1) \to \mathcal{S}_{\mathrm{E}}(G) \tag{4.5.1.4}$$

compatible with the left-translation action of G^1 through the right tensor factor on the source, and by left-translation on the target.

Formation of the series defining $\Phi_\gamma(f)(x)$ as a function of $x \in G$ for any $f \in \mathcal{S}_{\mathrm{E}}(G)$ converges pointwise over G to a $\gamma^{\mathbf{Z}}$-invariant function, hence a function on $G/\gamma^{\mathbf{Z}}$. To see this, we proceed as follows.

We call $\gamma_0 := |\gamma|_G > 1$, and, for each $g \in G^1$, $\phi_g(t) := f(\tau(t)g)$ as a Schwartz-Bruhat function of $t \in \mathbf{R}_{>0}$ of exponential decay.

The function $t \mapsto \phi(t) := \sup_{g \in G^1} \phi_g(t)$ is continuous over $\mathbf{R}_{>0}$ because G^1 is compact Hausdorff, and, for every integer $N \geq 1$, there exists $T_N > 0$ and a real number $C_N > 0$, such that $\phi(t)$ still satisfies the estimate $|\phi(t)| \leq C_N e^{-N|\log(t)|}$ for all $t \in \mathbf{R}_{>0}$ such that $|\log(t)| > T_N$.

The same method from the proof of Lemma 4.2.1.5 then shows that the series defining $\Phi_\gamma(f)(x)$ converges for every $x \in G$, since it is dominated from above by the series $\sum_{k \in \mathbf{Z}} |\phi(\gamma_0^k t)|$, for $\gamma_0 := |\gamma|_G > 1$. Upon replacing f with $\partial_\alpha^k f$ for each integer $k \geq 0$ and for $\alpha : \mathbf{R}_{>0} \to G$ the arbitrary continuous 1-parameter subgroup, the same holds for the series defining $\Phi_\gamma(\partial_\alpha^k f)$, and the convergence is uniform over the compact subsets of G: since G^1 is compact, the locally uniform mode of convergence is seen by means of the same method from the proof of Lemma 4.2.1.5, which discusses the case $G = \mathbf{R}_{>0}$.

Formation of ∂_α^k commutes with Φ_γ for each integer $k \geq 0$, and hence $\Phi_\gamma(f)$ is a smooth $\gamma^{\mathbf{Z}}$-invariant function on G (we refer the reader to the proof of Lemma 2.2.4.3, for a similar argument. Inducting on the degree $k \geq 1$ of ∂_α, we may rename $\partial_\alpha^{k-1} f$ as f and assume $k = 1$. Since $G \simeq \mathbf{R}_{>0} \times G^1$ as Hausdorff locally compact abelian groups, it is enough to treat the case of α with image contained in G^1, respectively $\alpha(\mathbf{R}_{>0}) \cap G^1 = \{e\}$, the neutral element.

In the former case, uniform convergence of the series defining Φ_γ on compact subsets, and uniform convergence over G of the incremental ratio $(u-1)^{-1}(f(\alpha(u)^{-1}x) - f(x))$ as $u \to 1$, for any $f \in \mathcal{S}_{\mathrm{E}}(G)$, imply the desired commutation of Φ_γ and ∂_α.

We therefore assume $\alpha(\mathbf{R}_{>0}) \cap G^1 = \{e\}$. In this case, the argument is closely analogous to the proof of Lemma 4.2.1.5.

For $k \subset G$ a compact subgroup such that $k^\perp$ is compactly generated, and such that f is k-invariant, then $\Phi_\gamma(f)$ is also k-invariant, since the actions of $\gamma^{\mathbf{Z}}$ and of k by left-translation on $\mathcal{S}_{\mathrm{E}}(G)$ commute. This shows that $\Phi_\gamma(f)$ is Schwartz-Bruhat over $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

Φ_γ is surjective because $G/\gamma^{\mathbf{Z}}$ is compact Hausdorff: we may therefore find a smooth compactly supported function $u_0 \in \mathcal{D}(G)$ such that $\Phi_\gamma(u_0) = 1$, the constant function with value 1 on $G/\gamma^{\mathbf{Z}}$, and for every $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$, the function $\psi := u_0 \cdot (\phi \circ \pi)$, for $\pi : G \to G/\gamma^{\mathbf{Z}}$ the quotient homomorphism, is smooth and compactly supported over G, hence of exponential decay, and we have $\Phi_\gamma(\psi) = \phi$ (we further refer the reader to [**Bou-II**, Ch.VII,§2,No.1,Lemma 1,INT VII.28] and to the footnote [**Bou-II**, Ch.VII,§2,No.1,Remark 1,INT VII.30] for similar arguments).

For the purposes of showing continuity of the linear operator Φ_γ, we observe that any convergent Cauchy net $\{f_i\}_{i \in I} \subset \mathcal{S}_{\mathrm{E}}(G)$ is eventually bounded [1]

(i.e. there exists $i_0 \in I$ such that $\{f_i\}_{i \in I, i \geq i_0}$ is bounded), and hence, up to replacing I with $I_0 := \{i \in I : i \geq i_0\}$ for some i_0 large enough and renaming I_0 as I, it is contained within the pre-image $\mathcal{S}_{\mathrm{E}}(\Gamma \cdot \tau(\mathbf{R}_{>0}), k)$ of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(k, \Gamma)$ under (4.5.1.12) (see Definition D.1.5.3 for the construction of the Hausdorff locally convex topological $\mathbf{C}$-vector space of Schwartz-Bruhat functions on a Hausdorff locally compact abelian group), where $k \subset G^1$ is a compact subgroup whose orthogonal $k^\perp \subset \widehat{G^1}$ is compactly generated, $\Gamma \subset G^1$ is an open

[1] Return here.

and compactly generated subgroup, and $k \subset \Gamma$, and $\mathcal{S}(k,\Gamma)$ denotes the $\mathbf{C}$-linear subspace of $\mathcal{S}(G^1)$ of those continuous functions that are supported over Γ, are k-invariant, and induce a unique continuous function on the elementary Lie group Γ/k that is Schwartz there.

Indeed, since G^1 is compact, we may choose $\Gamma = G^1$. The assertion on a fixed bounded subset $B \subset \mathcal{S}_{\mathrm{E}}(G)$ being contained in some $\mathcal{S}_{\mathrm{E}}(k,G)$ follows from Theorem D.3.3.7(3).

We may further denote by G_0^1 the quotient G^1/k and define $G_0 := (\tau(\mathbf{R}_{>0}) \cdot G^1)/k = G/k$, promoting it to an $\mathbf{R}_{>0}$-modulated group by defining the module $|\cdot|_{G_0}$ by k-invariance (being k contained in G^1, the module descends to $(\tau(\mathbf{R}_{>0}) \cdot G^1)/k$, since $|k|_G = 1$).

We let $\gamma^{\mathbf{Z}}$ acts by left-multiplication on the left cosets G/k, obtaining a continuous action of $\gamma^{\mathbf{Z}}$ on $\mathcal{S}_{\mathrm{E}}(G/k) \simeq \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(G_0^1)$ by left-translation that is compatible with the one inherited from the left-translation action of the subgroup $\gamma^{\mathbf{Z}} \subset G$ on $\mathcal{S}_{\mathrm{E}}(G)$ (i.e. even though $\gamma^{\mathbf{Z}}$ is not a subgroup of G/k anymore).

We note that, since G is abelian, we have the canonical identification of homogeneous spaces topologized with the respective quotient topologies: $\gamma^{\mathbf{Z}}\backslash G/k = G/(\gamma^{\mathbf{Z}} \cdot k)$.

G/k is now an elementary $\mathbf{R}_{>0}$-modulated abelian Lie group, with G_0^1 an elementary compact abelian Lie group, and the resulting operator $\Phi_{\gamma,0} : \mathcal{S}_{\mathrm{E}}(G/k) \to \mathcal{S}(G/(\gamma^{\mathbf{Z}} \cdot k))$ defined by averaging an element of $\mathcal{S}_{\mathrm{E}}(G/k)$ over the action of $\gamma^{\mathbf{Z}}$, is indeed valued in $\mathcal{S}(G/(\gamma^{\mathbf{Z}} \cdot k))$, which is seen upon repeating the argument from the foregoing, and promotes the following to a commutative diagram of linear operators:

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{E}}(G/k) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G) \\ {\scriptstyle \Phi_{\gamma,0}}\downarrow & & \downarrow{\scriptstyle \Phi_{\gamma}} \\ \mathcal{S}(G/(\gamma^{\mathbf{Z}} \cdot k)) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G/\gamma^{\mathbf{Z}}) \end{array} \tag{4.5.1.5}$$

where the horizontal arrows, sending functions on G/k, respectively $G/(\gamma^{\mathbf{Z}} \cdot k)$, to k-invariant functions on G, respectively $G/\gamma^{\mathbf{Z}}$ (in the latter case, $G/\gamma^{\mathbf{Z}}$ is identified with $\gamma^{\mathbf{Z}}\backslash G$ and the continuous action of k is by right-translation on the left cosets) are continuous.

It is therefore enough to show that $\Phi_{\gamma,0}$ is continuous (for any such k), to conclude that we have, for f the limit in $\mathcal{S}_{\mathrm{E}}(G)$ of the Cauchy net $\{f_i\}_{i\in I}$, also contained in $\mathcal{S}_{\mathrm{E}}(k,G)$ since this latter is a closed subspace, $\Phi_\gamma(f_i) = \Phi_{\gamma,0}(f_i) \to \Phi_{\gamma,0}(f) = \Phi_\gamma(f)$. [2]

We are using Theorem D.3.3.7(3) to deduce that the bounded subsets of $\mathcal{S}_I(G)$ are indeed contained in some $\mathcal{S}_I(k,G)$, the pre-image under the linear topological isomorphism $\mathcal{S}_I(G) \simeq \mathcal{S}_I(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(G^1)$ of $\mathcal{S}_I(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(k,G^1) \simeq \mathcal{S}_I(G/k)$.

We now return to continuity of $\Phi_{\gamma,0}$. The elementary Lie group $G/(\gamma^{\mathbf{Z}} \cdot k)$ is now compact Hausdorff, and the Hausdorff locally convex topology on $\mathcal{S}(G/(\gamma^{\mathbf{Z}} \cdot k))$ is now equivalent to the topology of uniform convergence on smooth functions and their derivatives of arbitrary order along any continuous, equivalently smooth, 1-parameter subgroup.

In this special case, the proof of continuity of $\Phi_{\gamma,0}$ then proceeds in close analogy with the proof of Lemma 4.2.1.5(1) and (2), using uniform differentiability of the elements of $\mathcal{S}(G/k)$.

We now return to the case of a general $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group G.

To show that the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G)$ is sent by Φ_γ onto a closed subspace of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$, we proceed as follows.

We define on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ the following operator:

$$\Psi_\chi : \mathcal{S}(G/\gamma^{\mathbf{Z}}) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$$

assigned by $\Psi_\chi(f)(x) := \int_{G^1} \chi(g) f(gx)\, \mathrm{d}\mu_1(g)$ as functions of $x \in G$, invariant under $\gamma^{\mathbf{Z}}$.

[2] Return here.

We define the operator $\Psi_\gamma : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$, as:

$$\Psi_\gamma(\phi)(x) := \sum_{k\in\mathbf{Z}} \phi(\gamma^k x) \tag{4.5.1.6}$$

as functions of $x \in G$, invariant under the action of $\gamma^{\mathbf{Z}}$.

The following diagram of linear operators:

$$\begin{array}{ccc} \mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G) & \xrightarrow{\Phi_\gamma} & \mathcal{S}(G/\gamma^{\mathbf{Z}}) \\ \downarrow{\scriptstyle \Phi_{\chi|_{G^1}}} & & \downarrow{\scriptstyle \Psi_\chi} \\ \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \xrightarrow[\Psi_\gamma]{} & \mathcal{S}(G/\gamma^{\mathbf{Z}}) \end{array} \tag{4.5.1.7}$$

is commutative, where $\Phi_{\chi|_{G^1}}$ is the operator:

$$\Phi_{\chi|_{G^1}} : f \mapsto \int_{G^1} \chi(g) f(gx)\, \mathrm{d}\mu_1(g) \tag{4.5.1.8}$$

and where the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ is mapped linearly and topologically isomorphically onto $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ by $\Phi_{\chi|_{G^1}}$, by Lemma 4.3.3.1 (up to multiplication by $\mu_1(G^1)$) since G^1 is compact and its left-translation action on $\mathcal{S}_{\mathrm{E}}(G)$ is uniformly continuous (see Lemma D.2.2.1).

The same argument as for Lemma 4.3.3.1 and Lemma D.2.2.1, using completeness of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$, compactness of G^1, and uniform continuity of its action by left-translation on $\mathcal{S}_{\mathrm{E}}(G)$, and hence on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ by left translation on the left cosets, shows that Ψ_χ is indeed valued in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ and has closed range (because its square is the multiplication operator by $\mu_1(G^1)$), which, by commutativity of the diagram in (4.5.1.7), shows that so is the linear image under Φ_γ of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$, so long as we show that Φ_γ in the topmost horizontal arrow is surjective. The surjectivity assertion was shown in the foregoing and holds by compactness of $G/\gamma^{\mathbf{Z}}$, which finishes the proof of (1).

(2) and (3) : The operator $\Phi_{\gamma,\chi}$ is the composition of the two operators $\Phi_{\chi|_{G^1}}$ from (4.5.1.8), and Φ_γ from (4.5.1.1), which, by (1), is continuous.

The former is, by Lemma 4.3.3.1, being G^1 compact Hausdorff and $\mathcal{S}_{\mathrm{E}}(G)$ a complete, bornological Hausdorff locally convex topological $\mathbf{C}$-vector space by Lemma D.3.1.2 and Theorem D.3.3.7(1) (and since the right-translation action of G^1 on $\mathcal{S}_{\mathrm{E}}(G)$ is uniformly sequentially continuous by Proposition D.1.9.2 and Proposition D.3.3.12), a continuous endomorphism of $\mathcal{S}_{\mathrm{E}}(G)$ with closed range.

Its range is the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \simeq \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ of the Hausdorff locally convex topological $\mathbf{C}$-vector space of exponentially decaying Schwartz-Bruhat functions on G that are $\chi|_{G^1}$-homogeneous with respect to the left-translation action by G^1.

By Theorem D.3.3.7(1), $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ is linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ under pullback along the chosen continuous homomorphic section τ to the module $|\cdot|_G$. In particular, it is a Fréchet space.

By commutation of the diagram from (4.5.1.7) and since $\Phi_{\chi|_{G^1}}$ has closed range by Lemma 4.3.3.1, and it is therefore relatively open, then in order to show that $\Phi_{\chi,\gamma}$ is relatively open, it is enough to show that Ψ_γ is relatively open. It has closed range by (1), and since $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is a strict LF-space, so is any of its closed subspaces. Ψ_γ is therefore relatively open by Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]).

(4) : The kernel $\ker(\Phi_{\chi,\gamma}) \subset \mathcal{S}_{\mathrm{E}}(G)$, a closed subspace by (1), contains the closure $V_{\chi,\gamma}$ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as ϕ varies over all the elements of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$, i.e. all exponentially decaying Schwartz-Bruhat functions on G that are $\chi|_{G^1}$-homogeneous with respect to the left-translation action by G^1.

The Mellin transform $M \circ \Phi_{\chi|_{G^1}} : \mathcal{S}_{\mathrm{E}}(G) \to A\mathcal{S}(\mathbf{C})$ (see Discussion 4.4.1.1, choosing $\pi = \gamma^{-1}$), sends $V_{\chi,\gamma} \subset \mathcal{S}_{\mathrm{E}}(G)$ (up to multiplication by the reciprocal of the μ_1-volume of G^1. Compare with Lemma 4.3.3.1) injectively onto the closed subspace $(1 - \chi(\gamma)|\gamma|_G^{-s})A\mathcal{S}(\mathbf{C})$ (by the Asymptotic Residue Theorem, Theorem B.3.2.1, from which it follows that it is sequentially closed, hence closed, being $A\mathcal{S}(\mathbf{C})$ a Fréchet space by Lemma D.3.1.2).

It follows that the orthogonal to $(1 - \chi(\gamma)|\gamma|_G^{-s})A\mathcal{S}(\mathbf{C})$ with respect to the duality $\langle A\mathcal{S}(\mathbf{C}), A\mathcal{S}(\mathbf{C})'_\beta\rangle$ is linearly and topologically generated by the continuous point-evaluation functionals supported at the points of the complex s-plane $\left\{\frac{2\pi i m}{\log(\gamma)} - it_0(\chi), m \in \mathbf{Z}\right\}$, and these continuous functionals form a topologically free countable set, as is seen by repeating the argument for Theorem 7.1.4.2(2) and (3) (i.e. using the Asymptotic Residue Theorem and boundedness over the vertical strips of bounded width of the complex s-plane, away from $i\mathbf{R}$, of the reciprocal of $1 - \chi(\gamma)|\gamma|_G^{-s}$, to show the existence of a bi-orthogonal sequence, for then deducing topological freeness from [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289]).

Such functionals are, each respectively, the image of the continuous linear functionals sending $\phi \in \mathcal{S}(G/\gamma^{\mathbf{Z}})$ to $a_{\chi,m}(\phi)$, the functional $a_{\chi,m}(\cdot)$ being defined in (4.4.2.6) of Lemma 4.4.2.6, and hence, being the latter set of functionals an unconditional Schauder basic sequence within $\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta$ by Lemma 4.4.2.6, the continuous linear map:

$$\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta \to (\mathcal{S}_{\mathrm{E}}(G/V_{\chi,\gamma})'_\beta$$

which is the continuous adjoint of Φ_γ, restricts to an injective continuous linear operator from the closure $W_{\chi,\gamma}$ of the set $\{e'_\xi, \xi \in (\gamma^{\mathbf{Z}})_\chi \subset (\gamma^{\mathbf{Z}})^\perp\}$ from (4.5.1.10).

By (1), we know that $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is itself continuous, and sends the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G)$ onto a closed subspace of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$.

By reflexivity of both source of Φ_γ (which is the quotient of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$, which is linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by Proposition D.3.3.12, by $V_{\chi,\gamma}$. Its reflexivity is seen from [**Scha**, Ch.III,§7,Thm.7.4,p.103], since $V_{\chi,\gamma}$ is closed and hence $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))/V_{\chi,\gamma}$ is nuclear Fréchet, hence Montel, hence reflexive) and target (which is the closed image of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ under Φ_γ in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$), in order to show that Φ_γ is injective it is enough to show that so is Φ''_γ, and by [**NB**, Thm.8.10.6(e)] this is indeed the case, since, by the foregoing, a topologically free set of strong (hence weak) topological generators for $(\mathcal{S}_{\mathrm{E}}(G)/V_{\chi,\gamma})'$, is in the image of Φ'_γ.

The continuous linear operator $\Phi_\gamma : \mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))/V_{\chi,\gamma} \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is therefore injective. It is also surjective onto its linear image, and relatively open by (3), hence a linear topological isomorphism onto its image.

This proves the assertions on $E_\gamma(\chi|_{G^1})$ being an absolute Schauder basis for $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))/V_{\chi,\gamma} \simeq \mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma})$.

(5) : (4) and its proof give more: the set $\{e'_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp\} \subset \mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta$ is an unconditional Schauder basis by Lemma 4.4.2.6, and by (4) (and its argument) it is mapped under Φ'_γ onto a strongly topologically free set of weak topological generators of the continuous dual of the quotient $\mathcal{S}_{\mathrm{E}}(G)/V_\gamma$, with V_γ the closure of the linear subspace of $\mathcal{S}_{\mathrm{E}}(G)$ linearly generated by the elements $\phi(x) - \phi(\gamma x)$ as ϕ varies over all the elements $\phi \in \mathcal{S}_{\mathrm{E}}(G)$.

To see this, we proceed as follows. We observe that by (4), the bi-orthogonal set to $\{e_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp_\chi\}$ can be chosen to be contained within $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ (a consequence of Lemma 4.3.3.1 applied to the operator $\Phi_\chi : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, $\Phi_\chi(\phi) := \int_{G^1} \chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)$, by compactness of G^1).

It follows that $\{e'_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp\}$ is indeed topologically free by [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289], because each subset $\{e'_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp_\chi\}$ is, and we have $\Phi_{\chi_1}(\mathrm{H}^0(\chi_2|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))) = 0$ if $\chi_1 \neq \chi_2$, for $\chi_1, \chi_2 \in \widetilde{G^1}$.

We deduce that the continuous linear operator $\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta \to (\mathcal{S}_{\mathrm{E}}(G)/V_\gamma)'_\beta$ is injective.

Its range is weakly dense. Indeed, $\bigoplus_{\chi\in\widetilde{G^1}} \mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G)$ is strongly dense, and hence so is its image under the open quotient map onto $\mathcal{S}_{\mathrm{E}}(G)/V_\gamma$. This implies that $\bigcap_{\chi\in\widetilde{G^1}} \ker(\Phi_\chi)$ is trivial in $\mathcal{S}_{\mathrm{E}}(G)$ (alternatively, this is so by Fourier Inversion, since each element of $\bigcap_{\chi\in\widetilde{G^1}} \ker(\Phi_\chi)$ has everywhere vanishing Fourier transform), and so $\{e'_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp\}$ separates the points of $\mathcal{S}_{\mathrm{E}}(G)/V_\gamma$, because each $\{e'_\xi : \xi \in (\gamma^{\mathbf{Z}})^\perp_\chi\}$ separates the points of $\mathcal{S}_{\mathrm{E}}(G)/V_{\chi,\gamma}$.

To conclude, we now use Theorem D.3.3.7(3).

By [**NB**, Thm.8.10.6(e)] applied to Φ'_γ and using semi-reflexivity of source and target, $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G)/V_\gamma \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is injective. Since Φ_γ is surjective, it is a continuous bijection. Its linear inverse sends bounded subsets to bounded subsets by Theorem D.3.3.7(3), because $\mathcal{S}_{\mathrm{E}}(G)$ is strict LF and Φ_γ is therefore open (and V_γ is closed since its intersection with each $\mathcal{S}_{\mathrm{E}}(k, G)$ is closed for each $k \subset G^1$ compact with compactly generated $k^\perp$).

This implies that $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G)/V_\gamma \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is a linear topological isomorphism.

(6) : The range of P_γ is $\ker(\Phi_\gamma)$, which is closed, since Φ_γ from (4.5.1.1) is continuous.

We let $\Phi_\chi : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ be the continuous operator $\Phi_\chi(\phi) := \int_{G^1} \chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)$. Φ_χ is surjective by Lemma 4.3.3.1 and induces a linear topological isomorphism $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \simeq \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, up to multiplication by the nonzero real number $\mu_1(G^1)$, with $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \simeq \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ the closed subspace of those Schwartz-Bruhat functions on G with exponential decay, that are $\chi|_{G^1}$-homogeneous with respect to the continuous left-translation action by G^1.

We have $\Phi_{\chi,\gamma} = \Phi_\gamma \circ \Phi_\chi$, and since the $\gamma^{\mathbf{Z}}$-action and the G^1-action by left-translation on $\mathcal{S}_{\mathrm{E}}(G)$ commute, we have $P_\gamma \circ \Phi_\chi = \Phi_\chi \circ P_\gamma$ (upon identifying the range of Φ_χ with the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G)) \subset \mathcal{S}_{\mathrm{E}}(G)$).

In particular, the range of $P_\gamma \circ \Phi_\chi = \Phi_\chi \circ P_\gamma$ is the intersection of $\ker(\Phi_\gamma)$ with $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$, which equals the intersection of $\ker(\Phi_{\chi,\gamma})$ and $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$. In particular, it is again a closed subspace of $\mathcal{S}_{\mathrm{E}}(G)$ and of $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$, equivalently of $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$. □

Remark 4.5.1.4. We note that in the proof of (5), we do not need to know that $\mathcal{S}_{\mathrm{E}}(G)$ is bornological, which is indeed the case under the second-countability assumption on G that is being made from Situation 4.4.2.3.

Recall that the canonical continuous linear operator $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1) \to \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(G^1)$ is a continuous bijection. In particular, the $\gamma^{\mathbf{Z}}$-periodization operator Φ_γ induces a linear operator $\Phi_\gamma : \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(G^1) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$, which is surjective.

It is also sequentially continuous, and the induced continuous linear bijection $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(G^1)/\ker(\Phi_\gamma) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ has sequentially continuous linear inverse, because so does the continuous linear bijection $\mathcal{S}_I(G) \to \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(G^1)$, as an application of Proposition C.2.3.11.

Since $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is bornological, [**NB**, Thm.13.2.7(a)] then applies.

Corollary 4.5.1.5 (Boundedness and equicontinuity). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the assumptions and choices from Situation 4.4.2.3.*

Then for each $\chi \in \widetilde{G^1}$, the set of representatives in $\widehat{G}$ chosen in Situation 4.4.2.3 for the continuous characters of G^1, then the set $\{a_{\chi,m}(\cdot)\}_{m\in\mathbf{Z}}$ from (4.4.2.6) is bounded in $\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta$, and bounded and equicontinuous in $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})'_\beta \subset \mathcal{S}(G/\gamma^{\mathbf{Z}})$, where $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ is the closed subspace of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ of those elements that are $\chi|_{G^1}$-homogeneous under the continuous left-translation action by G^1 on the left cosets.

Proof. By Proposition 4.5.1.1(3), the continuous linear operator $\Phi_{\chi,\gamma} : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to \mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ is surjective and open, with $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ is the closed range of the continuous linear operator $\Psi_\chi : \mathcal{S}(G/\gamma^{\mathbf{Z}}) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ defined by $\Psi_\chi(\phi)(x) := \frac{1}{\mu_1(G^1)} \int_{G^1} \chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)$ (its continuity, surjectivity onto the closed subspace of the $\chi|_{G^1}$-homogeneous elements in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ under the continuous left-translation action by G^1 on the left cosets, is seen as a consequence of Lemma 4.3.3.1).

By [**Kothe-II**, Ch.7,§33.3,(6),p.15] applied to $\Phi_{\chi,\gamma}$, its continuous adjoint $\Phi'_{\chi,\gamma} : \mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})'_\gamma \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})'_\gamma$ is a linear topological embedding (with respect to the topology of uniform convergence on the precompact subsets of $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$). Since both source and target are nuclear, by [**Tre**, Part III,Ch.50,Prop.50.2(50.12,p.519] we deduce that $\Phi'_{\chi,\gamma} : \mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})'_\beta \to \mathcal{S}(G)(\chi|_{G^1})'_\beta$ is a linear topological embedding.

The boundedness assertion follows from simple boundedness of $\{a_{\chi,m}(\cdot)\}_{m\in\mathbf{Z}}$, which is seen by means of the same argument provided in the proof of Corollary 4.2.1.7, using the Mellin transform and Theorem D.3.3.13.

By [**Scha**, Ch.III,§3.4,Corollary,p.82], since $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ is complete, every simply bounded subset in $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})'$ is bounded in the strong topology, and by [**Scha**, Ch.III,§4.2,Thm.p.83], which applies since $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ is barrelled (being, by Proposition 4.5.1.1(1), a quotient of a strict LF-space, or by [**Bru**, §9,Prop.11(a),p.61]) it is equicontinuous.

Its image in $\mathcal{S}(G/\gamma^{\mathbf{Z}})'_\beta$ under the continuous adjoint of Ψ_χ is also bounded. □

4.5.1.2. *Exponential system on* $\mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma})$.

NOTATION 4.5.1.1. Let G be an $\mathbf{R}_{>0}$-modulated group satisfying the assumptions from Situation 4.4.2.3. Let $\gamma \in G$ be such that $|\gamma|_G > 1$.

On $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ we find the continuous left-translation action of the compact group G^1 on the left cosets, and the operator $\Psi_\chi : f \mapsto \frac{1}{\mu_1(G^1)} \int_{G^1} \chi(g) f(gx)\, \mathrm{d}\mu_1(g)$ is a continuous idempotent operator valued in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ by Lemma 4.3.3.1, whose range is the linear subspace of those elements of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ that are $\chi|_{G^1}$-homogeneous under such continuous left-translation action.

We denote such subspace by:

$$\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})) = \mathrm{Im}(\Psi_\chi) \subset \mathcal{S}(G/\gamma^{\mathbf{Z}}) \tag{4.5.1.9}$$

a closed subspace, being Ψ_χ continuous and idempotent.

COROLLARY 4.5.1.6 (Sourcing Periodic Schauder Bases - II). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the conditions and choices from Situation 4.4.2.3.*

Any enumeration of the countable set comprised of the continuous functionals

$$e'_\xi : \phi \mapsto \int_G \phi(g)\xi(g)\, \mathrm{d}\mu(g) \tag{4.5.1.10}$$

as $\xi \in \widehat{G}$ varies over the countable subset $(\gamma^{\mathbf{Z}})^\perp_\chi \subset (\gamma^{\mathbf{Z}})^\perp$ of those elements of $(\gamma^{\mathbf{Z}})^\perp$ such that $\xi|_{G^1} = \chi|_{G^1}$, is an unconditional Schauder basis for the orthogonal of $\ker(\Phi_{\chi,\gamma})$ with respect to the duality $\langle \mathcal{S}_{\mathrm{E}}(G), \mathcal{S}_{\mathrm{E}}(G)'\rangle$, equipped with the strong topology on the dual $(\mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma}))'$.

Its dual basis in $\mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma})$ is absolute and maps bijectively under the operator Φ_γ from (4.5.1.1), onto the subset:

$$E_\gamma(\chi|_{G^1}) := \{\overline{\xi} : \xi \in (\gamma^{\mathbf{Z}})^\perp_\chi\} \subset \mathcal{S}(G/\gamma^{\mathbf{Z}}) \tag{4.5.1.11}$$

equipped with the induced enumeration.

PROOF. By Proposition 4.5.1.1(3) and (4), the operator $\Phi_{\chi,\gamma}$ induces a linear topological embedding:

$$\Phi_{\chi,\gamma} : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/\ker(\Phi_{\chi,\gamma}) \to \mathcal{S}(G/\gamma^{\mathbf{Z}}) \tag{4.5.1.12}$$

whose image is the closed subspace $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ of the elements of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ that are $\chi|_{G^1}$-homogeneous with respect to the continuous left-translation action of G^1 on the left cosets $G/\gamma^{\mathbf{Z}}$.

$E_\gamma(\chi|_{G^1})$ is an absolute Schauder basis for $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ by Lemma 4.4.2.6.

Since $E_\chi(\chi|_{G^1})$ is the bi-orthogonal set to the set $\{a_{\chi,m}(\cdot)\}_{(\chi,m)\in\widetilde{G^1}\times\mathbf{Z}}$ defined in (4.4.2.6), to finish the proof it is enough to show that indeed for each $\xi \in (\gamma^{\mathbf{Z}})^\perp_\chi$ there exists a unique pair $(\chi, m) \in \widetilde{G^1} \times \mathbf{Z}$ such

that $\Phi'_\gamma(a_{\chi,m}(\cdot)) = e'_\xi$. Indeed, each such ξ is uniquely of the form $\chi|\cdot|_G^{\frac{2\pi i m}{\log(|\gamma|_G)} - it_0(\chi)}$, following the notation and choices introduced in Situation 4.4.2.3. □

4.5.2. Periodic Schauder Bases: support conditions in the monogenic case. The following is an analog for general $\mathbf{R}_{>0}$-modulated groups satisfying the assumptions from Situation 4.4.2.3 of Proposition 4.2.2.2.

We let G be an $\mathbf{R}_{>0}$-modulated group satisfying the assumptions made in Situation 4.4.2.3.

For any compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, and for any compact subgroup $k \subset G^1$ whose orthogonal $k^\perp$ is compactly generated, we denote by $\mathcal{S}_K(k,G) \subset \mathcal{D}(G)$ the linear subspace of those smooth compactly supported functions on G that are k-invariant, and whose support is contained in $\tau(K)\cdot G^1 \subset G$.

By [**Sch-a**, Thm.II,p.66] and [**Scha**, Ch.II,§6,6.4,p.58] and the definition of the topology on $\mathcal{D}(G)$ in [**Bru**, §1], the topology induced on $\mathcal{S}_K(k,G)$ by the locally convex inductive limit topology on $\mathcal{D}(G)$ is the topology obtained by transport of structure from the Hausdorff locally convex Fréchet topology of uniform convergence on functions and their derivatives (directional, along arbitrary continuous, equivalently smooth, 1-parameter subgroups) of any order on $\mathcal{D}_Z(P)$ for $P := \tau(\mathbf{R}_{>0})\cdot G^1/k$, an abelian elementary Lie group since G^1 is compact, and $Z \subset P$ the compact subset $\tau(K)\cdot G^1/k$, i.e. the continuous linear inclusion $\mathcal{S}_K(k,G) \subset \mathcal{D}(G)$ is a linear topological embedding.

We further note that under the linear topological isomorphism $\mathcal{S}_{\mathrm{E}}(G) \simeq \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi \mathcal{S}(G^1)$ from Theorem D.3.3.7(3), for any compact subgroup $k \subset G^1$ with $k^\perp$ compactly generated, we have that $\mathcal{S}_K(k,G)$ is mapped linearly and topologically isomorphically onto $\mathcal{D}_K(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi \mathcal{S}(k,G^1)$. To see this, we use nuclearity of $\mathcal{D}_K(\mathbf{R}_{>0})$ (which is [**Tre**, Part III,Ch.51,Corollary,p.530]) and [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], to deduce that the linear inclusion $\mathcal{D}_K(\mathbf{R}_{>0})\otimes_\pi \mathcal{S}(k,G^1) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\otimes_\pi \mathcal{S}(G^1)$ is a linear topological embedding (by Proposition 4.2.2.2(1)), and it remains so upon passage to completions on source and target. The source is $\mathcal{D}_K(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi \mathcal{S}(k,G^1)$ and the target is $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi \mathcal{S}(G^1)$.

We further denote by $\mathcal{D}_K(G) \subset \mathcal{S}_{\mathrm{E}}(G)$ the linear subspace of those Schwartz-Bruhat functions on G of exponential decay, whose support is compact and contained within $\tau(K)\cdot G^1$.

Proposition 4.5.2.1 (Sourcing Periodic Schauder Bases - III). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the assumptions and choices from Situation 4.4.2.3.*

(1) *For any compact subgroup $k \subset G^1$ such that $k^\perp$ is compactly generated, and for any compact subset $K \subset \mathbf{R}_{>0}$ with non-empty interior, the linear subspace $\mathcal{S}_K(k,G) \subset \mathcal{S}_{\mathrm{E}}(G)$ is closed, and the subspace topology agrees with the topology on $\mathcal{S}_K(k,G)$ obtained by transport of structure from the Hausdorff locally convex Fréchet topology of uniform convergence on functions and their derivatives of any order on $\mathcal{D}_Z(P)$ for $P := \tau(\mathbf{R}_{>0})\cdot G^1/k$, an abelian elementary Lie group, and $Z \subset P$ the compact subset $\tau(K)\cdot G^1/k$.*

(2) *If $K = b\cdot[1/a,a]$ for $a,b>0$ real numbers, $a>1$, and $2\log(a) > \log(|\gamma|_G)$, then the quotient $G \to G/\gamma^{\mathbf{Z}}$ maps $\tau(K)\cdot G^1$ surjectively onto $G/\gamma^{\mathbf{Z}}$.*

For any compact subgroup $k \subset G^1$ such that $k^\perp \subset \widehat{G^1}$ is compactly generated, the induced continuous linear map:

$$\Phi_\gamma : \mathcal{S}_K(k,G)/(\ker(\Phi_\gamma)\cap \mathcal{S}_K(k,G)) \to \mathcal{S}(G/(\gamma^{\mathbf{Z}}\cdot k)) \tag{4.5.2.1}$$

is a linear topological isomorphism.

(3) *Under the assumption on k from (1) and on K from (2), let ϕ be an element in the intersection $\ker(\Phi_\gamma)\cap \mathcal{S}_K(k,G)$. Then there exists $\psi \in \mathcal{S}_{K_0}(k,G)$ such that:*
   (1) *For $A_0 := \max\{ab, a/b\}$, $K_0 \subset \mathbf{R}_{>0}$ is a compact subset contained in $[e^{-A}, e^A]$, with $A := |-\log(|\gamma|_G) + \log(A_0)|$.*
   (2) *$\psi(x) - \psi(\gamma x) = \phi(x)$ as Schwartz-Bruhat functions of $x \in G$, supported over K.*

PROOF. (1) : The subspace topology on $\mathcal{S}_K(k,G)$ is stronger than the Fréchet topology of uniform convergence on functions and their derivatives of arbitrary order (because it is generated by a larger family of seminorms, equivalently because the seminorms defining the Fréchet topology on $\mathcal{D}_Z(P)$ induce on $\mathcal{S}_K(k,G)$ continuous seminorms).

With respect to the subspace topology, $\mathcal{S}_K(k,G)$ is a sequentially closed subspace, since such topology preserves the support and k-invariance conditions upon formation of limits in $\mathcal{S}_{\mathrm{E}}(G)$ of Cauchy sequences of elements in $\mathcal{S}_K(k,G)$, and hence $\mathcal{S}_K(k,G)$ is sequentially closed in $\mathcal{S}_{\mathrm{E}}(G)$.

We note that $\mathcal{S}_K(k,G)$ is contained within the pre-image $\mathcal{S}_{\mathrm{E}}(k,G)$ in $\mathcal{S}_{\mathrm{E}}(G)$ of the closed subspace $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(k,G^1)$ of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(G^1)$ under the linear topological isomorphism from Theorem D.3.3.7(3), where $\mathcal{S}(k,G^1)$ is the closed subspace of those smooth functions on G^1 that are k-invariant, and indeed $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(k,G^1)$ is a closed subspace of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(G^1)$ by nuclearity of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, which holds by Lemma D.3.2.3(1), and by Proposition C.2.3.13(2).

It is therefore enough to show that $\mathcal{S}_K(k,G)$ is closed as a linear subspace of $\mathcal{S}_{\mathrm{E}}(k,G)$. Since $\mathcal{S}_{\mathrm{E}}(k,G)$ is a Fréchet space by [**Jar**, Part III,Ch.15,§15.1,Cor.4,p.26], it is sequential, and hence $\mathcal{S}_K(k,G)$ is closed, being sequentially closed within the larger ambient, and since $\mathcal{S}_{\mathrm{E}}(k,G)$ is itself closed within the larger ambient by the foregoing.

(2) : Let $x \in G/\gamma^{\mathbf{Z}}$ be an arbitrary point. The subset

$$\tau([1,|\gamma|_G)) \cdot G^1 \subset G$$

tassellates G under the proper and free action of the discrete subgroup $\gamma^{\mathbf{Z}} \subset G$, since $|\gamma|_G^{\mathbf{Z}} \cdot [1,|\gamma|_G) = \mathbf{R}_{>0}$.

Since $2\log(a) > \log(|\gamma|_G)$, there exists $b' \in \mathbf{R}_{>0}$ such that $b' \cdot [1,|\gamma|_G) \subset [1/a,a]$ (for instance, $b' := 1/a$) and hence $(b/a)\cdot[1,|\gamma|_G) \subset K$.

We note that $|\gamma|_G^{\mathbf{Z}} \cdot (b/a) \cdot [1,|\gamma|_G) = (b/a)\cdot|\gamma|_G^{\mathbf{Z}} \cdot [1,|\gamma|_G) = (b/a)\cdot\mathbf{R}_{>0} = \mathbf{R}_{>0}$, and hence there exists $y \in \tau((b/a)\cdot[1,|\gamma|_G))\cdot G^1 \subset \tau(K)\cdot G^1$ mapping to x under the quotient homomorphism $G \to G/\gamma^{\mathbf{Z}}$.

Under the given assumptions on K and k, the continuous linear operator $\mathcal{S}_K(k,G) \to \mathcal{S}(G/(\gamma^{\mathbf{Z}}\cdot k))$ is surjective, hence open by [**NB**, Thm.14.4.5], being source and target Fréchet spaces.

To see the surjectivity aspect, it is enough to observe that there exists a smooth compactly supported function $u_0 \in \mathcal{D}(G)$ whose support is contained within $\tau(K)\cdot G^1$, and such that $\Phi_\gamma(u_0) = 1$ as a continuous function on the compact group $G/\gamma^{\mathbf{Z}}$. It is here that we are using the assumption on K. Indeed, under the quotient homomorphism $G \to G/\gamma^{\mathbf{Z}}$, $\tau(K)\cdot G^1$ surjects onto $G/\gamma^{\mathbf{Z}}$. We let $K' \subset K$ be a compact subset with non-empty interior such that $\tau(K')\cdot G^1$ surjects onto $G/\gamma^{\mathbf{Z}}$. We choose $v_0 \in \mathcal{D}(\mathbf{R}_{>0})$ real-valued and nonnegative, supported over K and constant and identically 1 over K'.

We call $v := v_0 \otimes 1_{G^1}$, which is an element of $\mathcal{D}(G)$ with support contained in $\tau(K)\cdot G^1$. Then $\Phi_\gamma(v) = \Phi_{|\gamma|_G}(v_0)$, and hence $u_0 := v/\Phi_{|\gamma|_G}(v_0)$ is an example of a smooth compactly supported function on G supported over $\tau(K)\cdot G^1$ and satisfying $\Phi_\gamma(u_0) = 1$. It is even k-invariant, for $k \subset G^1$ a compact subgroup with compactly generated $k^\perp$.

Alternatively, given such u_0 with $\Phi_\gamma(u_0) = 1$, and upon choosing the volume-1 Haar measure μ_k on k, and the unique Haar measure $\mu_{G^1/k}$ prescribed as per Theorem D.1.2.1 by the choices μ_1 on G^1 and μ_k on k, the function on G defined by $u_1(x) := \int_k u_0(g^{-1}x)\,\mathrm{d}\mu_k(g)$ is a smooth compactly supported function on G whose support is contained within $\tau(K)\cdot G^1$, and such that it is k-invariant. An application of Dominated Convergence shows that $\Phi_\gamma(u_1)(x) = \int_k \Phi_\gamma(u_0)(g^{-1}x)\,\mathrm{d}\mu_k(g) = \mu_k(k) = 1$ for all $x \in G$.

It follows that the restriction of Φ_γ to $\mathcal{S}_K(k,G)$ also surjects onto $\mathcal{S}(G/(\gamma^{\mathbf{Z}}\cdot k))$.

The operator Φ_γ therefore induces a linear topological isomorphism as in (4.5.2.1).

(3) : We refer the reader to Discussion 4.4.1.1 for the notation on the Mellin transform in force below and throughout the rest of the argument.

By Proposition 4.5.1.1(5), there exists $\psi \in \mathcal{S}_{\mathrm{E}}(k,G)$ such that $\psi(x) - \psi(\gamma x) = \phi(x)$ as Schwartz-Bruhat functions on G of exponential decay. To show that ψ is compactly supported if ϕ is, and that its support

is contained in a compact subset of the form $\tau(K_0)\cdot G^1$ with $K_0\subset \mathbf{R}_{>0}$ of the desired form, it is enough to repeat the argument for Proposition 4.2.2.2(3): we conclude by the Paley-Wiener Theorem (see [**Ru-2**, Ch.19,Thm.19.3,p.375], which applies since we have in particular that the function of $t\in\mathbf{R}_{>0}$ defined by $\phi(\tau(t)g)$ is, for each $g\in G^1$, in $L^2((1/A_0,A_0),\mathrm{d}t/t)$ for the stated A_0). □

REMARK 4.5.2.2. Note that in the proof of Proposition 4.5.2.1 we are not claiming that $\mathcal{S}_{\mathrm{E}}(G)$ is a sequential space, nor are we claiming that in a strict LF-space, a subspace is closed if and only if its pre-image in each of the spaces in a defining sequence is closed.

In the special case at hand, $\mathcal{S}_K(k,G)$ is already entirely contained within $\mathcal{S}_{\mathrm{E}}(k,G):=\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(k,G^1)$, which is itself linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(k,G^1)$ since the tensor factors are Fréchet spaces (see Corollary C.2.3.12), and the projective tensor product topology on the latter is equivalent to the subspace topology inherited by $\mathcal{S}_{\mathrm{E}}(G)$. We use [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348] and nuclearity of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, to identify injective and projective tensor topologies; see also Proposition C.2.3.13(2).

We deduce that $\mathcal{S}_{\mathrm{E}}(k,G):=\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(k,G^1)\simeq\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(k,G^1)$ is metrizable, hence sequential, and it therefore suffices to prove sequential closedness of $\mathcal{S}_K(k,G)$ in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(k,G^1)$ to deduce closedness of $\mathcal{S}_K(k,G)$ in $\mathcal{S}_{\mathrm{E}}(k,G)$, and hence in $\mathcal{S}_{\mathrm{E}}(G)$, being $\mathcal{S}_{\mathrm{E}}(k,G)$ a closed subspace of $\mathcal{S}_{\mathrm{E}}(G)$ (since $\mathcal{S}(k,G^1)\subset\mathcal{S}(G^1)$ is).

THEOREM 4.5.2.3 (Sourcing Periodic Schauder Bases - IV). *Let G be an $\mathbf{R}_{>0}$-modulated group. We adopt the assumptions and choices from Situation 4.4.2.3. Let $\chi\in\widehat{G}$ be a continuous character. Assume χ is an element of the set $\widetilde{G^1}$ of representatives of the continuous characters of G^1 chosen in Situation 4.4.2.3.*

For any compact subset $K\subset\mathbf{R}_{>0}$ with non-empty interior, the continuous linear map:

$$\Phi_{\chi,\gamma}:\mathcal{S}_K(G)/(\ker(\Phi_{\chi,\gamma})\cap\mathcal{S}_K(G))\to\mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma}) \tag{4.5.2.2}$$

induced by the inclusion $\mathcal{S}_K(G)\subset\mathcal{S}_{\mathrm{E}}(G)$, is a linear topological isomorphism, compatible with passage to closed subspaces of k-invariant functions, for $k\subset G^1$ a compact subgroup with compactly generated $k^\perp$.

In particular:

(1) *the continuous functionals*

$$e'_{\chi,m}:\phi\mapsto\int_G\phi(g)\chi(g)|g|_G^{\frac{2\pi im}{\log(|\gamma|_G)}-it_0(\chi)}\,\mathrm{d}\mu(g)$$

for $m\in\mathbf{Z}$, defined in (4.5.1.10) *from Proposition 4.5.1.1, are an unconditional Schauder basis for the orthogonal of $\ker(\Phi_{\chi,\gamma})$ with respect to the duality $\langle\mathcal{S}_{\mathrm{E}}(G),\mathcal{S}_{\mathrm{E}}(G)'\rangle$, equipped with the strong topology on the dual $(\mathcal{S}_{\mathrm{E}}(G)/\ker(\Phi_{\chi,\gamma}))'$.*

(2) *The bi-orthogonal sequence in $\mathcal{S}_{\mathrm{E}}(G)$ can be chosen to be contained in $\mathcal{S}_K(G)$ for $K=b\cdot[1/a,a]$ and $a,b\in\mathbf{R}_{>0}$ and a satisfying $2\log(a)>\log(|\gamma|_G)$, and its image is an absolute Schauder basis for the quotient $\mathcal{S}_K(G)/(\ker(\Phi_{\chi,\gamma})\cap\mathcal{S}_K(G))$. Under this assumption on a,b, the image of the functionals $e'_{\chi,m}$ are an unconditional Schauder basis for the strong dual of $\mathcal{S}_K(G)/(\ker(\Phi_{\chi,\gamma})\cap\mathcal{S}_K(G))$.*

PROOF. Since χ is fixed, there exists a compact subgroup $k\subset G^1$ with $k^\perp$ compactly generated, equivalently finitely generated, such that $\chi(k)=1$, i.e. $\chi\in k^\perp$ (this is so, because G^1 is compact and second-countable, hence separable, and so $\widehat{G^1}$ is separable and discrete, hence countable, and hence a directed union of its finitely generated abelian subgroups Γ. We can then choose Γ so large as to have $\chi\in\Gamma$, and define $k:=\Gamma^\perp$ is then closed in G^1, hence compact, and we have $k^\perp=(\Gamma^\perp)^\perp=\Gamma$ by [**DE**, Prop.3.6.1(b)]).

All the assertions of the Theorem therefore follow from the combination of the following two assertions: Φ_γ induces the linear topological isomorphism in (4.5.2.1) from Proposition 4.5.2.1(2), since $\Phi_{\chi,\gamma}$ factors through the projector $\pi_k:\mathcal{S}_{\mathrm{E}}(G)\to\mathcal{S}_{\mathrm{E}}(k,G)$ defined as $\pi_k(\phi)(x):=\int_k\phi(gx)\,\mathrm{d}\mu_k(g)$ for μ_k the volume-1 Haar measure on k, which respects the closed subspace $\mathcal{S}_K(G)$, sending it onto $\mathcal{S}_K(k,G)$.

The functionals $e'_{\chi,m}$ from (4.5.1.10) are indeed the image under the continuous adjoint $\Phi'_{\chi,\gamma}$, of the unconditional Schauder basis $E_\gamma(\chi|_{G^1})$ from (4.5.1.11) of Corollary 4.5.1.6 (and hence from Lemma 4.4.2.6). They are therefore an unconditional Schauder basis for the strong dual of $\mathcal{S}_K(G)/(\ker(\Phi_{\chi,\gamma}) \cap \mathcal{S}_K(G)) = \mathcal{S}_K(k,G)/(\ker(\Phi_{\chi,\gamma}) \cap \mathcal{S}_K(k,G))$.

Its bi-orthogonal set is an absolute Schauder basis because its image under the linear topological isomorphism from (4.5.2.1) is. □

4.5.3. Monogenic periodization of I-exponentially decaying Schwartz-Bruhat functions. We deduce the following two Corollaries to Theorem D.1.7.1 from §D.1.7, the Distributional Poisson Summation Formula, which will make Theorem 4.6.2.1, Theorem 4.6.1.5 and Lemma 4.4.2.6 applicable:

COROLLARY 4.5.3.1 (Monogenic periodization on $\mathcal{S}_I(G)$). *Let G be a Hausdorff locally compact abelian group. Assume G is $\mathbf{R}_{>0}$-modulated. Let $\gamma \in G$ be such that $|\gamma|_G > 1$. Assume the choices and assumptions from Situation 4.4.2.3 are in force.*

Let $I \subset \mathbf{R}$ be a non-empty open connected interval.

If $0 \in I$, then the continuous surjective linear operator $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ from (4.5.1.1) *of Proposition 4.5.1.1 factors through a unique continuous surjective linear operator*

$$\Phi_\gamma : \mathcal{S}_I(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}}) \tag{4.5.3.1}$$

making the following diagram of continuous linear operators:

$$\begin{array}{ccc} \mathcal{S}_I(G) & \xrightarrow{\Phi_\gamma} & \mathcal{S}(G/\gamma^{\mathbf{Z}}) \\ \downarrow & & \| \\ \mathcal{S}(G) & \xrightarrow[\Phi_{\gamma^{\mathbf{Z}}}]{} & \mathcal{S}(G/\gamma^{\mathbf{Z}}) \end{array} \tag{4.5.3.2}$$

where $\mathcal{S}_I(G) \to \mathcal{S}(G)$ is the canonical continuous injective linear operator, and the bottom horizontal arrow is the continuous surjective open linear operator from (D.1.7.2) *of Theorem D.1.7.1 applied to $k = \gamma^{\mathbf{Z}}$.*

Upon denoting

$$P_\gamma : \mathcal{S}_I(G) \to \mathcal{S}_I(G),\ P_\gamma : \mathcal{S}(G) \to \mathcal{S}(G) \tag{4.5.3.3}$$

the continuous linear operator assigned by $P_\gamma(\phi)(x) := \phi(x) - \phi(\gamma x)$, we have:

1. *The kernel of Φ_γ in $\mathcal{S}_I(G)$ is the closure $V_{I,\gamma}$ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as $\phi \in \mathcal{S}_I(G)$ varies over all the elements of $\mathcal{S}_I(G)$, compatibly with passage to G^1-isotypes, i.e. $\ker(\Phi_\gamma) = P_\gamma \mathcal{S}_I(G)$.*
2. *The kernel of $\Phi_{\gamma^{\mathbf{Z}}} : \mathcal{S}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ from* (D.1.7.2) *of Theorem D.1.7.1 in the case $k = \gamma^{\mathbf{Z}}$ is the closure V_γ of the linear subspace generated by the elements $\phi(x) - \phi(\gamma x)$ as $\phi \in \mathcal{S}(G)$ varies over all the elements of $\mathcal{S}(G)$, i.e. $\ker(\Phi_{\gamma^{\mathbf{Z}}}) = P_\gamma \mathcal{S}(G)$, compatibly with passage to G^1-isotypes.*

PROOF. We define $\Phi_\gamma : \mathcal{S}_I(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ as the composition of $\Phi_\gamma : \mathcal{S}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ with the canonical injective linear operator $\mathcal{S}_I(G) \to \mathcal{S}(G)$, which exists by the assumption $0 \in I$ (see Theorem D.3.3.7(4)).

By Proposition 4.5.1.1(1), $\Phi_\gamma : \mathcal{S}_I(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is surjective since $\Phi_\gamma : \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G/\gamma^{\mathbf{Z}})$ is.

It follows that we have continuous linear bijections:

$$\mathcal{S}_{\mathrm{E}}(G)/P_\gamma \mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_I(G)/\ker(\Phi_\gamma) \to \mathcal{S}(G)/\ker(\Phi_{\gamma^{\mathbf{Z}}}) \tag{4.5.3.4}$$

where we used Proposition 4.5.1.1(5),(6) to deduce that we have $\ker(\Phi_\gamma) = P_\gamma \mathcal{S}_{\mathrm{E}}(G)$, and hence $P_\gamma \mathcal{S}_I(G) = \mathcal{S}_I(G) \cap V_\gamma$ (where the intersection denotes the pre-image of V_γ along the continuous linear inclusion $\mathcal{S}_I(G) \to \mathcal{S}(G)$).

Since we also have continuous linear bijections:

$$\mathcal{S}_{\mathrm{E}}(G)/P_\gamma\mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_I(G)/V_{I,\gamma} \to \mathcal{S}(G)/V_\gamma \tag{4.5.3.5}$$

we deduce that the canonical continuous linear operators:

$$\mathcal{S}_I(G)/V_{I,\gamma} \to \mathcal{S}_I(G)/\ker(\Phi_\gamma),\ \mathcal{S}(G)/V_\gamma \to \mathcal{S}_I(G)/\ker(\Phi_{\gamma^{\mathbf{Z}}}) \tag{4.5.3.6}$$

induced by the inclusions $V_{I,\gamma} \subset \ker(\Phi_\gamma) \subset \mathcal{S}_I(G)$, and $V_\gamma \subset \ker(\Phi_{\gamma^{\mathbf{Z}}})$, are also bijections. This gives $V_{I,\gamma} = \ker(\Phi_\gamma)$ and $V_\gamma = \ker(\Phi_{\gamma^{\mathbf{Z}}})$, as closed subspaces of $\mathcal{S}_I(G)$, respectively $\mathcal{S}(G)$.

Compatibilty of the formation of $V_{I,\gamma}$ and V_γ with passage to G^1-isotypes is a consequence of the commutation of the continuous left-translation actions on G by $\gamma^{\mathbf{Z}}$ and G^1. □

4.6. Periodic Schauder Bases in the presence of multiplicity

4.6.1. Bases in the presence of multiplicity. Throughout this subsection, we use Theorem 4.5.2.3 as an instrument to identify absolute Schauder bases on certain quotients of $\mathcal{S}_{\mathrm{E}}(G)$, for G an $\mathbf{R}_{>0}$-modulated abelian group.

Situation 4.6.1.1. Let G be an $\mathbf{R}_{>0}$-modulated group. Assume G is second-countable.

(1) Let τ be a choice of continuous homomorphic section to the module $|\cdot|_G$.
(2) Let μ_1 be a choice of Haar measure on G^1, and μ the unique Haar measure on G matching the Radon product $\frac{\mathrm{d}t}{t} \otimes \mu_1$ under push-forward along the isomorphism $G \simeq \mathbf{R}_{>0} \times G^1$ of Hausdorff locally compact abelian groups induced by the choice of τ.
(3) Let $N \geq 1$ be an integer, and $\gamma_1, \ldots, \gamma_N \in G$ be such that $|\gamma_k|_G > 1$ for all $1 \leq k \leq N$. We allow repetitions among the γ_k's. We will be writing $S = (\gamma_1, \ldots, \gamma_N)$ to mean the ordered list of the γ_k's, possibly with repetitions.
(4) We choose, for each continuous character $\omega \in \widehat{G^1}$, a continuous character $\chi \in \widehat{G}$ such that $\chi|_{G^1}$. We denote by $\widetilde{G^1}$ the subset of $\widehat{G}$ given by the chosen continuous characters χ.
(5) We let k be an integer such that $1 \leq k \leq N$, and let $\gamma_k \in G$ be as in (3). We call, for each $\chi \in \widetilde{G^1}$, $x_{0,k}(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_{0,k}(\chi)}$ as smooth functions of $t \in \mathbf{R}_{>0}$ and $t_{0,k}(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\gamma_k) = |\gamma_k|_G^{it_{0,k}(\chi)}$. In particular, $\chi|\cdot|_G^{\frac{-2\pi i m}{\log(|\gamma|_G)} - it_{0,k}(\chi)}$ is a parametrization of all the countably many elements of $(\gamma_k^{\mathbf{Z}})^\perp$, as (χ, m) vary over $\widetilde{G^1} \times \mathbf{Z}$.

Notation 4.6.1.1. We let G be a second-countable $\mathbf{R}_{>0}$-modulated group, and $\gamma \in G$ such that $|\gamma|_G > 1$. We record here notation first introduced in Corollary 4.5.1.6, to be used throughout the rest of this section.

We adopt the notation $\widetilde{G^1}$ from Situation 4.4.2.3, for a fixed choice of representatives for the elements of $\widehat{G^1}$, i.e. for each $\omega \in \widehat{G^1}$, one continuous character $\chi \in \widehat{G}$ such that $\chi|_{G^1} = \omega$. This is equivalent to choosing a holomorphic parametrization $\chi|\cdot|_G^s$ for the Riemann surface $\Omega(G)$ of continuous quasi-characters on G.

We define, for each $\chi \in \widetilde{G^1}$, the subset:

$$(\gamma^{\mathbf{Z}})^\perp_\chi \subset (\gamma^{\mathbf{Z}})^\perp \subset \widehat{G} \tag{4.6.1.1}$$

to be the subset of those continuous characters $\xi \in \widehat{G}$ such that $\xi(\gamma^{\mathbf{Z}}) = 1$, equivalently $\xi(\gamma) = 1$, and such that $\xi|_{G^1} = \chi|_{G^1}$.

Remark 4.6.1.2. Let G be a second-countable $\mathbf{R}_{>0}$-modulated group.

Since G^1 is compact, $\widehat{G^1}$ is discrete by [**DE**, Prop.3.1.5(a)].

We note that since G is second-countable, so is G^1, and hence $\widehat{G^1}$ is countable, and it is therefore the directed union of its finitely generated abelian subgroups, each of which is therefore open, closed, discrete. Finitely generated and compactly generated are equivalent conditions on the subgroups of $\widehat{G^1}$.

For each $\eta \in \widehat{G^1}$, there therefore exists a large enough finitely generated subgroup $\Gamma \subset \widehat{G^1}$ containing the character η. We define $k := \Gamma^\perp \subset G^1$. Since orthogonals of closed subgroups are closed subgroups (see [**DE**, Ch.3,§3.6,p.78]. For $\Gamma \subset \widehat{G}$, we have

$$\Gamma^\perp = \{x \in G : \chi(x) = 1, \text{ for all } \chi \in \Gamma\}$$

and hence $\Gamma^\perp$ is the intersection of the closed subgroups $\ker(\chi)$ as χ varies over all $\chi \in \Gamma$), $k \subset G^1$ is closed, hence compact, and $\eta(k) = 1$, i.e. $\eta \in k^\perp$ (since $k^\perp = \Gamma$ by [**DE**, Prop.3.6.1(b)]).

In particular, for each $\chi \in \widetilde{G^1}$, there exists a compact subgroup $k \subset G^1$ with finitely generated orthogonal $k^\perp$ such that $\chi(k) = 1$. This implies that we have:

$$(\gamma^{\mathbf{Z}})^\perp_\chi \subset (\gamma^{\mathbf{Z}} \cdot k)^\perp \tag{4.6.1.2}$$

for such χ.

4.6.1.1. *Polygenic periodization and absolute bases in the presence of multiplicity.*

LEMMA 4.6.1.3. *Let G be an $\mathbf{R}_{>0}$-modulated group. Assume the conditions and choices of Situation 4.6.1.1 are in force.*

Let $\gamma_1, \ldots, \gamma_N \in G$ be such that $|\gamma_k|_G > 1$ for all $1 \leq k \leq N$.

Let $\chi \in \widetilde{G^1}$ be a continuous character. Let $S_0 := (\gamma_1, \ldots, \gamma_{N-1})$.

(1) *The operator*

$$P_{S_0} : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.3}$$

defined as the continuous linear endomorphism assigned by the composition $P_{S_0} := \prod_{1 \leq k \leq N-1} P_{\gamma_k}$, where P_{γ_k} is defined by the assignment:

$$P_{\gamma_k}(\phi)(g) := \phi(x) - \phi(\gamma_k x) \tag{4.6.1.4}$$

on any $\phi \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, as functions of $x \in G$, has closed range.

(2) *The quotient*

$$\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.5}$$

is a separable nuclear Fréchet space.

(3) *The operator*

$$P_{\gamma_N} : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.6}$$

induces a linear topological isomorphism:

$$Q : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to P_{\gamma_N}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0 \cup \{\gamma_N\}}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.7}$$

where both source and target are separable nuclear Fréchet space, compatibly with the continuous left-translation actions of G on source and target.

PROOF. (1) : This is a consequence of Proposition 4.5.1.1(4).

(2) : $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is a nuclear Fréchet space by Proposition D.3.3.12 (it is linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, which is a nuclear Fréchet space by Lemma D.3.2.3(1)).

Since P_{S_0} has closed range in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ by (1), the subquotient in (4.6.1.5) is also a Fréchet space, nuclear by [**Scha**, Ch.III,§7.4,p.103].

(3) : The operator P_{γ_N} from (4.6.1.6) followed by the quotient, induces a surjective continuous linear operator

$$Q : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to P_{\gamma_N}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0 \cup \{\gamma_N\}}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.8}$$

which is open by [NB, Thm.14.4.5], being source and target Fréchet spaces. To show that $\ker(Q) = P_{S_0}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, we proceed as follows. Assume $\phi \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, is such that $P_{\gamma_N}(\phi) \in P_{S_0\cup\{\gamma_N\}}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

Then

$$\widetilde{M}(P_{\gamma_N}(\phi))(\chi|\cdot|_G^s) = (1-\chi(\gamma_N)|\gamma_N|_G^{-s})\widetilde{M}(\phi))(\chi|\cdot|_G^s)$$

as entire functions on the complex s-plane and elements of $A\mathcal{S}(\mathbf{C})$. We refer the reader to Discussion 4.4.1.1, for the definition of $\widetilde{M}(\phi)(\chi|\cdot|_G^s) = \int_{\mathbf{R}_{>0}} t^{s+ix_0(\chi)} \int_{G^1} \chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t}$.

Since $P_{\gamma_N}(\phi) \in P_{S_0\cup\{\gamma_N\}}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, we have:

$$(1-\chi(\gamma_N)|\gamma_N|_G^{-s})\widetilde{M}(\phi))(\chi|\cdot|_G^s) = \prod_{1\leq k\leq N}(1-\chi(\gamma_k)|\gamma_k|_G^{-s})\widetilde{M}(\psi)(\chi|\cdot|_G^s) \tag{4.6.1.9}$$

for some $\psi \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

By the Asymptotic Residue Theorem (see Theorem B.3.2.1), which applies because the meromorphic function $(1-\chi(\gamma_N)|\gamma_N|_G^{-s})^{-1}$ is periodic and with slow growth near its poles, which are all located within the vertical line $i\mathbf{R}$ of the complex s-plane, and since $\widetilde{M}$ factors through M as per (4.4.1.3) from Discussion 4.4.1.1, and M is a linear topological isomorphism by Theorem D.3.3.13, we deduce that we have $\phi = P_{S_0}(\psi)$, i.e. $\phi \in P_{S_0}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ (here we use Lemma 4.3.3.1 to identify $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, with the closed subspace $\mathrm{H}^0(\chi|_{G^1}, \mathcal{S}_{\mathrm{E}}(G))$ of the exponentially decaying Schwartz-Bruhat functions on G that are $\chi|_{G^1}$-homogeneous with respect to the continuous left-translation action by G^1, under the continuous linear operator $\phi \mapsto \frac{1}{\mu_1(G^1)}\int_{G^1}\chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)$)

It follows that the operator Q from (4.6.1.8) induces a linear topological isomorphism:

$$\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to P_{\gamma_N}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_0\cup\{\gamma_N\}}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.10}$$

which is the operator from (4.6.1.7). □

Lemma 4.6.1.4. *Let $N \geq 1$ be an integer, and $b_1, \ldots, b_N \in \mathbf{R}_{>0}$ real numbers. Assume $b_k > 1$ for each $1 \leq k \leq N$.*

The following conditions on the set $\{b_1, \ldots, b_N\}$ are equivalent:

(1) *There exists $b \in \mathbf{R}_{>0}$ with $b > 1$ and $b_k \in \mathbf{Q}\cdot\log(b)$ for all $1 \leq k \leq N$.*
(2) *The set of nonzero complex numbers:*

$$\{b_k^{\frac{2\pi i m}{\log(b_h)}}, 1 \leq k, h \leq N, m \in \mathbf{Z}\} \tag{4.6.1.11}$$

is finite.

Proof. Assume $\{b_k\}_{1\leq k\leq N} \subset \mathbf{Q}\cdot\log(b)$ for some $b \in \mathbf{R}_{>0}$ with $b > 1$. Then $\log(b_k) = c_k\cdot\log(b)$ for each $1 \leq k \leq N$, for rational numbers $c_k \in \mathbf{Q}$, $c_k \neq 0$.

We have, for each $1 \leq k \leq N$, $b_k = b^{c_k}$, and hence:

$$b_k^{\frac{2\pi i m}{\log(b_h)}} = b^{\frac{2\pi i m}{c_h\cdot\log(b)}c_k} = e^{\frac{2\pi i m c_k}{c_h}}$$

where c_k/c_h is a nonzero rational number. Since for each $1 \leq k \leq N$, there is a unique $c_k \in \mathbf{Q}$ such that $\log(b_k) = c_k\cdot\log(b)$, the set $F := \{c_k/c_h, 1 \leq h, k \leq N\}$ is a finite subset of $\mathbf{Q}$.

It follows that we have $\mathbf{Z}(1)\cdot F \subset (1/M)\mathbf{Z}(1)$ for a large enough integer $M \geq 1$, and hence $\exp(\mathbf{Z}(1)\cdot F)$ is a finite set. It follows that so is the set of nonzero complex numbers in (4.6.1.11).

Conversely, if the set in (4.6.1.11) is finite, then, upon calling F' the finite set $F' := \{\frac{\log(b_k)}{\log(b_h)}, 1 \leq h, k \leq N\} \subset \mathbf{R}$, then $\exp(\mathbf{Z}(1)\cdot F')$ is a finite set.

For each $c \in F'$, $\exp(\mathbf{Z}(1)\cdot c)$ is a finite subgroup of $\mathrm{U}(1)$, which forces $\mathbf{Z}(1)\cdot c \subset (1/M_c)\mathbf{Z}(1)$ for an integer $M_c \geq 1$. It follows that c is a nonzero rational number, and hence $F' \subset \mathbf{Q}$.

We define b to be the any of the b_k's, for instance one that minimizes their absolute value, and we have $\log(b_k) \in \mathbf{Q}\cdot\log(b)$ for each $1 \leq k \leq N$. □

We now use Lemma 4.6.1.3 and Theorem 4.5.2.3 to prove the existence of absolute Schauder bases on the subquotients:

$$\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$$

with $S = (\gamma_1, \ldots, \gamma_N)$ for $N \geq 1$ an integer, and, for each $1 \leq k \leq N$, for $\gamma_k \in G$ such that $|\gamma_k|_G > 1$. We define $b_k := \log(|\gamma_k|_G)$ for each $1 \leq k \leq N$.

We will be assuming that the logarithms of the real numbers b_k, for $1 \leq k \leq N$, $\mathbf{Q}$-linearly generate a 1-dimensional $\mathbf{Q}$-linear subspace of $\mathbf{R}$.

THEOREM 4.6.1.5 (Sourcing Periodic Schauder Bases - V). *Let G be an $\mathbf{R}_{>0}$-modulated group. Assume the conditions and choices from Situation 4.6.1.1 are in force. We denote by S the finite set $S := (\gamma_1, \ldots, \gamma_N)$. We define $b_k := \log(|\gamma_k|_G)$ for each $1 \leq k \leq N$.*

Assume that there exists a real number $b \in \mathbf{R}_{>1}$ such that $\log(b_k) \in \mathbf{Q} \cdot \log(b)$ for every $1 \leq k \leq N$.

Let $\chi \in \widetilde{G^1}$ be a continuous character. We define the discrete subset:

$$\mathcal{U}_{G,S}(\chi) := \bigcup_{1 \leq k \leq N} (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi} \tag{4.6.1.12}$$

contained within the connected component of the Riemann surface $\Omega(G)$ parametrized by the continuous character $\chi|_{G^1}$. We define on $\mathcal{U}_{G,S}(\chi)$ the multiplicity function $N : \mathcal{U}_{G,S}(\chi) \to \mathbf{N}$:

$$N(\xi) := \sum_{1 \leq k \leq N} 1_{(\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}}(\xi). \tag{4.6.1.13}$$

For any element $\xi \in \mathcal{U}_{G,S}(\chi)$, we write, for k any integer such that $1 \leq k \leq N$ and $\xi \in (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$, $n_{\xi,k}$ for the unique integer such that we have $\xi = \chi| \cdot |_G^{\frac{2\pi i n_{\xi,k}}{\log(b_k)} - it_{0,k}(\chi)}$.

We define, for each $\xi \in \bigcup_{1 \leq k \leq N} (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$ and each integer $0 \leq m \leq N(\xi) - 1$, the continuous functional:

$$D_{\xi,m} : \phi \mapsto \int_G \phi(g)\xi(g)(\log|g|_G)^m \, \mathrm{d}\mu(g) \tag{4.6.1.14}$$

on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

The countable set

$$E_S(\chi|_{G^1}) := \{D_{\xi,m} : \xi \in \bigcup_{1 \leq k \leq N} (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}, 0 \leq m \leq N(\xi) - 1\} \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})'_{\beta} \tag{4.6.1.15}$$

is contained in $(\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}))'_{\beta}$ and any of its enumerations is an unconditional Schauder basis for $(\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}))'_{\beta}$, whose bi-orthogonal set

$$\{\psi_{\xi,m} : \xi \in \bigcup_{1 \leq k \leq N} (\gamma_k^{\mathbf{Z}})^{\perp}, 0 \leq m \leq N(\xi) - 1\} \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.16}$$

equipped with the induced enumeration, is an absolute Schauder basis.

It further satisfies the following conditions:

(1) *Let q be any continuous seminorm on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$. Then there exists a polynomial $p \in \mathbf{C}[X]$ depending on q, such that, for any choice of integer k such that $1 \leq k \leq N$ and $\xi \in (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$:*

$$q(\psi_{\xi,m}) \ll_{q,k} p(n_{\xi,k}) \tag{4.6.1.17}$$

uniformly with respect to $0 \leq m \leq N(\xi) - 1$, as $|n_{\xi,k}| \to +\infty$.

(2) *For any $\phi \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, for any fixed integer $0 \leq m \leq N - 1$, and for any choice of integer k such that $1 \leq k \leq N$ and such that at least an element ξ of $(\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$ satisfies $N(\xi) \geq m$, the function $\xi \mapsto D_{\xi,m}(\phi)$ as a function of the integer $n_{\xi,k}$ as ξ varies over the elements of $(\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$ of multiplicity at least m, is an element of $\mathcal{S}(\mathbf{Z})$.*

PROOF. We first construct a bi-orthogonal set to $E_S(\chi|_{G^1})$. We proceed as in the proof of Theorem 7.1.4.2.

We choose $\phi \in A\mathcal{S}(\mathbf{C})$ such that $\phi(z_m) \neq 0$ for all $m \in \mathbf{Z}$, with $z_m := \frac{2\pi i m}{\log(b_k)} - it_{0,k}(\chi)$ with $1 \leq k \leq N$ any k such that $\chi|\cdot|_G^{z_m} \in (\gamma_k^{\mathbf{Z}})_\chi^\perp$.

We define:

$$L_{G,S,\phi}(s) := \prod_{1 \leq k \leq N} (1 - \chi(\gamma_k)|\gamma_k|_G^{-s})\phi(s) \tag{4.6.1.18}$$

an entire function with zeros at the zeros of $\prod_{1\leq k\leq N}(1-\chi(\gamma_k)|\gamma_k|_G^{-s})$, with exactly the same multiplicities (and possibly other zeros elsewhere).

We have

$$\frac{\partial^j}{\partial s^j} \prod_{1\leq k\leq N} (1-\chi(\gamma_k)|\gamma_k|_G^{-s}) = \sum_{k_1+\ldots+k_N=j} \frac{j!}{k_1!\cdots k_N!} \prod_{1\leq k\leq N} \partial_s^{k_r}(1-\chi(\gamma_k)|\gamma_k|_G^{-s})$$

and since $\partial_s^{k_r}(1-\chi(\gamma_k)|\gamma_k|_G^{-s}) = (-1)^{k_r-1}\chi(\gamma_k)(\log|\gamma_k|_G)^{k_r}|\gamma_k|_G^{-s}$ is bounded on vertical strips of bounded width within the complex s-plane, we deduce that both $\prod_{1\leq k\leq N}(1-\chi(\gamma_k)|\gamma_k|_G^{-s})$ and its holomorphic derivatives of any order, are bounded uniformly over each vertical strip of bounded width of the complex s-plane.

It follows that $L_{G,S,\phi}$ is an element of $A\mathcal{S}(\mathbf{C})$. We fix an integer $m \in \mathbf{Z}$, and an integer $0 \leq j \leq N(\chi|\cdot|_G^{z_m})-1$. We have implicitly chosen an integer $1 \leq k \leq N$ such that $\chi|\cdot|_G^{z_m} \in (\gamma_k^{\mathbf{Z}})_\chi^\perp$.

We define the function:

$$\lambda_{(\chi,m),j}(s) := \frac{1}{(s-z_m)^j} L_{G,S,\phi}(s) \tag{4.6.1.19}$$

a meromorphic function with only one removable singularity at $s = z_m$, which is a zero of order $N(\chi|\cdot|_G^{z_m}) - j$. Moreover, since the meromorphic function $\frac{s^M}{(s-z_m)^j}$ of $s \in \mathbf{C}$ has, for any integer $M \geq 0$, slow growth near its poles in the sense of Definition B.3.1.1, the Asymptotic Residue Theorem (see Theorem B.3.2.1) applied to it, upon integrating over two vertical lines bounding a large enough open vertical strip of bounded width containing $i\mathbf{R}$ in its interior, implies that the unique analytic continuation of $\lambda_{(\chi,m),j}$ to an entire function on the complex s-plane, which we also denote by $\phi_{(\chi,m),j}$, is Schwartz uniformly over the non-empty open vertical strips of bounded width of the complex s-plane, i.e. an element of $A\mathcal{S}(\mathbf{C})$.

There therefore exists a unique $\phi_{(\chi,m),j} \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ such that $\widetilde{M}(\phi_{(\chi,m),j})(\chi|\cdot|_G^s) = \lambda_{(\chi,m),j}(s)$, where $\widetilde{M}$ is the Mellin transform operator $\mathcal{S}_{\mathrm{E}}(G) \to A\mathcal{S}(\mathbf{C})$ defined as $\phi \mapsto \int_G \phi(g)\chi(g)|g|_G^s \, \mathrm{d}\mu(g)$ (see Discussion 4.4.1.1).

We note that we have $D_{\chi|\cdot|_G^{z_m},j'}(\phi_{(\chi,m),j}) \neq 0$ for all integers

$$N(\chi|\cdot|_G^{z_m}) - j \leq j' \leq N(\chi|\cdot|_G^{z_m}) - 1$$

and $D_{\chi|\cdot|_G^{z_m},j}(\phi_{(\chi,m'),j'}) = 0$ otherwise.

This implies that a bi-orthogonal set to the set $E_S(\chi|_{G^1})$ exists, and, for each fixed $m \in \mathbf{Z}$, the unique element in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ on which the value of $D_{\xi,m}$ is 1, is the image of an element of $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ obtained as a finite $\mathbf{C}$-linear combination of the finitely many elements $\phi_{(\chi,m),j}$ only, as j varies over the integers $1 \leq j \leq N(\xi)$.

Explicitly, we solve the $N(\xi) \times N(\xi)$ linear system, for $\xi = \chi|\cdot|_G^{z_m}$:

$$\sum_{1\leq j_0\leq N(\xi)} c_{j_0,j} \cdot D_{\xi,j}(\phi_{(\chi,m),j_0}) = 1 \tag{4.6.1.20}$$

$$\sum_{1\leq j_0\leq N(\xi)} c_{j_0,j} \cdot D_{\xi,h}(\phi_{(\chi,m),j_0}) = 0, \quad 0 \leq h \leq N(\xi)-1, h \neq j. \tag{4.6.1.21}$$

To see that this system indeed has a solution, we observe that the system is triangular. On the diagonal, we find the complex numbers $D_{\xi,N(\xi)-j_0}(\phi_{(\chi,m),j_0})$, all nonzero for each $1 \leq j_0 \leq N(\xi)$, and hence the system has a unique solution $(c_{1,j},\ldots,c_{N(\xi),j})$ (for instance, for $j = N(\xi)-1$, since $D_{\xi,N(\xi)-1}(\phi_{(\chi,m),1}) \neq 0$ and $D_{\xi,j'}(\phi_{(\chi,m),1}) = 0$ for all $0 \leq j' \leq N(\xi)-2$, we obtain $c_{1,N(\xi)-1} = 1/D_{\xi,N(\xi)-1}(\phi_{(\chi,m),1})$. The other coefficients $c_{2,N(\xi)-1},\ldots,c_{N(\xi),N(\xi)-1}$ are uniquely determined by the remaining $N(\xi)-1$ equations. We therefore have and $\psi_{\xi,N(\xi)-1} := \sum_{1\leq j_0\leq N(\xi)} c_{j_0,N(\xi)-1}\cdot\phi_{(\chi,m),j_0}$, and it is such that $D_{\xi,N(\xi)-1}(\psi_{\xi,N(\xi)-1}) = 1$, and $D_{\xi,j}(\psi_{\xi,N(\xi)-1}) = 0$ for all $0 \leq j \leq N(\xi)-2$).

We choose $N = 2$ and discuss this special case first. In particular, we have $S = (\gamma_1,\gamma_2)$.

We consider the following diagram whose top and middle rows are short linearly-exact sequences:

(4.6.1.22)

$$\begin{array}{ccccc}
P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \\
\downarrow{\scriptstyle P_{\gamma_1}^{-1}} & & \| & & \| \\
\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_2}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \xrightarrow{P_{\gamma_1}} & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \\
\downarrow{\scriptstyle \Phi_{\chi,\gamma_2}} & & & & \downarrow{\scriptstyle \Phi_{\chi,\gamma_1}} \\
\mathcal{S}_\chi(G/\gamma_2^{\mathbf{Z}}) & & & & \mathcal{S}_\chi(G/\gamma_1^{\mathbf{Z}})
\end{array}$$

We note that in fact the top and middle rows are topologically exact: the leftmost spaces are closed subspaces under the respective linear inclusions, and $P_{\gamma_1}^{-1}$ denotes the inverse linear topological isomorphism Q from (4.6.1.7) provided by Lemma 4.6.1.3.

The leftmost horizontal arrow in the middle row of (4.6.1.22) is the linear topological isomorphism Q from (4.6.1.7) of Lemma 4.6.1.3, followed by the continuous inclusion of $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ into $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, which is the leftmost horizontal arrow in (4.6.1.22): the former has closed range so long as the former does.

We note that the quotient topology and the subspace topology on $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ are equivalent by [**NB**, Thm.14.4.5(b)], since $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{(\gamma_1,\gamma_2)}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is closed with respect to the subspace topology, hence a Fréchet space with respect to such topology, and it is a Fréchet space with respect to the quotient topology, both assertions following from Lemma 4.6.1.3(2).

The closed subspace $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$, for $\gamma \in G$ such that $|\gamma|_G > 1$, is the closed range of the continuous operator $\Psi_\chi : \phi \mapsto \frac{1}{\mu_1(G^1)}\int_{G^1}\chi(g)\phi(gx)\,\mathrm{d}\mu_1(g)$ as functions of $x \in G/\gamma^{\mathbf{Z}}$, where the continuous action of G^1 is by left-translation on the left $\gamma^{\mathbf{Z}}$-cosets (compare with the proof of Proposition 4.5.1.1(1), and note that indeed Ψ_χ is valued in $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ because the left-translation action by G^1 on $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ is uniformly continuous, in the sense of Lemma 4.3.3.1, and the Lemma applies. $\mathcal{S}_\chi(G/\gamma^{\mathbf{Z}})$ agrees with the closed subspace of the $\chi|_{G^1}$-homogeneous elements under such continuous left-translation action by G^1).

We now define the subset:

$$E_{S/\gamma_1}(\chi|_{G^1}) := \{D_{\xi,0} : \xi \in (\gamma_2^{\mathbf{Z}})_\chi^\perp - (\gamma_1^{\mathbf{Z}})_\chi^\perp\} \cup \{D_{\xi,1} : \xi \in (\gamma_1^{\mathbf{Z}})_\chi^\perp \cap (\gamma_2^{\mathbf{Z}})_\chi^\perp\} \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})'_\beta \tag{4.6.1.23}$$

and we observe that $E_{S/\gamma_1}(\chi|_{G^1})$ is contained in the closed range of the continuous adjoint of the quotient $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, and maps injectively into the strong dual of $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

This implies that the subset of the set in (4.6.1.16):

$$F_{S/\gamma_1}(\chi|_{G^1}) := \{\psi_{\xi,0} : \xi \in (\gamma_2^{\mathbf{Z}})_\chi^\perp - (\gamma_1^{\mathbf{Z}})_\chi^\perp\} \cup \{\psi_{\xi,1} : \xi \in (\gamma_1^{\mathbf{Z}})_\chi^\perp \cap (\gamma_2^{\mathbf{Z}})_\chi^\perp\} \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.1.24}$$

is, modulo $P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, contained in $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$. In fact, the very design of each $\psi_{\xi,m}$ ensures that $F_{S/\gamma_1}(\chi|_{G^1})$ is itself contained within $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ (i.e. not merely modulo $P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$).

Indeed, the construction of each $\psi_{\xi,m}$ exhibits $\psi_{\xi,m}$ as a finite linear combination of the $\phi_{(\chi,n_\xi),k}$'s for $N(\xi)-m \leq k \leq N(\xi)$. In particular, each $\psi_{\chi,m}$ in the subset F_{S/γ_1}, is such that its Mellin transform

$\widetilde{M}(\psi_{\xi,m})(\chi|\cdot|_G^s)$ is divisible by $1-\chi(\gamma_1)|\gamma_1|_G^{-s}$, and hence, since $M : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to A\mathcal{S}(\mathbf{C})$ is a linear topological isomorphism by Theorem D.3.3.13 and in light of the relation between M and $\widetilde{M}$ explained in Discussion 4.4.1.1, we deduce that we have $F_{S/\gamma_1}(\chi|_{G^1}) \subset P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

For each $\xi \in (\gamma_2^{\mathbf{Z}})_\chi^\perp$, there exists a unique $m \in \mathbf{Z}$ such that $\xi = \chi|\cdot|_G^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_0(\chi)}$. We consider the functional $a_{\chi,m}$ from (4.4.2.6), which, recall, was defined as:

$$a_{\chi,m}(\phi) := \frac{1}{\log(|\gamma_2|_G)\mu_1(G^1)} \int_{G/\gamma^{\mathbf{z}}} \phi(g)\chi(g)|g|_G^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_0(\chi)} \, \mathrm{d}\overline{\mu}_{\gamma_2}(g)$$

for ϕ the arbitrary element in $\mathcal{S}(G/\gamma_2^{\mathbf{Z}})$, and $\overline{\mu}_{\gamma_2}$ is the unique Haar measure on $G/\gamma_2^{\mathbf{Z}}$ prescribed as per Theorem D.1.2.1 by the choice of μ on G and the counting measure on the discrete closed subgroup $\gamma_2^{\mathbf{Z}} \subset G$.

Assume $\xi \in (\gamma_1^{\mathbf{Z}})_\chi^\perp \cap (\gamma_2^{\mathbf{Z}})_\chi^\perp$. Then, by L'Hospital's Rule, the image of $a_{\chi,m}(\cdot)$ under the continuous adjoint of the composition $P_{\gamma_1}^{-1} \circ \Phi_{\chi,\gamma_1}$, is the continuous functional on $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ which is the restriction to $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ of the continuous functional $-\frac{1}{\log(|\gamma_1|_G)} D_{\xi,1}$ in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})'_\beta$, for $\xi = \chi|\cdot|_G^{\frac{2\pi im}{\log(|\gamma_1|_G)} - it_0(\chi)}$.

If $\xi \in (\gamma_2^{\mathbf{Z}})_\chi^\perp - (\gamma_1^{\mathbf{Z}})_\chi^\perp$, then the image of $a_{\chi,m}$ is the continuous functional:

$$\frac{1}{1-\chi(\gamma_1)|\gamma_1|_G^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)}} D_{\xi,0}$$

where $t_{0,1}(\chi) \in \mathbf{R}$ is the unique real number in $[0, \frac{2\pi}{\log(|\gamma_1|_G)})$ such that $\chi(\gamma_1)|\gamma_1|^{-it_{0,1}(\chi)} = 1$, and similarly for $t_{0,2}(\chi)$, for $\xi = \chi|\cdot|_G^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)}$.

It follows from Lemma 4.4.2.6, that the image of $F_{S/\gamma_1}(\chi|_{G^1})$ in the quotient $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is an absolute Schauder basis within its closure (because for each $\psi_{\xi,m}$, there exists a nonzero complex number $c_{\xi,m}$ such that $c_{\xi,m} \cdot \psi_{\xi,m}$ maps under the linear topological isomorphism $\Phi_{\chi,\gamma_1} \circ P_{\gamma_1}^{-1}$ to the absolute Schauder basis $e_{\chi,m}$ from (4.4.2.5)).

By Lemma 4.4.2.6, the countable subset:

$$F_{\gamma_1}(\chi|_{G^1}) := \{\psi_{\xi,0} : \xi \in (\gamma_1^{\mathbf{Z}})_\chi^\perp\} \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$$

endowed with an arbitrary choice of enumeration, maps under the composition of the quotient $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, followed by Φ_{χ,γ_1} to an absolute Schauder basis in $\mathcal{S}_\chi(G/\gamma_1^{\mathbf{Z}})$ (which, as a set, is the bi-orthogonal set to the set of functionals $a_{\chi,m}$ from (4.4.2.6) of Lemma 4.4.2.6, for $\psi_{\xi,0}$ mapping to the unique element ψ such that, for $\xi = \chi|\cdot|_G^{\frac{2\pi im}{\log(|\gamma_1|_G)} - it_{0,1}(\chi)}$, $a_{\chi,m}(\psi) = 1$ and $a_{\chi,m'}(\psi) = 0$ for all integers $m' \neq m$) and hence it maps to an absolute Schauder basis in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, since Φ_{χ,γ_1} is a linear topological isomorphism.

Since $\log(b_1), \log(b_2) \in \mathbf{Q} \cdot \log(b)$ for a real number $b > 1$, the set of complex numbers:

$$\{|\gamma_1|^{\frac{2\pi im}{\log(|\gamma_2|_G)}}\}_{m \in \mathbf{Z}} \tag{4.6.1.25}$$

is finite by Lemma 4.6.1.4.

As a consequence, the following set of nonzero complex numbers:

$$\left\{ \frac{1}{1-\chi(\gamma_1)|\gamma_1|^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)}} \right\}_{m \in A} \tag{4.6.1.26}$$

is also finite, where $A \subset \mathbf{Z}$ is the countable subset of those integers such that $1-\chi(\gamma_1)|\gamma_1|^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)}$ is nonzero, equivalently those integers such that $\chi|\cdot|_G^{\frac{2\pi im}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)} \in (\gamma_2^{\mathbf{Z}})_\chi^\perp - (\gamma_1^{\mathbf{Z}})_\chi^\perp$.

It follows that each continuous functional $a_{\chi,m}(\cdot) \in \mathcal{S}_\chi(G/\gamma_2^{\mathbf{Z}})'_\beta$ is mapped by the continuous adjoint $(P_{\gamma_1}^{-1} \circ \Phi_{\chi,\gamma_1})'$ of $P_{\gamma_1}^{-1} \circ \Phi_{\chi,\gamma_1}$ to $c_{\xi,h} \cdot D_{\xi,h}$ where h is either 0 or 1, and where $c_{\xi,1} := -\frac{1}{\log(b_1)}$, and $c_{\xi,0}$ is an element

of the finite set of nonzero complex numbers from (4.6.1.26), depending on whether $\xi = \chi|\cdot|_G^{\frac{2\pi i m}{\log(|\gamma_2|_G)} - it_{0,2}(\chi)}$ is either in $(\gamma_1^{\mathbf{Z}})^{\perp}_{\chi} \cap (\gamma_2^{\mathbf{Z}})^{\perp}_{\chi}$, or in $(\gamma_1^{\mathbf{Z}})^{\perp}_{\chi} - (\gamma_2^{\mathbf{Z}})^{\perp}_{\chi}$.

For every continuous seminorm q on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, its restriction to the closed subspace $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is also a continuous seminorm on such subspace, and since $\Phi_{\chi,\gamma_2} \circ P_{\gamma_1}^{-1}$ is a linear topological isomorphism, and the set in (4.6.1.25) is finite, we can maximize the implied constants provided by Lemma 4.4.2.6 over it, to incorporate the finitely many values of the multiplicative constants $c_{\xi,h}$, obtaining from Lemma 4.4.2.6(2) that for each $\psi_{\xi,m} \in F_{S/\gamma_1}(\chi|_{G^1})$, and denoting by $\psi_{\xi,m}$ also its image in the quotient $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, there exists a polynomial $p \in \mathbf{C}[X]$ depending on q such that the estimate:

$$q(\psi_{\xi,m}) \ll_q p(n_{\xi,k}) \tag{4.6.1.27}$$

as $|n_{\xi,k}| \to +\infty$, for $n_{\xi,k}$ the unique integer such that $\xi = \chi|\cdot|_G^{\frac{2\pi i n_{\xi,k}}{\log(|\gamma_k|_G)} - it_{0,k}(\chi)}$, $k = 1, 2$.

Changing the choice of k such that $\xi \in (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$, changes $n_{\xi,k}$ by a single integer translator (since the set of zeros of $1 - \chi(\gamma_k)b_k^{s-it_{0,k}(\chi)}$ is a torsor under the action by vertical translation by $\frac{2\pi i\mathbf{Z}}{\log(b_k)}$), and up to rendering the implied constant (4.6.1.27) dependent on k (by enlarging it at most N times, which in this special case $N = 2$ means at most twice), the estimate in (4.6.1.27) holds for any such choice of k.

We further obtain that the sequence of coefficients $D_{\xi,m}(\phi)$ is of rapid decay as a function of $n_{\xi,k}$, for each $\phi \in \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, and hence it remains so upon changing the choice of k such that $\xi \in (\gamma_k^{\mathbf{Z}})^{\perp}_{\chi}$, and so it is also when renormalized by a sequence valued in the set of any of the finitely many nonzero complex numbers from (4.6.1.26).

In particular, (1) and (2) hold for the image in $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ of the countable set $F_{S/\gamma_1}(\chi|_{G^1})$.

For any choice of enumeration of the countable set $F_{S/\gamma_1}(\chi|_{G^1})$ from (4.6.1.24), the finite rank continuous operators on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ defined as:

$$P_M := \sum_{0 \le k \le M} D_{\xi_k,m_k} \cdot \psi_{\xi_k,m_k}$$

converge pointwise on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$: indeed, for any $\phi \in \mathcal{S}_{\mathrm{E}}(G)$, $D_{\xi_k,m_k}(\phi)$ is a sequence of rapid decay, since the integers m_k are at most finitely many, and the at most N holomorphic derivatives of the Mellin transform $\widetilde{M}(\phi)(\chi|\cdot|_G^s)$ are Schwartz uniformly on every vertical strip of bounded width of the complex s-plane, and hence in particular their values on the points $\xi_k = \chi|\cdot|_G^{\frac{2\pi i n_k}{\log(b_{h_k})} - it_{0,h_k}(\chi)}$ decay rapidly.

Using (4.6.1.27), applied to the restriction to the closed subspace

$$P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$$

of an arbitrary continuous seminorm q on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, we obtain that the sums:

$$\sum_{0 \le k \le M} |D_{\xi_k,m_k}(\phi)| \cdot q(\psi_{\xi_k,m_k})$$

converge as $M \to +\infty$, for each $\phi \in \mathcal{S}_{\mathrm{E}}(G)$, and hence in particular the sequence $\{P_M\}_{M \ge 1}$ converges pointwise on the Fréchet space

$$\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}).$$

The pointwise limit P is therefore a linear idempotent operator on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ valued in $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

By [**Ru-3**, Part I,Ch.2,Thm.2.8,p.46], a form of the Uniform Boundedness Principle for Fréchet spaces, which applies since $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is a Fréchet space, P is continuous.

Since $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \subset \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ is therefore linearly and topologically complemented, with $\ker(P)$ linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, we deduce that the image in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ of the union of $F_{S/\gamma_1}(\chi|_{G^1})$ and $F_{\gamma_1}(\chi|_{G^1})$ in $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, is an absolute Schauder basis, satisfying (1) and (2).

It follows that its bi-orthogonal sequence, which, as a set, is the set $E_S(\chi|_{G^1})$ from (4.6.1.15), is an unconditional Schauder basis for the strong dual of $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

This finishes the proof in the special case $S = (\gamma_1, \gamma_2)$.

We now discuss the case of general $N \geq 1$. We define $S_1 := (\gamma_2, \ldots, \gamma_N) \subset S$.

The following diagram of continuous linear maps:

(4.6.1.28)

$$\begin{array}{ccccc} P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \\ \Big\downarrow{\scriptstyle P_{\gamma_1}^{-1}} & & \Big\| & & \Big\| \\ \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \xrightarrow{P_{\gamma_1}} & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \end{array}$$

has topologically exact rows, and $P_{\gamma_1}^{-1}$ is the inverse of the linear topological isomorphism Q from (4.6.1.7) provided by Lemma 4.6.1.3 (up to relabelling the elements of S).

On both $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ and $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$, the assertions on the images of $E_{(\gamma_1)}(\chi|_{G^1})$, respectively $E_{S_1}(\chi|_{G^1})$, and their respective bi-orthogonal sets, hold by inductive hypothesis.

Moreover, the image of the continuous functionals in $E_{S_1}(\chi|_{G^1})$ under the continuous adjoint of P_{γ_1}, is itself a Schauder basic sequence and a subset of $E_S(\chi|_{G^1})$ up to renormalizing each element by a sequence of nonzero complex numbers which again takes values in a finite set of complex numbers of the form $(\log(b_k))^r$ for integers $2 \leq k \leq N$, and $0 \leq r \leq N$, or special values of $(1-\chi(\gamma_1)|\gamma_1|_G^s)$ at imaginary numbers of the form $\frac{2\pi i m}{\log(b_k)} - it_{0,k}(\chi)$ for $2 \leq k \leq N$ and for integers $m \in \mathbf{Z}$ such that the resulting special value of $(1-\chi(\gamma_1)|\gamma_1|_G^s)$ is nonzero: this set of complex numbers is finite, by another application of Lemma 4.6.1.4.

In particular, upon changing the implied constants, the estimate in (4.6.1.17) is inherited by the elements of the set in (4.6.1.16) from the analogous estimate for S_1 in place of S and (γ_1) in place of S, respectively, both of which hold by inductive hypothesis.

To assemble the resulting basic sequence in $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ and $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ into the desired basic sequence, we repeat the argument for the construction of a continuous projector onto the former closed subspace, resting on the Uniform Boundedness Principle in its form from [**Ru-3**, Part I,Ch.2,Thm.2.8,p.46]. □

We illustrate Theorem 4.6.1.5 on an example:

EXAMPLE 4.6.1.6. Let $\gamma \in \mathbf{R}_{>0}$ such that $\gamma > 1$.

We describe the absolute Schauder basis provided by Theorem 4.6.1.5, in the special case $G = \mathbf{R}_{>0}$, and $S = (\gamma, \gamma)$ with $P_S = P_\gamma^2$.

The Mellin transform $\widetilde{M} = M$ (in this special case; see Discussion 4.4.1.1) sends $P_\gamma^2\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ onto the closed subspace $(1-\gamma^{-s})^2 A\mathcal{S}(\mathbf{C}) \subset A\mathcal{S}(\mathbf{C})$.

The following short exact sequence:

$$0 \to (1-\gamma^{-s})A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})^2 A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})^2 A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C}) \to 0 \quad (4.6.1.29)$$

is obtained under Mellin transform from (4.6.1.7) of Lemma 4.6.1.3, applied to $G = \mathbf{R}_{>0}$, $\chi = 1$, $S_0 = (\gamma, \gamma)$ and $\gamma_N = \gamma$.

It is topologically exact, i.e. the leftmost quotient in (4.6.1.29) equipped with the quotient topology, carries the subspace topology.

The linear topological isomorphism Q provided by Lemma 4.6.1.3 is, in this special case, the linear topological isomorphism:

$$P_\gamma^{-1} : (1-\gamma^{-s})A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})^2 A\mathcal{S}(\mathbf{C}) \to A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C}) \tag{4.6.1.30}$$

and the operator Φ_γ from (4.2.1.1) identifies the right side in (4.6.1.30) with $\mathcal{S}(\mathbf{R}_{>0}/\gamma^{\mathbf{Z}})$. This provides an absolute Schauder basis for $A\mathcal{S}(\mathbf{C})/(1-\gamma^{-s})A\mathcal{S}(\mathbf{C})$: the pre-image under Φ_γ of the exponential system, and an unconditional Schauder basis for the strong dual $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})/P_\gamma\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, as the image of the dual basis under Φ'_γ.

The pre-image under P_γ^{-1} from (4.6.1.30) and the image of the dual basis under its continuous adjoint, involves renormalizing the basis elements by the nonzero complex number $\log(\gamma)$ and its reciprocal. Indeed, P_γ^{-1} sends $\phi \in (1-\gamma^{-s})A\mathcal{S}(\mathbf{C})$ to the unique analytic continuation of $\frac{\phi(s)}{1-\gamma^{-s}}$, an element of $A\mathcal{S}(\mathbf{C})$, and point-evaluation at any $z_m := \frac{2\pi i m}{\log(\gamma)}$ gives, in terms of ϕ:

$$\delta_{z_m}\left(\frac{\phi(s)}{1-\gamma^{-s}}\right) = \frac{\phi'(z_m)}{\log(\gamma)\gamma^{-z_m}} = \frac{\phi'(z_m)}{\log(\gamma)}$$

which is the image of the dual basis element $a_m(\cdot)$ from (4.2.1.4) of Lemma 4.2.1.2 under the composition of Φ'_γ and $(P_\gamma^{-1})'$.

4.6.2. Support conditions in the presence of multiplicity. The following Proposition will allow us to source absolute Schauder bases comprised of compactly supported smooth functions on G, with some degree of control on the location of the support.

THEOREM 4.6.2.1 (Sourcing Compactly Supported Absolute Schauder Bases). *Let G be an $\mathbf{R}_{>0}$-modulated group. Assume the conditions and choices from Situation 4.6.1.1 are in force.*

Let $N \geq 1$ be an integer, and $S = (\gamma_1, \ldots, \gamma_N)$ an N-tuple of elements of G such that for every integer $1 \leq k \leq N$, we have $|\gamma_k|_G > 1$.

For each $\chi \in \widetilde{G^1}$, there exists a compact subset $K_S \subset G$ satisfying the following properties:

(1) $G^1 \cdot K_S = K_S$.
(2) $|K_S|_G \subset (0,1)$.
(3) *The continuous linear map:*

$$\mathcal{S}_{K_S}(G)(\chi|_{G^1}) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.2.1}$$

is surjective. In particular, the induced continuous linear map:

$$\mathcal{S}_{K_S}(G)(\chi|_{G^1})/(\mathcal{S}_{K_S}(G)(\chi|_{G^1}) \cap P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.2.2}$$

is a linear topological isomorphism.

PROOF. The following diagram:

(4.6.2.3)

$$\begin{array}{ccccc}
P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \\
\downarrow{\scriptstyle P_{\gamma_1}^{-1}} & & \| & & \| \\
\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \xrightarrow{P_{\gamma_1}} & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})
\end{array}$$

of continuous linear operators, with topologically exact rows and where $P_{\gamma_1}^{-1}$ is the inverse of the linear topological isomorphism Q from (4.6.1.7) provided by Lemma 4.6.1.3 (up to relabelling the elements of S) enables the following inductive argument.

We call $K_{(\gamma_1)} := \tau([1/b_1', b_1']) \cdot G^1$ for a choice of $b_1' > b_1 = \log(|\gamma_1|_G)$, and deduce from Proposition 4.5.2.1(2) and Proposition 4.5.1.1(4) and (6), that the continuous linear map:

$$\mathcal{S}_{K_{(\gamma_1)}}(G)(\chi|_{G^1})/(P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \cap \mathcal{S}_{K_{(\gamma_1)}}(G)(\chi|_{G^1})) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.2.4}$$

is a linear topological isomorphism. By induction on $N \geq 1$, we may and do assume that a compact subset $K_{S_1} \subset G$ exists such that $G^1 \cdot K_{S_1} = K_{S_1}$, and such that the continuous linear map:

$$\mathcal{S}_{K_{S_1}}(G)(\chi|_{G^1})/(P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \cap \mathcal{S}_{K_{S_1}}(G)(\chi|_{G^1})) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.2.5}$$

is a linear topological isomorphism. The linear topological isomorphisms from (4.6.2.5) and from (4.6.2.4) hold upon replacing $K_{(\gamma_1)}$, respectively K_{S_1}, with larger compact subsets.

The operator P_{γ_1} maps $\mathcal{S}_{K_{S_1}}(G)(\chi|_{G^1})$ into $\mathcal{S}_{K'_{S_1}}(G)(\chi|_{G^1})$ for K'_{S_1} the compact subset $K'_{S_1} := K_{S_1} \cup \bigcup_{2 \leq k \leq N} \gamma_k^{-1} \cdot K_{S_1}$. We therefore define $K'_S := K'_{S_1} \cup K_{(\gamma_1)}$ and replace $K_{(\gamma_1)}$ with K'_S.

The linear operator $P_{\gamma_1}^{-1}$ in the leftmost vertical arrow in (4.6.2.3) induces a linear topological isomorphism:

$$(P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \cap \mathcal{S}_{K'_S}(G)(\chi|_{G^1}))/(P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \cap \mathcal{S}_{K'_S}(G)(\chi|_{G^1})) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})/P_{S_1}\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \tag{4.6.2.6}$$

because its range contains the image of the linear topological isomorphism from (4.6.2.5), by design of K'_{S_1} and K'_S. It follows that (4.6.2.6), is a continuous bijection, and since source and target are Fréchet spaces, it is a linear topological isomorphism by [**NB**, Thm.14.4.5(b)].

Since all the quotients in (4.6.2.3) are continuous representations under the continuous left-translation action by G on all function spaces and their quotients, we choose $t \in (0,1)$ and call $\gamma^{-1} := \tau(t)$, and the left-translation action by γ^M on $\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$ sends $\mathcal{S}_{K'_S}(G)(\chi|_{G^1})$ linearly and topologically isomorphically onto $\mathcal{S}_{\gamma^M K'_S}(G)(\chi|_{G^1})$, compatibly with the closed subspace $\mathcal{S}_{K'_S}(G)(\chi|_{G^1}) \cap P_S\mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$.

We choose $M \geq 1$ large enough so as to have $\gamma^M \cdot K'_S \subset \tau((0,1)) \cdot G^1$, and rename this compact subset as K_S.

We have shown that K_S satisfies (3).

Since K_{S_1} is G^1-stable, as is $K_{(\gamma_1)}$ by construction, so is K_S, since the action of G^1 by left-translation commutes with the action of $\gamma_k^{\mathbf{Z}}$ for each $2 \leq k \leq N$, and $\gamma^{\mathbf{Z}}$. It follows that K_S satisfies (1) and (2). □

4.6.3. The case of $\mathcal{S}_I(G)$.

Corollary 4.6.3.1. *Let G be a Hausdorff locally compact abelian group. Assume G is $\mathbf{R}_{>0}$-modulated. Let $\gamma \in G$ be such that $|\gamma|_G > 1$. Assume the choices and assumptions from Situation 4.6.1.1 are in force.*

Let $I \subset \mathbf{R}$ be a non-empty open connected interval. Assume $0 \in I$.

Upon denoting $S = (\gamma_1, \ldots, \gamma_N)$ for $N \geq 1$ an integer and $\gamma_k \in G$ with $|\gamma_k|_G > 1$ for each $1 \leq k \leq N$ as in Situation 4.6.1.1, we write P_S for the composition of the operators P_{γ_k} from (4.5.3.3) on $\mathcal{S}_I(G)$ and $\mathcal{S}(G)$.

The canonical continuous linear operators:

$$\mathcal{S}_{\mathrm{E}}(G)/P_S\mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_I(G)/P_S\mathcal{S}_I(G) \to \mathcal{S}(G)/P_S\mathcal{S}(G) \tag{4.6.3.1}$$

are linear topological isomorphisms, compatible with passage to G^1-isotypes.

Proof. We write $S = (\gamma_1) \cup S_0$ with $S_0 = (\gamma_2, \ldots, \gamma_N)$.

Recall from Situation 4.6.1.1 that the notation $(\gamma_2, \ldots, \gamma_N)$ means the set $\{\gamma_2, \ldots, \gamma_N\}$ whose elements are counted with multiplicity. For example, for $N = 2$ and $\gamma_1 = \gamma_2 = \gamma$, $P_{(\gamma,\gamma)} = P_\gamma^2$ (see Example 4.6.1.6).

The following commutative diagram of continuous linear operators:

(4.6.3.2)
$$\begin{array}{ccccc}
P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G)/P_S\mathcal{S}_{\mathrm{E}}(G) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)/P_S\mathcal{S}_{\mathrm{E}}(G) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(G)/P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G) \\
\downarrow & & \downarrow & & \downarrow \\
P_{\gamma_1}\mathcal{S}_I(G)/P_S\mathcal{S}_I(G) & \longrightarrow & \mathcal{S}_I(G)/P_S\mathcal{S}_{\mathrm{E}}(G) & \longrightarrow & \mathcal{S}_I(G)/P_{\gamma_1}\mathcal{S}_I(G) \\
\downarrow & & \downarrow & & \downarrow \\
P_{\gamma_1}\mathcal{S}(G)/P_S\mathcal{S}(G) & \longrightarrow & \mathcal{S}(G)/P_S\mathcal{S}(G) & \longrightarrow & \mathcal{S}(G)/P_{\gamma_1}\mathcal{S}(G)
\end{array}$$

is induced by the continuous linear inclusions $\mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_I(G) \to \mathcal{S}(G)$ and by the linear topological embeddings $P_{\gamma_1}\mathcal{S}_{\mathrm{E}}(G) \subset \mathcal{S}_{\mathrm{E}}(G)$, $P_{\gamma_1}\mathcal{S}_I(G) \subset \mathcal{S}_I(G)$, and $P_{\gamma_1}\mathcal{S}(G) \subset \mathcal{S}(G)$.

By Corollary 4.5.3.1, the rightmost vertical arrows in (4.6.3.2) are linear topological isomorphisms.

The operators P_{γ_1} are continuous and with closed range by Proposition 4.5.1.1(5),(6) and Corollary 4.5.3.1.

Let E be any of the three Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}_{\mathrm{E}}(G)$, $\mathcal{S}_I(G)$ and $\mathcal{S}(G)$.

Let $\phi \in E$ be such that $P_{\gamma_1}(\phi) \in P_S(E)$. Following the same argument provided in Lemma 4.6.1.3 (applied upon relabelling γ_1 as γ_N) shows that we have $\phi \in P_S(E)$, from which we deduce that the leftmost, and hence the middle, vertical arrows are continuous linear bijections, by replacing S with S_0 and inducting on the integer $N \geq 1$, and by another application of Corollary 4.5.3.1 to treat the base case of the induction.

To see this, we observe that the canonical continuous linear operator $E \to \prod_{\chi \in \widehat{G^1}} E(\chi|_{G^1})$, the target being endowed with the product locally convex topology, induced by the operators $\phi \mapsto \int_{G^1} \chi(gx)\phi(x)\,\mathrm{d}\mu_1(g)$ (each continuous and with closed range by an application of Lemma 4.3.3.1) is injective (by Fourier Inversion, since the vanishing of an element of E under $E \to \prod_{\chi \in \widehat{G^1}} E(\chi|_{G^1})$ implies the vanishing of its Fourier transform, and hence $\phi = 0$; see the proof of Proposition 5.2.3.2 for a similar, related argument).

It follows that the continuous surjective linear operator $E/P_{S_0}(E) \to P_{\gamma_1}(E)/P_S(E)$ is also injective, since it is so on G^1-isotypes by the same argument from Lemma 4.6.1.3 applied to each $E(\chi|_{G^1})$.

We deduce that the middle vertical arrows in (4.6.3.2) are continuous linear bijections.

Since, by Theorem D.3.3.7(3), the bornological topology on $\mathcal{S}_{\mathrm{E}}(G)$ and $\mathcal{S}_I(G)$ is strict-LF (see Remark D.3.3.11), and since $\mathcal{S}(G)$, and hence $P_S\mathcal{S}(G)$, is itself a strict LF-space, the inverses of the continuous linear bijections which are the middle vertical arrows in (4.6.3.2) are bounded by Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8], applied to $E/P_S(E)$ re-topologized with its bornological topology, with respect to which $P_S(E)$ remains closed, since $P_S(E)$ is the kernel of the continuous operators Φ_γ, which remain continuous upon refinement of the topology of their source).

Since $\mathcal{S}(G)$ is bornological (by [**Scha**, Ch.II,§8,8.2,Cor.2,p.62]) so are all its quotients by its closed subspaces by [**Scha**, Ch.II,§8,8.2,Cor.1,p.62], and hence so is $\mathcal{S}(G)/P_S\mathcal{S}(G)$. It follows that the inverses to the continuous linear bijections $\mathcal{S}_{\mathrm{E}}(G)/P_S\mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}(G)/P_S\mathcal{S}(G)$ and $\mathcal{S}_I(G)/P_S\mathcal{S}_I(G) \to \mathcal{S}(G)/P_S\mathcal{S}(G)$ from (4.6.3.2), both bounded by the foregoing, are linear topological isomorphisms by [**NB**, Thm.3.2.7(a)].

It follows that the continuous linear bijection $\mathcal{S}_{\mathrm{E}}(G)/P_S\mathcal{S}_{\mathrm{E}}(G) \to \mathcal{S}_I(G)/P_S\mathcal{S}_I(G)$ is also a linear topological isomorphism.

Compatibility with passage to G^1-isotypes on source and target follows from commutation of the continuous left-translation actions on G by $\gamma_k^{\mathbf{Z}}$ for each $1 \leq k \leq N$, and G^1. □

CHAPTER 5

Spaces of adelic θ-functions and Idelic Z-functions

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5.1. The topological vector spaces of adelic θ-functions and Idelic Z-functions

Let F be a number field, S a finite set of places of F containing all the Archimedean places.

In this subsection we define the Hausdorff locally convex topological $\mathbf{C}$-vector space

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$$

of continuous co-invariants of the left-translation action by $F^\times$ on the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}_F)$ of Schwartz-Bruhat functions on $\mathbf{A}_F$. After defining it, $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ will be identified with a Hausdorff quotient of the $\mathbf{C}$-linear space of algebraic co-invariants

$$\mathrm{H}_0(F^\times, \mathcal{S}(\mathbf{A}_F)) = \mathrm{Tor}_0^{\mathbf{Z}[F^\times]}(\mathbf{Z}, \mathcal{S}(\mathbf{A}_F)) = \mathbf{Z} \otimes_{\mathbf{Z}[F^\times]} \mathcal{S}(\mathbf{A}_F)$$

of $\mathcal{S}(\mathbf{A}_F)$ under the left-translation action by the discrete subgroup $F^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$, which promotes $\mathcal{S}(\mathbf{A}_F)$ to a (continuous $\mathbf{Z}[F^\times]$-module. The ring map $\mathbf{Z}[F^\times] \to \mathbf{Z}$ is the augmentation map.

These will be spaces of global, respectively S-local, adelic θ-functions, topologized as quotients of $\mathcal{S}(\mathbf{A}_F)$, respectively $\mathcal{S}(\mathbf{A}_{F,S})$.

We will study the topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and prove several structural results in the spirit of Proposition 2.1.4.1 for $\mathcal{S}(\mathbf{A}_F)$, by relating $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ to its S-local analogs $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, in turn a Hausdorff quotient of the $\mathbf{C}$-linear space of algebraic co-invariants $\mathrm{H}_0(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ of the $\mathbf{Z}[\mathcal{O}_{F,S}^\times]$-module $\mathcal{S}(\mathbf{A}_{F,S})$.

The main ingredients in the arguments for the main results of this Chapter are the existence of the analytic continuation to the idelic zeta functions as per Theorem 3.2.2.1, the structural results on $\mathcal{S}(\mathbf{A}_F)$ and their S-local analogs, and the Asymptotic Residue Theorem (see Theorem B.3.2.1), applicable by known estimates on the order of growth of Hecke L-functions and related meromorphic functions uniformly on vertical strips of bounded width, recalled in Appendix B.

At least formally, there is some overlap between some of the results discussed throughout this Chapter, and some of the results from [**Me1**, §4], insofar as some of the statements from [**Me1**, §4.6,4.7] are formally analogous, and at times their proofs share some elements in common (with particular emphasis on the use of multiplier operators from [**Me1**, §4.7] in the proof of [**Me1**, §4.7,Lemma 4.20,p.20]. We will be using similar operators, constructed and discussed at length in §5.3.1 of this Chapter, to a similar effect throughout the proof of Theorem 5.4.1.1, even though the statement of Theorem 5.4.2.2 from §5.4.2, a consequence of Theorem 5.4.1.1, is different, though analogous, to [**Me1**, §4.7,Lemma 4.20,p.20]).

We will be referring the reader to the relevant contents from [**Me1**, §4] whenever appropriate, explaining the similarities and differences. An important difference that we record here from the outset, is that the results being discussed in this Chapter concern the Hausdorff locally convex topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, as well as their corresponding spaces of meromorphic functions on the Riemann surfaces $\Omega(C_F)$, respectively $\Omega(J_{F,S})$, under the Mellin transform, while those results from [**Me**] that find analogs throughout this Chapter, are concerned with the von Neumann bornologies underlying the topological vector spaces which are the subject of our study.

These are referred to as *bornological vector spaces* in [**Me1**] (see also [**Me2**] and [**Me3**]), with the caveat that, therein and in the quoted references, a bornological vector space is meant to be a complex vector space equipped with a bornology of bounded subsets, rather than a Hausdorff locally convex topological vector space which is bornological (see [**NB**, Ch.13,§13.2]). The latter class provides examples of the bornological vector spaces from the former.

We further refer the reader to Remark 5.2.3.4.

5.1.1. Definition of the global and S-local spaces of adelic θ-functions and Idelic Z-functions. For any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_F)$, respectively $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the resulting idelic zeta integral $Z(f,c)$, respectively $Z_S(f,c)$, as defined in (3.2.2.1), respectively in (3.3.1.1), is, recall:

$$Z(f,c) := \int_{\mathrm{GL}_1(\mathbf{A}_F)} f(g)c(g)\,\mathrm{d}\mu(g), \quad Z_S(f,c) := \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\,\mathrm{d}\mu_S(g)$$

for $c \in \Omega(C_F)_{>1}$, respectively $c \in \Omega(J_{F,S})_{>0}$, and Theorem 3.2.2.1, respectively Theorem 3.3.1.1 shows that $Z(f,c)$ and $Z_S(f,c)$ extend to meromorphic functions over $\Omega(C_F)$, respectively $\Omega(J_{F,S})$, whose only poles are located within the subsets $\{c = 1, c = |\cdot|_F\}$, respectively among the poles of the meromorphic function ρ_S over $\Omega(J_{F,S})$ from (3.3.1.3): they are at most simple, in the former case, and they are counted with multiplicity bounded above by the polar multiplicity of each pole of $\rho_S(c)$, in the latter.

The Theorems further provide equivalent ways to write $Z(f,c)$ and $Z_S(f,c)$ in terms of the θ-function operators Φ_F, $\Phi_{F,\omega}$, $\Phi_{F,S}$ and $\Phi_{F,S,\omega}$, where ω is the restriction of c to C_F^1, respectively $J_{F,S}^1$.

Namely, $Z(f,c)$ and $Z_S(f,c)$ are obtained as Mellin transforms over C_F, respectively $J_{F,S}$, of $\Phi_F(f)$, respectively $\Phi_{F,S}(f)$, up to a vertical translation on the complex s-plane uniquely determined by a choice of continuous character $\chi \in \widehat{C_F}$, respectively $\chi \in \widehat{J_{F,S}}$, extending to C_F, respectively $J_{F,S}$, the continuous

character $\omega \in \widehat{C_F^1}$, respectively $\omega \in \widehat{J_{F,S}^1}$, and on a choice of continuous homomorphic section to the idelic, respectively S-idelic, norm.

We now define linear spaces whose elements are adelic θ-functions, both global and S-local, respectively, and their corresponding spaces of meromorphic functions, and topologize them using the Hausdorff locally convex topology of the Schwartz-Bruhat spaces $\mathcal{S}(\mathbf{A}_F)$, respectively $\mathcal{S}(\mathbf{A}_{F,S})$.

DEFINITION 5.1.1.1 (Spaces of adelic θ-functions and Idelic Z-functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places.

(1) Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function. We denote by $Z(f,c)$ the analytic continuation to $\Omega(C_F)$ of the function of $c \in \Omega(C_F)_{>1}$ from (3.2.2.1), and refer to it as the *global idelic Z-function of f*.

We denote by:

$$Z : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F)) \tag{5.1.1.1}$$

the linear operator sending $f \in \mathcal{S}(\mathbf{A}_F)$ to the idelic L-function $Z(f,c)$ as an element of the complex vector space $\mathscr{M}(\Omega(C_F))$ of meromorphic functions on $\Omega(C_F)$ whose poles are contained within the subset $\{1, |\cdot|_F\} \subset \Omega(C_F)$. We refer to the linear operator Z from (5.1.1.1) as the *global Idelic Mellin transform*.

(2) Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function. We denote by $Z_S(f,c)$ the analytic continuation to $\Omega(J_{F,S})$ of the function of $c \in \Omega(J_{F,S})_{>0}$ from (3.3.1.1), and refer to it as the *local S-idelic L-function of f*.

We denote by:

$$Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S})) \tag{5.1.1.2}$$

the linear operator sending $f \in \mathcal{S}(\mathbf{A}_{F,S})$ to the S-idelic Z-function $Z_S(f,c)$ as an element of the complex vector space $\mathscr{M}(\Omega(J_{F,S}))$ of meromorphic functions on $\Omega(J_{F,S})$ whose poles are contained within the subset $\mathcal{U}_S$ of $\Omega(J_{F,S})$ of the poles of $\rho_S(c)$ from (3.3.1.3). We refer to the linear operator Z_S from (5.1.1.2) as the *S-local Idelic Mellin transform*.

(3) We define:

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) := \mathcal{S}(\mathbf{A}_F)/\ker(\Phi_F), \quad \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) := \mathcal{S}(\mathbf{A}_{F,S})/\ker(\Phi_{F,S}) \tag{5.1.1.3}$$

equipped with the quotient topologies.

We denote by

$$Z : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathscr{M}(\Omega(C_F)), \quad Z_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathscr{M}(\Omega(J_{F,S})) \tag{5.1.1.4}$$

the operators induced by Z from (5.1.1.1) by factorization through Φ_F, respectively Z_S from (5.1.1.2) by factorization through $\Phi_{F,S}$. We refer to the operators Z, respectively Z_S, from (5.1.1.4), as the global, respectively S-local, *Idelic Mellin transforms*.

(4) We define:

$$\mathcal{Z}_F := Z(\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))) \subset \mathscr{M}(\Omega(C_F)) \quad \mathcal{Z}_{F,S} := Z_S(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))) \subset \mathscr{M}(\Omega(J_{F,S})) \tag{5.1.1.5}$$

equipped with the linear topologies induced by the Mellin transforms Z, respectively Z_S.

REMARK 5.1.1.2. We note that the quotient topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, are locally convex by [**Scha**, Ch.II,§6,pp.54-55]. They are Hausdorff so long as $\ker(\Phi_F)$, respectively $\ker(\Phi_{F,S})$, are closed subspaces, which we have yet to prove (see Theorem 5.1.1.3 below).

The main result of this section is the following:

THEOREM 5.1.1.3. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. The Mellin transform operators Z on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and Z_S on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ are injective. Moreover:*

(1) *The kernels of Φ_F and $\Phi_{F,S}$ are closed as $\mathbf{C}$-linear subspaces of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}(\mathbf{A}_F)$, respectively $\mathcal{S}(\mathbf{A}_{F,S})$.*

(2) *For every $\omega \in \widehat{C_F^1}$, the kernel of the operator $\Phi_{F,\omega}$ is a closed $\mathbf{C}$-linear subspace of the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}_F)$.*

(3) *For every $\omega \in \widehat{J_{F,S}^1}$, the kernel of the operator $\Phi_{F,S,\omega}$ is a closed $\mathbf{C}$-linear subspace of the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}_{F,S})$.*

The Theorem will enable to study the quotient topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ induced by the operators Φ_F, respectively $\Phi_{F,S}$, and deduce that such topologies are Hausdorff, locally convex, and promote each of them to complete LF-spaces that are nuclear and Montel, whose strong duals are complete, nuclear and Montel.

We will be recalling these concepts from the theory of Hausdorff locally convex topological vector spaces further, as the need arises.

In order to prove Theorem 5.1.1.3, we need preparations on the isotypes of the continuous actions of C_F^1 on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, and $J_{F,S}^1$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, inherited from the continuous left-translation actions of $\mathrm{GL}_1(\mathbf{A}_F)$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S})$, on $\mathcal{S}(\mathbf{A}_F)$, respectively $\mathcal{S}(\mathbf{A}_{F,S})$, by restriction to the respective norm-one closed subgroups.

5.1.2. The kernels of the S-local and global adelic θ-function operators. In this subsection we establish preparations on the kernels of the global and local adelic θ-function operators, to prove Theorem 5.1.1.3, which asserts that their kernel is a closed subspace of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$.

This is the first instance of a structural result on the locally convex topological $\mathbf{C}$-vector spaces $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, whose arguments rest on the idelic (both S-local and global) Mellin transform.

Specifically, we will be deducing that the quotient topologies on the spaces of adelic θ-functions $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ inherited from the S-local and global Schwartz-Bruhat spaces $\mathcal{S}(\mathbf{A}_{F,S})$ and $\mathcal{S}(\mathbf{A}_F)$, both locally convex, are also Hausdorff.

The structural results on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ all initiate from here.

LEMMA 5.1.2.1 (Kernels of global θ-function and Z-function operators). *Let F be a number field, $\omega \in \widehat{C_F^1}$ a continuous character. Let $Z : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F))$ be the Idelic Mellin transform operator from* (5.1.1.1). *Then:*

(1) *We have, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{A}_F)$:*

$$\ker(Z) = \ker(\Phi_F). \tag{5.1.2.1}$$

(2) *We have, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{A}_F)$:*

$$\ker(\Phi_{F,\omega}) = \{f \in \mathcal{S}(\mathbf{A}_F) : Z(f,c)|_{\Omega(C_F)(\omega)} = 0\}. \tag{5.1.2.2}$$

PROOF. (1) : Let $f \in \mathcal{S}(\mathbf{A}_F)$ be a Schwartz-Bruhat function, and assume that its adelic θ-function satisfies $\Phi_F(f) = 0$. Then the function of $c \in \Omega(C_F)_{>1}$ given by:

$$Z(f,c) = \int_{C_F} \Phi_F(f)(g)c(g)\,\mathrm{d}\overline{\mu}(g)$$

vanishes for every $c \in \Omega(C_F)_{>1}$. Since $Z(f,c)$ is meromorphic over $\Omega(C_F)$ by Theorem 3.2.2.1(1), it vanishes identically over $\Omega(C_F)$ away from its poles by the Identity Theorem, and hence it is entire and identically zero.

Conversely, if $Z(f,c) = 0$ for all $c \in \Omega(C_F) - \{1, |\cdot|_F\}$, then for any choice of continuous homomorphic section τ to the idelic norm, the integral

$$Z(f,c) = \int_{\mathbf{R}_{>0}} c(\tau(t))\Phi_{F,\omega}(f)(\tau(t))\,\frac{\mathrm{d}t}{t}$$

vanishes for any $c \in \Omega(C_F)(\omega)_{>1}$ for any $\omega \in \widehat{C_F^1}$.

Since, if $\sigma(c) > 1$, the function $c(\tau(t))\Phi_{F,\omega}(f)(\tau(t))$ is Schwartz on $\mathbf{R}_{>0}$ by Theorem 2.2.5.4, then by Fourier inversion, since we have $Z(f, c|\cdot|_F^{ix}) = 0$ for every $x \in \mathbf{R}$, we deduce that we have $c(\tau(t))\Phi_{F,\omega}(f)(\tau(t)) = 0$ as continuous functions of $t \in \mathbf{R}_{>0}$. This holds for any $\omega \in \widehat{C_F^1}$, and by Lemma 4.3.2.2 we have $c \cdot \Phi_F(f) = 0$ as $\overline{\mu}$-measurable and square $\overline{\mu}$-integrable functions on C_F, $\overline{\mu}$-almost everywhere.

Since c is continuous and nowhere vanishing over C_F, we deduce that, as $\overline{\mu}$-measurable functions over C_F, we have $\Phi_F(f) = 0$, $\overline{\mu}$-almost everywhere. Since $\Phi_F(f)$ is continuous as a function on C_F by Theorem 2.2.3.4, then $\Phi_F(f) = 0$ holds everywhere on C_F.

(2) : If $f \in \ker(\Phi_{F,\omega})$, then $Z(f, c) = 0$ for every $c \in \Omega(C_F)(\omega)_{>1}$, since over this domain of the Riemann surface $\Omega(C_F)$, we have

$$Z(f, c) = \int_{\mathbf{R}_{>0}} c(\tau(t))\Phi_{F,\omega}(f)(\tau(t)) \frac{\mathrm{d}t}{t}$$

and hence $Z(f, c) = 0$ everywhere over $\Omega(C_F)(\omega) - \{1, |\cdot|_F\}$ by the Identity Theorem, and hence everywhere over $\Omega(C_F)(\omega)$.

Conversely, if $Z(f, c) = 0$ for every $c \in \Omega(C_F)(\omega)$, then, since $c|\cdot|_F^{ix} \in \Omega(C_F)(\omega)$ for every $x \in \mathbf{R}$, Fourier Inversion implies that $c(\tau(t))\Phi_{F,\omega}(f)(\tau(t)) = 0$ as Schwartz (by Theorem 2.2.5.4(4)) functions of $t \in \mathbf{R}_{>0}$, and hence $\Phi_{F,\omega}(f) = 0$, since c is smooth and nowhere vanishing over C_F, and hence so is its composition with any continuous 1-parameter subgroup. □

The following is an S-local analog of Lemma 5.1.2.1, characterizing the kernels of $\Phi_{F,S}$ and $\Phi_{F,S,\omega}$ in terms of the kernel of Z_S from (5.1.1.2).

Lemma 5.1.2.2 (Kernels of S-local θ-function and Z-function operators). *Let F be a number field, S a finite set of places containing all Archimedean places, $\omega \in \widehat{J_{F,S}^1}$ a continuous character. Let $Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S}))$ be the S-Idelic Mellin transform operator from* (5.1.1.2)*. Then:*

(1) *We have, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{A}_{F,S})$:*

$$\ker(Z_S) = \ker(\Phi_{F,S}). \tag{5.1.2.3}$$

(2) *We have, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{A}_{F,S})$:*

$$\ker(\Phi_{F,S,\omega}) = \{f \in \mathcal{S}(\mathbf{A}_{F,S}) : Z_S(f, c)|_{\Omega(J_{F,S})(\omega)} = 0\}. \tag{5.1.2.4}$$

Proof. (1) : Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function. If $\Phi_{F,S}(f) = 0$, then $Z_S(f, c) = 0$ for all $c \in \Omega(J_{F,S})_{>0}$, and hence $Z_S(f, c) = 0$ for every $c \in \Omega(J_{F,S})$ by the Identity Theorem.

Assume now $Z_S(f, c) = 0$ for all $c \in \Omega(J_{F,S})$. Over the domain $\Omega(J_{F,S})_{>0}$, the integral:

$$Z_S(f, c) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\, \mathrm{d}\mu_S(g)$$

is absolutely convergent and we may write it, by Theorem D.1.2.1, as:

$$Z_S(f, c) = \int_{\mathbf{R}_{>0}} c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t)) \frac{\mathrm{d}t}{t}$$

for any choice of continuous homomorphic section τ to the S-idelic norm, and for any $c \in \Omega(C_{F,S})(\omega)_{>0}$ for each $\omega \in \widehat{J_{F,S}^1}$.

Since, for any $x \in \mathbf{R}$, we have $c|\cdot|_F^{ix} \in \Omega(J_{F,S})(\omega)_{>0}$ for any $c \in \Omega(C_{F,S})(\omega)_{>0}$, and $c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t))$ is a Schwartz function on $\mathbf{R}_{>0}$ for any $\omega \in \widehat{J_{F,S}^1}$ and $c \in \Omega(J_{F,S})(\omega)_{>1}$ (because

$$(\log(t))^m\, c(\tau(t))\, \Phi_{F,S,\omega}(f)(\tau(t))$$

is bounded as a function of $t \in \mathbf{R}_{>0}$ for every integer $m \geq 0$ by Proposition 2.3.1.4(2) and Corollary 2.4.3.2, as are all of its $(t\partial_t)$-derivatives of any order, by Proposition 2.3.2.2 upon replacing f with $\partial_{\gamma_\tau} f$ using Lemma

2.3.2.1, for γ_τ the continuous 1-parameter subgroup of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ associated to τ), then by Fourier Inversion we have $c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t)) = 0$ as continuous functions of $t \in \mathbf{R}_{>0}$ for every $\omega \in \widehat{J^1_{F,S}}$. By Lemma 4.3.2.2, we deduce that we have $c\,\Phi_{F,S}(f) = 0$, $\overline{\mu}_S$-almost everywhere over $J_{F,S}$. Since c and $\Phi_{F,S}(f)$ are continuous functions on $J_{F,S}$, and since c is nowhere vanishing over $J_{F,S}$, we deduce that we have $\Phi_{F,S}(f) = 0$ everywhere over $J_{F,S}$.

(2) : Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function. If $\Phi_{F,S,\omega}(f) = 0$, then $Z_S(f,c) = 0$ for all $c \in \Omega(J_{F,S})(\omega)_{>0}$, and hence $Z_S(f,c) = 0$ for every $c \in \Omega(J_{F,S})(\omega)$ by the Identity Theorem.

Conversely, assume $Z_S(f,c) = 0$ for all $c \in \Omega(J_{F,S})(\omega)$. Over the domain $\Omega(J_{F,S})(\omega)_{>0}$, the integral:

$$Z_S(f,c) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\,\mathrm{d}\mu_S(g)$$

is absolutely convergent and we may write it, by Theorem D.1.2.1, as:

$$Z_S(f,c) = \int_{\mathbf{R}_{>0}} c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t))\,\frac{\mathrm{d}t}{t}$$

for any choice of continuous homomorphic section τ to the S-idelic norm, and for any $c \in \Omega(C_{F,S})(\omega)_{>0}$.

Since, for any $x \in \mathbf{R}$, we have $c|\cdot|_F^{ix} \in \Omega(J_{F,S})(\omega)_{>0}$ for any $c \in \Omega(C_{F,S})(\omega)_{>0}$, and $c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t))$ is a Schwartz function on $\mathbf{R}_{>0}$ for any $\omega \in \widehat{J^1_{F,S}}$ and $c \in \Omega(J_{F,S})(\omega)_{>0}$, then by Fourier Inversion we have $c(\tau(t))\Phi_{F,S,\omega}(f)(\tau(t)) = 0$ as continuous functions of $t \in \mathbf{R}_{>0}$, and hence, since $c(\tau(t))$ is continuous and nowhere vanishing over $\mathbf{R}_{>0}$, we have $\Phi_{F,S,\omega}(f) = 0$. □

5.1.3. Continuity of the Mellin transform operators. We recall that as a consequence of Theorem 3.2.2.1, the global Idelic Z-function operator $Z : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F))$ from (5.1.1.1), which we also termed global Idelic Mellin transform, is valued in the linear subspace of the $\mathbf{C}$-vector space of meromorphic functions $\mathscr{M}(\Omega(C_F))$ on the Riemann surface $\Omega(C_F)$, of those meromorphic functions on $\Omega(C_F)$ whose divisor is bounded below by the divisor $D := -(1) - (|\cdot|_F)$ on $\Omega(C_F)$.

We call such linear subspace $\mathscr{M}(\Omega(C_F), D)$ and topologize it with the topology inherited from $\mathscr{H}(\Omega(C_F))$, the space of entire functions on $\Omega(C_F)$ topologized with the topology of uniform convergence on the compact subsets of $\Omega(C_F)$, under the linear bijection:

$$\mathscr{M}(\Omega(C_F), D) \to \mathscr{H}(\Omega(C_F))$$

defined by sending each meromorphic function in $\mathscr{M}(\Omega(C_F), D)$ to the entire function obtained by multiplication by the entire function which is constant and equal to 1, on each connected component of $\Omega(C_F)$ other than $\Omega(C_F)(1)$, and, upon parametrizing $\Omega(C_F)(1)$ as $\{|\cdot|_F^s, s \in \mathbf{C}\}$, by $s(s-1)$.

This linear topology promotes $\mathscr{M}(\Omega(C_F), D)$ to a separable Fréchet space, by [**Jar**, Part I,Ch.5,§5.5,p.96] (separability being a consequence of countability of the dual group of C_F^1).

Similarly, as a consequence of Theorem 3.3.1.1, the S-Idelic Z-function operator $Z_S : \mathcal{S}(\mathbf{A}_{F,S})) \to \mathscr{M}(\Omega(J_{F,S}))$ from (5.1.1.2), which we also termed S-Idelic Mellin transform, is valued in the linear subspace of the $\mathbf{C}$-vector space $\mathscr{M}(\Omega(J_{F,S}))$ of meromorphic functions on $\Omega(J_{F,S})$, of those meromorphic functions whose divisor is bounded below by the divisor D_S of the meromorphic function $L_S(c) := \prod_{v\in S} L_v(c_v)$, for $L_v(c_v)$ the local Euler factor at $v \in S$ of the Hecke L-function $L(c)$ of F.

Recall from (3.3.1.32) that $L_S(c)$ is related to the correcting factor $\rho_S(c)$ in the S-local functional equation of (3.3.1.2) from Theorem 3.3.1.1, by the identity of meromorphic functions on $\Omega(J_{F,S})$:

$$\rho_S(c) = \epsilon_S(c)\frac{L_S(c)}{L_S(\widehat{c})}$$

where $\epsilon_S(c)$ is the entire nowhere vanishing function defined in (3.3.1.28) of Lemma 3.3.1.7, and $1/L_S(\widehat{c})$ is entire, with zeros at the image of the poles of $L_S(c)$ under the self-map $\Omega(J_{F,S}) \to \Omega(J_{F,S})$ assigned by $c \mapsto \widehat{c} := c^{-1}|\cdot|_F$.

We topologize $\mathscr{M}(\Omega(J_{F,S}), D_S)$ with the topology induced from the separable Fréchet space $\mathscr{H}(\Omega(J_{F,S}))$ of entire functions on $\Omega(J_{F,S})$, under the linear bijection $\mathscr{M}(\Omega(J_{F,S}), D_S) \to \mathscr{H}(\Omega(J_{F,S}))$ sending an element ϕ of $\mathscr{M}(\Omega(J_{F,S}), D_S)$ to $L_S(c)^{-1}\phi(c)$, an entire function over $\Omega(J_{F,S})$.

The topology on $\mathscr{H}(\Omega(J_{F,S}))$ is the topology of uniform convergence on compact subsets, which promotes $\mathscr{H}(\Omega(J_{F,S}))$ to a separable Fréchet space by [**Jar**, Part I,Ch.5,§5.5,p.96] (separability being a consequence of countability of the dual group of $J^1_{F,S}$).

The topologies described above on the spaces of meromorphic functions $\mathscr{M}(\Omega(C_F), D)$ and $\mathscr{M}(\Omega(J_{F,S}), D_S)$, respectively, agree with the topology termed *Holdgrün topology* in [**GE**, §3.1,p.290].

Indeed, the Hausdorff locally convex topologies defined on $\mathscr{M}(\Omega(C_F), D)$ and $\mathscr{M}(\Omega(J_{F,S}), D_S)$, promote each to a separable Fréchet space as a consequence of the Maximum Modulus Principle (see [**Cnw-b**, Ch.VI,§1,Thm.1.2,p.128]) and [**Cnw-b**, Ch.VII,§2,Thm.2.1,p.151], and the Holdgrün topology also promotes each of the two linear spaces $\mathscr{M}(\Omega(C_F), D)$ and $\mathscr{M}(\Omega(J_{F,S}), D_S)$ to Fréchet spaces (see [**GE**, §2.2,p.290] applied to $\Omega = \Omega(G)(\omega) \cup \{\infty\}$ for $G = C_F$, $G = J_{F,S}$, respectively, and for each continuous character of G^1, upon choosing a holomorphic parametrization $\{\chi|\cdot|_G^s, s \in \mathbf{C}\}$ for each, with $\delta(\infty) = 0$ and $\delta : \Omega - \{\infty\} \to \mathbf{N}$ supported on a countable discrete subset of the complex s-plane; see also [**Kothe-I**, Ch.6,§27,(3),p.373]).

By [**NB**, Thm.14.5.1] (noting that Fréchet spaces are Baire), the identity on $\mathscr{M}(\Omega(C_F), D)$ and $\mathscr{M}(\Omega(J_{F,S}), D_S)$, which is continuous with respect to the Höldgrun topology on the source, and the topology from the foregoing, on the target, is a linear topological isomorphism.

We further refer the reader to [**GE**] for more on Hausdorff locally convex topologies on the space of meromorphic functions on a complex domain, and how the respective induced topologies on the linear subspaces of meromorphic functions with prescribed location and order of poles, relate (and to [**GE**, Appendix,p.301] for an account on the problem of topologizing the linear space of meromorphic functions on a complex domain, with unconstrained divisor of poles).

LEMMA 5.1.3.1 (Meromorphic compact convergence continuity of the Idelic Mellin transforms). *Let F be a number field, S a finite set of places containing all the Archimedean places.*

(1) *The global Idelic Mellin transform operator $Z : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F), D)$ from (5.1.1.1) is continuous with respect to the Hausdorff locally convex topology of $\mathcal{S}(\mathbf{A}_F)$ on the source, and the topology of uniform convergence on the compact subsets of $\Omega(C_F) - \{1, |\cdot|_F\}$ on the target. It satisfies, for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$, the following identity:*

$$Z(\rho_F(g,\cdot), c) = c(g)Z(\cdot, c) \tag{5.1.3.1}$$

as continuous linear operators on $\mathcal{S}(\mathbf{A}_F)$, where $\rho_F : \mathrm{GL}_1(\mathbf{A}_F) \times \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$ is the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$.

(2) *The S-local Idelic Mellin transform operator $Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S}), D_S)$ (5.1.1.2) is continuous with respect to the Hausdorff locally convex topology of $\mathcal{S}(\mathbf{A}_F)$ on the source, and the topology of uniform convergence on the compact subsets of $\Omega(J_{F,S}) - D_S$ on the target, where D_S is the divisor on $\Omega(J_{F,S})$ of the meromorphic function $\rho_S(c)$ on $\Omega(J_{F,S})$ defined in (3.3.1.3). It satisfies, for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, the following identity:*

$$Z_S(\rho_{F,S}(g,\cdot), c) = c(g)Z_S(\cdot, c) \tag{5.1.3.2}$$

as continuous linear operators on $\mathcal{S}(\mathbf{A}_{F,S})$, where $\rho_{F,S} : \mathrm{GL}_1(\mathbf{A}_{F,S}) \times \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S})$ is the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$.

PROOF. Throughout this argument, for G a Hausdorff locally compact abelian group which is chosen to be either $\mathrm{GL}_1(\mathbf{A}_F)$, C_F, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, respectively $J_{F,S}$, we denote, upon writing $|\cdot|_G$ for either the idelic, or S-idelic norm, G^I, for $I \subset \mathbf{R}_{>0}$ a connected interval with non-empty interior, for the subset $G^I := \{g \in G : |g|_G \in I\}$.

We write $G^{>a}$ if $I = (a, +\infty)$ for a real number $a \in \mathbf{R}_{>0}$.

(1) : We choose $\chi \in \widehat{C_F}$ and parametrize $\Omega(C_F)(\chi)$ as $c = \chi|\cdot|_F^s$, $s \in \mathbf{C}$.

If $\chi|_{C_F^1}$ is trivial, then $\chi = |\cdot|_F^{ix_0}$ for a unique real number x_0, and we have

(5.1.3.3)

$$Z(f,c) = \int_{\mathrm{GL}_1(\mathbf{A}_F)^{>1}} f(g)\chi(g)|g|_F^s \,\mathrm{d}\mu(g) + \int_{\mathrm{GL}_1(\mathbf{A}_F)^{>1}} \widehat{f}(g)\overline{\chi}(g)|g|_F^{1-s} \,\mathrm{d}\mu(g) + \delta_{\chi|_{C_F^1}} \overline{\mu}_1(C_F^1) \left\{ \frac{\widehat{f}(0)}{s - ix_0 - 1} - \frac{f(0)}{s - ix_0} \right\}$$

by [**CF**, Ch.XV,Thm.4.2.1] and the proof of [**CF**, Ch.XV,Thm.4.4.1], where $\delta_{\chi|_{C_F^1}} = 0$ if $\chi|_{C_F^1} \neq 1$, and $\delta_{\chi|_{C_F^1}} \neq 0$ if $\chi|_{C_F^1} = 1$.

Formation of $f(0)$ and $\widehat{f}(0)$ is linear and continuous on $\mathcal{S}(\mathbf{A}_F)$, since the former is the tempered distribution δ_0 and the latter is $\widehat{\delta}_0$, its adelic Fourier transform, and the adelic Fourier transform is a linear topological automorphism of $\mathcal{S}(\mathbf{A}_F)$.

By Dominated Convergence, formation of the two integrals in (5.1.3.3) is sequentially continuous on $\mathcal{S}(\mathbf{A}_F)$, uniformly on the compact subsets of $\Omega(C_F) - \{1, |\cdot|_F\}$. To see this, we proceed as follows, using Theorem 2.2.5.4(6).

Let $K \subset \mathbf{C}$ be a compact subset with non-empty interior.

Let $f \in \mathcal{S}(\mathbf{A}_F)$, and $\{f_m\}_{m\geq 0} \subset \mathcal{S}(\mathbf{A}_F)$ a sequence converging to f in $\mathcal{S}(\mathbf{A}_{F,S})$. Then $\mathcal{B} := \{f_m\}_{m\geq 0}$ is a bounded subset of $\mathcal{S}(\mathbf{A}_F)$.

We let $N > 0$ be a fixed integer satisfying $N - 1 > \max\{|\mathrm{Re}(s)|, |1 - \mathrm{Re}(s)|, s \in K\}$.

By Theorem 2.2.3.4(4), there exist real numbers $C_{\mathcal{B},N}, T_{\mathcal{B},N} > 0$ such that for all $g \in C_F$ satisfying $|g|_F > T_{\mathcal{B},N}$, then:

$$|\Phi_F(u)(g)| \leq C_{\mathcal{B},N}|g|_F^{-N} \tag{5.1.3.4}$$

for all $u \in \mathcal{B}$. By Theorem 2.2.3.4(6), $T_{\mathcal{B},N}$ can be chosen to be $T_{\mathcal{B},N} = 1$ at the cost of replacing $C_{\mathcal{B},N}$ with a larger constant, still uniform over $\mathcal{B}$.

For each $u \in \mathcal{S}(\mathbf{A}_F)$, we have, by Theorem D.1.2.1, uniformly with respect to c varying over the vertical strips of bounded width of the connected components of $\Omega(C_F)$:

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)^{>1}} u(g)c(g) \,\mathrm{d}\mu(g) = \int_{C_F^{\geq 1}} \Phi_F(u)(g)c(g) \,\mathrm{d}\overline{\mu}(g). \tag{5.1.3.5}$$

We therefore have, for every integer $m \geq 0$:

$$\int_{\mathrm{GL}_1(\mathbf{A}_F)^{>1}} f_m(g)\chi(g)|g|_F^s \,\mathrm{d}\mu(g) = \int_{C_F^{\geq 1}} \Phi_F(f_m)(g)\chi(g)|g|_F^s \,\mathrm{d}\overline{\mu}(g) \tag{5.1.3.6}$$

where the integral in the right side of (5.1.3.6) converges to $\int_{C_F^{\geq 1}} \Phi_F(f)(g)\chi(g)|g|_F^s \,\mathrm{d}\overline{\mu}(g)$ by Dominated Convergence, applied to the sequence $\mathcal{B}$ using $C_{\mathcal{B},N}|g|_F^{-N}$ as majorant by (5.1.3.4), uniformly over K.

Another application of Theorem D.1.2.1 gives the sequential continuity of the formation of the first of the two integrals in the right side of (5.1.3.3). The same argument, and the same constant $C_{\mathcal{B},N}$, apply to the image of $\mathcal{B}$ under the Fourier transform (which is again a bounded subset of $\mathcal{S}(\mathbf{A}_F)$ since the Fourier transform is a linear topological automorphism of $\mathcal{S}(\mathbf{A}_F)$), proving the sequential continuity of the second of the two integrals from (5.1.3.3).

It follows that the operator $Z(\cdot, c) : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F))$ is sequentially continuous.

Since $\mathcal{S}(\mathbf{A}_F)$ is Hausdorff complete and a locally convex inductive limit of Fréchet spaces by Proposition 2.1.3.2(1), it is bornological by [**Scha**, Ch.II,§8,Cor.1,p.62], and by [**NB**, Thm.13.2.12] it is ultrabornological.

The Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathscr{M}(\Omega(C_F), D)$ is a Fréchet space, topologized in the manner discussed in the prelude to the Lemma.

In particular, it is webbed by [**NB**, Thm.14.6.2]. It follows that by [**NB**, Thm.14.7.3] the operator $Z(\cdot, c)$ is continuous (see also [**Jar**, Part I,Ch.5,§5.4,Thm.2,p.94], noting that Fréchet spaces are Baire, by [**Jar**, Part I,Ch.5,§5.1,Thm.1,p.87]).

The identity in (5.1.3.1) is a consequence of invariance of the Haar measure μ on $\mathrm{GL}_1(\mathbf{A}_F)$ for $c \in \Omega(C_F)_{>1}$, and it holds for all $c \in \Omega(C_F)$ by analytic continuation.

(2) : For every $c \in \Omega(J_{F,S})_{>0}$, the function $Z_S(f, c)$ is given by:

$$Z_S(f,c) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} f(g)c(g)\,\mathrm{d}\mu_S(g) = \int_{J_{F,S}} \Phi_{F,S}(f)(g)c(g)\,\mathrm{d}\overline{\mu}_S(g) \tag{5.1.3.7}$$

where the second identity holds over $\Omega(J_{F,S})_{>0}$ by Theorem D.1.2.1, uniformly as c varies over the compact subsets of $\Omega(J_{F,S})_{>0}$.

By Dominated Convergence, formation of the rightmost integral in (5.1.3.7) is sequentially continuous on $\mathcal{S}(\mathbf{A}_{F,S})$, uniformly on the compact subsets of $\Omega(J_{F,S})_{>0}$.

To see this, we proceed as in (1), upon using Theorem 2.3.1.2(4) in the same role as Theorem 2.2.3.4(6) in the proof of (1).

It follows that the linear operator $Z_S(\cdot, c) : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{H}(\Omega(J_{F,S})_{>0})$ is sequentially continuous, upon endowing the target with the topology of uniform convergence on the compact subsets of $\Omega(J_{F,S})_{>0}$.

Since $\mathcal{S}(\mathbf{A}_{F,S})$ is Hausdorff complete and a locally convex inductive limit of Fréchet spaces by Lemma 2.1.4.1, it is bornological by [**Scha**, Ch.II,§8,Cor.1,p.62], and by [**NB**, Thm.13.2.12] it is ultrabornological.

The complex vector space $\mathscr{H}(\Omega(J_{F,S})_{>0})$ of holomorphic functions over $\Omega(J_{F,S})_{>0}$, equipped with the topology of uniform convergence on the compact subsets of $\Omega(J_{F,S})_{>0}$ is a Fréchet space. In particular, it is webbed by [**NB**, Thm.14.6.2].

It follows that by [**NB**, Thm.14.7.3] the operator $Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{H}(\Omega(J_{F,S})_{>0})$ is continuous.

By the functional equation from (3.3.1.2) of Theorem 3.3.1.1, Z_S is continuous as an operator $Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S}), D_S)$.

The identity in (5.1.3.2) is a consequence of invariance of the Haar measure μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ for $c \in \Omega(J_{F,S})_{>0}$, and it follows for all $c \in \Omega(J_{F,S})$ away from the poles of $L_S(c)$, by analytic continuation. □

We now prove Theorem 5.1.1.3. Recall that the Theorem asserts that the kernels $\ker(\Phi_F)$, $\ker(\Phi_{F,S})$ of the adelic θ-function operators and their homogeneous counterparts $\Phi_{F,\omega}$, $\Phi_{F,S,\omega}$, for continuous characters $\omega \in \widehat{C^1_F}$, respectively $\omega \in \widehat{J^1_{F,S}}$, are closed subspaces of $\mathcal{S}(\mathbf{A}_F)$, respectively $\mathcal{S}(\mathbf{A}_{F,S})$, which implies that the Hausdorff locally convex quotient topologies on the spaces of global, respectively S-local, adelic θ-functions $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ are Hausdorff.

Proof of Theorem 5.1.1.3. (1) : Since, by Lemma 5.1.3.1(1), Z is continuous as a linear operator $\mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F), D)$, its kernel is closed. By Lemma 5.1.2.1, $\ker(\Phi_F) = \ker(Z)$ is closed as a linear subspace of $\mathcal{S}(\mathbf{A}_F)$. Similarly, by Lemma 5.1.3.1(2), Z_S is continuous as a linear operator $\mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S}), D_S)$, and hence its kernel is closed. By Lemma 5.1.2.2, $\ker(\Phi_{F,S}) = \ker(Z_S)$ is closed.

(2) : Let $\omega \in \widehat{C^1_F}$ be a continuous character. Then upon calling $j_\omega : \Omega(C_F)(\omega) \to \Omega(C_F)$ the open immersion, we have, by Lemma 5.1.2.1(2), $\ker(\Phi_{F,\omega}) = \ker(j^*_\omega Z)$, and since the image of j_ω is closed in $\Omega(C_F)$, the linear operator $j^*_\omega Z : \mathcal{S}(\mathbf{A}_F) \to \mathscr{M}(\Omega(C_F), D)(\omega))$ is continuous, and hence its kernel is closed.

(3) : Let $\omega \in \widehat{J^1_{F,S}}$ be a continuous character. Then upon calling $j_\omega : \Omega(J_{F,S})(\omega) \to \Omega(J_{F,S})$ the open immersion, we have, by Lemma 5.1.2.2(2), $\ker(\Phi_{F,S,\omega}) = \ker(j^*_\omega Z_S)$, and since the image of j_ω is closed in $\Omega(J_{F,S})$, the linear operator $j^*_\omega Z_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{M}(\Omega(J_{F,S}), D_S)(\omega))$ is continuous, and hence its kernel is closed. □

[1]

5.2. Structure Theorem I: topology of the spaces of adelic θ-functions

In this section we use the results proved in Lemma 5.1.3.1, Lemma 5.1.2.1 and Lemma 5.1.2.2, to establish the first structural result on the Hausdorff locally convex topology of the spaces of adelic θ-functions

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)), \quad \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})).$$

5.2.1. Topology of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$. We recall from Definition 5.1.1.1, that we defined, for a number field F and for S a finite set of places of F containing all the Archimedean places, the spaces of Adelic (both S-local and global) θ-functions $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ as the quotients:

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) := \mathcal{S}(\mathbf{A}_{F,S})/\ker(\Phi_{F,S}), \quad \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) := \mathcal{S}(\mathbf{A}_F)/\ker(\Phi_F)$$

where $\Phi_{F,S}$ and Φ_F are the θ-function operators from Definition 2.2.3.1 and Definition 2.3.1.1.

We proved in Theorem 5.1.1.3 that their kernels are closed in $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_F)$, and so we know that the quotient topology on the spaces of adelic θ-functions is Hausdorff.

The following is the main result on the topology of the spaces of adelic θ-functions:

THEOREM 5.2.1.1 (Structure Theorem I - topology of the spaces of adelic θ-functions). *Let F be a number field, S a finite set of places of F containing all Archimedean places.*

(1) *The Hausdorff locally convex topological vector spaces*

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)), \quad \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$$

are locally convex strict inductive limits of nuclear Fréchet spaces, hence strict LF-spaces. They are complete, nuclear and Montel. Their strong duals are complete, nuclear and Montel.

(2) *For S' a finite set of places of F containing S, the continuous linear operator $\pi_{S,S'}$ from* (2.1.4.3) *induces a unique continuous linear map:*

$$\overline{\pi}_{S,S'} : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{5.2.1.1}$$

compatible with enlargement of the finite set of places $S \subset S'$, and which is a linear topological embedding. The inductive system $\{\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))\}_{S\subset\Sigma_F}$ is strict, and the inductive limit locally convex topological vector space admits a unique continuous linear map:

$$\overline{\pi} : \operatorname*{colim}_{S\subset\Sigma_F} \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.1.2}$$

which is a linear topological isomorphism, compatible with the continuous actions of $C_{F,S}$ on the source, through its quotient $J_{F,S}$, and as an open subgroup of C_F on the target, for each $S \subset \Sigma_F$ containing S_∞.

(3) *Upon calling, for each finite set S of places of F containing S_∞:*

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.1.3}$$

the canonical continuous insertion operator induced by the linear topological isomorphism from (5.2.1.2)*, $\overline{\pi}_S$ is a linear topological embedding, compatible with the continuous action of $C_{F,S}$ on the source, through its quotient $J_{F,S}$, and as an open subgroup of C_F on the target.*

We will be proving the Theorem only at the end of this Chapter. This subsection is devoted to stating the Theorem and its analog for the isotypes of the continuous left-translation actions of $J_{F,S}^1$ and C_F^1 on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ respectively, for then establishing some initial properties and discussing a few reductions.

[1] Include version with I-exponentially decaying target.

2

PROOF. (1) : By Theorem 5.1.1.3, the $\mathbf{C}$-linear spaces $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, endowed with the quotient topology, are Hausdorff and locally convex, since the kernel of the surjective linear operators $\Phi_F : \mathcal{S}(\mathbf{A}_F) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\Phi_{F,S} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ are closed.

Since $\mathcal{S}(\mathbf{A}_F)$ is Hausdorff complete and a strict locally convex inductive limit of a sequence of nuclear Fréchet spaces (see Proposition 2.1.3.2), and the quotient of a Fréchet space by a closed subspace is a Fréchet space by [**Scha**, Ch.I,6.3,p.31], $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ is the locally convex inductive limit of a sequence nuclear Fréchet spaces. By [**Scha**, Ch.II,6.6], it is complete. By the same token, so is $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ (see Proposition 2.1.4.1).

By [**Scha**, Ch.III,Thm.7.4,p.103], they inherit nuclearity. By [**Scha**, Ch.II,§8,Cor.1,p.61] they are barrelled, and since they are complete and nuclear, they are both Montel spaces by [**Scha**, Ch.III,§7,Cor.2,p.101].

Since $\mathcal{S}(\mathbf{A}_F)$ is Hausdorff complete and a locally convex inductive limit of nuclear Fréchet spaces by Proposition 2.1.3.2(1), it is bornological by [**Scha**, Ch.II,§8,Cor.1,p.62].

Similarly, since $\mathcal{S}(\mathbf{A}_{F,S})$ is Hausdorff complete and a locally convex inductive limit of nuclear Fréchet spaces by Proposition 2.1.4.1, it is bornological by [**Scha**, Ch.II,§8,Cor.1,p.62]. By [**Scha**, Ch.IV,§6,6.1], their strong duals are Hausdorff and complete.

By [**Scha**, Ch.IV,5.9,p.147], they are Montel (see also [**MV**, Part IV,Ch.24,Thm.24.25,p.286]). Since $\Phi_F : \mathcal{S}(\mathbf{A}_F) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ is a quotient map, the adjoint linear map:

$$\Phi'_F : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta \to \mathcal{S}(\mathbf{A}_F)'_\beta \tag{5.2.1.4}$$

is continuous by [**NB**, Thm.8.11.3(d)], injective by [**NB**, Thm.8.10.6(e)], and a linear topological embedding by [**NB**, Thm.8.13.5] applied to $A = \Phi'_F$, combined with [**NB**, Thm.8.13.4] applied to $A = \Phi_F$, which is open, being a quotient map, and with [**NB**, Thm.8.13.2] applied to $A = \Phi_F$, which implies that Φ'_F is weakly relatively open.

Since $\mathcal{S}(\mathbf{A}_F)'_\beta$ is nuclear by Proposition 2.1.3.2(2), so is $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta$ by [**Scha**, Ch.III,§7,Thm.7.4,p.103]. A similar argument shows that $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ is nuclear.

(2) and (3) follow from Proposition 2.1.4.1: indeed, upon writing $\mathcal{S}(\mathbf{A}_F)$ as $\mathrm{colim}_{S\subset\Sigma_F}\mathcal{S}(\mathbf{A}_{F,S})$ the index set being the set of the finite subsets $S \subset \Sigma_F$ containing S_∞, the transition operators being the $\pi_{S,S'}$ from (2.1.4.12), the inductive limit is strict by Lemma 2.1.4.3 and the pre-image of the closed subspace $\ker(\Phi_F)$ under π_S is closed in $\mathcal{S}(\mathbf{A}_{F,S})$.

Such closed subspace agrees with $\ker(\Phi_{F,S})$. To see this, it is enough to show that it agrees with $\ker(Z_S)$ by Lemma 5.1.2.2. Since by Theorem 3.4.1.1 we have $Z(\pi_S(f_S), c) = Z_S(f_S, \psi_{F,S}(c))L_{\Sigma_F - S}(\chi, s)$ for $c = \chi|\cdot|_F^s$ unramified outside of S, then by Lemma 3.1.2.2(1), which applies since c is in the open subset $\Omega_S(C_F)(\chi)$, then $f_S \in \ker(\Phi_{F,S})$ if and only if $\pi_S(f_S) \in \ker(\Phi_F)$, which proves that we have $\pi_S^{-1}(\ker(\Phi_F)) = \ker(\Phi_{F,S})$. This identity holds as subspaces of $\mathcal{S}(\mathbf{A}_{F,S})$.

The induced operator $\overline{\pi}_{S,S'} : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'}))$ for any inclusion $S \subset S'$ is therefore injective, linear and continuous by functoriality of the quotient topology, and it induces on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ the subspace topology. To see this, we observe that we have $\overline{\Pi}_S(\ker(\Phi_F)) = \ker(\Phi_F) \cap \pi_S(\mathcal{S}(\mathbf{A}_{F,S})) = \overline{\pi}_S(\ker(\Phi_{F,S})) \subset \ker(\Phi_F)$, and so $\overline{\Pi}_S$ descends to a unique continuous linear operator $\overline{\Pi}_S : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ whose range is the image in $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ of $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$, i.e. $\overline{\pi}_S(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})))$, which is the identity on $\overline{\pi}_S(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})))$, and satisfies $\overline{\Pi}_S^2 = \overline{\Pi}_S$ on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. This proves (3), and it shows that $\overline{\pi}_S(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})))$ is a closed subspace of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. As such, $\overline{\pi}_S$ is a continuous linear bijection onto its image, with closed range, and hence a linear topological embedding. Since it factors through $\overline{\pi}_{S,S'}$ and $\overline{\pi}_{S'}$, this latter being a linear

[2]WORK 1: adjust this Theorem.

topological embedding by the same token, then the range of $\overline{\pi}_{S,S'}$, which is $\overline{\pi}_{S'}^{-1}(\overline{\Pi}_{S'}(\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))))$, is closed in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times,\mathcal{S}(\mathbf{A}_{F,S'}))$, and $\overline{\pi}_{S,S'}$ is a continuous linear bijection onto its image, with closed range, hence a linear topological embedding. This proves (2).

The same argument as in Corollary 2.1.4.4 shows that the net $\{\overline{\Pi}_S\}_{S\subset\Sigma_F}$ indexed by the set of finite subsets of Σ_F containing S_∞, filtered by inclusion, converges uniformly on the bounded subsets of $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ to the identity. □

5.2.2. Isotypes of $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$. [3]

From the discussion in §3.1.1 we have

$$\Omega(C_F)=\coprod_{\omega\in\widehat{C_F^1}}\Omega(C_F)(\omega),\quad \Omega(J_{F,S})=\coprod_{\omega\in\widehat{J_{F,S}^1}}\Omega(J_{F,S})(\omega) \tag{5.2.2.1}$$

where $\Omega(C_F)(\omega)$, respectively $\Omega(J_{F,S})(\omega)$, are defined in §3.1.1 and are bi-holomorphic to the complex plane upon choosing an idelic, respectively S-idelic quasi-character c_0 such that $c_0|_{C_F^1}=\omega$, respectively $c_0|_{J_{F,S}^1}=\omega$, to write any point in $\Omega(C_F)(\omega)$, respectively $\Omega(J_{F,S})(\omega)$, as $c=c_0|\cdot|_F^s$, respectively $c=c_0|\cdot|_{F,S}^s$, for a unique complex number s.

The decomposition of $\Omega(C_F)$ and $\Omega(J_{F,S})$ from (5.2.2.1) into connected components induces a $\mathbf{C}$-linear product decomposition:

$$\mathscr{M}(\Omega(C_F))=\prod_{\omega\in\widehat{C_F^1}}\mathscr{M}(\Omega(C_F)(\omega)),\quad \mathscr{M}(\Omega(J_{F,S}))=\prod_{\omega\in\widehat{J_{F,S}^1}}\mathscr{M}(\Omega(J_{F,S})(\omega)). \tag{5.2.2.2}$$

Recall that we have defined (see (5.1.1.1) from Definition 5.1.1.4) $\mathscr{M}(\Omega(C_F))$ to be the $\mathbf{C}$-linear space of all meromorphic functions on $\Omega(C_F)$ whose only poles are at most of order 1 and are contained within the subset $\{1,|\cdot|_F\}$, and $\mathscr{M}(\Omega(J_{F,S}))$ to be the $\mathbf{C}$-linear space of all meromorphic functions whose only poles $c\in\Omega(J_{F,S})$ are contained in the set $\mathfrak{U}_S$ of the poles of the meromorphic function $\rho_S(c)$ from (3.3.1.3), and such that the order of pole of c is at most $\mathrm{ord}_{c'=c}\,\rho_S(c')$.

We have endowed in Lemma 5.1.3.1 the linear spaces $\mathscr{M}(\Omega(C_F))$ and $\mathscr{M}(\Omega(J_{F,S}))$ with the topology of uniform convergence on compact subsets of $\Omega(C_F)-\{1,|\cdot|_F\}$ and $\Omega(J_{F,S})-\mathfrak{U}_S$.

We will retain this choice of topology on $\mathscr{M}(\Omega(C_F))$ and $\mathscr{M}(\Omega(J_{F,S}))$ throughout this subsection.

Definition 5.2.2.1. Let F be a number field, S a finite set of places containing all the Archimedean places.

We define:

(1) For each $\omega\in\widehat{C_F^1}$, the complex vector space of continuous co-invariants of the continuous representation ω of C_F^1 on $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$:

$$\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F)):=\mathcal{S}(\mathbf{A}_F)/\ker(\Phi_{F,\omega}) \tag{5.2.2.3}$$

(2) For each $\omega\in\widehat{J_{F,S}^1}$, the complex vector space of continuous co-invariants of the continuous representation ω of $J_{F,S}^1$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$:

$$\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S})):=\mathcal{S}(\mathbf{A}_{F,S})/\ker(\Phi_{F,S,\omega}). \tag{5.2.2.4}$$

We note that the continuous co-invariants $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F)))$ of the continuous representation ω of C_F^1, respectively $C_{F,S}^1$, on $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S})))$, are obtained as the quotient of $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$, by the closure of the $\mathbf{C}$-linear subspace $\mathbf{C}$-linearly generated by the elements $\{\phi(x)-\overline{\omega}(g)\phi(g^{-1}x),g\in C_F^1,\phi\in\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))\}$, respectively $\{\phi(x)-\overline{\omega}(g)\phi(g^{-1}x),g\in C_{F,S}^1,\phi\in\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))\}$.

[3] WORK 2: adjust this introduction.

This is a consequence of Lemma 4.3.3.1. Such description of the kernels of $\Phi_{F,S}$ and $\Phi_{F,S,\omega}$ and their global analog will be provided in Remark 5.2.3.4 at the end of this section.

5.2.3. First Structure Theorem on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S}))$. [4]

THEOREM 5.2.3.1 (First Structure Theorem - isotypes). *Let F be a number field, S a finite set of places of F containing all Archimedean places. Let $\omega \in \widehat{C_F^1}$ be a continuous character, unramified outside S, and let ω_S be the unique continuous character induced by $\omega|_{C^1_{F,S}}$ to $J^1_{F,S}$ under the quotient homomorphism $C_{F,S} \to J_{F,S}$.*

(1) *For S' a finite set of places of F containing S, the continuous linear operator $\pi_{S,S'}$ from (5.2.1.1) induces a unique continuous linear map:*

$$\overline{\pi}_{S,S'} : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega_{S'}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{5.2.3.1}$$

compatible with enlargement of the finite set of places S', and which is a linear topological embedding, compatible with the continuous actions of $C_{F,S}$ on the source, through its quotient $J_{F,S}$, and on target through the continuous homomorphism $C_{F,S} \to C_{F,S'}$. The inductive system $\{\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'}, \mathcal{S}(\mathbf{A}_{F,S'}))\}_{S\subset S'\subset\Sigma_F}$ is strict, and the inductive limit locally convex topological vector space admits a unique continuous linear map:

$$\overline{\pi} : \operatorname*{colim}_{S\subset\Sigma_F} \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.3.2}$$

which is a linear topological isomorphism.

(2) *The Hausdorff locally convex topological vector spaces $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$, for each finite set of places S' of F containing S, are locally convex inductive limits of nuclear Fréchet spaces, and hence LF-spaces. They are complete, nuclear, Montel spaces. Their strong duals are complete nuclear and Montel.*

(3) *Upon calling, for each S' finite set of places of F containing S, $\overline{\pi}_{S'}$ the canonical continuous insertion operator induced by the linear topological isomorphism from (5.2.3.2), there exists a continuous linear operator*

$$\overline{\Pi}_{S'} : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.3.3}$$

whose image is $\overline{\pi}_{S'}(\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$, which is the identity on $\overline{\pi}_{S'}(\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$, and satisfies $\overline{\Pi}^2_{S'} = \overline{\Pi}_{S'}$. The net $\{\overline{\Pi}_{S'}\}_{S\subset S'\subset\Sigma_F}$ indexed by the finite subsets $S' \subset \Sigma_F$ containing S, converges to the identity on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ in $\mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)))$, i.e. with respect to the topology of uniform convergence on the bounded subsets of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$.

PROOF. (1) : The continuous linear operator $\overline{\pi}_{S,S'}$ from (5.2.1.1) is compatible with the action of $C_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ on the source, which is through the quotient $J_{F,S}$, and $C_{F,S'}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ on the target, where $C_{F,S}$ acts on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ through the continuous injective homomorphism $C_{F,S} \to C_{F,S'}$ induced by $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \prod_{v\in S'-S} \mathcal{O}^\times_{F_v} \to \mathrm{GL}_1(\mathbf{A}_{F,S'})$, which is in turn defined by sending an S-idele to the S'-idele whose S-indexed components are unchanged, and the $(S'-S)$-indexed components are all 1. Since ω_S and $\omega_{S'}$ are each the restriction to $C^1_{F,S}$, respectively $C^1_{F,S'}$, descended to $J^1_{F,S}$, respectively $J^1_{F,S'}$, of a continuous character $\omega \in \widehat{C^1_F}$, then if $f \in \ker(\Phi_{F,S,\omega_S})$, we also have $\pi_{S,S'}(f) \in \ker(\Phi_{F,S',\omega_{S'}})$. Indeed, by Lemma 5.1.2.2, we have $Z_S(f,c) = 0$ for all $c \in \Omega(J_{F,S})(\omega_S)$, and

$$L_{S'}(\pi_{S,S'}(f), c) = Z_S(f,c) \prod_{v\in S'-S} (1 - c(\pi_v))^{-1} \tag{5.2.3.4}$$

for all $c \in \Omega(J_{F,S'})(\omega_{S'})$ (noting that every $c \in \Omega(J_{F,S'})(\omega_{S'})$ is unramified outside S). In the identity in (5.2.3.4), $Z_S(f,c)$ is $Z_S(f,c')$ evaluated at the quasi-character c' obtained by lifting c to $C_{F,S'}$ along the

[4] WORK 3: adjust this Theorem.

continuous quotient homomorphism $C_{F,S'} \to J_{F,S'}$, restricting it to $C_{F,S}$ along the continuous homomorphism $C_{F,S} \to C_{F,S'}$, and, being such restriction unramified outside S, since c is, factoring it through the quotient $J_{F,S}$. Such c' is denoted by c again in (5.2.3.4), as are its v-indexed components in the finite product $\prod_{v\in S'-S}(1-c(\pi_v))^{-1}$.

This forces $L_{S'}(\pi_{S,S'}(f),c)=0$ for all c by the Identity Theorem, and hence $\pi_{S,S'}(f)\in\ker(\Phi_{F,S',\omega_{S'}})$.

The continuous linear operator $\overline{\pi}_{S,S'}$ from (5.2.1.1) therefore descends to a continuous linear operator $\overline{\pi}_{S,S'}$ as in (5.2.3.1).

By (5.2.3.4), we deduce that $\overline{\pi}_{S,S'}$ is injective. Indeed, let $f\in\mathcal{S}(\mathbf{A}_{F,S})$ be such that $\overline{\pi}_{S,S'}(\Phi_{F,S,\omega_S}(f))=0$. Then $L_{S'}(\pi_{S,S'}(f),c)=0$ for all $c\in\Omega(J_{F,S'})(\omega_{S'})$, and by (5.2.3.4) and the Identity Theorem we deduce $Z_S(f,c)=0$ for all $c\in\Omega(J_{F,S})(\omega_S)$, which implies $f\in\ker(\Phi_{F,S,\omega_S})$ by Lemma 5.1.2.2.

Since $\overline{\pi}_{S,S'}$ from (5.2.1.1) is a linear topological embedding, and the linear topologies on $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'},\mathcal{S}(\mathbf{A}_{F,S'}))$ are the quotient topologies induced from $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_{F,S'})$, by Φ_{F,S,ω_S}, respectively $\Phi_{F,S',\omega_{S'}}$, we deduce that $\overline{\pi}_{S,S'}$ from (5.2.3.1) is a linear topological embedding. To see this, we observe that the idempotent $\overline{\Pi}_S$ from (**??**)[5] on $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ with range $\overline{\pi}_S(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S})))$ descends to each space $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ and has range $\overline{\pi}_S(\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S})))$, so long as ω is unramified outside S. It follows that $\overline{\pi}_S$ has closed range and it is a continuous linear bijection onto its image, and it is therefore a linear topological embedding, for every S such that ω is unramified outside S. This implies that $\overline{\pi}_{S,S'}$ also has closed range, given by $\overline{\pi}_{S'}^{-1}(\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F)))$, and being a continuous linear bijection onto its image, it is a linear topological embedding.

We deduce that the inductive system $\{\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'},\mathcal{S}(\mathbf{A}_{F,S}))\}_{S\subset S'\subset\Sigma_F}$ is strict, and the locally convex inductive limit admits a unique continuous linear operator $\overline{\pi}$ to $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$, since each $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times,\mathcal{S}(\mathbf{A}_{F,S'}))$ does compatibly with the continuous transition operators $\pi_{S,S'}$.

$\overline{\pi}$ is a continuous bijection. It is also a linear topological embedding: to see this, we observe that the linear topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ is the quotient topology induced from $\mathcal{S}(\mathbf{A}_F)$ by $\Phi_{F,\omega}$, and $\mathcal{S}(\mathbf{A}_F)$ is linearly topologically isomorphic to the strict locally convex inductive limit of the $\mathcal{S}(\mathbf{A}_{F,S})$ with the $\pi_{S,S'}$ from (2.1.4.2) as continuous transition operators, by Proposition 2.1.4.1. It follows that $\overline{\pi}$, pre-composed with the surjective continuous linear operator

$$\operatorname*{colim}_{S\subset S'\subset\Sigma_F}\mathcal{S}(\mathbf{A}_{F,S'})\to\operatorname*{colim}_{S\subset S'\subset\Sigma_F}\mathrm{H}_{0,\mathrm{cont}}(\omega_{S'},\mathcal{S}(\mathbf{A}_{F,S'}))$$

which is open by [**Gro-f**, Ch.4,§5,Thm.2(1),p.148] (see also [**SN**, §2,Cor.3,p.8]), equals the composition of the linear topological isomorphism π from (2.1.4.4) of Proposition 2.1.4.1, followed by $\Phi_{F,\omega}$, which is a quotient map. In particular, $\overline{\pi}$ is open, and hence a linear topological isomorphism.

(2) : Since $\mathcal{S}(\mathbf{A}_{F,S})$ is a strict locally convex inductive limit of nuclear Fréchet spaces, then its separated quotient $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$ is a locally convex inductive limit of nuclear Fréchet spaces (possibly non-strict). Indeed, by Lemma 5.1.2.2 the kernel of $\Phi_{F,S}$ is $\ker(Z_S)$, and by Lemma 5.1.2.1, $\ker(\Phi_F)$ is $\ker(Z)$. As in the proof of (2) from Theorem 5.2.1.1, $\pi_S^{-1}\ker(\Phi_F)=\ker(\Phi_{F,S})$, and by Lemma 5.1.2.2 and Lemma 5.1.2.1 formation of such kernels and pre-images under π_S is compatible with passage to ω-, respectively ω_S-isotypes.

This proves that the continuous linear operator $\overline{\pi}_S:\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))\to\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ is compatible with ω_S-, respectively ω-isotypes. By Theorem 5.2.1.1, such induced continuous linear operator $\overline{\pi}_S:\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))\to\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ is a linear topological embedding, and to see that $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ is an LF-space, by (1) it is enough to show that so is each $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$. Since $\mathcal{S}(\mathbf{A}_{F,S})$ is, and by Theorem 5.1.1.3(3) $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$ is a separated quotient of an LF-space, $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$ is an LF-space.

The same argument applies to $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$.

[5]Return here.

As such, $\mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S}))$ is Hausdorff, locally convex and complete. It is nuclear (by [**Scha**, Ch.III,Thm.7.4]) and barrelled (by [**Scha**, Ch.II,§8,Cor.2,p.61]). It is Montel by completeness combined with [**Tre**, Part III,Ch.50,Prop.50.2(5.12)], which applies by nuclearity (see also [**Scha**, Ch.III,§7,Cor.2,p.101]).

Being these properties stable under passage to locally convex inductive limits, they are inherited by $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ by (1). By [**Scha**, Ch.IV,§6,6.1,p.148] (which applies since $\mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ are bornological by [**Scha**, Ch.II,§8.8.2,Cor.2,p.62], being separated quotients of $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_F)$) and [**Scha**, Ch.IV,§5,5.9,p.147] (see also [**Jar**, Part II,Ch.11,§11.5,Prop.4(f),p230]), the strong duals $(\mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ and $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)))'_\beta$ are complete and Montel.

The strong duals $(\mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ and $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)))'_\beta$ are nuclear by [**Scha**, Ch.III,§7,Thm.7.4,p.103], because, being Φ_{F,S,ω_S}, respectively $\Phi_{F,\omega}$, are quotient maps, and hence the adjoints Φ'_{F,S,ω_S} and $\Phi'_{F,\omega}$ are linear topological embeddings: to see this, we use [**NB**, Thm.8.11.3(b)] to deduce that Φ_{F,S,ω_S} and $\Phi_{F,\omega}$ are weakly continuous, and being surjective, by [**NB**, Thm.8.13.2] their adjoints are weakly relatively open and injective. By Theorem 5.1.1.3, since $\ker(\Phi_{F,S,\omega_S})$ and $\ker(\Phi_{F,\omega})$ are closed, then by [**NB**, Thm.8.12.4(b)], the assumptions of [**NB**, Thm.8.13.4] are satisfied by $A := \Phi'_{F,S,\omega_S}$, respectively $A := \Phi'_{F,\omega}$, proving that they are both linear topological embeddings.

(3) : Since $\overline{\pi}_{S,S'}$ from (5.2.3.1), and hence the unique continuous linear topological embedding $\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$, are compatible with the local and global θ-function operators Φ_{F,S,ω_S} and $\Phi_{F,\omega}$ and the continuous linear topological embeddings π_S from (2.1.4.5) of Proposition 2.1.4.1, the existence and properties of $\overline{\Pi}_S$ follow from Lemma 2.1.4.3 and the convergence $\overline{\Pi}_S \to \mathrm{id}$ in $\mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)))$ is shown by means of the same argument as in Corollary 2.1.4.4, using closedness of the linear image of $\overline{\pi}_{S,S'}$ from (5.2.3.1) for every S and [**Scha**, Ch.II,§6,6.5,p.59] to deduce that every bounded subset $B \subset \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ is the image under $\overline{\pi}_{S_0}$ of a bounded subset $B_0 \subset \mathrm{H}_{0,\mathrm{cont}}(\omega_{S_0}, \mathcal{S}(\mathbf{A}_{F,S_0}))$ for a large enough S_0. □

6

We denote by

$$\rho_F : C_F \times \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.3.5}$$

and

$$\rho_{F,S} : J_{F,S} \times \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \tag{5.2.3.6}$$

the continuous actions of C_F, respectively $J_{F,S}$, on the Hausdorff locally convex topological vector spaces $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$.

Let ω be a continuous character of C_F^1, respectively $J_{F,S}^1$.

In Proposition 5.2.3.2 below, we will be proving that there exist continuous open linear projectors defined on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ surjective onto the closed subspace:

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))(\omega) := \{\phi \in \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) : \rho_{F,S}(g,\phi) = \omega(g)\phi \text{ for all } g \in J_{F,S}^1\} \tag{5.2.3.7}$$

and similarly on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, surjective onto the closed subspace:

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega) := \{\phi \in \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) : \rho_F(g,\phi) = \omega(g)\phi \text{ for all } g \in C_F^1\}. \tag{5.2.3.8}$$

Such projectors factor through the quotients $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$, therefore identifying them with (5.2.3.7) and (5.2.3.8), respectively.

We refer to the closed subspaces from (5.2.3.7) and (5.2.3.8) as the *ω-isotypes* of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$.

This leads to the following:

[6] WORK 4: Arrive here and stop for today.

PROPOSITION 5.2.3.2. *Let F be a number field, S a finite set of places containing all the Archimedean places. Let ω be a continuous character of C_F^1, respectively of $J_{F,S}^1$.*

(1) *The integrals:*

$$P_{F,\omega} := \int_{C_F^1} \overline{\omega(g)}\rho_F(g,\cdot)\,\mathrm{d}\overline{\mu}_1(g), \quad P_{F,S,\omega} := \int_{J_{F,S}^1} \overline{\omega(g)}\rho_{F,S}(g,\cdot)\,\mathrm{d}\overline{\mu}_{S,1}(g) \tag{5.2.3.9}$$

exist in the sense of Pettis, and define surjective open continuous linear operators:

(5.2.3.10)

$$P_{F,\omega} : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega), \quad P_{F,S,\omega} : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))(\omega)$$

inducing linear topological isomorphisms:

$$P_{F,\omega} : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega) \tag{5.2.3.11}$$

and:

$$P_{F,S,\omega} : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))(\omega). \tag{5.2.3.12}$$

(2) *The induced continuous linear operators:*

(5.2.3.13)

$$P_F : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \prod_{\omega \in \widehat{C_F^1}} \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)), \quad P_{F,S} : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \prod_{\omega \in \widehat{J_{F,S}^1}} \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$$

to the Hausdorff locally convex products, are continuous and injective.

(3) *The operators L and L_S induce continuous injective $\mathbf{C}$-linear operators:*

$$Z : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \to \mathscr{M}(\Omega(C_F)(\omega)), \quad Z_S : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathscr{M}(\Omega(J_{F,S})(\omega)) \tag{5.2.3.14}$$

compatible with the $\mathbf{C}$-linear decompositions of $\mathscr{M}(\Omega(C_F))$ and $\mathscr{M}(\Omega(J_{F,S}))$ from (5.2.2.2).

If ω is unramified outside S, the following diagram of continuous linear operators between Hausdorff locally convex topological vector spaces:

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{\overline{\pi}_S} & \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \\ \Big\downarrow{\scriptstyle P_{F,S,\omega_S}} & & \Big\downarrow{\scriptstyle P_{F,\omega}} \\ \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow[\overline{\pi}_S]{} & \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \end{array} \tag{5.2.3.15}$$

commutes, where ω_S is the continuous character obtained from ω by restriction to $C_{F,S}^1$, descended to the quotient $J_{F,S}^1$.

REMARK 5.2.3.3. We note that the constraint on ω being unramified outside S is necessary to ensure that $\omega|_{C_{F,S}^1}$ descends to a character ω_S of $J_{F,S}^1$ along the quotient homomorphism $C_{F,S}^1 \to J_{F,S}^1$. This happens precisely when ω is unramified outside S.

We note that the products in Lemma 5.2.3.2 are endowed with the product topology, which is Hausdorff locally convex by [**Scha**, Ch.II,p.52].

PROOF. [7] (1) : Upon defining, for $G := C_F^1$, respectively $G := J_{F,S}^1$, both compact Hausdorff abelian groups, and upon choosing μ_G to be $\overline{\mu}_1$, respectively $\overline{\mu}_{S,1}$, each induced by a choice of continuous homomorphic section to the idelic norm, respectively the S-idelic norm, the continuous G-action on $E := \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, respectively $E := \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ defined as $\rho_G := \overline{\omega}(\cdot)\rho_F$, respectively $\rho_G := \overline{\omega}(\cdot)\rho_{F,S}$, Lemma 4.3.3.1 applies, since by Theorem 5.2.1.1(1), $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ are Hausdorff locally

[7] Return here.

convex and complete, hence quasi-complete by [**Scha**, Ch.III,§4,Corollary,p.85], and the continuous G-action defined in the foregoing is uniformly continuous, since the operator L from (5.1.1.1), respectively L_S from (5.1.1.2), combined with Lemma 5.1.2.2 showing that L, respectively L_S, factors through E, exhibits it as being isomorphic to a diagonal multiplication action by the image of $\overline{\omega}$ in $\mathrm{U}(1)$ (see Lemma 5.1.3.1), hence uniformly continuous since such action on any Hausdorff locally convex $\mathbf{C}$-linear space satisfies the uniform continuity condition of Lemma 4.3.3.1 by absolute homogeneity of any seminorm on a $\mathbf{C}$-linear space.

Lemma 4.3.3.1 therefore gives (1), since both $P_{F,\omega}$ and $P_{F,S,\omega}$ are of the form Φ_G.

(2) : The operators P_F and $P_{F,S}$ from (5.2.3.13) are injective because if $\phi \in \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ is such that $P_F(\phi) = 0$, then by Lemma 4.3.2.2 $\phi = 0$ in $L^2(C_F, \overline{\mu})$, and hence $\phi = 0$ in $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ since $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \subset L^2(C_F, \overline{\mu})$ as $\mathbf{C}$-linear spaces, as the elements of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ are continuous functions on C_F, square $\overline{\mu}$-integrable by Proposition 2.2.3.4. The same argument shows that $P_{F,S}$ is injective, upon using Proposition 2.3.1.2 to see that $|\cdot|_{F,S}^{\varepsilon}\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ is a $\mathbf{C}$-linear subspace of $L^2(C_{F,S}, \overline{\mu}_S)$ for every real number $\varepsilon > 0$. Indeed, if $P_{F,S}(\phi) = 0$, then $|\cdot|_{F,S}^{\varepsilon} P_{F,S}(\phi) = 0$ and so $|\cdot|_{F,S}^{\varepsilon}\phi = 0$ as an element of $|\cdot|_{F,S}^{\varepsilon}\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, and hence $\phi = 0$.

We define, for any $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$:

$$Z(\phi, c) := Z(\Phi_{F,\omega}(f), c),\ c \in \Omega(C_F)$$

for some $f \in \mathcal{S}(\mathbf{A}_F)$ such that $\Phi_{F,\omega}(f) = \phi$. This assignment is well-defined by design of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$. Continuity is a consequence of the fact that the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ is the quotient topology of the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, and so continuity of L is a consequence of continuity of L.

Similarly, we define, for any $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$:

$$Z_S(\phi) := Z_S(\Phi_{F,S,\omega}(f))$$

for some $f \in \mathcal{S}(\mathbf{A}_{F,S})$ such that $\Phi_{F,S,\omega}(f) = \phi$, and L_S is continuous because L_S is.

Commutativity of the diagram in (5.2.3.15) is a consequence of the identities $\Phi_{F,\omega} = P_{F,\overline{\omega}} \circ \Phi_F$ and $\Phi_{F,S,\omega_S} = P_{F,S,\overline{\omega_S}} \circ \Phi_{F,S}$, and compatibility of Φ_F, $\Phi_{F,S}$ with the continuous linear operators π_S on $\mathcal{S}(\mathbf{A}_{F,S})$ and $\overline{\pi}_S$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$. □

Remark 5.2.3.4. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

(1) Let $V_{F,S}$ be the $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_{F,S})$ generated by the elements $f_\xi(x) := f(x) - f(\xi^{-1}x)$ as functions of $x \in \mathbf{A}_{F,S}$, as $f \in \mathcal{S}(\mathbf{A}_{F,S})$ and $\xi \in \mathcal{O}_{F,S}^\times$. The kernel of $\Phi_{F,S}$ is the closure of V_S with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S})$ induced by $\Phi_{F,S}$.

(2) Let V_F be the $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_F)$ generated by the elements $f_\xi(x) := f(x) - f(\xi^{-1}x)$ as functions of $x \in \mathbf{A}_F$, as $f \in \mathcal{S}(\mathbf{A}_F)$ and $\xi \in F^\times$. The kernel of Φ_F is the closure of V_F with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_F)$ induced by Φ_F.

Both assertions are a consequence of the fact that upon endowing $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ with the quotient topology inherited from $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_F)$, then the kernel of $\Phi_{F,S}$ and the kernel of Φ_F are the closure of 0 in $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_F)$, with respect to the topology induced by $\Phi_{F,S}$, respectively Φ_F. This is a consequence of the definition of quotient topology. To see (1) and (2) it is therefore sufficient to observe that we have $V_{F,S} \subset \ker(\Phi_{F,S})$, and $V_F \subset \ker(\Phi_F)$.

Similar assertions hold for $\ker(\Phi_{F,S,\omega})$, respectively $\ker(\Phi_{F,\omega})$, for $\omega \in \widehat{J_{F,S}^1}$, respectively $\omega \in \widehat{C_F^1}$.

This justifies to some extent the use of the notation $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, for the quotients of $\mathcal{S}(\mathbf{A}_{F,S})$ and $\mathcal{S}(\mathbf{A}_F)$ by $\ker(\Phi_{F,S})$ and $\ker(\Phi_F)$.

A full justification for the use of such notation is indeed available: we note that both Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ are bornological. Upon

forgetting their topologies and retaining only their von Neumann bornology, i.e. the bornology of all bounded subsets, we obtain the two underlying "bornological spaces" in the sense of [Me3]. As such, each is indeed the degree-zero homology of the bornologically smooth (in the sense of [Me3, Prop.4.7,p.22]) $\mathbf{C}[\mathcal{O}_{F,S}^\times]$-module $\mathcal{S}(\mathbf{A}_{F,S})$, respectively of the bornologically smooth (in the sense of [Me3, Prop.4.7,p.22]) $\mathbf{C}[F^\times]$-module $\mathcal{S}(\mathbf{A}_F)$, in the usual sense of derived functor homology and cohomology.

Both $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ are then seen to be smooth $\mathcal{D}(J_{F,S})$-, respectively smooth $\mathcal{D}(C_F)$-modules, where $\mathcal{D}(J_{F,S})$ and $\mathcal{D}(C_F)$ are the convolution algebras of test functions on $J_{F,S}$ and C_F (in the sense of [Bru, §4], seen as bornological algebras as in [Me3, §4]). We will only need the structure of continuous modules over such algebras, and the smoothness (in the sense of linear topological representations) will play an indirect role through the existence and continuity of certain infinitesimal generators. It will be our preference to retain the topologies throughout, to access robust and well established literature on nuclear operators.

The kernels of $\Phi_{F,S}$ and Φ_F turn out to be the bornological closure of the linear subspaces $V_{F,S}$ and V_F in $\mathcal{S}(\mathbf{A}_{F,S})$, respectively $\mathcal{S}(\mathbf{A}_F)$. In particular, our $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ recover, upon forgetting their topologies and only retaining their von Neumann bornologies, the bornological spaces $\mathcal{S}(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$ and $\mathcal{S}(\mathbf{A}_F)/F^\times$ of [Me1, §4.5].

We will never be using the fact that both $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, as bornological spaces in the sense of [HN] (see also [Me2]), are indeed degree-zero homologies in the classical sense of homological algebra, although within the context of bornological spaces (see [Me3, Thm.4.15,p.28]).

We will only be using, throughout the remainder of this Chapter, that each is equipped with a linear surjective map from $\mathcal{S}(\mathbf{A}_{F,S}) \otimes_{\mathbf{C}[\mathcal{O}_{F,S}^\times]} \mathbf{C} = \mathrm{H}_0(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $\mathcal{S}(\mathbf{A}_F) \otimes_{\mathbf{C}[F^\times]} \mathbf{C} = \mathrm{H}_0(F^\times, \mathcal{S}(\mathbf{A}_F))$. Indeed, we have:

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes_{\mathbf{C}[\mathcal{O}_{F,S}^\times]} \mathbf{C} = \mathcal{S}(\mathbf{A}_{F,S})/V_{F,S}, \quad \mathcal{S}(\mathbf{A}_F) \otimes_{\mathbf{C}[F^\times]} \mathbf{C} = \mathcal{S}(\mathbf{A}_F)/V_F$$

and, as discussed in the foregoing, $V_{F,S} \subset \ker(\Phi_{F,S})$ and $V_F \subset \ker(\Phi_F)$. This will facilitate calculations.

Remark 5.2.3.5. We recall from Proposition 5.2.3.2 that $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$, for $\omega \in \widehat{C_F^1}$, is identified with the ω-isotype:

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega)$$

under the continuous open linear operator $P_{F,\omega}$ on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, which factors through $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ and surjects onto $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega)$.

The restriction of L to the closed subspace $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega)$ is, however, not the operator that L induces on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$, but it is so only after multiplication by the scalar $\overline{\mu}_1(C_F^1)$.

A similar observation holds for the ω-isotypes in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ for any $\omega \in \widehat{J_{F,S}^1}$.

5.2.4. Interaction with the Fourier Transform. Let F be a number field, S a finite set of places of F containing the Archimedean places.

The S-adelic Fourier transform is a linear topological automorphism on $\mathcal{S}(\mathbf{A}_{F,S})$. We denote it $\mathcal{F}_S : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S})$.

In this subsection we prove that it descends to a linear topological automorphism of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, which, as S tends to Σ_F, tends to the inversion automorphism on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$.

Theorem 5.2.4.1 (Equivariant Fourier Transform). *Let F be a number field, S a finite set of places of F containing the Archimedean places.*

(1) *There exists a unique linear topological automorphism:*

$$\mathcal{F}_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \tag{5.2.4.1}$$

satisfying $\mathcal{F}_S^2 = \mathrm{id}$*, making the following diagram of continuous linear operators between Hausdorff locally convex topological vector spaces:*

$$\begin{array}{ccc} \mathcal{S}(\mathbf{A}_{F,S}) & \xrightarrow{\quad\mathcal{F}_S\quad} & \mathcal{S}(\mathbf{A}_{F,S}) \\ {\scriptstyle\Phi_{F,S}}\downarrow & & \downarrow{\scriptstyle\Phi_{F,S}} \\ \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow[\mathcal{F}_S]{} & \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})) \end{array} \tag{5.2.4.2}$$

commute, compatibly with enlargement of S *and passage to* ω_S*-isotypes (exchanging the* ω_S *and the* $\overline{\omega}_S$*-isotypes), for any continuous character* $\omega \in \widehat{C_F^1}$*.*

(2) *There exists a unique linear topological automorphism:*

$$\mathcal{F} : \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)) \tag{5.2.4.3}$$

satisfying $\mathcal{F}^2 = \mathrm{id}$*, making the following diagram of continuous linear operators between Hausdorff locally convex topological vector spaces:*

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{\overline{\pi}_S} & \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)) \\ {\scriptstyle\mathcal{F}_S}\downarrow & & \downarrow{\scriptstyle\mathcal{F}} \\ \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow[\overline{\pi}_S]{} & \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)) \end{array} \tag{5.2.4.4}$$

commute, compatibly with enlargement of S *and passage to* ω*- and* ω_S*-isotypes, for any continuous character* $\omega \in \widehat{C_F^1}$*.*

Proof. (1) : By Lemma 5.1.2.2, we have $\ker(\Phi_{F,S}) = \ker(Z_S)$. Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be such that $\Phi_{F,S}(f) = 0$. We have, for $c \in \Omega(J_{F,S})_{>0}$:

$$\int_{C_{F,S}} \Phi_{F,S}(\widehat{f})(g)c(g)\,\mathrm{d}\overline{\mu}_S(g) = \rho_S(c)Z_S(f, \widehat{c})$$

by (3.3.1.2) from Theorem 3.3.1.1.

Since $f \in \ker(\Phi_{F,S})$, we have $Z_S(f, \widehat{c}) = 0$ for all $\widehat{c} \in \Omega(J_{F,S})$, and hence $Z_S(\widehat{f}, c) = 0$ over $\Omega(J_{F,S})_{>0}$, and so $Z_S(\widehat{f}, c) = 0$ everywhere over $\Omega(J_{F,S})$ by the Identity Theorem. We deduce that we have $\mathcal{F}_S(f) = \widehat{f} \in \ker(\Phi_{F,S})$, and so $\mathcal{F}_S$ induces a unique continuous linear operator $\mathcal{F}_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ making the diagram in (5.2.4.2) commute. The same argument applies to $\mathcal{F}_S^{-1}$. Since $-1 \in \mathcal{O}_{F,S}^{\times}$, the induced continuous linear operator $\mathcal{F}_S^{-1}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ satisfies $\mathcal{F}_S^2 = \mathrm{id}$. Uniqueness implies compatibility with enlargement of S and passage to isotypes.

(2) : By Lemma 5.1.2.1, we have $\ker(\Phi_F) = \ker(Z)$. Let $f \in \mathcal{S}(\mathbf{A}_F)$ be such that $\Phi_F(f) = 0$. We have, for $c \in \Omega(C_F)_{>1}$:

$$\int_{C_F} \Phi_F(\widehat{f})(g)c(g)\,\mathrm{d}\overline{\mu}(g) = Z(f, \widehat{c})$$

by (3.2.2.2) from Theorem 3.2.2.1.

Since $f \in \ker(\Phi_F)$, we have $Z(f, \widehat{c}) = 0$ for all $\widehat{c} \in \Omega(C_F)$, and hence $Z(\widehat{f}, c) = 0$ over $\Omega(C_F)_{>1}$, and so $Z(\widehat{f}, c) = 0$ everywhere over $\Omega(C_F)$ by the Identity Theorem. We deduce that we have $\mathcal{F}(f) = \widehat{f} \in \ker(\Phi_F)$, and so $\mathcal{F}$ induces a unique continuous linear operator $\mathcal{F} : \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ making the diagram in (5.2.4.4) commute. The same argument applies to $\mathcal{F}^{-1}$. Since $-1 \in F^{\times}$, the induced continuous linear operator $\mathcal{F}^{-1}$ on $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$ satisfies $\mathcal{F}^2 = \mathrm{id}$. Uniqueness implies compatibility with enlargement of S and passage to isotypes. The assertion on $\mathcal{F}_S$ exchanging the ω_S and the $\overline{\omega}_S$-isotypes follows from the functional equation of (3.3.1.2) from Theorem 3.3.1.1. □

5.3. Weil's Principal Value Operators

In this section we discuss continuous operators on $\mathcal{S}(\mathbf{A}_{F,S})$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$, that have the effect of removing the poles of the S-idelic Zeta functions $Z_S(f,c)$ on $\Omega(J_{F,S})$ in a controlled manner, enabling a detailed study of the topology of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ and its isotypes, which will in turn enable the remaining of the Structure Theorems in this Chapter to be stated and proved, and upon which several constructions in sequel Chapters rest.

5.3.1. Non-Archimedean Principal Value Operators. Let F be a number field.

LEMMA 5.3.1.1. *Let F be a number field, v a finite place, π_v a choice of uniformizer at v. The linear operator:*

$$P_v : \mathcal{S}(F_v) \to \mathcal{S}(F_v) \tag{5.3.1.1}$$

defined by $P_v(f) := f - \rho_v(\pi_v, f)$, does not depend on the choice of π_v, and it is a linear topological embedding. Its range is the kernel of the continuous functional $\delta_0 \in \mathcal{S}(F_v)'_\beta$.

PROOF. The range of P_v is closed by Proposition C.2.5.1, since $\mathcal{S}(F_v)$ carries the finest locally convex topology by Proposition 2.1.2.4. It is contained in the kernel of $\delta_0 \in \mathcal{S}(F_v)'_\beta$. Assume $f \in \mathcal{S}(F_v)$ is such that $f(0) = 0$.

Since δ_0 is $\mathcal{O}_{F_v}^{\times}$-invariant, we may write its kernel, up to linear topological isomorphism, as $\bigoplus_{\omega \in \widehat{\mathcal{O}_{F_v}^{\times}}} \ker(\delta_0)(\omega)$, compatibly with the linear and topological decomposition $\mathcal{S}(F_v) \simeq \bigoplus_{\omega \in \widehat{\mathcal{O}_{F_v}^{\times}}} \mathcal{S}(F_v)(\omega)$, and since P_v is compatible with both decompositions and multiplication by ω^{-1} is a bounded invertible linear operator on $\ker(\delta_0)$ permuting the linear topological direct summands, we may assume without loss of generality that f is $\mathcal{O}_{F_v}^{\times}$-invariant.

Since f is compactly supported away from $0 \in F_v$, there exists an integer $N \geq 0$ such that the support of f is contained within $\bigcup_{-N \leq m \leq N} \pi_v^m \mathcal{O}_{F_v}$. The constant value c_m of f over $\pi_v^m \mathcal{O}_{F_v} - \pi_v^{m+1} \mathcal{O}_{F_v}$ is such that $f - c_m \cdot 1_{\pi_v^m \mathcal{O}_{F_v}^{\times}}$ is supported over $\bigcup_{-N \leq m \leq N} \pi_v^m \mathcal{O}_{F_v} - \pi_v^m \mathcal{O}_{F_v}^{\times}$, and we may iteratively write $f = \sum_{-N \leq m \leq N} c_m \cdot 1_{\pi_v^m \mathcal{O}_{F_v}^{\times}}$. We may therefore assume $f = 1_{\pi_v^m \mathcal{O}_{F_v}^{\times}}$ for some $m \in \mathbf{Z}$. Indeed we have, since $\mathcal{O}_{F_v}^{\times} = \mathcal{O}_{F_v} - \pi_v \mathcal{O}_{F_v}$, $f = P_v(1_{\mathcal{O}_{F_v}})$. □

We recall here the definition of the operators:

$$P_v : \mathcal{S}(F_v) \to \mathcal{S}(F_v) \tag{5.3.1.2}$$

for v any finite place of F, defined in (5.3.1.1).

We recall from Theorem 3.4.3.3 that for each non-Archimedean place v of F, and for each $c_v \in \Omega(F_v^{\times})$, there exists a unique local eigendistribution $\Delta'_v \in \mathcal{S}(F_v)'_\beta$ (see Theorem 3.4.3.3(1)) such that:

$$\langle \Delta'_v(c_v), 1_{\mathcal{O}_{F_v}^{\times}} \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)} = 1 \tag{5.3.1.3}$$

if c_v is unramified at v, and

$$\langle \Delta'_v(c_v), c_v^{-1} 1_{\mathcal{O}_{F_v}^{\times}} \rangle_{\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)} = 1 \tag{5.3.1.4}$$

if c is ramified at v.

Note that by Theorem 3.4.3.3(1.c), for $f_v = \overline{\chi} \cdot 1_{\mathcal{O}_{F_v}^{\times}}$ with $\chi \in \widehat{F_v^{\times}}$ a continuous character, we have $Z_v(f_v, c_v) = \mu_v(\mathcal{O}_{F_v}^{\times})$ for $c_v \in \Omega(F_v^{\times})(\chi)$, and $Z_v(f_v, c_v) = 0$ otherwise.

For $\pi_v \in F_v$ a choice of uniformizer at v, P_v is defined by sending $f \in \mathcal{S}(F_v)$ to the Schwartz-Bruhat function of $x \in F_v$ defined by $f(x) - f(\pi_v^{-1} x)$. Upon denoting by $\rho_v : F_v^{\times} \times \mathcal{S}(F_v) \to \mathcal{S}(F_v)$ the continuous left-translation action of $F_v^{\times}$ on $\mathcal{S}(F_v)$, we have $P_v(f) = f - \rho_v(\pi_v, f)$.

Lemma 5.3.1.1 described the range of P_v: it is the kernel of $\delta_0 \in \mathcal{S}(F_v)'_\beta$. The kernel of P_v is trivial: indeed, if $f(x) = f(\pi_v^{-1}x)$ for all $x \in F_v$, then we also have $f(\pi_v^{\mathbf{Z}}x) = f(x)$ for all $x \in F_v$, and since f is compactly supported, say over $K_v \subset F_v$, and $\pi_v^{\mathbf{Z}}K_v \cap K_v$ is finite, we see that f must vanish identically, since for each $x \in K_v$ there exists $m \in \mathbf{Z}$ such that $\pi_v^m x \notin K_v$.

Lemma 5.3.1.1 proves that P_v is relatively open (in fact, a linear topological embedding onto its closed range), thanks to the Hausdorff locally convex topology on $\mathcal{S}(F_v)$ being the finest locally convex topology (see Proposition 2.1.2.4).

We introduce the following:

Definition 5.3.1.2. Let F be a number field, v a finite place of F.

We define:

$$\mathrm{PV}_v : \mathcal{S}(F_v) \to \mathcal{S}(F_v) \tag{5.3.1.5}$$

to be the unique continuous linear operator uniquely determined by the following conditions: for any continuous character $\omega \in \widehat{\mathcal{O}_{F_v}^\times}$:

(1) $\mathrm{PV}_v(\mathcal{S}(F_v)(\omega)) \subset \mathcal{S}(F_v)(\omega)$.
(2) PV_v induces the identity operator on $\mathcal{S}(F_v)(\omega)$ if ω is non-trivial.
(3) $\mathrm{PV}_v = P_v$ on $\mathcal{S}(F_v)(\omega)$ if ω is trivial.

We refer to PV_v as *Weil's Principal Value operator* on $\mathcal{S}(F_v)$.

Lemma 5.3.1.3. *Let F be a number field, v a finite place of F. Then Weil's Principal Value operator $\mathrm{PV}_v : \mathcal{S}(F_v) \to \mathcal{S}(F_v)$ exists and is continuous. It is a linear topological embedding with closed range, compatible with passage to isotypes.*

Proof. Since $\mathcal{O}_{F_v}^\times$ is compact Hausdorff, and $\mathcal{S}(F_v)$ is Hausdorff, locally convex, complete, with $\mathcal{O}_{F_v}^\times$ acting on $\mathcal{S}(F_v)$ uniformly continuously (see Lemma 2.1.2.10) by left-translation, Lemma 4.3.3.1 applies to give that the operator $f \mapsto \Phi_{v,\omega}(f)(x) := \mu_v(\mathcal{O}_{F_v}^\times)^{-1} \int_{\mathcal{O}_{F_v}^\times} \overline{\omega}(g) f(g^{-1}x)\, \mathrm{d}\mu_v(g)$ is continuous, linear, with closed range, and surjective onto $\mathrm{H}^0(\omega, \mathcal{S}(F_v))$. Moreover, $\Phi_{v,\omega}$ is idempotent on $\mathrm{H}^0(\omega, \mathcal{S}(F_v))$, a closed subspace of $\mathcal{S}(F_v)$.

It follows that $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ splits off as a linear topological direct summand, i.e. $\mathcal{S}(F_v) = \mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \oplus W$, where W is the closure of $\mathrm{H}^0(\omega, \mathcal{S}(F_v))$ for all non-trivial continuous characters ω of $\mathcal{O}_{F_v}^\times$.

On W, indeed $\bigoplus_{\omega \neq 1}(\mathrm{Id} : \mathrm{H}^0(\omega, \mathcal{S}(F_v)) \to \mathrm{H}^0(\omega, \mathcal{S}(F_v))) = \mathrm{Id}_W$, the identity, has the same effect as $\mathrm{PV}_v|_W$, if PV_v exists. On $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$, P_v has the same effect as PV_v, if PV_v exists.

$\mathrm{PV}_v := P_v \oplus \mathrm{Id}_W$ extends by continuity to $\mathcal{S}(F_v)$, since the finite direct sum $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \oplus W$ is already complete by [**Scha**, Ch.II,§6,Thm.6.2,p.55].

This definition of PV_v indeed satisfies all the requirements, and any other continuous linear operator on $\mathcal{S}(F_v)$ that does, agrees with PV_v on both $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ and W, and hence agrees with PV_v.

To show that PV_v is a linear topological embedding with closed range, it is enough to show that it is injective and with closed range. This reduces to showing that P_v satisfies the same conditions, as a continuous linear operator on $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$, which is indeed the case by Lemma 5.3.1.1 and closedness of $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ as a P_v-stable linear subspace of $\mathcal{S}(F_v)$. □

Definition 5.3.1.4. Let F be a number field. Let S be a finite set of places of F containing all the Archimedean places. Let $S_0 \subset S$ be a finite subset of finite places.

We define:

(1) *Weil's S-local Principal Value operator*:

$$\mathrm{PV}_{S,S_0} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S}) \tag{5.3.1.6}$$

as the composition of the linear topological isomorphism:

$$\mathcal{S}(\mathbf{A}_{F,S}) \simeq \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f} \mathcal{S}(F_v) \tag{5.3.1.7}$$

provided by (2.1.4.1) from Proposition 2.1.4.1, followed by the tensor product of the identity $\mathrm{Id}_{\mathcal{S}(\mathbf{A}_{F,S_\infty})}$ on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ and the unique continuous linear operator on $\bigotimes_{v\in S_f}\mathcal{S}(F_v)$ obtained as the tensor product of the identity $\mathrm{Id}_{\mathcal{S}(F_v)}$ if $v \in S - S_0$, and of PV_v if $v \in S_0$.

(2) *Weil's Global Principal Value operator*:

$$\mathrm{PV}_{S_0} : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F) \tag{5.3.1.8}$$

as the tensor product of PV_{S,S_0} with the identity on $\mathcal{S}(\widehat{\prod}_{\Sigma_F - S}(F_v, \mathcal{O}_{F_v}))$.

By design of PV_{S_0}, we have that formation of PV_{S_0} and PV_{S,S_0} from (5.3.1.6) and (5.3.1.8) is compatible with the transition operators $\pi_{S,S'}$ from (2.1.4.2) of Proposition 2.1.4.1, and π_S.

We note that since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is nuclear by Proposition 2.1.4.1(3), we have, by Proposition 2.1.4.1(2) and [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], that PV_{S,S_0} and PV_{S_0} are, respectively, linear topological embeddings with closed (as a consequence of Lemma 5.3.1.3 and [**Jar**, Part III,Ch.15,§15.2,Prop.1,p.326]) range, by Lemma 5.3.1.3.

Formation of PV_{S,S_0} commutes with the continuous left-translation action of $\mathcal{O}^\times_{F,S}$ on $\mathcal{S}(\mathbf{A}_{F,S})$, and it is compatible with the operators $\pi_{S,S'}$ from (5.2.1.1) for any finite subset $S' \subset \Sigma_F$ containing S. It follows that PV_{S,S_0} descends to a unique continuous linear operator:

$$\mathrm{PV}_{S,S_0} : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})) \tag{5.3.1.9}$$

and similarly PV_{S_0} on $\mathcal{S}(\mathbf{A}_F)$ descends to a unique continuous linear operator:

$$\mathrm{PV}_{S_0} : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \tag{5.3.1.10}$$

both compatible with formation of $J^1_{F,S}$-, respectively C^1_F-isotypes.

LEMMA 5.3.1.5. *Let F be a number field, S a finite set of places of F containing all the Archimedean places, $S_0 \subset S$ a non-empty finite subset of finite places. The operators PV_{S_0} from* (5.3.1.9) *and* (5.3.1.10) *are linear topological embeddings with closed range, compatible with passage to isotypes, and whose formation is compatible with the transition operators $\overline{\pi}_{S,S'}$ for any finite subset $S' \subset \Sigma_F$ containing S, and $\overline{\pi}_S$, provided by Theorem 5.2.1.1.*

PROOF. To see this, it is enough to observe that it is injective by the Identity Theorem, since $L_S \circ \mathrm{PV}_{S,S_0}$, respectively $L \circ \mathrm{PV}_{S_0}$, is the operator sending $Z(f,c)$ for some $f \in \mathcal{S}(\mathbf{A}_{F,S})$, respectively $f \in \mathcal{S}(\mathbf{A}_F)$, as meromorphic functions on $\Omega(J_{F,S})$, respectively $\Omega(C_F)$, to the products $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})Z_S(f,\chi|\cdot|^s_{F,S})$, respectively $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})Z(f,\chi|\cdot|^s_F)$, where, for $\chi \in \widehat{J_{F,S}}$, respectively $\chi \in \widehat{C_F}$, S_χ is the finite set of places of F contained in S, respectively of F, at which χ is ramified.

PV_{S,S_0}, respectively PV_{S_0}, is therefore a linear topological embedding onto its closed range, since it is injective and its composition with $\Phi_{F,S}$, respectively Φ_F, is the composition of the relatively open continuous linear operators PV_{S,S_0} from (5.3.1.6), respectively PV_{S_0} from (5.3.1.8), followed by $\Phi_{F,S}$, respectively Φ_F. As such, each PV_{S,S_0} from (5.3.1.9) and PV_{S_0} from (5.3.1.10) is a linear topological embedding.

Since the continuous insertion operator $\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ induced by the operators from (5.2.1.1) is, by the First Structure Theorem (see Theorem 5.2.1.1), a linear topological embedding with closed range into $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, and formation of PV_{S,S_0}, PV_{S_0} and $\overline{\pi}_S$ is compatible, we see that the range of PV_{S_0} in $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ is the image under $\overline{\pi}_S$ of the range of PV_{S,S_0} in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, and it is therefore closed.

Compatibility with passage to isotypes is a consequence of Lemma 5.3.1.3. □

5.3.2. Preparations on tempered distributions supported at the origin. We record here results describing the topology on the space of tempered distributions on a finite product of Archimedean local fields, whose support is contained in a union of standard hyperplanes.

We begin with the two cases of interest, $\mathbf{R}, \mathbf{C}$.

We recall here that the bi-dual of a Hausdorff locally convex topological $\mathbf{C}$-vector space is defined to be the continuous dual of the strong dual; see [**Jar**, Part II,Ch.11,§11.2]. This convention, adopted throughout this work, is also in force throughout [**Kothe-I, Kothe-II**] (see [**Kothe-I**, Ch.V,§23,p.298]), as is in [**Scha**, Ch.IV,p.143], from which we shall be referring on occasion throughout this subsection, concerning topologies on the bi-dual.

Recall that the support of a tempered distributions $T \in \mathcal{S}(\mathbf{R}^m)$ for each integer $m \geq 0$, is defined to be the smallest closed subset $\mathrm{supp}(T)$ of $\mathbf{R}^m$ such that for any $x \in \mathbf{R}^m - \mathrm{supp}(T)$, there exists an open neighborhood $U \subset \mathbf{R}^m - \mathrm{supp}(T)$ of x such that for any $f \in \mathcal{S}(\mathbf{R}^m)$ whose support is entirely contained in U, then $\langle T, f\rangle_{\mathcal{S}(\mathbf{R}^m)'_\beta, \mathcal{S}(\mathbf{R}^m)} = 0$.

We refer the reader to [**Horm**, Ch.II,§2.2,Def.2.2.2], which applies by definition to elements of $\mathcal{S}(\mathbf{R}^m)'_\beta$ too, since $\mathcal{S}(\mathbf{R}^m)'_\beta$ is a $\mathbf{C}$-linear subspace of $\mathcal{D}(\mathbf{R}^m)'_\beta$ equipped with a finer topology, and to Definition D.1.8.2, which shows how to generalize this definition to tempered distributions Hausdorff locally compact abelian groups.

THEOREM 5.3.2.1. *Let $T \in \mathcal{S}(\mathbf{R})'_\beta$ be supported at $0 \in \mathbf{R}$. Then there exists an integer $N \geq 0$, complex numbers $c_0, \ldots, c_N$, such that, as elements of $\mathcal{S}(\mathbf{R})'_\beta$, we have:*

$$T = \sum_{0 \leq k \leq N} c_k \delta_0^{(k)}. \tag{5.3.2.1}$$

(1) *Let $\mathcal{S}_{00}(\mathbf{R})$ be the closed subspace defined as $\bigcap_{k\geq 0} \ker(\delta_0^{(k)})$. The orthogonal in $\mathcal{S}(\mathbf{R})'_\beta$ of $\mathcal{S}_{00}(\mathbf{R})$ with respect to the duality $\langle \mathcal{S}(\mathbf{R}), \mathcal{S}(\mathbf{R})'_\beta\rangle$ is a linear subspace of $\mathcal{S}(\mathbf{R})'$ linearly isomorphic to the continuous dual of the quotient $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$, which, topologized with the strong topology on the continuous dual of the quotient, is linearly and topologically isomorphic to the $\mathbf{C}$-linear space underlying the polynomial algebra $\mathbf{C}[X]$ equipped with the finest locally convex topology, equivalently the topology induced by the family of seminorms $\{p_N\}_{N\geq 0}$ with:*

$$p_N(c_0 + c_1 X + \ldots + c_N X^N) := \sup_{0\leq k\leq N} |c_k| \tag{5.3.2.2}$$

the isomorphism being uniquely determined by the assignment $X^N \mapsto \frac{1}{N!}\delta_0^{(N)}$, and matching the degree of a polynomial with the finite order of the resulting distribution supported at $0 \in \mathbf{R}$.

(2) *The quotient $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$, equipped with the quotient topology, is linearly and topologically isomorphic to the $\mathbf{C}$-linear space underlying the formal power series algebra $\mathbf{C}[\![X]\!]$ equipped with the Hausdorff locally convex topology induced by the family of seminoms $\{q_N\}_{N\geq 0}$:*

$$q_N(\sum_{k\geq 0} c_k X^k) := \sup_{0\leq k\leq N} |c_k| \tag{5.3.2.3}$$

the isomorphism being defined by sending each $f \in \mathcal{S}(\mathbf{R})'_\beta$ to the formal power series $\sum_{k\geq 0} \frac{f^{(k)}(0)}{k!} X^k$.

REMARK 5.3.2.2. Note that the strong topology of the quotient topology of $\mathcal{S}_{00}(\mathbf{R})^\perp$ is equivalent to the subspace topology of the strong topology on $\mathcal{S}(\mathbf{R})'$, since $\mathcal{S}(\mathbf{R})$ is quasi-normable and its bounded subsets are pre-compact, by nuclearity (see [**Jar**, Part II,Ch.10,§10.7,Cor.3]).

See Lemma 4.2.1.3. There is, in particular, no ambiguity on how to topologize $\mathcal{S}_{00}(\mathbf{R})^\perp$.

PROOF. By [**Horm**, Ch.VII,Thm.7.1.18,p.168], since, if $T \in \mathcal{D}(\mathbf{R})'_\beta$ is supported at $0 \in \mathbf{R}$, then its restriction (see [**Horm**, Ch.II,§2.2,p.41]) to $\mathbf{R} - \{0\}$ is homogeneous of degree $a \in \mathbf{R}$ (see [**Horm**,

Ch.III,Def.3.2.2,p.74]) for any $a \in \mathbf{R}$, and hence T is tempered and supported at the origin. If $T \in \mathcal{S}(\mathbf{R})'_\beta$ is supported at the origin, then, since the continuous injective linear operator $\mathcal{D}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$ induces a continuous injective (by denseness of $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$; see [**Horm**, Ch.VII,Lemma 7.1.8,p.163]) linear operator $\mathcal{S}(\mathbf{R})'_\beta \to \mathcal{D}(\mathbf{R})'_\beta$, T is also a distribution supported at 0.

The continuous injective linear operator $\mathcal{S}(\mathbf{R})'_\beta \to \mathcal{D}(\mathbf{R})'_\beta$ therefore induces a linear bijection from the space of tempered distributions supported at $0 \in \mathbf{R}$, and the space of distributions supported at $0 \in \mathbf{R}$.

We note that every distribution $T \in \mathcal{D}(\mathbf{R})'_\beta$ supported at $0 \in \mathbf{R}$ is of finite order (see [**Horm**, Ch.II,Example 2.1.2,p.34]). By [**Horm**, Ch.II,Thm.2.3.4,p.46], $T = \sum_{0 \leq k \leq N} c_k \delta_0^{(k)}$, for N the finite order of T.

(1) : Let $T \in \mathcal{S}(\mathbf{R})'_\beta$ be a tempered distribution such that $T(\mathcal{S}_{00}(\mathbf{R})) = 0$, i.e. an element of $\mathcal{S}_{00}(\mathbf{R})^\perp \subset \mathcal{S}(\mathbf{R})'_\beta$. Let $x \in \mathbf{R}$, $x \neq 0$. For any open neighborhood $U \subset \mathbf{R}$ of x such that $0 \notin U$, we let $\phi \in \mathcal{D}(\mathbf{R})$ be supported within U.

Then $\phi \in \mathcal{S}_{00}(\mathbf{R})$, and hence $T(\phi) = 0$. It follows that T is supported at 0, and hence of the form as in (5.3.2.1). Conversely, every tempered distribution of the form as in (5.3.2.1) is an element of $\mathcal{S}_{00}(\mathbf{R})^\perp$.

This shows that indeed the assignment $X^N \mapsto \frac{1}{N!}\delta_0^{(N)}$ is a linear isomorphism $\mathbf{C}[X] \to \mathcal{S}_{00}(\mathbf{R})^\perp$, and the degree N of a polynomial matches the order of the resulting tempered distribution supported at $0 \in \mathbf{R}$.

The linear operator $\mathbf{C}[X] \to \mathcal{S}_{00}(\mathbf{R})^\perp$ is continuous, upon endowing the source with the finest locally convex topology, and the target with the strong topology on the continuous dual on the quotient $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$, noting that by [**NB**, Thm.8.12.1(a)], the continuous dual of $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$ is linearly isomorphic to $\mathcal{S}_{00}(\mathbf{R})^\perp$.

It is therefore a continuous bijection.

The source is a strict LF-space, and the target is the strong dual of the Fréchet space $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$.

We call $F := \mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R})$ the quotient Fréchet space.

Since both $\mathcal{S}(\mathbf{R})$ and $\mathcal{S}_{00}(\mathbf{R})$ are nuclear Fréchet (the former by Proposition 2.1.1.6, the latter by [**Scha**, Ch.III,§7,Thm.7.4,p.103]), so is F (by [**Scha**, Ch.III,§7,Thm.7.4,p.103], since F is a separated quotient of a nuclear Fréchet space), and hence F is Montel (see [**Tre**, Part III,Ch.50,Prop.50.2,(50.12),p.519]), hence reflexive (see [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.11] and the subsequent discussion).

We let $\{U_k\}_{k \geq 0}$ be a countable decreasing sequence of closed absolutely convex 0-neighborhoods, and p_k the Minkowski functional of U_k for each integer $k \geq 0$, such that $\{U_k\}_{k \geq 0}$ is a countable 0-neighborhood base.

We define F_k to be the completion of the quotient $F/p_k^{-1}(0)$ with respect to the normed topology generated by the norm that p_k induces on the quotient $F/p_k^{-1}(0)$. In particular, the range of the continuous linear operator $F \to F_k$ is dense.

By the proof of the "only if" implication of [**Val-a**, §5,Thm.3,p.343], which applies since, being F nuclear and Fréchet, all its quotients by a closed subspace are also nuclear Fréchet, and hence Montel, hence reflexive, we may and do assume that F_k is a reflexive Banach space.

In particular, by [**Jar**, Part II,Ch.11,§11.5,Cor.6(b),p.229], $(F_k)'_\beta$ is also reflexive.

The canonical continuous linear operator $F \to \varprojlim_{k \geq 0} F_k$ is a linear topological isomorphism. The projective limit is reduced in the sense of [**Scha**, Ch.IV,§4,p.139].

The Mackey dual $F'_{\tau(F',F)}$ is therefore linearly and topologically isomorphic to the locally convex inductive limit $\varinjlim_{k \geq 0} (F_k)'_{\tau(F'_k,F_k)}$ where the transition continuous operators are the continuous adjoints of the continuous operators $F_{k+1} \to F_k$ for each integer $k \geq 0$, in turn induced by the containments $U_{k+1} \subset U_k$.

By reflexivity of $(F_k)'_\beta$, $\tau(F'_k, F_k) = \beta(F'_k, F_k)$ by [**Jar**, Part II,Ch.11,§11.4,Prop.1.(2),p.227]. In particular, $(F_k)'_{\tau(F'_k,F_k)}$ is a (reflexive) Banach space, since it is linearly and topologically isomorphic to $(F_k)'_\beta$, and hence, by [**Scha**, Ch.IV,§4,Thm.4.4,p.139], $F'_{\tau(F',F)}$ is linearly and topologically isomorphic to the locally convex inductive limit $\varinjlim_{k \geq 0} (F'_k)_\beta$, a countable locally convex inductive limit of Banach spaces, hence an LF-space.

Since, by [**Jar**, Part II,Ch.11,§11.4,Prop.1(2),p.227], $\beta(F', F) = \tau(F', F)$, being F reflexive, then F'_β is an LF-space.

The continuous operator $\mathbf{C}[X] \to F'_\beta$ is therefore a linear topological isomorphism by [**SN**, §2,Cor.3,p.8].

It follows that the composition of the linear topological isomorphisms $\mathbf{C}[X] \to F'_\beta \simeq \mathcal{S}_{00}(\mathbf{R})^\perp$, which is the linear operator $\mathbf{C}[X] \to \mathcal{S}_{00}(\mathbf{R})^\perp$, is indeed a linear topological isomorphism.

(2) : That the continuous linear operator $\mathcal{S}(\mathbf{R})/\mathcal{S}_{00}(\mathbf{R}) \to \mathbf{C}[\![X]\!]$ is a linear topological isomorphism follows from (1) and [**Tre**, Part II,Ch.22,Thm.22.1(b),p.228] (see also [**Tre**, Part I,Ch.10,Example III,p.91] and [**Tre**, Part I,Ch.13,Example I,p.130]), which gives that its continuous adjoint is a linear topological isomorphism, by [**NB**, Thm.8.13.5] and [**NB**, Thm.14.4.5(b)]. □

THEOREM 5.3.2.3. *Let $m \geq 0$ be an integer. We denote by $x_k : \mathbf{R}^m \to \mathbf{R}$, $1 \leq k \leq m$, the global smooth coordinates on $\mathbf{R}^m$. We denote by H_k the standard hyperplane $x_k = 0$. Let $Z \subset \{1, \ldots, m\}$ be a non-empty subset, equipped with the induced total ordering.*

For $x \in \mathbf{R}^m$, we denote by x_Z the vector in $\mathbf{R}^{|Z|}$ whose coordinates are $(x_k)_{k \in Z}$, and by x_{Z^c} the vector in $\mathbf{R}^{m-|Z|}$ whose coordinates are $(x_k)_{k \notin Z}$, adopting the same notation for the multi-indices α_Z, α_{Z^c} associated to $\alpha \in \mathbf{N}^m$.

Let $T \in \mathcal{D}(\mathbf{R}^m)'_\beta$ be compactly supported within the closed subset $H_Z := \bigcap_{k \in Z} H_k$. Assume T is of finite order $N \geq 0$.

Let $R_{H_Z} : \mathcal{D}(\mathbf{R}^m) \to \mathcal{D}(\mathbf{R}^{m-|Z|})$ be the linear operator assigned by sending f to the smooth function $R_{H_Z}(f)$ obtained by restriction of f to $H_Z \simeq \mathbf{R}^{m-|Z|}$.

Then:

(1) *R_{H_Z} is valued in $\mathcal{D}(\mathbf{R}^{m-|Z|})$ and continuous, and the same assignment defines a continuous linear operator $R_{H_Z} : \mathcal{S}(\mathbf{R}^m) \to \mathcal{S}(\mathbf{R}^{m-|Z|})$.*

(2) *There exist N^m distributions $T_\alpha \in \mathcal{D}(\mathbf{R}^{m-|Z|})'_\beta$, for $\alpha \in \mathbf{N}^m$ such that $|\alpha| \leq N$, compactly supported and of order $N - |\alpha|$, and such that for all $\phi \in \mathcal{D}(\mathbf{R}^m)$, we have:*

$$\langle T, \phi \rangle_{\mathcal{D}(\mathbf{R}^m)'_\beta, \mathcal{D}(\mathbf{R}^m)} = \sum_{\alpha \in \mathbf{N}^m : |\alpha| \leq N} \langle R'_{H_Z}(T_\alpha), \partial_x^\alpha \phi \rangle_{\mathcal{D}(\mathbf{R}^m)'_\beta, \mathcal{D}(\mathbf{R}^m)}. \tag{5.3.2.4}$$

(3) *For any $\phi \in \mathcal{D}(\mathbf{R}^m)$ of the form $\phi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{D}(\mathbf{R}^{|Z|})$ and $\psi_2 \in \mathcal{D}(\mathbf{R}^{m-|Z|})$, the identity in* (5.3.2.4) *reads:*

$$\langle T, \phi \rangle_{\mathcal{D}(\mathbf{R}^m)'_\beta, \mathcal{D}(\mathbf{R}^m)} = \sum_{\alpha \in \mathbf{N}^m : |\alpha| \leq N} (\partial_{x_Z}^{\alpha_Z} \delta_0)(\psi_1) \cdot \langle \partial_{x_{Z^c}}^{\alpha_{Z^c}} T_\alpha, \psi_2 \rangle_{\mathcal{D}(\mathbf{R}^{m-|Z|})'_\beta, \mathcal{D}(\mathbf{R}^{m-|Z|})}. \tag{5.3.2.5}$$

(4) *For $T_\alpha \in \mathcal{D}(\mathbf{R}^{m-|Z|})'_\beta$ the distributions of order $N - |\alpha|$ from* (1)*, we have:*

$$T = \sum_{\alpha \in \mathbf{N}^m : |\alpha| \leq N} (\partial_{x_Z}^{\alpha_Z} \delta_0) \otimes (\partial_{x_{Z^c}}^{\alpha_{Z^c}} T_\alpha). \tag{5.3.2.6}$$

(5) *T is tempered and so are the T_α from* (5.3.2.6) *and their derivatives, and* (5.3.2.6) *holds in $\mathcal{S}(\mathbf{R}^m)'_\beta$.*

PROOF. (1) : The definition of R_{H_Z} makes sense on $C^\infty(\mathbf{R}^m)$. The restriction to H_Z of a smooth function on $\mathbf{R}^m$ is indeed smooth on H_Z, since H_Z is a global smooth complete intersection in $\mathbf{R}^m$ defined as the vanishing locus of the coordinates $x_k : \mathbf{R}^m \to \mathbf{R}$ for $k \in Z$, and formation of the derivatives $\partial_{x_k}^{\alpha_k}$ for $k \in Z$ commutes with R_{H_Z} on each $f \in C^\infty(\mathbf{R}^m)$.

For $\gamma, \gamma' \in \mathbf{N}^{m-|Z|}$, we define $\alpha := (0, \gamma) \in \mathbf{N}^m$ and $\beta := (0, \gamma')$, with 0 denoting the origin in $\mathbf{N}^{|Z|}$. Then we have:

$$\sup_{x \in H_Z} |x_Z^\gamma \partial_{x_Z}^{\gamma'} f(x_Z, 0)| \leq \sup_{x \in \mathbf{R}^m} |x^\alpha \partial_x^\beta f(x)| \tag{5.3.2.7}$$

which implies that $R_{H_Z}(\mathcal{S}(\mathbf{R}^m)) \subset \mathcal{S}(\mathbf{R}^{m-|Z|})$ and is continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{R}^m)$ and $\mathcal{S}(\mathbf{R}^{m-|Z|})$, since this is defined as being generated by the countable family of seminorms from (D.1.10.6).

If $\phi \in \mathcal{D}(\mathbf{R}^m)$ is supported on the compact subset $K \subset \mathbf{R}^m$, then $R_{H_Z}(\phi)$ is smooth and it is supported on the compact subset K_Z obtained as the compact image of K under the continuous projection $\mathbf{R}^m \to H_Z$. In particular, $R_{H_Z}(\mathcal{D}(\mathbf{R}^m)) \subset \mathcal{D}(\mathbf{R}^{m-|Z|})$.

The same estimate in (5.3.2.7) applied to $\gamma = 0$ and to $f = \phi$ supported on K, implies that R_{H_Z} is continuous as an operator $\mathcal{D}(\mathbf{R}^m) \to \mathcal{D}(\mathbf{R}^{m-|Z|})$.

(2) : This is [**Horm**, Ch.II,§2.3,Thm.2.3.5,p.47].

(3) and (4) : If $\phi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{D}(\mathbf{R}^{|Z|})$ and $\psi_2 \in \mathcal{D}(\mathbf{R}^{m-|Z|})$, then we have:

$$\langle R'_{H_Z}(T_\alpha), \partial_x^\alpha \phi\rangle_{\mathcal{D}(\mathbf{R}^m)'_\beta, \mathcal{D}(\mathbf{R}^m)} = \langle T_\alpha, R_{H_Z}(\partial_x^\alpha \phi)\rangle_{\mathcal{D}(\mathbf{R}^m)'_\beta, \mathcal{D}(\mathbf{R}^m)} \tag{5.3.2.8}$$

$$= \langle T_\alpha, \delta_0(\partial_{x_Z}^{\alpha_Z}(\psi_1)) \cdot \partial_{x_{Z^c}}^{\alpha_{Z^c}} \psi_2\rangle_{\mathcal{D}(\mathbf{R}^{m-|Z|})'_\beta, \mathcal{D}(\mathbf{R}^{m-|Z|})} \tag{5.3.2.9}$$

$$= \delta_0(\partial_{x_Z}^{\alpha_Z}(\psi_1)) \cdot \langle T_\alpha, \partial_{x_{Z^c}}^{\alpha_{Z^c}} \psi_2\rangle_{\mathcal{D}(\mathbf{R}^{m-|Z|})'_\beta, \mathcal{D}(\mathbf{R}^{m-|Z|})} \tag{5.3.2.10}$$

and hence, upon combining (5.3.2.8) with (5.3.2.4), we deduce that both (5.3.2.5) and (5.3.2.6) hold, since $\mathcal{D}(\mathbf{R}^{|Z|}) \otimes \mathcal{D}(\mathbf{R}^{m-|Z|}) \subset \mathcal{D}(\mathbf{R}^m)$ is sequentially dense by [**Tre**, Part III,Ch.39,Thm.39.2,p.406], hence dense.

(5) : Compactly supported distributions are tempered, and the continuous injective operator $\mathcal{D}(\mathbf{R}^m) \to \mathcal{S}(\mathbf{R}^m)$ is compatible with the dualities $\langle \mathcal{D}(\mathbf{R}^m), \mathcal{D}(\mathbf{R}^m)'_\beta\rangle$ and $\langle \mathcal{S}(\mathbf{R}^m), \mathcal{S}(\mathbf{R}^m)'_\beta\rangle$ (i.e. its adjoint R_{H_Z} is strongly continuous, by [**NB**, Thm.8.11.3(d)]. It follows that (5.3.2.6) holds in $\mathcal{S}(\mathbf{R}^m)'_\beta$. □

REMARK 5.3.2.4. Note that $\mathcal{D}(\mathbf{R}^{|Z|}) \otimes \mathcal{D}(\mathbf{R}^{m-|Z|})$ is indeed mapped onto a sequentially dense linear subspace of $\mathcal{D}(\mathbf{R}^m)$ under the canonical linear operator $\mathcal{D}(\mathbf{R}^{|Z|}) \otimes \mathcal{D}(\mathbf{R}^{m-|Z|}) \to \mathcal{D}(\mathbf{R}^m)$, by [**Tre**, Part III,Ch.39,Thm.39.2,p.406], but such operator only induces a linear topological isomorphism from the inductive tensor product of $\mathcal{D}(\mathbf{R}^{|Z|})$ and $\mathcal{D}(\mathbf{R}^{m-|Z|})$ onto $\mathcal{D}(\mathbf{R}^m)$, and not from their projective tensor product (nor injective, being this latter linearly and topologically isomorphic to the projective tensor product, as $\mathcal{D}(\mathbf{R}^{|Z|})$ is nuclear).

THEOREM 5.3.2.5. *Let $T \in \mathcal{S}(\mathbf{C})'_\beta$ be supported at $0 \in \mathbf{C}$. Then there exists an integer $N \geq 0$, complex numbers $c_{k,k'}$ with $(k,k') \in \mathbf{N}^2$ such that $k + k' \leq N$, such that, as elements of $\mathcal{S}(\mathbf{C})'_\beta$, we have:*

$$T = \sum_{\alpha \in \mathbf{N}^2, |\alpha| \leq N} c_{k,k'} \partial_z^k \partial_{\bar{z}}^{k'} \delta_0. \tag{5.3.2.11}$$

(1) *Let $\mathcal{S}_{00}(\mathbf{C})$ be the closed subspace defined as $\bigcap_{(k,k') \in \mathbf{N}^2} \ker(\partial_z^k \partial_{\bar{z}}^{k'} \delta_0)$. The orthogonal in $\mathcal{S}(\mathbf{C})'_\beta$ of $\mathcal{S}_{00}(\mathbf{C})$ with respect to the duality $\langle \mathcal{S}(\mathbf{C}), \mathcal{S}(\mathbf{C})'_\beta\rangle$ is a linear subspace of $\mathcal{S}(\mathbf{C})'_\beta$ linearly isomorphic to the continuous dual on the quotient $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$, and, topologized with the strong topology on the continuous dual of the quotient, it is linearly and topologically isomorphic to the $\mathbf{C}$-linear space underlying the polynomial algebra $\mathbf{C}[X,Y]$ equipped with the finest locally convex topology, equivalently the topology induced by the family of seminorms $\{p_N\}_{N \geq 0}$ with:*

$$p_N\Big(\sum_{(k,k') \in \mathbf{N}^2, k+k' \leq N} c_{k,k'} X^k Y^{k'}\Big) := \sup_{(k,k') \in \mathbf{N}^2, k+k' \leq N} |c_{k,k'}| \tag{5.3.2.12}$$

the isomorphism being uniquely determined by the assignment $X^k Y^{k'} \mapsto \frac{1}{k!(k')!} \partial_z^k \partial_{\bar{z}}^{k'} \delta_0$, and matching the degree of a polynomial with the finite order of the resulting distribution supported at $0 \in \mathbf{C}$.

(2) *The quotient $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$, equipped with the quotient topology, is linearly and topologically isomorphic to the $\mathbf{C}$-linear space underlying the formal power series algebra $\mathbf{C}[\![X,Y]\!]$ equipped with the Hausdorff locally convex topology induced by the family of seminoms $\{q_N\}_{N \geq 0}$:*

$$q_N\Big(\sum_{(k,k') \in \mathbf{N}^2} c_{k,k'} X^k Y^{k'}\Big) := \sup_{(k,k') \in \mathbf{N}^2, k+k' \leq N} |c_{k,k'}| \tag{5.3.2.13}$$

the isomorphism being defined by sending each $f \in \mathcal{S}(\mathbf{C})'_\beta$ to the formal power series $\sum_{(k,k') \in \mathbf{N}^2} \frac{\partial_z^k \partial_{\bar{z}}^{k'} f(0)}{k!(k')!} X^k Y^{k'}$.

PROOF. We let (x,y) be global smooth coordinates on $\mathbf{R}^2$ such that $(x,y)\mapsto z:=x+iy$ is an isomorphism $\mathbf{R}^2\to\mathbf{C}$ of Hausdorff locally compact abelian groups. In particular, it induces a linear topological isomorphism $\mathcal{S}(\mathbf{C})\to\mathcal{S}(\mathbf{R}^2)$.

We have $2\partial_z=\partial_x-i\partial_y$ and $2\partial_{\overline{z}}=\partial_x+i\partial_y$ as continuous operators on $\mathcal{S}(\mathbf{C})\simeq\mathcal{S}(\mathbf{R}^2)$.

We have $\delta_0=\delta_{0,x}\otimes\delta_{0,y}$, upon denoting $\delta_{0,x}:=R'_{H_{x=0}}(D_{0,x})$ and $\delta_{0,y}:=R'_{H_{y=0}}(D_{0,y})$, where $D_{0,x}$ denotes the evaluation at $x=0$ functional on $\mathcal{S}(\mathbf{R})$, and similarly for $D_{0,y}$, and $R_{H_{x=0}}$ and $R_{H_{y=0}}$ are the continuous operators $\mathcal{S}(\mathbf{R}^2)\to\mathcal{S}(\mathbf{R})$ from Theorem 5.3.2.3.

To see that indeed $\delta_0=\delta_{0,x}\otimes\delta_{0,y}$ in $\mathcal{S}(\mathbf{C})'_\beta$, it is enough to test the identity on Schwartz-Bruhat functions on $\mathbf{C}$ pulled back from the lines $H_{x=0}$ and $H_{y=0}$. Let $f(x,y)=\psi_1(x)\otimes\psi_2(y)$ be any such function, as a function of $(x,y)\in\mathbf{R}^2$, with $\psi_k\in\mathcal{S}(\mathbf{R})$ for $k=1,2$.

Then $\delta_0(f)=f(0,0)=\psi_1(0)\psi_2(0)=(\delta_{0,x}\otimes\delta_{0,y})(f)$. Since $\mathcal{S}(\mathbf{R})\otimes\mathcal{S}(\mathbf{R})\subset\mathcal{S}(\mathbf{R}^2)$ is dense by [**Tre**, Part III,Ch.51,Thm.51.6(51.3),p.530], we indeed have $\delta_0=\delta_{0,x}\otimes\delta_{0,y}$ in $\mathcal{S}(\mathbf{R}^2)'_\beta$.

We now let T be a tempered distribution on $\mathbf{C}$ supported at the origin. We rename as T its pull-back to $\mathcal{S}(\mathbf{R}^2)'_\beta$ along the isomorphism $\mathbf{R}^2\to\mathbf{C}$.

Then T is of finite order $N\geq 0$. Indeed, let $K\subset\mathbf{R}^2$ be a compact subset with non-empty interior. If $0\in K$, then for any compact subset $K'\subset K$ with non-empty interior and any smooth compactly supported function $\rho_{K'}\in\mathcal{D}(\mathbf{R}^2)$ whose support is contained in K' and contains 0 in its interior, and such that it is identically 1 on an open neighborhood of 0, we have:

$$\langle T,\phi\rangle_{\mathcal{S}(\mathbf{R}^2)'_\beta,\mathcal{S}(\mathbf{R}^2)}=\langle T,\rho_{K'}\cdot\phi\rangle_{\mathcal{S}(\mathbf{R}^2)'_\beta,\mathcal{S}(\mathbf{R}^2)}. \tag{5.3.2.14}$$

Indeed, upon choosing a smooth compactly supported function ρ supported over an open neighborhood of K with compact closure, and whose value on K is identically 1, then $\rho-\rho_{K'}$ vanishes over an open neighborhood of $0\in\mathbf{R}^2$, and hence on its compact closure by continuity.

Since T is supported at 0, we have $\langle T,(\rho-\rho_{K'})\cdot\phi\rangle_{\mathcal{S}(\mathbf{R}^2)'_\beta,\mathcal{S}(\mathbf{R}^2)}=0$. This proves that (5.3.2.14) holds for all $\phi\in\mathcal{D}(\mathbf{R}^2)$.

Since $\mathcal{D}(\mathbf{R}^2)\subset\mathcal{S}(\mathbf{R}^2)$ is dense, then (5.3.2.14) holds for all $\phi\in\mathcal{S}(\mathbf{R}^2)$ by continuity of T. By [**Horm**, Ch.II,§2.1,Thm.2.1.5,p.35], T is of finite order. We let $N\geq 0$ be its order.

By Theorem 5.3.2.3(4), there exist tempered distributions $T_{(k,k')}\in\mathcal{S}(\mathbf{R})'_\beta$, with $k+k'\leq N$, of order $N-k-k'$, and such that:

$$T=\sum_{(k,k')\in\mathbf{N}^2,k+k'\leq N}(\partial_x^k\delta_{0,x})\otimes(\partial_y^{k'}T_{(k,k')}) \tag{5.3.2.15}$$

as elements of $\mathcal{S}(\mathbf{R}^2)'_\beta$. Since T is supported at $0\in\mathbf{R}^2$, then each $T_{(k,k')}$ is supported at $0\in\mathbf{R}$, and it is of order $N-k-k'$. The support condition can be seen as follows. We let $\psi_{1,k}\in\mathcal{D}(\mathbf{R})$ be such that the support of $\psi_{1,k}$ contains 0 in its interior, and $\psi_{1,k}^{(j)}(0)=1$ for $j=k$ and vanishes for all integers $0\leq j\leq N$, $j\neq k$ (such $\psi_{1,k}$ indeed exists. For instance, we may define, for each integer $k>1$, $\phi_k(x):=\rho_{[-1,1]}(x)\cdot e^{x^k}$ for $\rho_{[-1,1]}$ a smooth compactly supported function whose support is contained in $[-1,1]$ and which is identically 1 over $[-1/2,1/2]$. Then $\phi_k^{(j)}(0)=0$ for all integers $1\leq j\leq k-1$, and is nonzero for $j=0,k$. For $k=1$, the same assignment ϕ_k satisfies $\phi_k^{(j)}(0)\neq 0$ for all integers $j\geq 0$, and for $k=0$, $\phi_k(0)\neq 0$ and $\phi_k^{(j)}(0)=0$ for all integers $j>0$. This shows that the closed subspace of Schwartz functions vanishing at 0 with order of zero exactly N, has codimension exactly $N+1$. Equivalently, that $\{\delta_{0,x}^{(k)},0\leq k\leq N\}$, is a $\mathbf{C}$-linear independent subset of $\mathcal{S}(\mathbf{R})'_\beta$).

We now let $\rho\in\mathcal{D}(\mathbf{R})$ be supported within an open subset of $\mathbf{R}$ whose compact closure does not contain $0\in\mathbf{R}$. Then $\phi_k:=\psi_{1,k}\otimes\rho$ is supported on a compact subset contained within an open subset of the form $U_1\times U_2$, with $0\in U_1\subset\mathbf{R}$ and $0\notin U_2\subset\mathbf{R}$.

We have $\langle T, \phi_k \rangle_{\mathcal{D}(\mathbf{R}^2)'_\beta, \mathcal{S}(\mathbf{R}^2)} = 0$, and by (5.3.2.15) this identity gives

$$\sum_{k' \in \mathbf{N}, 0 \leq k' \leq N-k} \langle \partial_y^{k'} T_{(k,k')}, \rho \rangle_{\mathcal{S}(\mathbf{R})'_\beta, \mathcal{S}(\mathbf{R})} = 0 \tag{5.3.2.16}$$

for each $0 \leq k \leq N$. We deduce that $T_{(N,k')}$ is supported at $0 \in \mathbf{R}$ and is of order 0. It follows that there exists a unique complex number $c_{(N,0)}$ such that $T_{(N,0)} = c_{(N,0)} \delta_{0,y}$.

Moreover, upon applying (5.3.2.16) with $k = N-1$, we deduce that $T - c_{(N,0)}(\partial_x^N \delta_{0,x}) \otimes \delta_{0,y} - \sum_{k'=0,1} \partial_y^{k'} T_{(N-1,k')}$ is again supported at $0 \in \mathbf{R}^2$, and that $T_{(N-1,0)}$ is supported at $0 \in \mathbf{R}$ if and only if $T_{(N-1,1)}$ is.

Descending induction on $N \geq 0$ therefore reduces the assertion to the case $N = 0$. If T is of order $N = 0$, then so is the unique $T_{(0,0)}$ satisfying (5.3.2.15), by Theorem 5.3.2.3. By testing T on ϕ_0, we deduce that $T_{(0,0)}$ is supported at $0 \in \mathbf{R}$, and of order 0. There therefore exists a complex number $c_{(0,0)}$ such that $T_{(0,0)} = c_{(0,0)} \cdot \delta_{0,y}$.

This shows that there exist unique complex numbers $c_{(k,k')}$ such that:

$$T = \sum_{(k,k') \in \mathbf{N}^2, k+k' \leq N} c_{(k,k')} (\partial_x^k \delta_{0,x}) \otimes (\partial_y^{k'} \delta_{0,y}) \tag{5.3.2.17}$$

$$= \sum_{(k,k') \in \mathbf{N}^2, k+k' \leq N} c_{(k,k')} \partial_x^k \partial_y^{k'} (\delta_{0,x} \otimes \delta_{0,y}) \tag{5.3.2.18}$$

$$= \sum_{(k,k') \in \mathbf{N}^2, k+k' \leq N} c_{(k,k')} \partial_x^k \partial_y^{k'} \delta_0 \tag{5.3.2.19}$$

as elements of $\mathcal{S}(\mathbf{R}^2)'_\beta$.

Upon evaluating on the identity in (5.3.2.17) the linear topological isomorphism $\mathcal{S}(\mathbf{R}^2)'_\beta \simeq \mathcal{S}(\mathbf{C})'_\beta$ induced by $(x, y) \mapsto z := x + iy$, we obtain (5.3.2.11).

(1) : Let $T \in \mathcal{S}(\mathbf{C})'_\beta$ be a tempered distribution such that $T(\mathcal{S}_{00}(\mathbf{C})) = 0$, i.e. an element of $\mathcal{S}_{00}(\mathbf{C})^\perp \subset \mathcal{S}(\mathbf{C})'_\beta$. Let $z \in \mathbf{C}$, $z \neq 0$. For any open neighborhood $U \subset \mathbf{C}$ of z such that $0 \notin U$, we let $\phi \in \mathcal{D}(\mathbf{C})$ be supported within U.

Then $\phi \in \mathcal{S}_{00}(\mathbf{C})$, and hence $T(\phi) = 0$. It follows that T is supported at 0, and hence of the form as in (5.3.2.11). Conversely, every tempered distribution of the form as in (5.3.2.11) is an element of $\mathcal{S}_{00}(\mathbf{C})^\perp$.

This shows that indeed the assignment $X^k Y^{k'} \mapsto \frac{1}{k!(k')!} \partial_z^k \partial_{\bar{z}}^{k'} \delta_0$ is a linear isomorphism $\mathbf{C}[X, Y] \to \mathcal{S}_{00}(\mathbf{C})^\perp$, and the degree N of a monomial matches the order of the resulting tempered distribution supported at $0 \in \mathbf{C}$.

The linear operator $\mathbf{C}[X, Y] \to \mathcal{S}_{00}(\mathbf{C})^\perp$ is continuous, upon endowing the source with the finest locally convex topology, hence a continuous bijection.

The source is a strict LF-space, and the target is the strong dual of the nuclear Fréchet space $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$ (we refer the reader to [**NB**, Thm.8.12.1(a)] for the linear isomorphism between $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$ and $\mathcal{S}_{00}(\mathbf{C})^\perp$).

We call $F := \mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$ the quotient Fréchet space.

Since both $\mathcal{S}(\mathbf{C})$ and $\mathcal{S}_{00}(\mathbf{C})$ are nuclear Fréchet (the former by Proposition 2.1.1.6, the latter by [**Scha**, Ch.III,§7,Thm.7.4,p.103]), so is F (by [**Scha**, Ch.III,§7,Thm.7.4,p.103], since F is a separated quotient of a nuclear Fréchet space), and hence F is Montel (see [**Tre**, Part III,Ch.50,Prop.50.2,(50.12),p.519]), hence reflexive (see [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.11] and the subsequent discussion). It is, in fact, totally reflexive, because every quotient by a closed subspace of F inherits nuclearity and is a Fréchet space, hence Montel, hence reflexive (alternatively, note that any nuclear Fréchet space is the countable projective limit of a sequence of separable Hilbert spaces; see [**Scha**, Ch.III,§7.3,Cor.3,p.103]).

The same argument as for the rest of (1) from Theorem 5.3.2.3(1), resting on the proof of the "only if" implication of [**Val-a**, §5,Thm.3,p.343], then applies, and gives that F'_β is an LF-space.

The continuous operator $\mathbf{C}[X,Y] \to F'_\beta$ is therefore a linear topological isomorphism by [**Gro-f**, Ch.4,§5,Thm.2(1),p.148] (see also [**SN**, §2,Cor.3,p.8]).

It follows that the composition of the linear topological isomorphisms $\mathbf{C}[X,Y] \to F'_\beta \simeq \mathcal{S}_{00}(\mathbf{C})^\perp$, which is the linear operator $\mathbf{C}[X,Y] \to \mathcal{S}_{00}(\mathbf{C})^\perp$, is indeed a linear topological isomorphism with respect to the stated topologies.

We further refer the reader to Lemma 4.2.1.3 for the equivalence between the strong topology on the continuous dual of $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C})$, and the subspace topology inherited from the strong topology on $\mathcal{S}(\mathbf{C})'$.

(2) : We topologize $\mathcal{S}_{00}(\mathbf{C})^\perp$ with the strong topology on the continuous dual of the quotient. The adjoint of the linear topological isomorphism $\mathbf{C}[X,Y] \to \mathcal{S}_{00}(\mathbf{C})^\perp$ is the linear operator sending $f \in \mathcal{S}(\mathbf{C})$ to its Taylor expansion at $z=0$ as a formal power series, i.e. $\sum_{(k,k')\in\mathbf{N}^2, k+k'\leq N} \frac{\partial_z^k \partial_{\overline{z}}^{k'} f(0)}{k!(k')!} X^k Y^{k'}$. This assignment is well-defined modulo $\mathcal{S}_{00}(\mathbf{C})$, since the coefficients $\partial_z^k \partial_{\overline{z}}^{k'} f(0)$ all vanish for every $f \in \mathcal{S}_{00}(\mathbf{C})$, and hence it indeed induces a linear operator $\mathcal{S}(\mathbf{C})/\mathcal{S}_{00}(\mathbf{C}) \to \mathbf{C}[\![X,Y]\!]$.

To show that it is a linear topological isomorphism with respect to the topology induced on $\mathbf{C}[\![X,Y]\!]$ by the countable family of seminorms defined in (5.3.2.13), by [**NB**, Thm.8.13.5] it is enough to show that this is a Hausdorff locally convex topology such that $(\mathbf{C}[\![X,Y]\!])'_\beta$ is linearly and topologically isomorphic to $\mathbf{C}[X,Y]$ equipped with the finest locally convex topology. This follows from [**Tre**, Part II,Ch.22,Thm.22.1(b),p.228]. □

5.3.3. Smoothness and rapid decay of extensions by zero. We recall that we defined, in Definition D.3.3.1 of §D.3 from Appendix D, the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}_I(G)$ of Schwartz-Bruhat functions on a Hausdorff locally compact $\mathbf{R}_{>0}$-modulated abelian group G with I-exponential decay, for $I \subset \mathbf{R}$ a non-empty connected interval.

These are functions $f : G \to \mathbf{C}$ such that $|\cdot|_G^a f \in \mathcal{S}(G)$ for every $a \in I$.

Example 5.3.3.1. Recall from Definition D.2.1.1 that a Hausdorff locally compact abelian group G is *modulated* if it is equipped with a continuous proper homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$ whose kernel G^1 is (therefore) a compact Hausdorff abelian subgroup. G is $\mathbf{R}_{>0}$-modulated if $|\cdot|_G$ is surjective.

To fix ideas, the reader may consider the case $G = \mathbf{R}_{>0}$, with $|\cdot|_G$ the identity homomorphism, in which case $f : G \to \mathbf{C}$ has I-exponential decay if and only if $t^a f(t) \in \mathcal{S}(\mathbf{R}_{>0})$. Upon performing the change of variables $t = e^x$, i.e. upon defining the module homomorphism $|x|_{\mathbf{R}} := e^x$, then $\mathcal{S}_I(\mathbf{R})$ is the $\mathbf{C}$-linear space of smooth functions $f : \mathbf{R} \to \mathbf{C}$ such that $e^{ax} f(x)$ decays faster than any positive power of $|x|^{-1}$ as $|x| \to +\infty$ for every $a \in I$, hence the lexicon "I-exponential decay". $\mathcal{S}_{\log(I)}(\mathbf{R})$ is linearly and topologically isomorphic to $\mathcal{S}_I(\mathbf{R}_{>0})$ with the above module homomorphisms.

We refer the reader to Theorem D.3.3.7 for the properties of the Hausdorff locally convex topology on $\mathcal{S}_I(G)$.

We will now use only the special cases $G = \mathbf{R}^\times, \mathbf{C}^\times$, and $I = (0, +\infty)$, respectively $I = [-m, +\infty)$ for each integer $m \geq 0$.

We note that we have injective continuous linear inclusions $\mathcal{S}(\mathbf{R}) \to \mathcal{S}_{(0,+\infty)}(\mathbf{R}^\times)$ and $\mathcal{S}(\mathbf{C}) \to \mathcal{S}_{(0,+\infty)}(\mathbf{C}^\times)$.

Proposition 5.3.3.2. *Let $m \geq 0$ be an integer. Then:*

(1) *pointwise multiplication by x, respectively z, $\overline{z}$, denoted M_x, respectively $M_z, M_{\overline{z}}$, is a bounded linear operator on $\mathcal{D}(\mathbf{R}^\times)$, $\mathcal{S}_{[a,+\infty)}(\mathbf{R}^\times)$ for any $a \in \mathbf{R} \cup \{-\infty\}$, respectively $\mathcal{D}(\mathbf{C}^\times)$, $\mathcal{S}_{[a,+\infty)}(\mathbf{C}^\times)$ for any $a \in \mathbf{R} \cup \{-\infty\}$, and it is boundedly invertible on $\mathcal{D}(\mathbf{R}^\times)$, $\mathcal{S}_{(-\infty,+\infty)}(\mathbf{R}^\times)$, respectively $\mathcal{D}(\mathbf{C}^\times)$, $\mathcal{S}_{(-\infty,+\infty)}(\mathbf{C}^\times)$.*
(2) *Let $m \geq 0$ be an integer.*
   (1) *$\partial_x^m := (M_x^{-1} \circ (x\partial_x))^m$ is a continuous linear endomorphism on $\mathcal{D}(\mathbf{R}^\times)$ with closed range. It extends to a continuous operator on $\mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$ valued in $\mathcal{S}_{(a+m,+\infty)}(\mathbf{R}^\times)$ for every real number a, and induces a continuous linear endomorphism of $\mathcal{S}_{(-\infty,+\infty)}(\mathbf{R}^\times)$.*

For each $\phi \in \mathcal{S}_{(0,+\infty)}(\mathbf{R}^\times)$, ϕ is a smooth function on $\mathbf{R}^\times$ (i.e. on $\mathbf{R}^\times$ as an open subset of $\mathbf{R}$), and $\partial_x\phi$ is computed pointwise by the ordinary derivative of ϕ as a smooth function on $\mathbf{R}^\times \subset \mathbf{R}$.

(2) *$\partial_z := M_z^{-1} \circ (z\partial_z)$, $\partial_{\overline{z}} := M_{\overline{z}}^{-1} \circ (\overline{z}\partial_{\overline{z}})$ are continuous linear endomorphisms on $\mathcal{D}(\mathbf{C}^\times)$ with closed range.*

∂_z and $\partial_{\overline{z}}^m$ extend to continuous operators on $\mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times)$ valued $\mathcal{S}_{(a+m,+\infty)}(\mathbf{C}^\times)$ for every real number $a < 0$, and induce continuous linear endomorphisms on $\mathcal{S}_{(-\infty,+\infty)}(\mathbf{C}^\times)$.

For each $\phi \in \mathcal{S}_{(0,+\infty)}(\mathbf{C}^\times)$, ϕ is a smooth function over $\mathbf{C}^\times$ (i.e. on $\mathbf{C}^\times$ as an open domain of the complex plane), and $\partial_z\phi$ and $\partial_{\overline{z}}\phi$ are computed pointwise by the ordinary holomorphic, respectively anti-holomorphic, derivatives of ϕ as a smooth function on $\mathbf{C}^\times \subset \mathbf{C}$.

PROOF. (1) : Multiplication by x is bounded and boundedly invertible on $\mathcal{D}(\mathbf{R}^\times)$.

Indeed, since $\mathbf{R}^\times$ and $\mathbf{C}^\times$ are abelian Lie groups, we topologize $\mathcal{D}(\mathbf{R}^\times)$ and $\mathcal{D}(\mathbf{C}^\times)$ with the strict locally convex inductive limit topology of the Fréchet space topologies on $\mathcal{D}_K(\mathbf{R}^\times)$, respectively $\mathcal{D}_K(\mathbf{C}^\times)$ (see also [**Sch-a**, Ch.III,§1,Thm.II,p.66], for a direct definition using a separating family of seminorms, for then identifying the resulting locally convex topology with the strict locally convex inductive limit).

We further refer the reader to [**Fol**, Ch.9,§9.1,p.282].

All the results on the Hausdorff locally convex topological vector space $\mathcal{D}(\mathbf{R}^m)$ from [**Sch-a**, Ch.III,§1] apply to $\mathcal{D}(\mathbf{R}^\times)$ and $\mathcal{D}(\mathbf{C}^\times)$.

By [**Sch-a**, Ch.III,§1,Thm.IV,p.69], for $B \subset \mathcal{D}(\mathbf{R}^\times)$ a bounded subset, there exists a compact subset $K \subset \mathbf{R}^\times$ and a bounded subset $B_0 \subset \mathcal{D}_K(\mathbf{R}^\times)$ such that B is the image of B_0 under the linear topological embedding $\mathcal{D}_K(\mathbf{R}^\times) \to \mathcal{D}(\mathbf{R}^\times)$.

If $\varphi \in \mathcal{D}_K(\mathbf{R}^\times)$, then $M_x(\varphi)$ is the function $x\varphi(x)$ on $\mathbf{R}^\times$, and it is still compactly supported with support contained in K, i.e. M_x preserves the subspaces $\mathcal{D}_K(\mathbf{R}^\times)$. It is therefore enough to show that M_x is bounded and boundedly invertible on $\mathcal{D}_K(\mathbf{R}^\times)$, for $K \subset \mathbf{R}^\times$ an arbitrary compact subset with non-empty interior.

A similar reduction is valid for M_z and $M_{\overline{z}}$ on $\mathcal{D}(\mathbf{C}^\times)$, being the Hausdorff locally convex topology on $\mathcal{D}(G)$ for G an abelian Lie group, the strict inductive limit of the topologies on each $\mathcal{D}_K(G)$, for $K \subset G$ a compact subset with non-empty interior.

For any $\varphi \in B_0$, we have

$$\sup_{x\in K} |\partial_x^m(x\varphi(x))| \leq \sup_{x\in K} |x\partial_x^m\varphi(x)| + \sup_{x\in K} |\partial_x^{m-1}\varphi(x)| \leq \sup_{x\in K} |x| \cdot \|\varphi^{(m)}\|_u + \|\varphi^{(m-1)}\|_u$$

and hence M_x is bounded on $\mathcal{D}_K(\mathbf{R}^\times)$. Similarly, M_z and $M_{\overline{z}}$ are both bounded on $\mathcal{D}_K(\mathbf{C}^\times)$ for any compact subset $K \subset \mathbf{C}^\times$ with non-empty interior. By [**NB**, Thm.6.5.2] (see also [**Scha**, Ch.II,§8.3,p.62]), M_x, M_z and $M_{\overline{z}}$ are continuous.

They are therefore all bounded and continuous (by [**NB**, Thm.13.2.7]) linear endomorphisms on $\mathcal{D}(\mathbf{R}^\times)$, respectively $\mathcal{D}(\mathbf{C}^\times)$. Since they are surjective, then by Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]) they are linear topological automorphisms, being their source and target (strict) LF-spaces.

(2) : Let $\{f_i\}_{i\in I} \subset \mathcal{D}(\mathbf{R}^\times)$ be a net in $\mathcal{D}(\mathbf{R}^\times)$ converging to f. By [**Sch-a**, Ch.III,§2,Thm.IV,p.69] we may and do assume that there exists a compact subset $K \subset \mathbf{R}^\times$ such that $\{f_i\}_{i\in I} \cup \{f\} \subset \mathcal{D}_K(\mathbf{R}^\times)$, since, up to replacing the directed set I with a subset of the form $I_0 := \{i \in I : i \geq i_0\}$ for some $i_0 \in I$, and renaming I_0 as I, we may assume that $\{f_i\}_{i\in I}$ is a bounded subset in $\mathcal{D}(\mathbf{R}^\times)$, being a Cauchy net (see [**NB**, Thm.3.4.2]. Cauchy nets are not necessarily eventually bounded, [8]

since for any 0-neighborhood V there exists $i_0 \in I$ such that $f_i - f_j \in V$ for all $i, j \geq i_0$, and hence upon choosing $i_1 \geq i_0$, we have, for all $i, j \geq i_1$ that $f_i \in f_{i_1} + V$ holds, and $\{f_i\}_{i\in I, i\geq i_1}$ is bounded.

[8] Return here.

Since $\mathcal{D}_K(\mathbf{R}^\times)$ is a Fréchet space, there exists a cofinal sequence of elements of I such that the resulting sequence $\{F_n := f_{i_n}\}_{n\geq 0}$ converges to f in $\mathcal{D}_K(\mathbf{R}^\times)$. If each f_i is of the form $x\partial_x u_i$ for some $u_i \in \mathcal{D}(\mathbf{R}^\times)$, then since each u_i is compactly supported, we must have $u_i \in \mathcal{D}_K(\mathbf{R}^\times)$, since $f \in \mathcal{D}_K(\mathbf{R}^\times)$ (and hence u_i is constant away from the support of f, which forces the support of u_i to be contained in K).

We have

$$\partial_x^N \int_0^x F_n(y)\,\frac{\mathrm{d}y}{y} \to \partial_x^N \int_0^x f(y)\,\frac{\mathrm{d}y}{y}$$

by Dominated Convergence, since $F_n \to f$ in $\mathcal{D}_K(\mathbf{R}^\times)$, the mode of convergence being uniform over K for each integer $N \geq 0$, and hence u_{i_n} converges to some $u \in \mathcal{D}_K(\mathbf{R}^\times)$ (see [**Sch-a**, Ch.III,§1,p.64]), and $\partial_x u = f/x$. It follows that $f = x\partial_x u$, i.e. f is in the range of $x\partial_x$. A similar argument shows that $z\partial_z$ and $\overline{z}\partial_{\overline{z}}$ both have closed range in $\mathcal{D}(\mathbf{C}^\times)$.

Since M_x, M_z and $M_{\overline{z}}$ are linear topological automorphisms, they have closed range, and hence the continuous linear operators ∂_x, ∂_z and $\partial_{\overline{z}}$ also have closed range.

Continuity of the operators $M_{x^{-1}} : \mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times) \to \mathcal{S}_{(a+1,+\infty)}(\mathbf{R}^\times)$ and $M_{z^{-1}}, M_{\overline{z}^{-1}} : \mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times) \to \mathcal{S}_{(a+1,+\infty)}(\mathbf{C}^\times)$ follows from the design of the Hausdorff locally convex topology on sources and targets (see Definition D.3.3.1), which, recall, is the projective topology on $\mathcal{S}_I(G)$, G a Hausdorff locally compact abelian group, $I \subset \mathbf{R}$ a non-empty connected interval, induced by the family of operators $T_a : \mathcal{S}_I(G) \to \mathcal{S}(G)$ sending $\phi \in \mathcal{S}_I(G)$ to $|\cdot|_G^a \cdot \phi$, with $|\cdot|_G$ the module of the modulated group G.

For $G = \mathbf{R}^\times$, then if $f_i \to f$ is a convergent J-indexed net in $\mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$, $\{M_{x^{-1}}(f_i)\}_{i\in J}$ is contained in $\mathcal{S}_{(a+1,+\infty)}(\mathbf{R}^\times)$, and $T_b(M_{x^{-1}}(f_i))$ is, for each $i \in J$ and as a function of $x \in \mathbf{R}^\times$, $x \mapsto |x|^b \cdot x^{-1} f_i(x)$, which is Schwartz-Bruhat on $\mathbf{R}^\times$ since $x = \mathrm{sgn}(x)|x|$, and multiplication by unitary characters is continuous on $\mathcal{S}(\mathbf{R}^\times)$.

It follows that $T_b(M_{x^{-1}}(f_i)) \to T_b(M_{x^{-1}}(f))$ for every $b > a+1$, i.e. $M_{x^{-1}}$ is continuous on $\mathcal{S}_{(a,+\infty)}$ and valued in $\mathcal{S}_{(a+1,+\infty)}$. A similar argument shows the same property holds for $M_{z^{-1}}$ and $M_{\overline{z}^{-1}}$ on $\mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times)$.

Since $x\partial_x$, $z\partial_z$ and $\overline{z}\partial_{\overline{z}}$ are all continuous endomorphisms of $\mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$, respectively $\mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times)$ for any $a \in \mathbf{R}$, since they are continuous differential operators on $\mathcal{S}(\mathbf{R}^\times)$ respectively $\mathcal{S}(\mathbf{C}^\times)$, and by design of the Hausdorff locally convex topology on $\mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$, respectively $\mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times)$, the remaining assertions on ∂_x^m, ∂_z^m and $\partial_{\overline{z}}^m$ follow by composition.

For each $y \in \mathbf{R}^\times$, and $\phi \in \mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$, for $a \in \mathbf{R}$ a real number, we note that the limit:

$$\lim_{x\to 1} \frac{\phi(x^{-1}y) - \phi(y)}{x^{-1} - 1} \tag{5.3.3.1}$$

exists and is finite, and its formation is continuous as a function of $y \in \mathbf{R}^\times$ because $|\cdot|_{\mathbf{R}}^a \phi \in \mathcal{S}(\mathbf{R}^\times)$.

Indeed, the limit:

$$\lim_{x\to 1} \frac{|x^{-1}y|_{\mathbf{R}}^b \phi(x^{-1}y) - |y|_{\mathbf{R}}^a \phi(y)}{x^{-1} - 1} \tag{5.3.3.2}$$

exists since it is performed along a continuous multiplicative 1-parameter subgroup of $\mathbf{R}^\times$, and $|\cdot|_{\mathbf{R}}^b \phi \in \mathcal{S}(\mathbf{R}^\times)$ for any $b > a$. It equals:

$$|y|_{\mathbf{R}}^b \lim_{x\to 1} \frac{|x^{-1}|_{\mathbf{R}}^b \cdot \phi(x^{-1}y) - \phi(y)}{x^{-1} - 1} \tag{5.3.3.3}$$

which in turn equal the sum of $|y|_{\mathbf{R}}^b \lim_{x\to 1} \frac{|x^{-1}|_{\mathbf{R}}^a - 1}{x^{-1}-1} \phi(x^{-1}y)$, and the limit in (5.3.3.1) (multiplied by $|y|_{\mathbf{R}}^b$) which therefore also exists, since the former does upon choosing $b > 1$ (which we can do since $\phi \in \mathcal{S}_{(a,+\infty)}$.

The limit in (5.3.3.1) therefore exists, it is finite, and it equals:

$$y \lim_{x\to 1} \frac{\phi(x^{-1}y) - \phi(y)}{x^{-1}y - y} \tag{5.3.3.4}$$

and hence at each $y \in \mathbf{R}^\times$ the limit $\lim_{x\to 1} \frac{\phi(x^{-1}y)-\phi(y)}{x^{-1}y-y}$, equivalently (since ϕ is continuous on $\mathbf{R}^\times$) $\lim_{y'\to y} \frac{\phi(y')-\phi(y)}{y'-y}$, exists and is finite, and its formation is continuous as a function of $y \in \mathbf{R}^\times$, since y is nowhere vanishing and continuous away from $0 \in \mathbf{R}$.

We deduce that any $\phi \in \mathcal{S}_{(a,+\infty)}(\mathbf{R}^\times)$ is smooth on $\mathbf{R}^\times$ as a function on a non-empty open subset of the real line. A similar argument applies to $\mathcal{S}_{(a,+\infty)}(\mathbf{C}^\times)$, and we deduce that the operators ∂_x, ∂_z and $\partial_{\overline{z}}$ are computed pointwise on functions by the ordinary smooth, holomorphic and anti-holomorphic derivatives, respectively.

(Note that continuity of $z\partial_z$ and $\overline{z}\partial_{\overline{z}}$, using which the same argument in the foregoing to establish continuity of ∂_x and compute it pointwise, carries over to ∂_z and $\partial_{\overline{z}}$, can also be seen on $\mathcal{S}(\mathbf{C}^\times)$ as follows: upon writing $z = re^{i\theta}$ in polar coordinates, from which we have $\partial_z = \frac{e^{-i\theta}}{2}(\partial_r - iM_{r^{-1}}\partial_\theta)$, and by Theorem D.3.3.7(3), it is enough to show continuity of ∂_θ on $\mathcal{S}(\mathrm{U}(1))$, which holds since ∂_θ is the directional derivatives along the 1-parameter subgroup $\exp(i\theta) : \mathbf{R} \to \mathrm{U}(1)$, and use continuity of ∂_x, which we established in the foregoing). □

LEMMA 5.3.3.3. *Let $p, q \geq 0$ be integers, and $I \subset \mathbf{R}$ an open connected non-empty interval. We call $(x_k)_{1\leq k\leq p}$ a choice of global smooth coordinates on $\mathbf{R}^p$, and $\{(z_m, \overline{z}_m)\}_{1\leq m\leq q}$ a choice of global holomorphic and anti-holomorphic coordinates on $\mathbf{C}^q$.*

We denote by $|\cdot|_{r,k}$ the real absolute value on the kth factor in $\mathbf{R}^p$, and $|\cdot|_{c,m}$ the square of the Euclidean norm on the mth factor in $\mathbf{C}^q$.

We denote, for $\underline{k} \subset \{1,\dots,p\}$ and $\underline{m} \subset \{1,\dots,q\}$ totally ordered subsets, and for $(\underline{a},\underline{b}) \in I^{|\underline{k}|} \times I^{|\underline{m}|}$:

$$|\cdot|^{\underline{a},\underline{b}}_{\underline{k},\underline{m}} := \prod_{1\leq k\leq p} |\cdot|^{a_k}_{r,k} \cdot \prod_{1\leq m\leq q} |\cdot|^{b_m}_{c,m} : (\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q \to \mathbf{R}_{>0}. \tag{5.3.3.5}$$

We define

$$\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.6}$$

to be the linear subspace of $\mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ of those Schwartz-Bruhat functions ϕ on $(\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q$ such that

$$|\cdot|^{\underline{a},\underline{b}}_{\underline{k},\underline{m}} \cdot \phi \in \mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.7}$$

endowing $\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ with the projective topology induced by the linear operators

$$T_{(\underline{k},\underline{m}),(\underline{a},\underline{b})} : \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \to \mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.8}$$

defined by $T_{(\underline{k},\underline{m}),(\underline{a},\underline{b})}(\phi) := |\cdot|^{\underline{a},\underline{b}}_{\underline{k},\underline{m}} \cdot \phi$.

Then:

1. *$\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ is a separable nuclear Fréchet space.*
2. *the canonical linear map:*

$$\mathcal{S}_I(\mathbf{R}^\times)^{\otimes^p_\pi} \otimes_\pi \mathcal{S}_I(\mathbf{C}^\times)^{\otimes^q_\pi} \to \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.9}$$

   is a linear topological embedding, inducing a linear topological isomorphism from the completion of the source.
3. *We define:*

$$\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q) \subset \mathcal{S}(\mathbf{R}^p \times \mathbf{C}^q) \tag{5.3.3.10}$$

   the closed linear subspace of those Schwartz functions $f \in \mathcal{S}(\mathbf{R}^p \times \mathbf{C}^p)$ such that all partial derivatives with respect to the p smooth coordinates, and all holomorphic and anti-holomorphic derivatives with respect to the q complex coordinates, of any order, vanish on each of the hyperplanes $x_k = 0$, $z_m = 0$, for each $1 \leq k \leq p$, $1 \leq m \leq q$.

If $I = (-\infty, +\infty)$, then the canonical linear restriction operator $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q) \to \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ is continuous and a linear topological isomorphism. In particular, multiplication by any continuous quasi-character of $(\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q$ extends uniquely to a bounded and boundedly invertible automorphism of $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$, which, on functions and pointwise over the open subset $(\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q \subset \mathbf{R}^p \times \mathbf{C}^q$, is computed by pointwise multiplication of smooth functions.

PROOF. (1) : We begin by observing that, as in the proof of Lemma D.3.2.3(1), the Hausdorff locally convex topology on $\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ is equivalent to the projective topology induced by a subfamily of the operators $T_{(\underline{k},\underline{m}),(\underline{a},\underline{b})}$: it is enough to assume $(\underline{a},\underline{b})$ vary over $\mathbf{Z}^{|\underline{k}|} \times \mathbf{Z}^{|\underline{m}|}$.

In particular, the operator:

$$T : \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \to \prod_{(\underline{a},\underline{b}) \in \mathbf{Z}^{|\underline{k}|} \times \mathbf{Z}^{|\underline{m}|}} \prod_{(\underline{k},\underline{m})} \mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.11}$$

induced by the $T_{(\underline{k},\underline{m}),(\underline{a},\underline{b})}$ from (5.3.3.8), is a linear topological embedding with closed range (see the proof of Lemma D.3.2.3(2)).

Since countable products of complete metrizable Hausdorff locally convex spaces are complete metrizable and Hausdorff by [**Scha**, Ch.I,§6,p.30] and [**Scha**, Ch.II,§5,5.3,p.52], the target Hausdorff locally convex topological $\mathbf{C}$-vector space in (5.3.3.11) is a Fréchet space (separable, since the product is countable and each factor is $\mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$, which is a separable Fréchet space), so is the source, i.e. $\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$, since it is a closed subspace of a separable Fréchet space.

Nuclearity is inherited by subspaces and arbitrary products of nuclear Hausdorff locally convex topological $\mathbf{C}$-vector spaces are nuclear (see [**Scha**, Ch.III,§7,Thm.7.4,p.103]), proving nuclearity of $\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$.

(2) : We prove that the canonical linear map:

$$\mathcal{S}_I(\mathbf{R}^\times)^{\otimes_\pi^p} \otimes_\pi \mathcal{S}_I(\mathbf{C}^\times)^{\otimes_\pi^q} \to \mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.12}$$

is continuous with respect to the projective tensor product topology on the source, and induces a linear topological isomorphism from the completion, where each $\mathbf{R}^\times$ is modulated by $|\cdot|_{r,k}$, and each $\mathbf{C}^\times$ is modulated by $|\cdot|_{c,k}$, onto the linear subspace

$$\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$$

of those Schwartz-Bruhat functions $\phi((x_k),(z_m))$ such that $\prod_{1\le k\le p} |\cdot|_{r,k}^{a_k} \cdot \prod_{1\le m\le q} |\cdot|_{c,m}^{b_m} \cdot \phi \in \mathcal{S}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ for all $a_k, b_m \in I$ and all $1 \le k \le p$ and $1 \le m \le q$.

By [**Jar**, Part III,Ch.15,§15.1,Rem.6(b),p.326] and [**Jar**, Part III,Ch.15,§15.2,Cor.4,p.328], formation of the projective tensor product topology is associative on finitely many tensor factors, and stable under arbitrary permutations of finitely many tensor factors.

By Theorem D.3.3.7(3), upon choosing a continuous homomorphic section τ_r and τ_r to the real Euclidean, respectively to the complex, absolute values, we have linear topological isomorphisms:

$$\mathcal{S}_I(\mathbf{R}^\times) \simeq \mathcal{S}(\{\pm 1\}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}_I(\mathbf{R}_{>0}), \quad \mathcal{S}_I(\mathbf{C}^\times) \simeq \mathcal{S}(\mathrm{U}(1)) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}_I(\mathbf{R}_{>0}). \tag{5.3.3.13}$$

The same argument as for the proof of Lemma D.2.2.2(1), shows that the canonical linear map:

$$\mathcal{S}(\{\pm 1\}^p \times \mathrm{U}(1)^q) \otimes_\pi \mathcal{S}_{I^{p+q}}(\mathbf{R}_{>0}) \to \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q) \tag{5.3.3.14}$$

is continuous and induces a linear topological isomorphism from the completion of the source. We note that indeed the argument from the proof of Lemma D.2.2.2(1) carries over, because the Hausdorff locally convex spaces $\mathcal{S}(\{\pm 1\}^p) \simeq \mathbf{C}^{2p}$, $\mathcal{S}(\mathrm{U}(1)^q)$, and $\mathcal{S}_{I^{p+q}}(\mathbf{R}_{>0})$, are nuclear Fréchet, and Proposition C.2.3.11 can be accessed as in the proof of Lemma D.2.2.2(1), to show continuity of the linear operator from (5.3.3.14).

However, since $\mathcal{S}(\{\pm 1\})$ and $\mathcal{S}(\mathrm{U}(1)^q)$ are also Fréchet spaces, the argument from [**Tre**, Part III,Ch.51,Thm.51.6,p.531] also carries over (the key difference is that since all tensor factors and the target are sequential, separate continuity of bi-linear forms from the product of two Fréchet spaces into an arbitrary Hausdorff locally convex

topological $\mathbf{C}$-vector space, is sufficient for continuity to hold; see [**Bou-TVS**, Ch.III,§5,Cor.1,TVS III.30]. Lemma D.2.2.2 allows $\mathcal{S}(G^1)$ to be only an LF-space, hence not necessarily metrizable).

We further refer the reader to the argument for Theorem D.3.3.7(3).

The following diagram of continuous linear operators:

$$\begin{array}{ccc} \mathcal{S}(\{\pm 1\})\,\widehat{\otimes}_\pi^{\,p}\,\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))\,\widehat{\otimes}_\pi^{\,q}\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi^{\,p+q} & \longrightarrow & \mathcal{S}_I(\mathbf{R}^\times)\,\widehat{\otimes}_\pi^{\,p}\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{C}^\times)\,\widehat{\otimes}_\pi^{\,q} \\ \downarrow & & \downarrow \\ \mathcal{S}(\{\pm 1\}^p\times \mathrm{U}(1)^q)\,\widehat{\otimes}_\pi\mathcal{S}_{I^{p+q}}(\mathbf{R}_{>0}^{p+q}) & \longrightarrow & \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p\times(\mathbf{C}^\times)^q) \end{array} \tag{5.3.3.15}$$

commutes, and we have shown that the bottom horizontal arrow is a linear topological isomorphism.

By Lemma D.1.8.1, combined with [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], which applies since the tensor factors are nuclear by Theorem D.1.5.7 and, for $\mathcal{S}_{I^{p+q}}(\mathbf{R}_{>0}^{p+q})$, by the same argument as in part (1), using nuclearity of $\mathcal{S}(\mathbf{R}_{>0}^{p+q})$ which is shown in Proposition 2.1.1.6 (see also [**Tre**, Part III,Ch.51,p.530,Corollary]), the leftmost vertical arrow is a linear topological isomorphism, so long as the canonical linear map:

$$\mathcal{S}_I(\mathbf{R}_{>0})^{\otimes_\pi^{p+q}} \to \mathcal{S}_{I^{p+q}}(\mathbf{R}_{>0}^{p+q}) \tag{5.3.3.16}$$

is continuous and induces a linear topological isomorphism from the completion of the source.

If so, then upon combining the linear topological isomorphisms from (5.3.3.13) with [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], which applies by nuclearity of the tensor factors, the top horizontal arrow is also a linear topological isomorphism, which proves that so is the rightmost vertical arrow. This proves (2).

We are therefore reduced to showing that the canonical linear operator from (5.3.3.16) is continuous with respect to the projective tensor product topology on the source, and a linear topological embedding with dense range.

We rename $p+q$ as p, and treat the case $p=2$ for simplicity, being the general case established in a similar manner.

It is therefore enough to prove that the canonical linear map:

$$\mathcal{S}_I(\mathbf{R}_{>0})\otimes\mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0}) \tag{5.3.3.17}$$

with $\mathbf{R}_{>0}$ modulated by the standard Euclidean absolute value $|\cdot|_r$, is continuous with respect to the projective tensor product topology on the source, and it extends to a linear topological isomorphism from the completion.

The Hausdorff locally convex topology on $\mathcal{S}_I(\mathbf{R}_{>0})$ is the projective topology induced by the linear operators $T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0})$ defined, for each $a\in I$, by $T_a(\phi) := |\cdot|_r^a\,\phi$ (see Definition D.3.3.1).

Equivalently, it is the topology induced by the unique linear map:

$$T := \prod_{a\in I} T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \prod_{a\in I}\mathcal{S}(\mathbf{R}_{>0}).$$

By the description of the 0-neighborhoods of the projective tensor product topology from [**Jar**, Part III,Ch.15,§15.1,p.324] (see also the discussion preceding (C.2.3.14)), and by [**Jar**, Part III,Ch.15,§15.4,Thm.1,p.332], the canonical linear map:

$$(\prod_{a\in I}\mathcal{S}(\mathbf{R}_{>0}))\otimes_\pi(\prod_{a\in I}\mathcal{S}(\mathbf{R}_{>0})) \to \prod_{(a,b)\in I\times I}(\mathcal{S}(\mathbf{R}_{>0})\otimes\mathcal{S}(\mathbf{R}_{>0}))$$

is a linear topological embedding, and by [**Tre**, Part III,Ch.51,Thm.51.6(51.3),p.530] the canonical linear operator $\mathcal{S}(\mathbf{R}_{>0})\otimes\mathcal{S}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ is a linear topological embedding of the projective tensor product topology on the source, extending to a linear topological isomorphism $\mathcal{S}(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$.

In particular and by design of the Hausdorff locally convex topologies on $\mathcal{S}_I(\mathbf{R}_{>0})$ and $\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$, we deduce that the canonical linear operator from (5.3.3.13) is a linear topological embedding with respect to

the projective tensor product topology on the source, and hence extends to a linear topological embedding from the completion of the source, into $\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$.

$\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ is a nuclear Fréchet space. Indeed, we may select a $\mathbf{Z}\times\mathbf{Z}$-indexed countable subfamily of the $T_{a,b}:\phi\mapsto|\cdot|_{r,1}^{a}|\cdot|_{r,2}^{b}\phi\in\mathcal{S}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ with a,b ranging over countable sequences $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ such that $\bigcup_{k\in\mathbf{Z}}(a_k,a_{k+1})=\mathbf{R}$ and $\bigcup_{k\in\mathbf{Z}}(b_k,b_{k+1})=\mathbf{R}$ without changing the Hausdorff locally convex topology on $\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$, which proves metrizability, and completeness and nuclearity are inherited by [**Scha**, Ch.III,§7,Thm.7.4,p.103], since $\prod_{a,b}T_{a,b}$ realizes $\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ as a closed subspace of the nuclear Fréchet space $\prod_{\{(a_k,b_k),k\geq 0\}}(\mathcal{S}(\mathbf{R}_{>0})\times\mathcal{S}(\mathbf{R}_{>0}))$.

It is therefore enough to show that $\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})\to\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ has dense range: being the source complete, the range is closed, and hence agrees with $\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$, which, by [**NB**, Thm.14.4.5], implies that the linear topological embedding $\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})\to\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ is indeed a linear topological isomorphism.

Smooth compactly supported functions of the form $\phi\otimes\psi$ with $\phi\in\mathcal{D}(\mathbf{R}_{>0})$ and $\psi\in\mathcal{D}(\mathbf{R}_{>0})$ linearly generate a dense subspace of the source by [**Jar**, Part III,Ch.15,§15.2,Prop.3(a),p.327] and Lemma D.3.6.1.

By [**Tre**, Part III,Ch.39,Thm.39.2,p.409], $\mathcal{D}(\mathbf{R}_{>0})\otimes\mathcal{D}(\mathbf{R}_{>0})\subset\mathcal{D}(\mathbf{R}_{>0}\times\mathcal{D}(\mathbf{R}_{>0})$ is sequentially dense, and so it is enough to show that $\mathcal{D}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})\subset\mathcal{S}_{I\times I}(\mathbf{R}_{>0}\times\mathbf{R}_{>0})$ is dense. This is seen by means of the same argument as in the proof of Lemma D.3.6.1.

(3) : The following diagram of linear operators is commutative:

$$\begin{array}{ccc}\mathcal{S}_{00}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}_{00}(\mathbf{C})^{\otimes_\pi^q} & \longrightarrow & \mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q)\\ \downarrow & & \downarrow\\ \mathcal{S}_I(\mathbf{R}^\times)^{\widehat{\otimes}_\pi^p}\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{C}^\times)^{\widehat{\otimes}_\pi^q} & \longrightarrow & \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p\times(\mathbf{C}^\times)^q)\end{array}\tag{5.3.3.18}$$

and all the tensor factors are nuclear Fréchet spaces.

The linear operator

$$\mathcal{S}_{00}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}_{00}(\mathbf{C})^{\otimes_\pi^q}\to\mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q)$$

is a linear topological embedding (to see this, we observe that $\mathcal{S}_{00}(\mathbf{R})$ and $\mathcal{S}_{00}(\mathbf{C})$ are closed subspaces of $\mathcal{S}(\mathbf{R})$, respectively $\mathcal{S}(\mathbf{C})$, and so is $\mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q)$ as a subspace of $\mathcal{S}(\mathbf{R}^p\times\mathbf{C}^q)$. Since $\mathcal{S}(\mathbf{R})$, $\mathcal{S}(\mathbf{C})$ and $\mathcal{S}(\mathbf{R}^p\times\mathbf{C}^q)$ and all their closed subspaces are nuclear, and $\mathcal{S}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}(\mathbf{C})^{\otimes_\pi^q}\to\mathcal{S}(\mathbf{R}^p\times\mathbf{C}^q)$ is continuous and a linear topological embedding with dense range by [**Tre**, Part III,Ch.51,Thm.51.6(51.3),p.530], then [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348] applies).

By (2), the bottom horizontal arrow in (5.3.3.18) is a linear topological isomorphism.

Since the topmost horizontal arrow is continuous and injective (because each of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}_{00}(\mathbf{R})$ and $\mathcal{S}_{00}(\mathbf{C})$ are closed in $\mathcal{S}(\mathbf{R})$ and $\mathcal{S}(\mathbf{C})$, respectively, and hence nuclear by [**Scha**, Ch.III,§7,Thm.7.4,p.103], which implies by [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348] that $\mathcal{S}_{00}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}_{00}(\mathbf{C})^{\otimes_\pi^q}\to\mathcal{S}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}(\mathbf{C})^{\otimes_\pi^q}$ is a linear topological embedding, which, followed by the linear topological embedding $\mathcal{S}(\mathbf{R})^{\otimes_\pi^p}\otimes_\pi\mathcal{S}(\mathbf{C})^{\otimes_\pi^q}\to\mathcal{S}(\mathbf{R}^p\times\mathbf{C}^q)$ from [**Tre**, Part III,Ch.51,Thm.51.6(51.3),p.530], agrees with the composition of the top horizontal in (5.3.3.18) followed by the linear topological embedding $\mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q)\to\mathcal{S}(\mathbf{R}^p\times\mathbf{C}^q)$), then the rightmost vertical arrow is a linear topological isomorphism so long as we prove the following two assertions:

(1) the rightmost vertical arrow is continuous.
(2) the two linear operators

$$\mathcal{S}_{00}(\mathbf{R})\to\mathcal{S}_I(\mathbf{R}^\times),\quad\mathcal{S}_{00}(\mathbf{C})\to\mathcal{S}_I(\mathbf{C}^\times)\tag{5.3.3.19}$$

are continuous and surjective.

Indeed, being source and target of the top horizontal and rightmost vertical arrows in (5.3.3.18) Fréchet spaces, surjectivity implies that they are both linear topological isomorphisms by [**NB**, Thm.14.4.5].

Since all the tensor factors in source and target of the leftmost vertical arrow in (5.3.3.18) are nuclear, by [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348] indeed the second of the two assertions above implies that it is a linear topological embedding with dense range.

We are therefore reduced to showing the two assertions in (1) and (2) above.

We first observe that (1) follows from (2). Indeed, we know that the topmost horizontal arrow in (5.3.3.18) is a linear topological embedding, hence inducing a linear topological embedding:

$$\mathcal{S}_{00}(\mathbf{R})^{\widehat{\otimes}_\pi^p}\widehat{\otimes}_\pi\mathcal{S}_{00}(\mathbf{C})^{\widehat{\otimes}_\pi^q}\to\mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q) \tag{5.3.3.20}$$

and upon showing (2), then the continuous linear map:

$$\mathcal{S}_{00}(\mathbf{R})^{\widehat{\otimes}_\pi^p}\widehat{\otimes}_\pi\mathcal{S}_{00}(\mathbf{C})^{\widehat{\otimes}_\pi^q}\to\mathcal{S}_I(\mathbf{R}^\times)^{\widehat{\otimes}_\pi^p}\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{C}^\times)^{\widehat{\otimes}_\pi^q}$$

is also a linear topological isomorphism, and hence the composition of the leftmost vertical arrow followed by the bottom horizontal arrow in (5.3.3.18), is a linear topological isomorphism, hence surjective.

This implies that the linear topological embedding from (5.3.3.20) is surjective, and hence a linear topological isomorphism, which implies both continuity and surjectivity of the rightmost vertical arrow, hence proving (1).

Continuity of $\mathcal{S}_{00}(\mathbf{R})\to\mathcal{S}_I(\mathbf{R}^\times)$, $\mathcal{S}_{00}(\mathbf{C})\to\mathcal{S}_I(\mathbf{C}^\times)$ and $\mathcal{S}_{00}(\mathbf{R}^p\times\mathbf{C}^q)\to\mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p\times(\mathbf{C}^\times)^q)$ is seen as follows.

We use Proposition D.3.2.5 and [**CF**, Ch.XV,Thm.2.4.1].

We observe that for any $f\in\mathcal{S}_{00}(\mathbf{R})$, respectively $f\in\mathcal{S}_{00}(\mathbf{C})$, we have, using the same continuous homomorphic sections τ_r, τ_c to the modules $|\cdot|_r$ and $|\cdot|_c$:

$$L_r(f,(s,\omega)):=\int_{\mathbf{R}_{>0}}t^s(f(\tau_r(t))+\omega(-1)f(-\tau_r(t))\,\frac{\mathrm{d}t}{t},\ \ L_c(f,(s,\omega)):=\int_{\mathbf{R}_{>0}}t^s\int_{\mathrm{U}(1)}\omega(e^{i\theta})f(\tau_c(t)e^{i\theta})\,(2|\mathrm{d}\theta|)\,\frac{\mathrm{d}t}{t}$$

are entire functions of s, as ω varies over all continuous characters of $\{\pm1\}$, respectively $\mathrm{U}(1)$.

We have the inequalities, for any real number $a>0$, and for any $\alpha,\beta\in\mathbf{N}$:

$$\sup_{x\in\mathbf{R}^\times}|(1+|\log|x|_r|^\alpha)(x\partial_x)^\beta|x|_r^af(x)|\le\sup_{x\in\mathbf{R}}|(1+|x|_r^\alpha)|x|_r^{a+\beta}\partial_x^\beta f(x)| \tag{5.3.3.21}$$

and:

$$\sup_{z\in\mathbf{C}^\times}|(1+|\log|z|_c|^\alpha)(z\partial_z)^\beta(\overline{z}\partial_{\overline{z}}^\gamma|z|_c^af(z)|\le\sup_{z\in\mathbf{C}}|(1+|z|_c^\alpha)|z|_c^{a+\beta+\gamma}\partial_z^\beta\partial_{\overline{z}}^\gamma f(z)|. \tag{5.3.3.22}$$

By the functional equation from [**CF**, Ch.XV,Thm.2.4.1] and the polynomial growth of $\rho_v(c)$ for v Archimedean (see the proof of Theorem 3.3.1.5), the inequalities in (5.3.3.21) and (5.3.3.22) combined with Asymptotic Residue Theorem (see Theorem B.3.2.1) implies the sequential continuity of $L_r(\cdot,(s,\omega))$, respectively $L_c(\cdot,(s,\omega))$, as linear operators on $\mathcal{S}_{00}(\mathbf{R})$, respectively $\mathcal{S}_{00}(\mathbf{C})$, valued in $A\mathcal{S}(\mathbf{C})\widehat{\otimes}_\pi\mathcal{S}(\widehat{\{\pm1\}})$, respectively $A\mathcal{S}(\mathbf{C})\widehat{\otimes}_\pi\mathcal{S}(\widehat{\mathrm{U}(1)})$, and being these Fréchet spaces, both $L_r(\cdot,(s,\omega))$ and $L_c(\cdot,(s,\omega))$ are continuous (this is so because Fréchet spaces are sequential).

The linear operators $L_r(\cdot,(s,\omega))$ and $L_c(\cdot,(s,\omega))$ are the compositions of the linear inclusions $\mathcal{S}_{00}(\mathbf{R})\to\mathcal{S}_I(\mathbf{R}^\times)$ and $\mathcal{S}_{00}(\mathbf{C})\to\mathcal{S}_I(\mathbf{C}^\times)$, followed by the Mellin transform (see Proposition D.3.2.5), which is a linear topological isomorphism $\mathcal{S}_I(G)\to A\mathcal{S}(\mathbf{C})\widehat{\otimes}_\pi\mathcal{S}(G^1)$ by Theorem D.3.3.7, for $I=(-\infty,+\infty)$ and G an $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group, and hence both linear inclusions $\mathcal{S}_{00}(\mathbf{R})\to\mathcal{S}_I(\mathbf{R}^\times)$ and $\mathcal{S}_{00}(\mathbf{C})\to\mathcal{S}_I(\mathbf{C}^\times)$ are continuous.

To establish the surjectivity assertions in (2), we use Proposition 5.3.3.2, which gives that $\mathcal{S}_I(\mathbf{R}^\times)$ and $\mathcal{S}_I(\mathbf{C}^\times)$ are stable under ordinary smooth, respectively holomorphic and anti-holomorphic, derivation of any order, and that this operation is continuous in each case respectively.

We let $\phi \in \mathcal{S}_I(\mathbf{R}^\times)$. We need to show that the extension by zero of ϕ to $\mathbf{R}$ is smooth at $0 \in \mathbf{R}$. This will prove that such extension by zero is Schwartz over $\mathbf{R}$ (the reader may compare this step in the argument, with the proof of Lemma B.2.1.4, where boundedness and convergence to zero of $\partial_x^m \phi$ as $|x| \to 0^+$ is obtained by direct calculation).

We know that, for every integer $m \geq 0$, $\partial_x^m \phi$ is continuous on $\mathbf{R}^\times$ and tends to 0 as $|x| \to 0$. We have yet to show that the derivative of ϕ exists at $0 \in \mathbf{R}$ and agrees with the limit value of $\partial_x^m \phi$ (which is zero).

This is seen by observing that we have, over $(-A, A)$:

$$\left|\frac{\phi(x)}{x}\right| \leq \frac{1}{2A}\int_{-A}^{A} |(\partial_x \phi)(y)|\,\mathrm{d}y \leq \frac{C}{2A}\int_{-A}^{A} |y|^M\,\mathrm{d}y \tag{5.3.3.23}$$

for some integer $M > 1$ and a real number $C > 0$, which gives the desired result, as $A \to 0^+$. To argue that we have $|\partial_x \phi(y)| \leq C|y|^M$ for $y \in (-A, A)$, we used boudedness of M_x and continuity of ∂_x on $\mathcal{S}_I(\mathbf{R}^\times)$, to deduce that $\partial_x \phi$ is of the form $x^M u$ for some $u \in \mathcal{S}_I(\mathbf{R}^\times)$, and hence $x^M u$ tends to 0 as $|x| \to 0^+$, giving boundedness of u over $(-A, A)$ by some $C > 0$, and the bound $|\partial_x \phi(y)| \leq C|y|^M$ over $(-A, -\varepsilon) \cup (\varepsilon, A)$, propagated to $(-A, A)$ by continuity.

The middle improper integral in (5.3.3.23) therefore converges and the inequalities in (5.3.3.23) hold.

Upon replacing ϕ with $\partial_x^m \phi$ for any $m \geq 0$, which is possible by Proposition 5.3.3.2, we deduce that the derivative of the extension by zero of ϕ to $\mathbf{R}$ exists at 0 and vanishes, therefore showing that such extension by zero is smooth and Schwartz. This shows surjectivity of the restriction operator $\mathcal{S}_{00}(\mathbf{R}) \to \mathcal{S}_I(\mathbf{R}^\times)$.

A similar argument shows that the restriction operator $\mathcal{S}_{00}(\mathbf{C}) \to \mathcal{S}_I(\mathbf{C}^\times)$, where the middle integral in (5.3.3.23) is replaces with the line integral from the origin to an arbitrary point z along an arbitrary smooth rectifiable curve $\gamma : [0, 1] \to \mathbf{C}$. This proves that the directional derivative along γ' of the extension by zero of ϕ exists at $z = 0$ and vanishes there, proving smoothness at $z = 0$.

To conclude the proof of (3), we observe that, since we have shown that the linear restriction operator $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q) \to \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$ is a linear topological isomorphism, since continuous quasi-characters of $(\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q$ are of the form $\bigotimes_{1 \leq k \leq p} c_k \otimes \bigotimes_{1 \leq m \leq q} c'_m$, for c_k a continuous quasi-character of $\mathbf{R}^\times$ for each $1 \leq k \leq p$, and c'_m a continuous quasi-character of $\mathbf{C}^\times$ for $1 \leq m \leq q$, then, using the linear topological isomorphisms $\mathcal{S}_{00}(\mathbf{R}) \mathbin{\widehat{\otimes}_\pi^p} \mathcal{S}(\mathbf{C}) \mathbin{\widehat{\otimes}_\pi^q} \to \mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$ and $\mathcal{S}_I(\mathbf{R}^\times) \mathbin{\widehat{\otimes}_\pi^p} \mathcal{S}_I(\mathbf{C}^\times) \mathbin{\widehat{\otimes}_\pi^q} \to \mathcal{S}_{I^{p+q}}((\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q)$, it is enough to treat the cases $(p, q) = (1, 0)$, $(0, 1)$, in order to show that pointwise multiplication by continuous quasi-characters of $(\mathbf{R}^\times)^p \times (\mathbf{C}^\times)^q$ on $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$ is valued in $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$, continuous and (hence) continuously invertible.

Continuous unitary characters on $\mathbf{R}^\times$, respectively $\mathbf{C}^\times$, are smooth, bounded, with smooth and bounded $(x\partial_x)^m$-derivatives, respectively $(z\partial_z)^m$ and $(\overline{z}\partial_{\overline{z}})^m$-derivatives, for every integer $m \geq 0$. Continuity and continuous invertibility of M_x on $\mathcal{S}_I(\mathbf{R}^\times)$ and of M_z, $M_{\overline{z}}$ on $\mathcal{S}_I(\mathbf{C}^\times)$, therefore show, by composition, that indeed multiplication on $\mathcal{S}_{00}(\mathbf{R})$, respectively $\mathcal{S}_{00}(\mathbf{C})$, by a continuous unitary character of $\mathbf{R}^\times$, respectively $\mathbf{C}^\times$, is a continuous linear operator. The case of arbitrary continuous quasi-characters then follows from Proposition 5.3.3.2(1). □

We conclude with the following observation:

PROPOSITION 5.3.3.4. *We parametrize $\Omega(\mathbf{R}^\times)$ and $\Omega(\mathbf{C}^\times)$ as per* (3.1.2.3) *from* (3.1.2.5) *upon choosing the canonical continuous homomorphic section to the real absolute value on $\mathbf{R}^\times$, and the continuous homomorphic section $t \mapsto t^{1/2}$ to the complex absolute value on $\mathbf{C}^\times$.*

Recall that we set:

$$\rho_r(s, 1) := \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}, \quad \rho_r(s, \mathrm{sgn}) := i\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)} \tag{5.3.3.24}$$

and, for each integer $n \in \mathbf{Z}$:

$$\rho_c(s, c_n) := (-1)^{|n|} \frac{(2\pi)^{1-s}\Gamma\left(s + \frac{|n|}{2}\right)}{(2\pi)^s\Gamma\left((1-s) + \frac{|n|}{2}\right)} \tag{5.3.3.25}$$

where $c_n : \mathrm{U}(1) \to \mathrm{U}(1)$ is the continuous character $\theta \mapsto e^{in\theta}$, and the subscript r reads "real" and the subscript c reads "complex".

We further define:

(5.3.3.26)

$$L_r(f, (s, \omega)) := \int_{\mathbf{R}_{>0}} t^s (f(\tau_r(t)) + \omega(-1) f(-\tau_r(t)) \frac{\mathrm{d}t}{t}, \;\; L_c(f, (s, \omega)) := \int_{\mathbf{R}_{>0}} t^s \int_{\mathrm{U}(1)} \omega(e^{i\theta}) f(\tau_c(t) e^{i\theta}) \, (2|\mathrm{d}\theta|) \, \frac{\mathrm{d}t}{t}$$

as meromorphic functions on $\Omega(\mathbf{R}^\times)$, respectively $\Omega(\mathbf{C}^\times)$.

Then, for $n, p, q \geq 0$ integers, the continuous operators:

$$M_{x^n} : \mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R}), \;\; M_{z^p \overline{z}^q} : \mathcal{S}(\mathbf{C}) \to \mathcal{S}(\mathbf{C}) \tag{5.3.3.27}$$

satisfy the following properties:

(1) *As meromorphic functions on $\Omega(\mathbf{R}^\times)$, the following identity holds:*

$$L_r(f, (s, \omega)) = L_r(f, (s + n, \mathrm{sgn}(x)^{\epsilon(n)}\omega)) \tag{5.3.3.28}$$

where $\epsilon(n) = 0$ if n is even, and $\epsilon(n) = 1$ if n is odd. We further have:

$$L_r(M_{x^n} f, (s, \omega)) = \rho_r(s + n, \mathrm{sgn}^{\epsilon(n)}\, \omega) L_r(\widehat{f}, (1 - s - n, \mathrm{sgn}(x)^{\epsilon(n)}\omega)). \tag{5.3.3.29}$$

(2) *As meromorphic functions on $\Omega(\mathbf{C}^\times)$, the following identity holds for every integer $n \in \mathbf{Z}$:*

$$L_c(M_{z^p \overline{z}^q} f, (s, c_n)) = L_c(f, (s + p/2 + q/2, c_{n+p-q})). \tag{5.3.3.30}$$

We further have:

$$L_c(M_{z^p \overline{z}^q} f, (s, c_n)) = \rho_c(s + p/2 + q/2, c_{n+p-q}) L_c(\widehat{f}, (1 - s - p/2 - q/2, c_{-n-p+q})). \tag{5.3.3.31}$$

Proof. (1) : We compute:

$$L_r\left(M_{x^n} f, (s, \omega)\right) = \int_{\mathbf{R}^\times} (M_{x^n} f)(x) \omega(x) |x|_r^s \frac{|\mathrm{d}x|}{|x|_r} \tag{5.3.3.32}$$

$$= \int_{\mathbf{R}^\times} f(x) \mathrm{sgn}(x)^{\epsilon(n)} \omega(u) |u|_r^{s+n} \frac{|\mathrm{d}u|}{|u|_r} \tag{5.3.3.33}$$

$$= L_r(f, (s + n, \mathrm{sgn}(x)^{\epsilon(n)}\omega)). \tag{5.3.3.34}$$

We also have, by the local functional equation from [**CF**, Ch.XV,Thm.2.4.1]:

$$L_r(M_{x^n} f, (s, \omega)) = \rho_r(s + n, \mathrm{sgn}^{\epsilon(n)}\, \omega) L_r(\widehat{f}, (1 - s - n, \mathrm{sgn}(x)^{\epsilon(n)}\omega)) \tag{5.3.3.35}$$

where $\epsilon(n) = 0$ if n is even, and, and $\epsilon(n) = 1$ if n is odd.

(2) : Recall that the complex absolute value $|\cdot|_c$ is the square of the complex Euclidean norm.

We further recall the following calculation:

$$L_c(M_{z^p\overline{z}^q}f,(s,c_n)) = \int_{\mathbf{C}^\times}(M_{z^p\overline{z}^q}f(z))c_n(z)|z|_c^s\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|_c} \tag{5.3.3.36}$$

$$= \int_{\mathbf{C}^\times} z^p\overline{z}^q f(z)c_n(z)|z|_c^s\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|_c} \tag{5.3.3.37}$$

$$= \int_{\mathbf{C}^\times} f(z)c_{n+p-q}(z)|z|_c^{s+p/2+q/2}\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|_c} \tag{5.3.3.38}$$

$$= L_c(f,(s+p/2+q/2,c_{n+p-q})). \tag{5.3.3.39}$$

We also have:

$$L_c(M_{z^p\overline{z}^q}f,(s,c_n)) = \rho_c(s+p/2+q/2,c_{n+p-q})L_c(\widehat{f},(1-s-p/2-q/2,c_{-n-p+q})). \tag{5.3.3.40}$$

All identities hold for $s\in\mathbf{C}_{>0}$, and hold as meromorphic functions over the complex s-plane by analytic continuation. □

We will be viewing M_{x^n} and $M_{z^p\overline{z}^q}$ as analogs for Archimedean local fields, of the operators PV_v for v non-Archimedean. The reason is that their application resolves the singularities of the Archimedean local L-functions, while PV_v does so for the non-Archimedean local L-functions.

5.3.4. Weil's Principal Value Operators and holomorphic S-idelic Zeta functions. We now discuss how to produce examples of S-idelic Zeta functions from Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$, with a prescribed divisor of poles. This theme will be reprised in the subsequent two Chapters, which will discuss poles and zeros in depth: Schauder bases arising from poles of S-idelic Zeta functions and zeros of global idelic Zeta functions are one of the principal instruments throughout the trace calculation which is the main subject of study of this work.

Specifically, before we learn how to construct Schwartz-Bruhat functions that satisfy prescribed divisorial conditions, we will now discuss how to resolve the singularities of their L-functions.

The Definition below exhibits a rich source of Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$ whose Mellin transform L_S is entire. We will then use Weil's Principal Value Operators to modify an arbitrary Schwartz-Bruhat function on $\mathbf{A}_{F,S}$ into one whose only poles arise from its Archimedean component, and these, we will classify completely at the end of this subsection.

This completes the preparations for the Second Structrue Theorem (see Theorem 5.4.1.1), which is in turn the stepping stone for the remaining Structure Theorems throughout this Chapter.

Definition 5.3.4.1. For F a number field and S a finite set of places of F containing all the Archimedean places.

We choose, for each real place v of F, an isomorphism of topological fields $F_v\simeq\mathbf{R}$, and for each complex place v of F, an isomorphism of topological fields $F_v\simeq\mathbf{C}$.

We define the closed linear subspace $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})\subset\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ as the pre-image along the pullback by the induced isomorphism of Hausdorff locally compact abelian groups $\mathbf{A}_{F,S_\infty}\simeq\mathbf{R}^{r_1}\times\mathbf{C}^{r_2}$, of the closed subspace $\mathcal{S}_{00}(\mathbf{R}^{r_1}\times\mathbf{C}^{r_2})\subset\mathcal{S}(\mathbf{R}^{r_1}\times\mathbf{C}^{r_2})$.

We define the following linear subspace of $\mathcal{S}(\mathbf{A}_{F,S})$:

$$\mathcal{S}_{00}(\mathbf{A}_{F,S}) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)\subset\mathcal{S}(\mathbf{A}_{F,S}). \tag{5.3.4.1}$$

We refer to $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ as the subspace of *polar-free Schwartz-Bruhat functions on* $\mathbf{A}_{F,S}$.

Lemma 5.3.4.2. *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

(1) *The linear subspace* $\mathcal{S}_{00}(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ *is closed, stable under the continuous left-translation action of* $\mathrm{GL}_1(\mathbf{A}_{F,S})$ *on* $\mathcal{S}(\mathbf{A}_{F,S})$*, and, as a subspace of* $\mathcal{S}(\mathbf{A}_{F,S})$*, independent of the choice of isomorphisms of topological fields* $F_v \simeq \mathbf{R}$*, for* v *real, and* $F_v \simeq \mathbf{C}$*, for* v *complex.*

(2) *For every* $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$*,* $Z_S(f,c)$ *is entire over* $\Omega(J_{F,S})$*.*

(3) *pointwise multiplication of each element of* $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ *by any continuous quasi-character* $c \in \Omega(\mathrm{GL}_1(\mathbf{A}_{F,S}))$ *invariant under* $\mathcal{O}_{F,S}^{\times}$*, is valued in* $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ *and defines a continuous, linear and continuously invertible endomorphism* M_c *on* $\mathcal{S}_{00}(\mathbf{A}_{F,S})$*.*

PROOF. (1) : The linear subspace $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \subset \mathcal{S}(\mathbf{A}_{F,S_\infty})$ is closed by Lemma 5.3.3.3(3), and since $\mathcal{S}(F_v^{\times}) \subset \mathcal{S}(F_v)$ is closed for each $v \in S_f$ since $\mathcal{S}(F_v)$ carries the finest locally convex topology by Lemma 2.1.2.4(1), then by [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], which applies since each tensor factor is nuclear, $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is closed, as a consequence of completeness of the projective tensor product in (5.3.4.1) which is seen as for the proof of Proposition 2.1.4.1, whose method applies since $\bigotimes_{v\in S_f} \mathcal{S}(F_v^{\times})$ also carries the finest locally convex topology (see Lemma C.2.5.2).

Stability under the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathcal{S}(\mathbf{A}_{F,S})$ is a consequence of the fact that both $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$ is stable under the continuous left-translation action of $(\mathbf{R}^{\times})^p \times (\mathbf{C}^{\times})^q$ on $\mathcal{S}(\mathbf{R}^p \times \mathbf{C}^q)$, and $\mathcal{S}(F_v^{\times})$ is stable under the continuous left-translation action of $F_v^{\times}$ on $\mathcal{S}(F_v)$ for any finite place $v \in S$.

Any isomorphism of topological fields $F_v \simeq \mathbf{R}$, if v is real, and $F_v \simeq \mathbf{C}$, if v is complex, induces an isomorphism of Hausdorff locally compact abelian groups $i : \mathbf{A}_{F,S_\infty} \simeq \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$. We choose, for each Archimedean place v, two such isomorphisms i_v, i'_v.

We call $\iota : \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \to \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ the isomorphism of Hausdorff locally compact abelian groups induced by $i_v \circ i_v^{-1}$ for each Archimedean place v of F.

i is therefore smooth, being source and target abelian Lie groups, and preserves the $\mathbf{C}$-algebras of differential operators on source and target.

Each is isomorphic to the tensor product of the algebras of differential operators on each of the factors $\mathbf{R}$ and $\mathbf{C}$, and since ι is diagonal, and it is the identity on each real component, and either the identity or complex conjugation on each complex component, it preserves the subspace $\mathcal{S}_{00}(\mathbf{R}^{r_1} \times \mathbf{C}^{r_2})$.

It follows that the vanishing conditions defining $\mathcal{S}_{00}(\mathbf{R}^{r_1} \times \mathbf{C}^{r_2})$ as a subspace of $\mathcal{S}(\mathbf{R}^{r_1} \times \mathbf{C}^{r_2})$, define on $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ vanishing conditions defining $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ that do not depend on the choice of isomorphisms $F_v \simeq \mathbf{R}$ and $F_v \simeq \mathbf{C}$.

(2) : Let $f = \bigotimes_{v\in S} f_v$ be a Schwartz-Bruhat function over $\mathbf{A}_{F,S}$, with $f_v \in \mathcal{S}(F_v)$ for each place $v \in S$.

Assume $Z_S(f,c)$ is entire over $\Omega(J_{F,S})$. As meromorphic functions on $\Omega(J_{F,S})$, we have $Z_S(f,c) = \prod_{v\in S} Z_v(f_v, c_v)$ where $c_v := \psi_v(c)$, with $\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^{\times})$ the surjective local analytic isomorphism defined by restricting the continuous quasi-character c of $J_{F,S}$ to the closed subgroup $F_v^{\times} \subset J_{F,S}$.

Lemma 3.3.2.1 and Lemma 3.3.2.2 provide us with formulas for the residues of $Z_v(f_v, c_v)$ for each place $v \in S$. Since $Z_S(f,c)$ is entire, then for every Archimedean place v of F contained in S we have, if v is real, $\delta_0^{(k)}(f_v) = 0$ for every integer $k \geq 0$, from (3.3.2.6) and (3.3.2.7) of Lemma 3.3.2.1(1), and $\partial_z^k \partial_{\bar{z}}^{k'} \delta_0(f_v) = 0$ for every pair of integers $(k,k') \in \mathbf{N}^2$ if v is complex, in light of (3.3.2.8) and (3.3.2.9) of Lemma 3.3.2.1(2).

By Lemma 3.3.2.2, we further have $\delta_0(f_v) = 0$ for every non-Archimedean place v of F contained in S. It follows that we have $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$.

Since L_S is continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S})$ and its closed subspaces, on the source, and the Hausdorff locally convex topology on the linear space $\mathscr{M}(\Omega(J_{F,S}))$ of meromorphic functions on the Riemann surface $\Omega(J_{F,S})$ of uniform convergence on the compact subsets of $\Omega(J_{F,S}) - \mathcal{U}_S$, where $\mathcal{U}_S$ is the subset of $\Omega(J_{F,S})$ of the poles of the meromorphic function $\rho_S(c)$ from (3.3.1.3) (see Lemma 5.1.3.1(2)), then the closure in $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ of the linear subspace linearly generated by the $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ of the above form is comprised of elements $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ such as $Z_S(f,c)$ is also entire.

By the design of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, we deduce from [**Jar**, Part III,Ch.15,§15.2,Prop.3(a),p.327] that such closure is $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, which proves (2).

(3) : Let $c \in \Omega(\mathrm{GL}_1(\mathbf{A}_{F,S}))$ be an $\mathcal{O}_{F,S}^\times$-invariant continuous quasi-character. Equivalently, a continuous quasi-character in $\Omega(J_{F,S})$, composed with the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S}) \to J_{F,S}$. We extend c by zero to a function on $\mathbf{A}_{F,S}$, which is nowhere vanishing on the open subset $\mathrm{GL}_1(\mathbf{A}_{F,S}) \subset \mathbf{A}_{F,S}$.

We call c_v the restriction of c to $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$, embedded as a closed subgroup through the insertion homomorphism into the v-indexed coordinate, with w-indexed coordinates defined to be 1 for every $w \in S - \{v\}$.

Let $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ be of the form $f_\infty \otimes \bigotimes_{v \in S_f} f_v$ for $f_\infty \in \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ and $f_v \in \mathcal{S}(F_v^\times)$. We note that $c_v \cdot f_v$ is also Schwartz- Bruhat over F_v, because c_v is locally constant and f_v is locally constant and compactly supported, and it is therefore an element of the closed subspace $\mathcal{S}(F_v^\times)$.

Multiplication by c_v is continuous on $\mathcal{S}(F_v^\times)$ for each finite place $v \in S$, since $\mathcal{S}(F_v^\times)$ carries the finest locally convex topology. It is in fact bounded, since c_v is bounded on every compact subset of $F_v^\times$.

We now call c_∞ the restriction of c to $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$. Any $f_\infty \in \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ vanishes at each of the hyperplanes $\{x_v = 0\}$ for each Archimedean place v of F, and hence $c_\infty \cdot f_\infty$ is a smooth function on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ vanishing on each $\{x_v = 0\}$.

pointwise multiplication by any quasi-character of $\mathrm{GL}_1(\mathbf{R}^p \times \mathbf{C}^q)$ of any element of $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$ is valued in $\mathcal{S}_{00}(\mathbf{R}^p \times \mathbf{C}^q)$, linear, continuous and continuously invertible by Lemma 5.3.3.3(3).

Since $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is an LF-space, by Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]) it suffices to show that M_c is surjective, in order to show that it is a linear topological isomorphism, in which case its inverse is computed by pointwise multiplication by c^{-1}.

For each finite place $v \in S$, multiplication by c_v is surjective on $\mathcal{S}(F_v^\times)$, since multiplication by c_v^{-1} is also continuous (in fact, bounded) and hence multiplication by c_v is boundedly invertible on $\mathcal{S}(F_v^\times)$.

This shows that every $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ of the form $f_\infty \otimes \bigotimes_{v \in S_f} f_v$ for $f_\infty \in \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ and $f_v \in \mathcal{S}(F_v^\times)$ for each finite place v of F, is in the image of M_c for every $\sigma(c) \geq 0$, and since M_c is linear and continuous and these Schwartz-Bruhat functions linearly generate a dense subspace of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, by [**Jar**, Part III,Ch.15,§15.2,Prop.3(a),p.327], then M_c is surjective, hence a linear topological isomorphism on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$. □

LEMMA 5.3.4.3. *Let $I := (-\infty, +\infty) = \mathbf{R}$, and let $\phi \in \mathcal{S}_I(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function with I-exponential decay.*

Let F be a number field, S a finite set of places of F containing all the Archimedean places.

There exists $f \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that:

$$\Phi_{F,S,1}(f) = \phi. \tag{5.3.4.2}$$

PROOF. We choose a continuous homomorphic section τ to the S-idelic norm.

The canonical linear map:

$$\mathcal{S}(\mathbf{A}_{F,S}) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v \in S_f} \mathcal{S}(F_v^\times) \to \mathcal{S}_I(\mathbf{R}^\times)^{\widehat{\otimes}_\pi^{r_1}} \widehat{\otimes}_\pi \mathcal{S}_I(\mathbf{C}^\times)^{\widehat{\otimes}_\pi^{r_2}} \otimes_\pi \bigotimes_{v \in S_f} \mathcal{S}(F_v^\times) \tag{5.3.4.3}$$

defined as the unique extension by continuity of the tensor product of the linear topological isomorphism

$$\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \to \mathcal{S}_{00}(\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}) \to \mathcal{S}_I((\mathbf{R}^\times)^{r_1}) \widehat{\otimes}_\pi \mathcal{S}_I((\mathbf{C}^\times)^{r_2}) \tag{5.3.4.4}$$

provided by Lemma 5.3.3.3(3) and (2), with the identity operator on $\bigotimes_{v \in S_f} \mathcal{S}(F_v^\times)$, is itself a linear topological isomorphism, since the tensor factors are nuclear, and [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348] applies.

We note that the linear topological isomorphism from (5.3.4.3) is compatible with the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on the source and of $\mathrm{GL}_1(\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}) \times \prod_{v \in S_f} F_v^\times$ on the target.

By Lemma D.2.2.2(1), we further have a linear topological isomorphism:

$$\mathcal{S}_I((\mathbf{R}^\times)^{r_1})\,\widehat{\otimes}_\pi\mathcal{S}_I((\mathbf{C}^\times)^{r_2}) \to \mathcal{S}(\{\pm 1\})^{\widehat{\otimes}_\pi^{r_1}}\,\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))^{\widehat{\otimes}_\pi^{r_2}}\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}} \tag{5.3.4.5}$$

and the tensor product with the identity operator on $\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)$ is a linear topological isomorphism:

$$\mathcal{S}_{00}(\mathbf{A}_{F,S}) \to \mathcal{S}(\{\pm 1\})^{\widehat{\otimes}_\pi^{r_1}}\,\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))^{\widehat{\otimes}_\pi^{r_2}}\,\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}\,\otimes_\pi\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times) \tag{5.3.4.6}$$

by [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348], again using nuclearity of the tensor factors to apply the Corollary, which, followed by the tensor product of the identity operator on the first $2r_1+2r_2$ tensor factors, with the tensor product of the linear topological isomorphisms $\mathcal{S}(F_v^\times)\simeq\mathcal{S}(\mathcal{O}_{F_v}^\times)\,\widehat{\otimes}_\pi\mathcal{S}(\pi_v^{\mathbf{Z}})$ for a choice of uniformizer π_v at v and for each finite place $v\in S$, is the linear topological isomorphism (by another application of [**Jar**, Part III,Ch.16,§16.2,Cor.3,p.348]):

$$\mathcal{S}_{00}(\mathbf{A}_{F,S}) \to \mathcal{S}(\{\pm 1\})^{\widehat{\otimes}_\pi^{r_1}}\,\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))^{\widehat{\otimes}_\pi^{r_2}}\,\widehat{\otimes}_\pi\bigotimes_{v\in S_f}\mathcal{S}(\mathcal{O}_{F_v}^\times)\otimes_\pi\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}\,\widehat{\otimes}_\pi\bigotimes_{v\in S_f}\mathcal{S}(\pi_v^{\mathbf{Z}}) \tag{5.3.4.7}$$

also compatible with the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on the source, and the continuous action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on the target uniquely determined by the actions through its quotients, on each tensor factor, extended by continuity.

By Theorem D.1.2.1, the operator $\Phi_{F,S,1}$ agrees, up to linear topological automorphisms of the source, on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, with the unique extension by continuity of the tensor product:

$$\Phi_{\{\pm 1\}}^{\otimes_\pi^{r_1}}\otimes_\pi\Phi_{\mathrm{U}(1)}^{\otimes_\pi^{r_2}}\otimes_\pi\bigotimes_{v\in S_f}\Phi_{\mathcal{O}_{F_v}^\times}\otimes_\pi\Phi_{\log}\otimes_\pi\bigotimes_{v\in S_f}\Phi_v \tag{5.3.4.8}$$

where:

(1) $\Phi_{\log}$ is the operator sending $\psi\in\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}$ to $\sum_{\xi\in\mathcal{O}_F^\times/\mu_F}\phi((|\xi_\infty x_\infty|_v)_{v\in S_\infty})$ where μ_F is the cyclic subgroup of $\mathcal{O}_F^\times$ of the roots of unity contained in F. The subscript y_∞ denotes the projection of $y\in\mathrm{GL}_1(\mathbf{A}_{F,S})$ onto $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.
(2) All the the other operators Φ_G are $\widehat{\delta_1}$ (i.e. the Fourier transform, in the sense of the Fourier transform on the Schwartz-Bruhat space $\mathcal{S}(G)$ of a Hausdorff locally compact abelian group G, of the evaluation-at-the-neutral-element continuous functional $\delta_1\in\mathcal{S}(G)'_\beta$, a tempered distribution; see §D.1).

The identification of the operators $\Phi_{F,S,1}$ and of the operator from (5.3.4.8) is up to a linear topological automorphism of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ only, which is the following one: we choose $\{g_1,\dots,g_{N_S}\}$ are representatives in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ of the N_S ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$ that are trivialized in $\mathrm{Pic}(\mathcal{O}_{F,S})$, and for $f\in\mathcal{S}_{00}(\mathbf{A}_{F,S})$ of the form $f_\infty\otimes\bigotimes_{v\in S_f}f_v$, with $f_v\in\mathcal{S}(F_v^\times)$ for each finite place $v\in S$, we have:

$$\Phi_{F,S,1}(f)(x)=\sum_{1\le i\le N_S}\int_{g\in E_{S_\infty,0}}\sum_{\xi\in\mathcal{O}_F^\times}f_\infty(\xi(g_i)_\infty g x_\infty)\,\mathrm{d}\mu_{S_\infty,1}(g)\cdot\sum_{\xi'\in\mathcal{O}_{F,S}^\times/\mathcal{O}_F^\times}\prod_{v\in S_f}\int_{g_v\in\mathcal{O}_{F_v}^\times}\sum_{m_v\in\mathbf{Z}}f_v(\xi' g_v(g_i)_v x_v)\,\mathrm{d}\mu_v(g_v) \tag{5.3.4.9}$$

writing $x\in\mathbf{A}_{F,S}$ as $(x_\infty,(x_v)_{v\in S_f})$, where $E_{S_\infty,0}$ is the fundamental domain from (A.2.4.5) for the continuous left translation action of $\mathcal{O}_F^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.

Since $(\prod_{v\in S_f}\mathcal{O}_{F_v}^\times)\cdot(\mathcal{O}_{F,S}^\times/\mathcal{O}_F^\times)$ has finite index h_F in $\prod_{v\in S_f}F_v^\times$, we may choose the g_i's so as to be of the form $(1,(g_{i,v})_{v\in S_f})$, i.e. each of their projections to $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is 1.

Upon making this choice, the value, for each $x\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, of the integral:

$$\int_{g\in E_{S_\infty,0}}\sum_{\xi\in\mathcal{O}_F^\times}f_\infty(\xi(g_i)_\infty g x_\infty)\,\mathrm{d}\mu_{S_\infty,1}(g) \tag{5.3.4.10}$$

does not depend on i and equals $\int_{g\in E_{S_\infty,0}}\sum_{\xi\in\mathcal{O}_F^\times} f_\infty(\xi g x_\infty)\,\mathrm{d}\mu_{S_\infty,1}(g)$. The innermost sub-sum over $\mathcal{O}_F^\times/\mu_F$ is, we now recognize, the value of $\Phi_{\log}$ on f_∞. The integral over $\mu_F\cdot E_{S_\infty,0}$ can now be recognized as being, for each $x_\infty\in\mathbf{A}_{F,S_\infty}$, the operator $\Phi_{\{\pm1\}}^{\otimes_\pi^{r_1}}\otimes_\pi\Phi_{\mathrm{U}(1)}^{\otimes_\pi^{r_2}}$ (see the construction of P_S from (A.2.4.8)), and hence the integral in (5.3.4.10) (upon setting $(g_i)_\infty=1$) agrees with the continuous extension to $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ of $\Phi_{\{\pm1\}}^{\otimes_\pi^{r_1}}\otimes_\pi\Phi_{\mathrm{U}(1)}^{\otimes_\pi^{r_2}}\otimes_\pi\Phi_{\log}$ on $\mathcal{S}(\{\pm1\})^{\widehat{\otimes}_\pi^{r_1}}\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))^{\widehat{\otimes}_\pi^{r_2}}\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}$.

The second factor in (5.3.4.9) can now be recognized as being, by another application of Theorem D.1.2.1, $\bigotimes_{v\in S_f}(\Phi_{\mathcal{O}_{F_v}^\times}\otimes\Phi_v)$ on $\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)$.

Making the choice of integrating over $E_{S_\infty,0}$ and the choice of the $\{g_1,\dots,g_{N_S}\}$ amounts to pre-composing $\Phi_{F,S,1}$ by a linear topological automorphism on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ (which, in fact and upon inspection, can be seen to be the identity).

We may choose the continuous homomorphic section τ to the S-idelic norm to have image in the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ (for example, by choosing it to be of the form $\tau=\tau_{v_0}$, for v_0 a fixed choice of Archimedean place of F; see Definition A.2.2.5, where τ_{v_0} is denoted s_{F,v_0}).

The linear space $\mathcal{D}(\mathbf{R}_{>0}^{r_1+r_2})$ is contained in $\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}$ (up to the linear topological isomorphism to the nuclear Fréchet space $\mathcal{S}_{I^{r_1+r_2}}(\mathbf{R}_{>0}^{r_1+r_2})$ defined similarly as in Lemma 5.3.3.3(2) (see also the proof of Lemma 5.3.3.3(3)), compatibly with the continuous left-translation action of $\mathbf{R}_{>0}^{r_1+r_2}$ on itself, and by [**Tre**, Part III,Ch.39,Thm.39.2], the choice of continuous homomorphic section τ to the S-idelic norm induces a continuous linear operator $\mathcal{D}(\mathbf{R}^{r_1+r_2-1})\otimes\mathcal{D}(\mathbf{R}_{>0})$ with sequentially dense range into $\mathcal{D}(\mathbf{R}_{>0}^{r_1+r_2})$, where

$$(\mathbf{R}_{>0}^{r_1+r_2})^1:=\ker\Big(\prod_{1\le k\le r_1}\prod_{1\le m\le r_2}|\cdot|_{r,k}|\cdot|_{c,m}:\mathbf{R}^{r_1+r_2}\to\mathbf{R}_{>0}\Big)$$

acts by continuous left-translation on $\mathbf{R}_{>0}^{r_1+r_2-1}$ up to the isomorphism $(\mathbf{R}_{>0}^{r_1+r_2})^1\simeq\mathbf{R}_{>0}^{r_1+r_2-1}$, and $\mathbf{R}_{>0}$ acts on $\mathcal{D}(\mathbf{R}_{>0})$ by continuous left-translation.

We now observe that $\Phi_{\log}$ can therefore further be written on $\mathcal{D}(\mathbf{R}^{r_1+r_2-1})\otimes\mathcal{S}_I(\mathbf{R}_{>0})\subset\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}$ (the containment being intended up to linear topological automorphism on the source) as $\Phi_{r_1+r_2-1}\otimes\mathrm{Id}$.

We then choose φ_0 in $\mathcal{D}(\{\pm1\})\simeq\mathbf{C}^2$ such that $\Phi_{\{\pm1\}}(\varphi_0)=1$, φ_1 in $\mathcal{D}(\mathrm{U}(1))$ such that $\Phi_{\mathrm{U}(1)}(\varphi_1)=1$, $\varphi_v\in\mathcal{D}(\mathcal{O}_{F_v}^\times)$ such that $\Phi_{\mathcal{O}_{F_v}^\times}(\varphi_v)=1$, and $\varphi_{\log}\in\mathcal{D}(\mathbf{R}^{r_1+r_2-1})$ such that $\Phi_{r_1+r_2-1}(\varphi_{\log})=1$. We therefore have that $\varphi_{\log}\otimes\phi$ is mapped by an equivariant linear topological isomorphism to an element $\psi\in\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2}}$, such that $\Phi_{\log}(\psi)=\phi$, where $\phi\in\mathcal{S}_I(\mathbf{R}_{>0})$ is our original Schwartz-Bruhat function on $\mathbf{R}_{>0}$ with exponential decay.

In the former cases, φ_0,φ_1 and φ_v for each $v\in S_f$ exist by compactness of each group, respectively, and in the latter case, $\varphi_{\log}$ exists because $\mathcal{O}_F^\times/\mu_F$ acts discretely, properly and freely, with co-compact quotient, on $\mathbf{R}^{r_1+r_2-1}\simeq\{x\in\mathbf{R}_{>0}^{r_1+r_2}:\prod_{1\le k\le r_1}|x|_{r,k}\cdot\prod_{1\le k\le r_2}|x|_{c,k}=1\}$.

Similarly, we choose $\psi_v\in\mathcal{S}(\pi_v^{\mathbf{Z}})$ such that $\Phi_v(\psi_v)=1$, which we can do because $\pi_v^{\mathbf{Z}}$ acts properly, freely and with co-compact quotient (a single point) on itself.

We now define:

$$\varphi_0^{\otimes_\pi^{r_1}}\otimes\varphi_1^{\otimes_\pi^{r_2}}\otimes\bigotimes_{v\in S_f}\varphi_v\otimes\psi\otimes\bigotimes_{v\in S_f}\psi_v \tag{5.3.4.11}$$

and element of $\mathcal{S}(\{\pm1\})^{\widehat{\otimes}_\pi^{r_1}}\widehat{\otimes}_\pi\mathcal{S}(\mathrm{U}(1))^{\widehat{\otimes}_\pi^{r_2}}\widehat{\otimes}_\pi\bigotimes_{v\in S_f}\mathcal{S}(\mathcal{O}_{F_v}^\times)\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})^{\widehat{\otimes}_\pi^{r_1+r_2-1}}\widehat{\otimes}\bigotimes_{v\in S_f}\mathcal{S}(\pi_v^{\mathbf{Z}})\widehat{\otimes}_\pi\mathcal{S}_I(\mathbf{R}_{>0})$.

The element from (5.3.4.11) corresponds under the linear topological isomorphism from (5.3.4.6) to a unique Schwartz-Bruhat function $f\in\mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that $\Phi_{F,S,1}(f)=\phi$. □

Proposition 5.3.4.4. *Let F be a number field. Let S be a finite set of places of F containing all the Archimedean places. Let $\chi\in\widehat{J_{F,S}}$ be a continuous character.*

The following conditions on Schwartz-Bruhat functions $f\in\mathcal{S}(\mathbf{A}_{F,S})$ are equivalent:

(1) *$Z_S(f,c)$ is entire over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$.*
(2) *There exists $f_0 \in \mathcal{S}_{00}(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ such that $Z_S(f,c) = Z_S(f_0,c)$ over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$.*
(3) *$\Phi_{F,S,\chi|_{J^1_{F,S}}}(f) \in \mathcal{S}_{(-\infty,\infty)}(J_{F,S})(\chi|_{J^1_{F,S}})$.*

PROOF. Assume that (1) holds. If $Z_S(f,c)$ is entire over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$, then the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1, combined with the polynomial growth of $\rho_S(c)$ (see Theorem 3.3.1.5) over the open vertical strips of bounded width of connected component parametrized by $\chi|_{J^1_{F,S}}$ of the Riemann surface $\Omega(J_{F,S}) - \mathcal{U}_S$, implies, by the Asymptotic Residue Theorem (see Theorem B.3.2.1), that $Z_S(f,c)$ is Schwartz uniformly over the relative vertical strips of Bounded width within $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$, and hence $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f) \in \mathcal{S}_{(-\infty,\infty)}(J_{F,S})(\chi|_{J^1_{F,S}})$ by Proposition D.3.2.5 (applied to the pull-back of $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$ along a choice of continuous homomorphic section τ to the S-idelic norm, for then using Theorem D.3.3.7(3)), showing that (3) holds.

If (3) holds, then by Lemma 5.3.4.3 combined with Lemma 5.3.3.3(3), which asserts that multiplication by continuous characters of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ is a bounded and boundedly invertible operator on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, there exists $f_1 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that, for τ a fixed choice of continuous homomorphic section to the S-idelic norm, $\Phi_{F,S,1}(f_1)(\tau(t)) = \Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(\tau(t))$, and hence $f_0 := \overline{\chi}|\cdot|_{F,S}^{it_0} \cdot f_1$, for $t_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{it_0}$, satisfies $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f_0) = \Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$.

This shows that (2) holds.

If (2) holds, then by Lemma 5.3.4.2(2), $Z_S(f_0,c)$ is entire over $\Omega(J_{F,S})$, and over the connected component parametrized by $\chi|_{J^1_{F,S}}$, it agrees with $Z_S(f,c)$, which is therefore itself entire over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$. □

We have the following result, explaining how to use Weil's Principal Value Operators PV_{S,S_0} to resolve some of the singularities of the L-functions of Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$.

PROPOSITION 5.3.4.5. *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$. Then:

(1) *We have, for any finite subset $S_0 \subset S_f$:*

$$Z_S(\mathrm{PV}_{S,S_0}(f),c) = \frac{Z_S(f,c)}{\prod_{v\in S_0} L_v(c_v)} \tag{5.3.4.12}$$

as meromorphic functions of $c \in \Omega(J_{F,S})$.

(2) *The poles of $Z_S(\mathrm{PV}_{S_f}(f),c)$ are located at the points*

$$\mathcal{U}_{S_\infty} = \bigcup_{v\in S_\infty} \psi_v^{-1}(\mathcal{U}_v) \tag{5.3.4.13}$$

where $\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^\times)$ is the surjective local analytic isomorphism restricting a continuous quasi-character of $J_{F,S}$ to $F_v^\times \subset J_{F,S}$, and $\mathcal{U}_v$ is the set of poles in $\Omega(F_v^\times)$ of the local Euler factor $L_v(c_v)$ of the Hecke L-function $L(c)$. $\mathcal{U}_{S_\infty}$ is equipped with the multiplicity function $N_{S_\infty} : \mathcal{U}_{S_\infty} \to \mathbf{N}$ defined as:

$$N_{S_\infty}(c) := \sum_{v\in S_\infty} \mathrm{ord}_{c'=c} L_v(c'_v). \tag{5.3.4.14}$$

(3) *For any continuous character $\chi \in \widehat{J_{F,S}}$, ramified at the finite places in $S_\chi \subset S$ and unramified at each place in $S - S_\chi$, the linear images in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ under the continuous operator $\Phi_{F,S,\chi|_{J^1_{F,S}}}$ of the closed subspaces:*

$$\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f} \mathcal{S}(F_v^\times), \quad \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f - S_\chi} \mathcal{S}(F_v^\times) \otimes_\pi \bigotimes_{v\in S_\chi} \mathcal{S}(F_v) \tag{5.3.4.15}$$

agree with $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$.

The meaning of Proposition 5.3.4.5 is that the effect of the operator PV_{S,S_f} on a Schwartz-Bruhat function $f\in\mathcal{S}(\mathbf{A}_{F,S})$ is to remove all the singularities of $Z_S(f,c)$ arising from the non-Archimedean local factors of the Hecke L-function $L(c)$, leaving only those arising from the Archimedean local factors, counted with their multiplicity, in place.

Proof. (1) : The identity in (5.3.4.12) holds by Theorem 3.4.3.3, where (3.4.3.13) provides the formula for the effect of PV_v if c is ramified at v, and (3.4.1.6) from Theorem 3.4.1.1 provides it in the unramified case.

(2) : The assertion on the location and multiplicity of the poles of $Z_S(\mathrm{PV}_{S_f}(f),c)$ follows from (1) combined with (3.3.1.2) from Theorem 3.3.1.1.

(3) : By (1), it is enough to show that for any $f_v\in\mathcal{S}(F_v)$, with $v\in S_\chi$, and for any $f_\infty\in\mathcal{S}(\mathbf{A}_{F,S_\infty})$, $f_v\in\mathcal{S}(F_v^\times)$ for any $v\in S_f-S_\chi$, there exists a large enough integer $N_v\geq 0$ and a complex number c_v for each $v\in S_\chi$, such that upon calling $f'_v:=f_v-c_v1_{\pi_v^{N_v}\mathcal{O}_{F_v}}$, then $f'_v\in\mathcal{S}(F_v^\times)$, and we have, upon calling $f:=f_\infty\otimes\bigotimes_{v\in S_\chi}f_v\otimes\bigotimes_{v\in S_f-S_\chi}f_v$ and $f:=f_\infty\otimes\bigotimes_{v\in S_\chi}f'_v\otimes\bigotimes_{v\in S_f-S_\chi}f_v$, we have, as meromorphic functions over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$:

$$Z_S(f,c)=Z_S(f',c).$$

It is enough to show that $Z_v(f_v,c_v)=Z_v(f'_v,c_v)$ for all $c_v\in\Omega(F_v^\times)(\chi_v|_{\mathcal{O}_{F_v}^\times})_{>0}$.

Since f_v is Schwartz-Bruhat over F_v, there exists a large enough integer $N_v\geq 0$ such that f_v is constant over the compact open subgroup $\pi_v^{N_v}\mathcal{O}_{F_v}$. We let c_v its constant value there. We then define $f'_v:=f_v-c_v1_{\pi_v^{N_v}\mathcal{O}_{F_v}}$.

We note that $1_{\pi_v^{N_v}\mathcal{O}_{F_v}}$ is an $\mathcal{O}_{F_v}^\times$-invariant function. As such, since c_v is ramified, $Z_v(1_{\pi_v^{N_v}\mathcal{O}_{F_v}},c_v)=0$ for $\sigma(c_v)>1$, and hence $Z_v(f_v,c_v)=Z_v(f'_v,c_v)$ for $\sigma(c_v)>1$. □

5.4. Structure Theorem II: S-local Čech sequence

We require a description of the Hausdorff locally convex quotient topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ inherited from $\mathcal{S}(\mathbf{A}_{F,S})$, for each continuous character ω of $J^1_{F,S}$, in terms of an explicit family of seminorms on a certain subspace of the space of meromorphic functions $\mathscr{M}(\Omega(J_{F,S}))$.

This is the purpose of the Second Structure Theorem (see Theorem 5.4.1.1), proved in this section.

5.4.1. Second Structure Theorem. We recall from (5.1.1.1) of Definition 5.1.1.4 that the operator:

$$Z_S:\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S},\mathcal{S}(\mathbf{A}_{F,S}))\to\mathscr{M}(\Omega(J_{F,S})) \tag{5.4.1.1}$$

is injective and continuous with respect to the Hausdorff locally convex topology of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S},\mathcal{S}(\mathbf{A}_{F,S}))$ on the source, and the topology of uniform convergence on the compact subsets of $\Omega(J_{F,S})-\mathcal{U}_S$ on the target, where $\mathcal{U}_S$ is the set of the poles of the meromorphic function ρ_S from (3.3.1.3) of Theorem 3.3.1.1.

In this subsection, we study stronger topologies on $\mathscr{M}(\Omega(J_{F,S}))$ that induce under L_S the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$, for each continuous unitary character ω of $J^1_{F,S}$.

We are now ready to state and prove the Second Structure Theorem, providing a description of the topology of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$.

Theorem 5.4.1.1 (Second Structure Theorem). *Let F be a number field, S a finite set of places containing all the Archimedean places. Let τ be a choice of continuous homomorphic section to the S-idelic norm.*

We write each $g\in J_{F,S}$ uniquely as $\tau(t)g_0$ for $t\in\mathbf{R}_{>0}$ and $g_0\in J^1_{F,S}$, under the resulting isomorphism of Hausdorff locally compact abelian groups $J_{F,S}\simeq\mathbf{R}_{>0}\times J^1_{F,S}$.

(1) *For any $\omega\in\widehat{J^1_{F,S}}$, the continuous linear map:*

$$U_{S,\tau,\omega}:\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))\to A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1}) \tag{5.4.1.2}$$

defined by $U_{S,\tau,\omega}(f) := (Z_S(f,c), Z_S(\widehat{f},\widehat{c}))$ *for* c *corresponding to the pair* $(|\cdot|_{\mathbf{R}}^s,\omega) \in \Omega(\mathbf{R}_{>0})_{>0} \times \{\omega\}$ *under the bi-holomorphism from* (3.1.2.5) *induced by* τ*, is a linear topological embedding, compatible with the continuous left-translation actions of* $J_{F,S}$ *on the source, and the continuous left-translation action of* $J_{F,S}$ *on the target defined as* $(g,(\phi(s),\psi(s))) \mapsto (t^s\omega(g_0)\phi(s), t^{1-s}\overline{\omega}(g_0)\psi(s))$ *as functions of* $s \in \mathbf{C}_{>0}$*.*

Its range is closed, and it is the linear subspace:

$$W_{F,S,\omega} := \{(\phi,\psi) \in A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) : \phi(s)|_{\mathbf{C}_{(0,1)}} = \rho_{S,\tau}(s)\psi(s)|_{\mathbf{C}_{(0,1)}}\} \tag{5.4.1.3}$$

where $\rho_{S,\tau}(s)$ *is the meromorphic function on the complex* s*-plane obtained from* $\rho_S(c)$ *of* (3.3.1.3) *by pre-composing* ρ_S *with the bi-holomorphism* $\Omega(\mathbf{R}_{>0}) \times \{\omega\} \to \Omega(J_{F,S})(\omega)$ *provided by* (3.1.2.5)*.*

(2) *Let* ω *be a continuous character of* $J^1_{F,S}$*. Then* $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ *is a separable nuclear Fréchet space.*

Remark 5.4.1.2. We note that, in describing the continuous left-translation action of $J_{F,S}$ on the target of $U_{S,\tau,\omega}$, it is understood that $J_{F,S}$ acts through the isomorphism $J_{F,S} \to \mathbf{R}_{>0} \times J^1_{F,S}$ induced by the choice of τ.

We could choose a holomorphic parametrization

$$\{\chi|\cdot|_{F,S}^s, (\chi,s) \in \widetilde{J^1_{F,S}} \times \mathbf{C}\}$$

for the Riemann surface $\Omega(J_{F,S})$, where $\widetilde{J^1_{F,S}}$ is a choice of lift to $J_{F,S}$ for each continuous character ω of $J^1_{F,S}$, equivalently a choice of base point for each connected component of $\Omega(J_{F,S})$, and upon calling $x_0(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0(\chi)}$ as smooth functions on $\mathbf{R}_{>0}$, for each $\chi \in \widetilde{J^1_{F,S}}$, then the composition of $U_{S,\tau,\omega}$ with the vertical translation on the complex half s-plane $\mathbf{C}_{>0}$ by $ix_0(\chi)$, and on the complex half s-plane $\mathbf{C}_{<1}$ by $-ix_0(\chi)$, is the operator sending $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$ to $(Z_S(f,\chi|\cdot|_{F,S}^s), Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1-s})$.

The action of $g \in J_{F,S}$ described on $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$ in the statement of the Proposition, then matches the multiplication action by the pair of entire functions $(\chi(g)|g|_{F,S}^s, \overline{\chi}(g)|g|_{F,S}^{1-s})$.

We further refer the reader on Discussion 4.4.1.1 from Chapter 5 of this work.

We begin by showing that $U_{S,\omega,\tau}$ from (5.4.1.2) is continuous.

Lemma 5.4.1.3. *Let* F *be a number field,* S *a finite set of places containing all the Archimedean places,* $\omega \in \widehat{J^1_{F,S}}$ *a continuous character. Let* τ *be a choice of continuous homomorphic section to the* S*-idelic norm.*

The linear operator $U_{S,\tau,\omega}$ *from* (5.4.1.2) *is continuous and injective.*

Proof. We rename $U_{S,\tau,\omega}$ as U_S for simplicity.

Let $\{f_m\}_{m\geq 0}$ be a sequence of Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$ converging in $\mathcal{S}(\mathbf{A}_{F,S})$ to f. By Proposition 2.3.1.4, since $\{f_m\}_{m\geq 0} \cup \{f\}$ is contained in a bounded subset $B \subset \mathcal{S}(\mathbf{A}_{F,S})$, being a Cauchy sequence, there exist real numbers $T_N > 0$ and $C_N > 0$ such that:

$$|\Phi_{F,S,\omega}(u)(g)| \leq C_N|g|_{F,S}^{-N} \tag{5.4.1.4}$$

for all $g \in J_{F,S}$ such that $|g|_{F,S} > T_N$, and for all $u \in B$.

For every real number $a > 0$, the singularity of the function $t \mapsto t^a$ is $(\frac{\mathrm{d}t}{t})$-integrable over $(0,T_N)$, and by (2.4.3.5) from Corollary 2.4.3.2 and the uniformity properties of the implied constant in (2.4.3.5) and of the constant T_ε, for $\varepsilon = a$, then by Dominated Convergence we deduce that we have:

$$\lim_{m\to+\infty} \int_{(0,T_N)} t^a \int_{J^1_{F,S}} |\Phi_{F,S,\omega}(f_m)(\tau(t)g_0) - \Phi_{F,S,\omega}(\tau(t)g_0)|^p \,\mathrm{d}\overline{\mu}_{S,1}(g_0)\,\frac{\mathrm{d}t}{t} = 0 \tag{5.4.1.5}$$

for each real number $p \geq 1$, being $J^1_{F,S}$ compact Hausdorff.

By the estimate from (5.4.1.4) and the uniformity properties of the constants $C_N, T_N > 0$, by Dominated Convergence we also have:

$$\lim_{m\to+\infty}\int_{(T_N,+\infty)} t^a \int_{J^1_{F,S}} |\Phi_{F,S,\omega}(f_m)(\tau(t)g_0) - \Phi_{F,S,\omega}(\tau(t)g_0)|^p \,\mathrm{d}\overline{\mu}_{S,1}(g_0)\,\frac{\mathrm{d}t}{t} = 0 \tag{5.4.1.6}$$

for each real number $p \geq 1$.

Upon combining (5.4.1.5) and (5.4.1.6), we deduce that the linear operator U_S is sequentially continuous.

Since, by Proposition 2.1.4.1(2), $\mathcal{S}(\mathbf{A}_{F,S})$ is a strict LF-space, it is bornological by [**Scha**, Ch.II,§8,Cor.2,p.62], and by [**NB**, Thm.13.2.12] it is ultrabornological. $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$ is Hausdorff locally convex nuclear and Fréchet by Lemma D.3.1.2. In particular, it is webbed by [**NB**, Thm.14.6.2].

By [**NB**, Thm.14.7.3], U_S is continuous. Injectivity is a consequence of the Identity Theorem: indeed, if for some $f \in \mathcal{S}(\mathbf{A}_{F,S})$ we have $Z_S(f,c) = 0$ over $\Omega(J_{F,S})(\omega)_{>0}$, then $Z_S(f,c) = 0$ over $\Omega(J_{F,S})(\omega)$, which, by (5.1.2.4) from Lemma 5.1.2.2, implies that we have $\Phi_{F,S,\omega}(f) = 0$, and hence U_S is injective. □

Before we prove Theorem 5.4.1.1, we establish a special case, resting on a concrete construction:

Proposition 5.4.1.4 (Second Structure Theorem, polar-free case). *Let F be a number field, S a finite set of places containing all the Archimedean places.*

Let τ be a choice of continuous homomorphic section to the S-idelic norm. We write each $g \in J_{F,S}$ uniquely as $\tau(t)g_0$ for $t \in \mathbf{R}_{>0}$ and $g_0 \in J^1_{F,S}$, under the resulting isomorphism of Hausdorff locally compact abelian groups $J_{F,S} \simeq \mathbf{R}_{>0} \times J^1_{F,S}$.

For any $\omega \in \widehat{J^1_{F,S}}$, the continuous linear operator from (5.4.1.2)*:*

$$U_{S,\tau,\omega} : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_{00}(\mathbf{A}_{F,S})) \to A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) \tag{5.4.1.7}$$

defined by $U_{S,\tau,\omega}(f) := (Z_S(f,c), Z_S(\widehat{f},\widehat{c}))$ for c corresponding to the pair $(|\cdot|^s_{\mathbf{R}}, \omega) \in \Omega(\mathbf{R}_{>0})_{>0} \times \{\omega\}$ under the bi-holomorphism from (3.1.2.5) *induced by τ, is a linear topological embedding, compatible with the continuous left-translation actions of $J_{F,S}$ on the source, and the continuous left-translation action of $J_{F,S}$ on the target defined as $(g, (\phi(s), \psi(s))) \mapsto (t^s\omega(g_0)\phi(s), t^{1-s}\overline{\omega}(g_0)\psi(s))$ as functions of $s \in \mathbf{C}_{>0}$.*

Its range is closed, and it is the linear subspace:

$$V_{F,S,\omega} \subset W_{F,S,\omega} := \{(\phi,\psi) \in A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) : \phi(s)|_{\mathbf{C}_{(0,1)}} = \rho_{S,\tau}(s)\psi(s)|_{\mathbf{C}_{(0,1)}}\} \tag{5.4.1.8}$$

where $\rho_{S,\tau}(s)$ is the meromorphic function on the complex s-plane obtained from $\rho_S(c)$ of (3.3.1.3) *by pre-composing ρ_S with the bi-holomorphism $\Omega(\mathbf{R}_{>0}) \times \{\omega\} \to \Omega(J_{F,S})(\omega)$ provided by* (3.1.2.5)*, and such that the unique analytic continuation of ϕ to the complex s-plane is entire.*

Proof. By Proposition 5.3.4.4, indeed $U_{S,\tau,\omega}$ is valued in $V_{F,S,\omega}$, when restricted to $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, and maps $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$ surjectively onto $V_{F,S,\omega}$.

We now prove this surjectivity assertion.

Let $(\phi,\psi) \in W_{F,S,\omega}$ be an arbitrary element, and denote by $\widetilde{\phi}(s)$ the unique analytic continuation of ϕ to a meromorphic function on the complex s-plane, whose existence is ensured by ϕ being holomorphic over $\mathbf{C}_{>0}$, ψ being holomorphic over $\mathbf{C}_{<1}$, and the functional equation $\phi(s) = \rho_{S,\tau}(s)\psi(s)$ over $\mathbf{C}_{(0,1)}$, with $\rho_{S,\tau}$ meromorphic over $\mathbf{C}$ and satisfying $\rho_{S,\tau}(\sigma + it) = O(|t|^{\sigma-1/2})$ for $\sigma \in (0,1)$ and as $|t| \to +\infty$ (see Theorem 3.3.1.5(3) applied to $k = 0$), uniformly for $\sigma \in [\varepsilon, A)$ for any $0 < \varepsilon < A$, from which it follows that $\rho_{S,\tau}(s)\psi(s) \in A\mathcal{S}(\mathbf{C}_{(0,1)})$.

The poles of $\widetilde{\phi}$ are therefore located at the poles of $\rho_{S,\tau}$ and their order of pole is bounded above by the order of pole of each of the corresponding poles of $\rho_{S,\tau}$, respectively.

The function $\widetilde{\phi}$ is entire if and only if ψ has a zero at $s = s_0$ of multiplicity at least $\mathrm{ord}_{s=s_0}\rho_{S,\tau}(s)$, this latter being the order of pole of $\rho_{S,\tau}$ at $s = s_0$.

The point-evaluation functionals $(\phi,\psi)\mapsto(\partial_s^k\psi)(s_0)$ are all continuous for each $s_0\in\mathbf{C}_{<1}$ and each integer $k\geq 0$, since they are continuous with respect to the topology of uniform convergence on compact subsets on $\mathcal{S}(\mathbf{C}_{<1})$, which is coarser than the Schwartz-Bergman topology on $A\mathcal{S}(\mathbf{C}_{<1})$.

It follows that $V_{F,S,\omega}$ is the intersection of the kernel of a countable collection of continuous functionals on $W_{F,S,\omega}$, and it is therefore closed in $W_{F,S,\omega}$. As such, by [**Scha**, Ch.I,§6,6.3,p.31] it is a Fréchet space.

By the Asymptotic Cauchy Integral Formula (see Theorem B.2.2.4; see also the proof of Lemma D.3.1.2) applied, for each integer $N\geq 0$, to $s^N\phi(s)$ on any vertical line within $\mathbf{C}_{>0}$, and to $s^N\psi(s)$ on any vertical line contained in $\mathbf{C}_{<0}$, and the functional equation satisfied by all elements of $W_{F,S,\omega}$ as per (5.4.1.8), for any $(\phi,\psi)\in V_{F,S,\omega}$ we deduce that the unique analytic continuation $\widetilde{\phi}$ of ϕ to the complex s-plane, is entire and Schwartz uniformly over all vertical strips of bounded width of the complex s-plane.

It is therefore an element of $A\mathcal{S}(\mathbf{C})$. Its inverse Fourier transform along the vertical line $i\mathbf{R}$, denoted $f_0(t)\in\mathcal{S}(\mathbf{R}_{>0})$, is therefore of exponential decay over $\mathbf{R}_{>0}$, by Proposition D.3.2.5.

By Lemma 5.3.4.3, there exists a Schwartz-Bruhat function $f_1\in\mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that $\Phi_{F,S,1}(f_1)(\tau(t))=\varphi_0(t)$ as Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ with exponential decay.

Since multiplication by unitary characters of $J_{F,S}$ is a continuous and continuously invertible operator on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, then upon choosing $\chi\in\widehat{J_{F,S}}$ such that $\chi|_{J^1_{F,S}}=\omega$, we let $x_0\in\mathbf{R}$ be the unique real number such that $\chi(\tau(t))=|t|_{\mathbf{R}}^{ix_0}$ as smooth functions of $t\in\mathbf{R}_{>0}$, (and hence the S-idelic quasi-character corresponding under (3.3.1.3) from (3.1.2.5) to the pair $(|\cdot|_{\mathbf{R}}^s,\omega)$ is also parametrized by $\chi|\cdot|_{F,S}^z$ for $z+ix_0=s$) and $f_0:=\overline{\chi}|\cdot|_{F,S}^{ix_0}\cdot f_1$ is also an element of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ (see Lemma 5.3.4.2).

We therefore have:

$$\Phi_{F,S,\omega}(f_0)(\tau(t))=\varphi_0(t)$$

as smooth functions of $t\in\mathbf{R}_{>0}$.

By design of φ_0 and f, we have $Z_S(f_0,c)=Z_S(f,\chi|\cdot|_{F,S}^{s-ix_0})=\phi(s)$ where c corresponds under (3.1.2.5) to the pair $(|\cdot|_{\mathbf{R}}^s,\omega)$.

(We note that, upon observing that the action of $J^1_{F,S}$ on $V_{F,S,\omega}$ is through multiplication by $\overline{\omega}$, then $\varphi_0\in\mathcal{S}_{(-\infty,+\infty)}(\mathbf{R}_{>0})$ corresponds under pullback along τ to a unique element of $\mathcal{S}_{(-\infty,+\infty)}(J_{F,S})(\chi|_{J^1_{F,S}})$, and hence Proposition 5.3.4.4 applies).

By the functional equation from (3.3.1.2) of Lemma 3.3.1.1, it follows that we have:

$$\phi(s)=\rho_{S,\tau}(s)Z_S(\widehat{f},\widehat{c}) \tag{5.4.1.9}$$

with c corresponding to the pair $(|\cdot|_{\mathbf{R}}^s,\omega)$ under the bi-holomorphism from (3.1.2.5). By uniqueness of the analytic continuation of ϕ, we deduce that we have $\psi(s)=Z_S(\widehat{f},\widehat{c})$.

We deduce that the arbitrary $(\phi,\psi)\in V_{F,S,\omega}$ is in the image of $U_{S,\tau,\omega}$, and hence $U_{S,\tau,\omega}$ is surjective onto $V_{F,S,\omega}$.

This proves the surjectivity assertion, i.e. we have shown that $U_{S,\tau,\omega}$ maps $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ surjectively onto $V_{F,S,\omega}$, a closed subspace of $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$.

We note that since $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is closed in $\mathcal{S}(\mathbf{A}_{F,S})$ and $\Phi_{F,S,\omega}$ has closed kernel by Theorem 5.1.1.3, then $\ker(\Phi_{F,S,\omega})\cap\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is also closed in $\mathcal{S}(\mathbf{A}_{F,S})$ and in $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, and this intersection agrees with the linear subspace of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ given by $\ker(\Phi_{F,S,\omega}|_{\mathcal{S}_{00}(\mathbf{A}_{F,S})})$.

It follows that $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ equipped with the quotient topology, is a complete, Hausdorff locally convex topological $\mathbf{C}$-vector space.

Since $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is a closed subspace of $\mathcal{S}(\mathbf{A}_{F,S})$, and $\mathcal{S}(\mathbf{A}_{F,S})$ is a strict LF-space by Proposition 2.1.4.1(2), $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ and hence $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ equipped with the quotient topology are also LF-spaces.

By Lemma 5.4.1.3, $U_{S,\tau,\omega}$ from (5.4.1.7) is continuous, upon endowing its source $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ with the quotient topology, since it is obtained as the composition of $U_{S,\tau,\omega}$ from (5.4.1.2) with the continuous linear inclusion $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))\subset\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$.

By Lemma 5.4.1.3, $U_{S,\tau,\omega}$ is a continuous bijection from the LF-space $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$ to the Fréchet space $V_{F,S,\omega}$. It is therefore a linear topological isomorphism by Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]). □

Proof of Theorem 5.4.1.1. We will be deducing the Theorem in its general case, from the special case discussed in Proposition 5.4.1.4.

We first note that by Lemma 5.4.1.3, $U_{S,\tau,\omega}$ is continuous.

(1) : We rename $U_{S,\tau,\omega}$ from (5.4.1.2) as U_S for simplicity.

To show that U_S is indeed a linear topological embedding, we will be using the Principal Value Operators of Weil, discussed in §5.3.3 and in Proposition 5.3.4.5, as well as the multiplication operators from Proposition 5.3.3.4, and Proposition 5.3.4.4, to reduce to replacing $\mathcal{S}(\mathbf{A}_{F,S})$ with the closed subspace $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, and $W_{F,S,\omega}$ with the closed subspace $V_{F,S,\omega}$ of those (ϕ, ψ) such that the unique meromorphic function $\widetilde{\phi}$ on the complex s-plane obtained by analytic continuation of ϕ from $\mathbf{C}_{>0}$ to $\mathbf{C}$ by the functional equation $\phi(s) = \rho_{S,\tau}(s)\psi(s)$, valid over $\mathbf{C}_{(0,1)}$, is entire.

This case is already settled in Proposition 5.4.1.4, and will finish the proof of Theorem 5.4.1.1.

We first verify that indeed U_S is valued in $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$.

To see this, we observe that, for a choice of continuous homomorphic section τ to the S-idelic norm, the Fourier transform

$$Z_S(f, c) = \int_{\mathbf{R}_{>0}} t^{a+ix} \left(\int_{J^1_{F,S}} \omega(g) \Phi_{F,S}(f)(g) \, \mathrm{d}\overline{\mu}_{S,1}(g) \right) \frac{\mathrm{d}t}{t}$$

for $a \in \mathbf{R}_{>0}$, and $c|\cdot|^{-a}_{F,S}$ corresponding to the pair (ω, x) under the resulting isomorphism of Hausdorff locally compact abelian topological groups $\widehat{J_{F,S}} \simeq \widehat{J^1_{F,S}} \times \mathbf{R}_{>0}$, is valued in $\mathcal{S}(\widehat{J_{F,S}}) \simeq \mathcal{S}(\mathbf{R}) \overline{\otimes} \mathcal{S}(\widehat{J^1_{F,S}})$. This holds for any $a > 0$, and so $\Phi_{F,S}$ composed with the Fourier transform on $J^1_{F,S}$, is valued in the Hausdorff locally convex topological $\mathbf{C}$-vector space of $(0, +\infty)$-exponentially decaying Schwartz-Bruhat functions on $J_{F,S}$, i.e. $\mathcal{S}_{(0,+\infty)}(\mathbf{R}_{>0}) \overline{\otimes} \mathcal{S}(\widehat{J^1_{F,S}})$ (see Definition B.2.1.1(1), and Theorem D.3.3.7(3). This is the completed inductive tensor product, for which we refer the reader to (C.2.3.23) and the subsequent discussion. We further refer the reader to Lemma D.1.8.1(1)).

This is so by Proposition 2.3.1.2 and Corollary 2.4.3.2.

The Fourier transform on the elements $\mathcal{S}_{(0,+\infty)}(\mathbf{R}_{>0})$ extends uniquely to a holomorphic function on $\mathbf{C}_{>0}$ (i.e. the Mellin transform of the same element of $\mathcal{S}_{>0}(\mathbf{R}_{>0})$) which is Schwartz uniformly on vertical strips of bounded width by Lemma B.2.1.5. This shows that L_S, followed by the restriction operator to the connected component of the Riemann surface $\Omega(J_{F,S})$ parametrized by $\omega \in \widehat{J^1_{F,S}}$, takes values in $A\mathcal{S}(\mathbf{C}_{>0})$.

The same argument applies to $Z_S(\widehat{f}, \widehat{c})$ for any $f \in \mathcal{S}(\mathbf{A}_{F,S})$, and hence U_S is valued in $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$.

We choose a holomorphic parametrization of the connected component $\Omega(J_{F,S})(\omega)$, i.e. we choose a continuous character $\chi \in \widehat{J_{F,S}}$ and parametrize the connected component as $\chi|\cdot|^s_{F,S}$ for s a global holomorphic coordinate. Up to linear topological isomorphism, U_S is therefore isomorphic to the operator $f \mapsto (Z_S(f, \chi|\cdot|^s_{F,S}), Z_S(\widehat{f}, \overline{\chi}|\cdot|^{1-s}_{F,S}))$.

We now observe that by Proposition 5.3.4.5(1), combined with the fact that reciprocal non-Archimedean Euler factors of the Hecke L-function $L(c)$ are entire, and, with respect to any holomorphic parametrization $\chi|\cdot|^s_{F,S}$ of $\Omega(J_{F,S})(\omega)$ (i.e. for any given choice of $\chi \in \widehat{J_{F,S}}$ such that $\chi|_{J^1_{F,S}} = \omega$), have holomorphic derivatives of any order that are bounded any vertical strip of bounded width of the complex s-plane, the Principal Value Operator PV_{S,S_f} of Weil defined in Definition 5.3.1.4 on $\mathcal{S}(\mathbf{A}_{F,S})$ and descended to $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, extends to the ambient $W_{F,S,\omega}$ to give a linear topological embedding on $W_{F,S,\omega}$ with closed range in $W_{F,S,\omega}$.

Indeed, let (ϕ,ψ) be the arbitrary element of $W_{F,S,\omega}$. We then define:

$$\widetilde{\mathrm{PV}}_{S,S_f}(\phi(s),\psi(s)) := \left(\frac{\phi(s)}{\prod_{v\in S_f} L_v(\chi_v|\cdot|_v^s)}, \frac{\psi(s)}{\prod_{v\in S_f} L_v(\chi_v|\cdot|_v^s)}\right). \tag{5.4.1.10}$$

We note that the Fourier transform of $\mathrm{PV}_{S,S_f}(f)$ indeed satisfies

$$Z_S(\mathcal{F}_S(\mathrm{PV}_{S,S_f}(f)),\overline{\chi}|\cdot|_{F,S}^{1-s}) = \prod_{v\in S_f} L_v(\chi_v|\cdot|_v^s)^{-1} Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1-s}).$$

This follows from the calculation:

$$\widehat{1_{\mathcal{O}_{F_v}^\times}} = \widehat{1_{\mathcal{O}_{F_v}}} - \widehat{1_{\pi_v\mathcal{O}_{F_v}}} = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2} 1_{\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}} - |\pi_v|_v N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2}\cdot 1_{\pi_v^{-1}\mathfrak{d}_{F_v/\mathbf{Q}_p}} \tag{5.4.1.11}$$

for any choice of uniformizing element π_v at $v\in S_f$.

The operator $\widetilde{\mathrm{PV}}_{S,S_f}$ therefore promotes the following diagram of continuous linear operators:

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{U_S} & W_{F,S,\omega} \\ \downarrow{\scriptstyle \mathrm{PV}_{S,S_f}} & & \downarrow{\scriptstyle \widetilde{\mathrm{PV}}_{S,S_f}} \\ \mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow[U_S]{} & W_{F,S,\omega} \end{array} \tag{5.4.1.12}$$

to a commutative diagram whose vertical arrows are linear topological embeddings with closed range. Note that $\widetilde{\mathrm{PV}}_{S,S_f}$ acts by multiplication by an entire function with holomorphic derivatives that are bounded on vertical strips of bounded width, and hence the range of $\widetilde{\mathrm{PV}}_{S,S_f}$ is sequentially closed since functions in this class are bounded multipliers on $A\mathcal{S}(\mathbf{C}_I)$ for any connected open interval $I\subset\mathbf{R}$, by [**Sch-a**, Ch.VII,§5,p.243].

By Proposition 5.3.4.5(3), the closed subspace $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ agrees with the linear image of $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi \bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)$ under $\Phi_{F,S,\omega}$.

It is therefore enough to show that U_S is surjective from $\Phi_{F,S,\omega}(\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi \bigotimes_{v\in S_f}\mathcal{S}(F_v^\times))$ onto $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$.

We already know from Proposition 5.4.1.4 that the restriction of U_S to $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$, i.e. the linear image of $\mathcal{S}_{00}(\mathbf{A}_{F,S})\otimes_\pi \bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)$, is surjective onto the closed subspace $V_{F,S,\omega}$. We note that $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ is contained in $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ and $V_{F,S,\omega}$ is contained in $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$, both as closed subspaces, and both by Proposition 5.3.4.5(1) and (3).

It therefore suffices to show that the induced continuous linear map:

$$\overline{U}_S : \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S})) \to \widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})/V_{F,S,\omega}. \tag{5.4.1.13}$$

is surjective.

The idea is to now exhibit a separated filtration on source and target of $\overline{U}_S$ from (5.4.1.13), compatible with $\overline{U}_S$, by closed subspaces, with finite dimensional subquotients, such that the Hausdorff locally convex topology on the source of $\overline{U}_S$ is equivalent to the projective topology induced by the quotient maps to the finite dimensional subquotients, and such topology is complete, for then showing that $\overline{U}_S$ induces $\mathbf{C}$-linear surjections (in fact isomorphisms) on the subquotients.

We choose an enumeration of $\Sigma_{F,r}=\{v_1,\ldots,v_{r_1}\}$ and $\Sigma_{F,c}=\{w_1,\ldots,w_{r_2}\}$, respectively, and order the tuples $(\underline{n},\underline{p})$ lexicographically according to the chosen enumerations, i.e. $(\underline{n},\underline{p})\le(\underline{n}',\underline{p}')$ if and only if for every $\underline{n}\le\underline{n}'$ under the lexicographical ordering on $\mathbf{N}^{r_1}$, and $\underline{p}\le\underline{p}'$ under the lexicographical ordering on $\mathbf{N}^{r_2}$.

We choose an integer tuple $(\underline{n},\underline{p})$ and define:

$$\mathcal{U}_{\le(\underline{n},\underline{p})} := \bigcup_{1\le k\le r_1} \psi_{v_k}^{-1}(\mathcal{U}_{v_k}^{\le n_{v_k}}) \cup \bigcup_{1\le k\le r_2} \psi_{w_k}^{-1}(\mathcal{U}_{w_k}^{\le p_{v_k}}) \tag{5.4.1.14}$$

where $\mathcal{U}_{v_k}^{\leq n_{v_k}}$ denotes the set of poles of the Euler factor $L_v(c_v)$ of the Hecke L-function $L(c)$ in $\Omega(F_v^\times)(\omega_v)$ whose ordinate $\sigma(c_{v_k})$ satisfies $\sigma(c_{v_k}) \geq -2n_{v_k} - \epsilon(c_{v_k})$ for $\epsilon(c_{v_k}) = 0$ if ω_{v_k} is trivial, and $\epsilon(c_{v_k}) = 1$ if ω_{v_k} is the sign character, and where $\mathcal{U}_{w_k}^{\leq p_{w_k}}$ denotes the set of poles of the Euler factor $L_v(c_v)$ of the Hecke L-function $L(c)$ in $\Omega(F_v^\times)(\omega_v)$ whose ordinate $\sigma(c_{w_k})$ satisfies $\sigma(c_{w_k}) \geq -p_{w_k} - \epsilon(c_{w_k})$ where $\epsilon(c_{w_k}) = |n|/2$ if $\omega_{w_k} = c_n \in \widehat{\mathrm{U}(1)}$.

We define the two filtrations:

$$F_S^{(\underline{n},\underline{p})} \subset \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})),\ \widetilde{F}_S^{(\underline{n},\underline{p})} \subset \widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega}) \tag{5.4.1.15}$$

of those elements $\Phi_{F,S,\omega}(f) \in \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ whose S-idelic Mellin transform $Z_S(f,c)$ (which, recall from Lemma 5.1.2.2(2), only depends on f through $\Phi_{F,S,\omega}(f)$) has no poles over $\mathcal{U}_{\leq(\underline{n},\underline{p})}$, respectively of those elements (ϕ,ψ) of $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$ such that the unique analytic continuation of ϕ to the complex s-plane has no poles over $\mathcal{U}_{\leq(\underline{n},\underline{p})}$.

We begin by observing that both $F_S^{(\underline{n},\underline{p})}$ and $\widetilde{F}_S^{(\underline{n},\underline{p})}$ are the countable intersection of the kernels of continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $W_{F,S,\omega}$.

The functionals on $W_{F,S,\omega}$ are defined as follows: for (ϕ,ψ) the arbitrary element of $W_{F,S,\omega}$ and $\widetilde{\phi}(s)$ the unique analytic continuation of ϕ to the complex s-plane which parametrizes the connected component of $\Omega(J_{F,S})$ over ω under the chosen holomorphic parametrization $\chi|\cdot|_{F,S}^s$, for $c \in \mathcal{U}_{\leq(\underline{n},\underline{p})}$ parametrized by $\chi|\cdot|_{F,S}^{s_c}$ a pole of order N_c of $\rho_S(\chi|\cdot|_{F,S}^s)$ we define:

$$a_m(c)(\phi,\psi) := \lim_{s\to s_c}(s-s_c)^m\widetilde{\phi}(s)$$

the mth coefficient of the Laurent series expansion of $\widetilde{\phi}$ at $s = s_c$, for each integer $1 \leq m \leq N_c$.

This induces a functional on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ by composition with the operator U_S. Since U_S is continuous, it is enough to show that a_m is continuous on $W_{F,S,\omega}$, and we will deduce from this that indeed both $F_S^{(\underline{n},\underline{p})}$ and $\widetilde{F}_S^{(\underline{n},\underline{p})}$ are the countable intersections $\bigcap_{(\underline{n}',\underline{p}')\leq(\underline{n},\underline{p})}\bigcap_{c\in\mathcal{U}^{\leq(\underline{n}',\underline{p}')}} a_m(c)$, respectively $\bigcap_{(\underline{n}',\underline{p}')\leq(\underline{n},\underline{p})}\bigcap_{c\in\mathcal{U}^{\leq(\underline{n}',\underline{p}')}} a_m(c)\circ U_S$.

Note that since for each $(\underline{n}',\underline{p}') \leq (\underline{n},\underline{p})$ we have $\mathcal{U}^{\leq(\underline{n}',\underline{p}')} \subset \mathcal{U}^{\leq(\underline{n},\underline{p})}$, we have:

$$\widetilde{F}_S^{(\underline{n},\underline{p})} = \bigcap_{c\in\mathcal{U}_{\leq(\underline{n},\underline{p})}} a_m(c),\quad F_S^{(\underline{n},\underline{p})} = \bigcap_{c\in\mathcal{U}_{\leq(\underline{n},\underline{p})}} a_m(c).$$

To conclude that $\widetilde{F}_S^{(\underline{n},\underline{p})}$ and $F_S^{(\underline{n},\underline{p})}$ are closed subspaces, it is enough to show that the functionals $a_m(c)$ are continuous on $W_{F,S,\omega}$. Since the topology on $W_{F,S,\omega}$ induced by the product topology on $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$ induces, under the linear operator sending (ϕ,ψ) to $\widetilde{\phi} \in \mathscr{M}_{\mathcal{U}}(\mathbf{C})$, the linear space of meromorphic functions on the complex s-plane whose poles are located at the points of $\bigcup_{(\underline{n},\underline{p})}\mathcal{U}^{\leq(\underline{n},\underline{p})}$, a Hausdorff locally convex topology that is finer than the topology of uniform convergence on compact subsets away from the poles, it is enough to show that each of the functionals $a_m(\chi|\cdot|_{F,S}^s)$ is continuous on $\mathscr{M}_{\mathcal{U}}(\mathbf{C})$ with respect to the Hausdorff locally convex topology of uniform convergence on compact subsets away from $\mathcal{U} \subset \mathbf{C}$ (using the chosen holomorphic parametrization for $\Omega(J_{F,S})(\omega)$ to parametrize the points in $\mathcal{U}$).

Continuity of each $a_m(\chi|\cdot|_{F,S}^s)$ indeed holds with respect to this topology by Cauchy's Residue Theorem (see [**Ahl**, Ch.4,§5,Thm.17,p.150]) applied to $(s-s_c)^{m-1}\widetilde{\phi}$ and combined with Dominated Convergence, applied to the integral of $(s-s_c)^{m-1}\widetilde{\phi}$ along a rectifiable simple closed curve enclosing s_c.

We deduce that each $F_S^{(\underline{n},\underline{p})}$ and $\widetilde{F}_S^{(\underline{n},\underline{p})}$ are closed and of finite codimension (since each $\mathcal{U}^{\leq(\underline{n},\underline{p})} \subset \Omega(J_{F,S})(\omega)$ contains only finitely many points) in their respective ambients $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$.

We now show that we have:

$$\bigcap_{(\underline{n},\underline{p})} F_S^{(\underline{n},\underline{p})} = \Phi_{F,S,\omega}(\mathcal{S}_{00}(\mathbf{A}_{F,S})). \tag{5.4.1.16}$$

Let $(\underline{n},\underline{p})$ be an arbitrary $(\Sigma_{F,r},\Sigma_{F,c})$-indexed tuple of nonnegative integers.

Let $f \in \mathrm{PV}_{S,S_f}(\mathcal{S}(\mathbf{A}_{F,S}))$ be such that $\Phi_{F,S,\omega}(f) \in F_S^{(\underline{n},\underline{p})}$. Then $Z_S(f,c)$ has no poles over $\mathcal{U}^{\leq(\underline{n},\underline{p})}$ within the connected component $\Omega(J_{F,S})(\omega)$.

Since all the poles that $Z_S(f,c)$ can have on the connected component $\Omega(J_{F,S})(\omega)$ are contained in the countable union:

$$\bigcup_{(\underline{n},\underline{p})} \left\{ \bigcup_{1\leq k\leq r_1} \psi_{v_k}^{-1}(\mathcal{U}_{v_k}^{\leq n_{v_k}}) \cup \bigcup_{1\leq k\leq r_2} \psi_{w_k}^{-1}(\mathcal{U}_{w_k}^{\leq p_{v_k}}) \right\}$$

we deduce that $Z_S(f,c)$ is entire over the connected component $\Omega(J_{F,S})(\omega)$.

By Proposition 5.3.4.4(3), there exists $f_1 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that $\Phi_{F,S,\omega}(f_1) = \Phi_{F,S,\omega}(f)$, which shows that the closed subspace in the left side of (5.4.1.16) is contained in $\Phi_{F,S,\omega}(\mathcal{S}_{00}(\mathbf{A}_{F,S}))$.

Similarly, within $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$, the closed subspace:

$$\bigcap_{(\underline{n},\underline{p})} \widetilde{F}_S^{(\underline{n},\underline{p})} \subset \widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega}) \tag{5.4.1.17}$$

is the closed subspace of those elements (ϕ,ψ) such that the resulting unique analytic continuation $\widetilde{\phi}$ of ϕ to the complex s-plane has no poles in the subset $\mathcal{U}_{\leq(\underline{n},\underline{p})}$ from (5.4.1.14) for any $(\underline{n},\underline{p})$.

We therefore also have $\bigcap_{(\underline{n},\underline{p})} \widetilde{F}_S^{(\underline{n},\underline{p})} = V_{F,S,\omega}$.

By design of $F_S^{(\underline{n},\underline{p})}$, we have:

$$F_S^{(\underline{n},\underline{p})} = U_S^{-1}(\widetilde{F}_S^{(\underline{n},\underline{p})}) \tag{5.4.1.18}$$

for any $(\underline{n},\underline{p})$.

We define:

$$H := \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}_{00}(\mathbf{A}_{F,S})) \tag{5.4.1.19}$$

endowed with the quotient topology and the induced filtration $\overline{F}_S^{(\underline{n},\underline{p})}$, and similarly:

$$\widetilde{H} := \widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})/V_{F,S,\omega} \tag{5.4.1.20}$$

endowed with the quotient topology and the filtration induced by $\widetilde{F}_S^{(\underline{n},\underline{p})}$, which we denote $\mathscr{F}_S^{(\underline{n},\underline{p})}$.

Each $\overline{F}_S^{(\underline{n},\underline{p})}$ carries the quotient topology, and it has finite codimension in H. Indeed, $\overline{F}_S^{(\underline{n},\underline{p})} = \overline{U}_S^{-1}(\mathscr{F}_S^{(\underline{n},\underline{p})})$, since $U_S^{-1}(V_{F,S,\omega}) = \Phi_{F,S,\omega}(\mathcal{S}_{00}(\mathbf{A}_{F,S}))$ by Proposition 5.3.4.4(3),and $U_S(\Phi_{F,S,\omega}(\mathcal{S}_{00}(\mathbf{A}_{F,S}))) = V_{F,S,\omega}$ by Proposition 5.4.1.4, and so the linear map $\overline{F}_S^{(\underline{n},\underline{p})} \to \mathscr{F}_S^{(\underline{n},\underline{p})}$ induced by $\underline{U}_S$ (which, in turn, is the operator $H \to \widetilde{H}$ induced by U_S), is injective, showing that each $\overline{F}_S^{(\underline{n},\underline{p})}$ has finite codimension in H, and is therefore closed with respect to the subspace topology inherited from H, by [**NB**, Thm.7.3.5].

A similar argument implies that the quotient topology and the subspace topology on $\mathscr{F}_S^{(\underline{n},\underline{p})}$ inherited from $\widetilde{H}$ agree, and $\mathscr{F}_S^{(\underline{n},\underline{p})}$ is closed in $\widetilde{H}$.

The filtration $\{\overline{F}_S^{(\underline{n},\underline{p})}\}_{(\underline{n},\underline{p})}$ induces on H a locally convex topology: the projective topology induced by the collection of all quotient maps $H \to H/\overline{F}_S^{(\underline{n},\underline{p})}$. This topology is Hausdorff, because $\bigcap_{(\underline{n},\underline{p})} \overline{F}_S^{(\underline{n},\underline{p})} = 0$, as a consequence of (5.4.1.16).

It agrees with the projective limit topology induced by the $H \to H/\overline{F}_S^{(\underline{n},\underline{p})}$: indeed, the ordering on the integer tuples $(\underline{n},\underline{p})$, promotes the collection of quotient maps $H \to H/\overline{F}_S^{(\underline{n},\underline{p})}$ to a cofiltered diagram, and the projective limit topology on H is metrizable. Its completion $\widehat{H}$ is a Fréchet space, surjecting on the complete Hausdorff locally convex topological $\mathbf{C}$-vector space H by [**Jar**, Part I,Ch.3,§3.4,Thm.2,p.61]. Since $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ is an LF-space, so are its closed subspaces and its quotient by closed subspaces, and hence so is H.

By Köthe's Open Mapping Theorem (see [**Gro-f**, Ch.4,§5,Thm.2(1),p.148], and [**SN**, §2,Cor.3,p.8]), the continuous surjective linear operator $\widehat{H} \to H$ is open, hence a linear topological isomorphism. The same argument applies to $\widetilde{H}$, which is therefore seen to carry the projective limit topology induced by the surjective quotient maps $\widetilde{H} \to \widetilde{H}/\mathscr{F}_S^{(\underline{n},\underline{p})}$ with finite dimensional range.

We have proved that $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega})$ fit into the following commutative diagram of continuous linear operators:

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_{00}(\mathbf{A}_{F,S})) & \xrightarrow{U_S} & V_{F,S,\omega} \\ \downarrow & & \downarrow \\ \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{U_S} & \widetilde{\mathrm{PV}}_{S,S_f}(W_{F,S,\omega}) \\ \downarrow & & \downarrow \\ H & \xrightarrow[\overline{U}_S]{} & \widetilde{H} \end{array} \tag{5.4.1.21}$$

The vertical columns in (5.4.1.21) are exact in the strict sense (i.e. the topmost Hausdorff locally convex topological $\mathbf{C}$-vector spaces are closed in the middle spaces, respectively, and the bottom vertical arrows are surjective quotient linear operators).

By Proposition 5.4.1.4, we have that the topmost horizontal arrow is a linear topological isomorphism. The middle horizontal arrow is continuous and injective, and so is the bottom horizontal arrow.

In order to show that the middle horizontal arrow is a linear topological isomorphism, it is enough to show that $\overline{U}_S$ is.

To see this, it is enough to show that the induced maps on the subquotients:

$$H/\overline{F}_S^{(\underline{n},\underline{p})} \to \widetilde{H}/\mathscr{F}_S^{(\underline{n},\underline{p})} \tag{5.4.1.22}$$

which we already know to be injective, are also surjective, being source and target finite dimesional.

To this end, it is enough to exhibit Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$ whose image under $\Phi_{F,S,\omega}$ has prescribed poles and orders of poles at the points of each $\mathcal{U}^{\leq(\underline{n},\underline{p})}$.

Each local character $\omega_v = \chi_v|_{\{u\in F_v^\times : |u|_v = 1\}}$ for each Archimedean place v, uniquely determines a Schwartz-Bruhat function f_v on F_v whose local L-function is the Γ-factor at v. We refer the reader to [**CF**, Ch.XV,§2.5,pp.316-319] and [**RV**, Ch.7,§7.1,pp.249-253] for the explicit definition of such functions, satisfying the self-duality properties in [**CF**, Ch.XV,§2.5,pp.316-319] with respect to the real and complex Fourier transforms.

Enough many Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$ such that their images under $\Phi_{F,S,\omega}$ and the quotient map to $H/\overline{F}_S^{(\underline{n},\underline{p})}$ composed with the injective linear map to $\widetilde{H}/\mathscr{F}_S^{(\underline{n},\underline{p})}$ linearly generate $\widetilde{H}/\mathscr{F}_S^{(\underline{n},\underline{p})}$, are obtained as follows.

We consider a tuple of nonnegative integers $(\underline{n},\underline{p}) = ((n_v)_{v\in\Sigma_{F,r}},(p_v)_{v\in\Sigma_{F,c}})$ where $\Sigma_{F,r}$ denotes the real places of F, and $\Sigma_{F,c}$ denotes the complex places, and define the linear map:

$$M_{(\underline{n},\underline{p})} := \bigotimes_{v\in\Sigma_{F,r}} M_{x_v^{2n_v}} \otimes \bigotimes_{v\in\Sigma_{F,c}} M_{z_v^{p_v}\overline{z}_v^{p_v}} : \mathcal{S}(\mathbf{A}_{F,S_\infty}) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}). \tag{5.4.1.23}$$

We further define

$$M^S_{(\underline{n},\underline{p})} := M_{(\underline{n},\underline{p})} \otimes \bigotimes_{v\in S_f - S_\chi} 1_{\mathcal{O}_{F_v}^\times} \otimes \bigotimes_{v\in S_\chi} \omega_v^{-1} 1_{\mathcal{O}_{F_v}^\times} : \mathcal{S}(\mathbf{A}_{F,S_\infty}) \to \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \bigotimes_{v\in S_f} \mathcal{S}(F_v^\times)$$

and, upon calling $f_\infty := \bigotimes_{v\in S_\infty} f_v$ for $f_v \in \mathcal{S}(F_v)$ the above choice of "self-dual" Schwartz-Bruhat function, the Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$ obtained as $M^S_{(\underline{n}',\underline{p}')}(f_\infty)$ for each $(\underline{n}',\underline{p}') \leq (\underline{n},\underline{p})$, satisfy the desired spanning properties (see Proposition 5.3.3.4).

This proves that $\overline{U}_S$ is surjective, hence a linear topological isomorphism, which proves that the middle horizontal arrow in (5.4.1.21) is also a linear topological isomorphism. In turn, we proved that this suffices to show that U_S from (5.4.1.2) is a linear topological isomorphism, finishing the proof of Theorem 5.4.1.1.

(2) : $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ is a nuclear Fréchet space since U_S is a linear topological embedding of $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ into a Fréchet space, making $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ metrizable, hence Fréchet. It is nuclear by Theorem 5.2.3.1. It is separable by Lemma D.3.1.2. □

Remark 5.4.1.5. We recall from the First Structure Theorem (see Theorem 5.2.1.1), that the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ is complete, nuclear, Montel, and its strong dual satisfies the same properties. It is an LF-space. From Theorem 5.4.1.1(4) we have that $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_{F,S}))$ is a Fréchet space,. In particular, $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ is itself realized as a *strict* LF-space, improving on Theorem 5.2.1.1. We shall see that the transition operators $\overline{\pi}_S$ are eventually surjective, and hence, since they are linear topological embeddings, $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))$ will be realized as itself a nuclear Fréchet space.

5.4.2. S-local Čech sequences. Let F be a number field, S a finite set of places containing all the Archimedean places of F. The Fourier transform $\mathcal{F}_S$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$ defined in (5.2.4.1) is continuous with respect to the normed topology on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$ induced by the continuous seminorm:

$$\|\phi\|_{2,S} := \left(\int_{J_{F,S}} |\phi(g)|^2\,|g|_{F,S}\,\mathrm{d}\overline{\mu}_S(g)\right)^{1/2}. \tag{5.4.2.1}$$

This can be seen as a consequence of Theorem 3.3.1.5: indeed, for any $f \in \mathcal{S}(\mathbf{A}_{F,S})$, we have:

$$Z_S(f,c) = \rho_S(c) Z_S(\widehat{f},\widehat{c})$$

by (3.3.1.2) from Theorem 3.3.1.1, and since $\rho_S(c) \in \mathrm{U}(1)$ by Theorem 3.3.1.5(2) for any $c \in \Omega(J_{F,S})$ with $\sigma(c) = 1/2$, we deduce that $\mathcal{F}_S$ is in fact unitary with respect to $\|\cdot\|_{2,S}$.

The extension of $\mathcal{F}_S$ to the completion $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$ with respect to $\|\cdot\|_{2,S}$, which we denote by:

$$L^2\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S})) \tag{5.4.2.2}$$

is a linear topological automorphism and a unitary isometry, compatible with passage to $J^1_{F,S}$-isotypes.

We conclude this section by showing that the Fourier transform on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$ extends to other Hausdorff locally convex topological $\mathbf{C}$-vector spaces of smooth functions on $J_{F,S}$, other than the Hilbert space $L^2\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$.

Principal tool to conduct this analysis is the S-idelic Mellin transform L_S (see (5.4.1.1)) and the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1.

As an application, we will be reformulating, in the next subsection, the Second Structure Theorem (see Theorem 5.4.1.1) using spaces of Schwartz-Bruhat functions on $J_{F,S}$ with I-exponential decay (see Definition D.3.3.1 and Theorem D.3.3.7), by means of the S-local Mellin transform L_S.

PROPOSITION 5.4.2.1 (S-local Fourier transform and I-exponential decay). *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character. Let $I \subset (0,1)$ be an open connected interval.*

Then:

(1) *The Fourier transform $\mathcal{F}_S$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ extends by continuity to a linear topological isomorphism:*

$$\mathcal{F}_S : \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \to \mathcal{S}_{1-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}). \tag{5.4.2.3}$$

(2) *If $1/2 \in I$, then the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}_I(J_{F,S})$ of Schwartz-Bruhat functions on $J_{F,S}$ with I-exponential decay, admits a continuous injective linear operator into $L^2\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$, and the Fourier transform $\mathcal{F}_S$ on*

$$L^2\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

restricts to $\mathcal{F}_S$ from (5.4.2.3) on $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$.

PROOF. (1) : We choose a continuous homomorphic section $\tau : \mathbf{R}_{>0} \to J_{F,S}$ to the S-idelic norm, and we denote by x_0 the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$. For $\phi \in \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$, the Fourier transform

$$\mathcal{F}(\phi)(x) = \int_{\mathbf{R}_{>0}} \phi(\tau(t)) t^{ix} \, \frac{\mathrm{d}t}{t}$$

extends to $M_I(\phi)(s) = \int_{\mathbf{R}_{>0}} \phi(\tau(t)) t^s \, \frac{\mathrm{d}t}{t}$ by Theorem D.3.3.7 (see also Proposition D.3.2.5) for $\mathrm{Re}(s) \in I$, with $M(\phi)(s)$ holomorphic over $\mathbf{C}_I$ and Schwartz uniformly on every vertical strip of bounded width strictly contained within $\mathbf{C}_I$.

We let $\{\phi_m\}_{m \geq 0} \in \mathcal{D}(J_{F,S})(\chi|_{J^1_{F,S}})$ be such that $\phi_m \to \phi$ in $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$, which is possible by Lemma D.3.6.1.

It follows that on every vertical line strictly contained within $\mathbf{C}_I$, parametrized as $\sigma + ix_0 + i\mathbf{R}$, $M_I(\phi_m) \to M_I(\phi)$ in the sense of Schwartz functions on the real line.

Since each ϕ_m is of the form $\Phi_{F,S}(\psi_m)$ for a smooth compactly supported test function on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ extended by zero to a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$, we have $M_I(\phi_m)(s + ix_0) = Z_S(\psi_m, \chi|\cdot|^s_{F,S})$ as entire functions on the complex s-plane, since $\phi_m \in \mathcal{S}_{\mathrm{E}}(J_{F,S})(\chi|_{J^1_{F,S}})$, and hence $M_I(\phi_m) = M_{\mathbf{R}}(\phi_m)$ is entire on the complex s-plane by Proposition D.3.2.5.

By the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1, we have, for each integer $m \geq 0$ and as meromorphic functions on the complex s-plane:

$$Z_S(\psi_m, \chi|\cdot|^s_{F,S}) = \rho_S(\chi|\cdot|^s_{F,S}) \, Z_S(\widehat{\psi_m}, \overline{\chi}|\cdot|^{1-s}_{F,S}) \tag{5.4.2.4}$$

For every real number $\sigma \in I$, we therefore have:

$$Z_S(\widehat{\psi_m}, \overline{\chi}|\cdot|^{1-\sigma-ix}_{F,S}) = \frac{Z_S(\psi_m, \chi|\cdot|^{\sigma+ix}_{F,S})}{\rho_S(\chi|\cdot|^{\sigma+ix}_{F,S})} = \frac{M_I(\phi_m, \sigma + ix + ix_0)}{\rho_S(\chi|\cdot|^{\sigma+ix}_{F,S})} \tag{5.4.2.5}$$

as meromorphic functions on the complex $(\sigma, x) = s$-plane. Indeed, $\rho_S(\chi|\cdot|^s_{F,S})$ is nowhere vanishing over $\mathbf{C}_{(0,1)}$ and grows at most polynomially over each vertical line of $\mathbf{C}_{(0,1)}$, as do all of its holomorphic derivatives (see Proposition 3.3.1.5).

We define $\mathcal{F}_S(\phi)$ to be the limit in $\mathcal{S}_{1-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ of the functions $\Phi_{F,S}(\widehat{\psi_m}) = \mathcal{F}_S(\Phi_{F,S}(\psi_m)) = \mathcal{F}_S(\phi_m)$. We have just shown that $\mathcal{F}_S$ is continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ on the source, and the Hausdorff locally convex topology on $\mathcal{S}_{1-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ on the target, and since it has a continuous inverse given by $\mathcal{F}_S$ itself, which is its own inverse on the dense linear subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, we deduce that $\mathcal{F}_S$ is a linear topological isomorphism as in (5.4.2.3).

(2) : Since $1/2 \in I$, for any $\phi \in \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ we have $|\cdot|^{1/2}_{F,S}\,\phi \in \mathcal{S}(J_{F,S})(\chi|_{J^1_{F,S}})$. In particular, $\phi \in L^2(J_{F,S}, |\cdot|_{F,S}\,\overline{\mu}_S)(\chi|_{J^1_{F,S}})$. Since $|\cdot|^{1/2}_{F,S}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \subset L^2(J_{F,S}, \overline{\mu}_S)$ is dense, since it contains $C^\infty_c(J_{F,S})$, we deduce that the continuous linear inclusion $|\cdot|^{1/2}_{F,S}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \subset L^2(J_{F,S}, |\cdot|_{F,S}\,\overline{\mu}_S)(\chi|_{J^1_{F,S}})$ extends by continuity to a linear topological embedding

$$L^2\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to L^2(J_{F,S}, |\cdot|_{F,S}\,\overline{\mu}_S)(\chi|_{J^1_{F,S}})$$

with dense range, hence a linear topological isomorphism between Hilbert spaces (in fact, up to pre-composing by $c_S\,\mathrm{Id}$, a surjective isometry).

We deduce that the injective continuous linear operator $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \to L^2(J_{F,S}, |\cdot|_{F,S}\,\overline{\mu}_S)(\chi|_{J^1_{F,S}})$ factors through $L^2\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$. By (1), the result follows upon observing that $\mathcal{F}_S$ on $L^2\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ restricts to $\mathcal{F}_S$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$. □

Let F be a number field, S a finite set of places of F containing all the Archimedean places.

We can now rephrase the Second Structure Theorem (see Theorem 5.4.1.1) into the following form:

THEOREM 5.4.2.2 (S-local Čech sequences). *Let F be a number field, S a finite set of places of F containing the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character. Let τ be a choice of continuous homomorphic section to the S-idelic norm. We define $x_0 \in \mathbf{R}$ to be the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$.*

We define the interval $I := (0, +\infty) \subset \mathbf{R}$, noting that we have $1 - I = (-\infty, 1)$, and $I \cap (1-I) = (0,1)$. The following diagram of continuous linear operators:

$$\begin{array}{ccccc}
\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{(\mathrm{Id},\mathcal{F}_S)} & \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \times \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) & \xrightarrow{\mathrm{pr}_1 - \widetilde{\mathrm{pr}}_2} & \mathcal{S}_{I\cap(1-I)}(\mathbf{R}_{>0}) \\
\| & & \downarrow{\scriptstyle (M_{I,\tau}, M_{I,\tau})} & & \downarrow{\scriptstyle T_{ix_0}\circ M_{I\cap(1-I)}} \\
\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow[(L_S, L_S\circ\mathcal{F}_S)]{} & A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{>0}) & \xrightarrow[\mathrm{pr}_1 - \widetilde{J}\circ\mathrm{pr}_2]{} & A\mathcal{S}(\mathbf{C}_{I\cap(1-I)})
\end{array} \tag{5.4.2.6}$$

for:

1. *$M_{I,\tau} : \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \to A\mathcal{S}(\mathbf{C}_I)$ is the Mellin transform operator from Theorem D.3.3.7 (see also Proposition D.3.2.5), defined as $M_{I,\tau}(\phi) := \int_{\mathbf{R}_{>0}} \phi(\tau(t)) t^{s+ix_0}\,\frac{\mathrm{d}t}{t}$ for $\mathrm{Re}(s) \in I$, and similarly for $M_{I,\tau}$ on $\mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$.*
2. *$J_\tau : \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \to \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ the linear topological isomorphism uniquely determined by sending $\phi \in \mathcal{S}_{1-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ as a function of $\tau(t)g_0$ for unique $t \in \mathbf{R}_{>0}$ and $g_0 \in J^1_{F,S}$, to the function $t^{-1}\phi(\tau(t)^{-1}g_0)$, which is an element of $\mathcal{S}_{1-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$.*
3. *$\widetilde{J} : A\mathcal{S}(\mathbf{C}_I) \to A\mathcal{S}(\mathbf{C}_{1-I})$ the linear topological isomorphism sending an element $\phi \in A\mathcal{S}(\mathbf{C}_I)$ to the holomorphic function $\widetilde{J}(\phi)(s) := \rho_S(\chi|\cdot|^s_{F,S})\phi(1-s)$ on $\mathbf{C}_{1-I}$.*
4. *$\widetilde{\mathrm{pr}}_2$ is the projection onto $\mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, followed by the operator $M^{-1}_{I,\tau} \circ M_{\rho_S} \circ M_{I,\tau}$, where M_{ρ_S} is the multiplication operator by $\rho_S(\chi|\cdot|^s_{F,S})$ on $A\mathcal{S}(\mathbf{C}_I)$.*

(5) *The operator* $\mathrm{pr}_1 - \widetilde{\mathrm{pr}}_2$ *is the operator sending* $(\phi, \psi) \in \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \times \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$*, to the pair* $(\phi, \widetilde{\psi} := \widetilde{\mathrm{pr}}_2(\phi, \psi))$*, further sent to the pair of elements of* $\mathcal{S}_{I\cap(1-I)}(\mathbf{R}_{>0})$ *defined by* $(\phi(\tau(t)), \widetilde{\psi}(\tau(t)))$*, further sent to their difference.*

(6) T_{ix_0} *is the vertical translation by* ix_0 *linear topological automorphism on* $A\mathcal{S}(\mathbf{C}_{I\cap(1-I)})$*.*

is commutative, the vertical arrows are linear topological isomorphisms, the leftmost horizontal arrows are linear topological embeddings with closed range, and both rows are exact on the left and in the middle.

We refer to the short-exact sequences of (5.4.2.6) as the *S-local Čech sequences*. Indeed, it exhibits the isotype $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ as a space of meromorphic functions on the connected component of $\Omega(J_{F,S})$ parametrized by $\chi|_{J^1_{F,S}}$, upon choosing the holomorphic parametrization $\chi|\cdot|^s_{F,S}$ for such connected component.

Remark 5.4.2.3. Let F be a number field, S a finite set of finite places of F containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character.

Theorem 5.4.2.2 is a way of rephrasing Theorem 5.4.1.1, the dictionary being the Mellin transform $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I)$ for any non-empty connected open interval $I \subset \mathbf{R}$ (up to a vertical translation by ix_0 for a unique $x_0 \in \mathbf{R}$, through the global holomorphic coordinate $\mathbf{C}_I$ prescribed by the equation $\chi(\tau(t)) = t^{ix_0}$ for all $t \in \mathbf{R}_{>0}$.

We define $I := (0, +\infty) \subset \mathbf{R}$.

We will, however, highlight here how it implies that the continuous linear map:

$$(\mathrm{Id}, \mathcal{F}_S) : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) \times \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \tag{5.4.2.7}$$

sending ϕ to $(\phi, \mathcal{F}_S(\phi))$, is a linear topological embedding with closed range.

Proof. By Proposition 5.4.2.1 and Theorem D.3.3.13, the Theorem follows from the Second Structure Theorem (see Theorem 5.4.1.1). □

5.5. Structure Theorem III: strong duals and Eigendistributions

Let F be a number field, S a finite set of places containing all Archimedean places of F.

In this section we prove a Structure Theorem describing the strong duals of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ in terms of Weil's eigendistributions, providing the tools to prove the remaining of the Structure Theorems, resting on group homology calculations.

5.5.1. Eigendistributions and the strong duals of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. Building on work of Weil on eigendistributions (see [**Weil-a**]), we assign the following:

Definition 5.5.1.1. Let F be a number field, S a finite set of places of F containing all Archimedean places.

(1) We say a tempered distribution $T \in \mathcal{S}(\mathbf{A}_{F,S})'_\beta$ is an *S-local eigendistribution of weight* $c \in \Omega(J_{F,S})$ if and only if

$$\langle T, \rho_{F,S}(g, f)\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta, \mathcal{S}(\mathbf{A}_{F,S})} = c(g)\langle T, f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta, \mathcal{S}(\mathbf{A}_{F,S})} \tag{5.5.1.1}$$

for every $f \in \mathcal{S}(\mathbf{A}_{F,S})$ and $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$. Equivalently, if and only if:

$$\rho'_{F,S}(g, T) = c(g)T \tag{5.5.1.2}$$

for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, where $\rho'_{F,S}$ is the adjoint action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$. We define $\mathcal{S}(\mathbf{A}_{F,S})'_\beta(c)$ to be the closed subspace of eigendistributions of weight c, endowed with the subspace topology.

(2) We say a tempered distribution $T \in \mathcal{S}(\mathbf{A}_F)'_\beta$ is a *global eigendistribution of weight* $c \in \Omega(C_F)$ if and only if

$$\langle T, \rho_F(g,f)\rangle_{\mathcal{S}(\mathbf{A}_F)'_\beta,\mathcal{S}(\mathbf{A}_F)} = c(g)\langle T, f\rangle_{\mathcal{S}(\mathbf{A}_F)'_\beta,\mathcal{S}(\mathbf{A}_F)} \tag{5.5.1.3}$$

for every $f \in \mathcal{S}(\mathbf{A}_F)$ and $g \in \mathrm{GL}_1(\mathbf{A}_F)$. Equivalently, if and only if:

$$\rho'_F(g,T) = c(g)T \tag{5.5.1.4}$$

for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$, where ρ'_F is the adjoint action of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}(\mathbf{A}_F)'_\beta$. We define $\mathcal{S}(\mathbf{A}_F)'_\beta(c)$ to be the closed subspace of eigendistributions of weight c, endowed with the subspace topology.

In a similar vein as in [**Weil-a**], we now classify S-local eigendistributions of weight $c \in \Omega(J_{F,S})$. We will later classify the global eigendistributions of weight $c \in \Omega(C_F)$.

THEOREM 5.5.1.2. *Let F be a number field, S a finite set of places of F containing all Archimedean places.*

(1) *For $\Delta'(c)$ the eigendistribution of weight c defined as:*

$$\Delta'(c) := \bigotimes_{v\in S} \Delta'_v(c) \tag{5.5.1.5}$$

for Δ'_v the local eigendistribution constructed in [**Weil-a**, §1]*, we have:*

$$\mathcal{S}(\mathbf{A}_{F,S})'_\beta(c) = \mathbf{C}\cdot\Delta'(c). \tag{5.5.1.6}$$

Moreover, formation of $\Delta'(c)$ is holomorphic as a function of $c \in \Omega(J_{F,S})$.

(2) *Let T be a fixed element of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$. Let T be an S-local eigendistribution of weight $c \in \Omega(J_{F,S})$. There exists a unique $T_0 \in \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$, such that the adjoint action $\rho'_{F,S}$ of $J_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ satisfies:*

$$\rho'_{F,S}(g,T_0) = c(g)T_0 \tag{5.5.1.7}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, and such that:

$$\langle T_0, \Phi_{F,S}(f)\rangle_{\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S},\mathcal{S}(\mathbf{A}_{F,S}))'_\beta,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S},\mathcal{S}(\mathbf{A}_{F,S}))} = \langle T, f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta,\mathcal{S}(\mathbf{A}_{F,S})}. \tag{5.5.1.8}$$

Moreover, upon denoting by $\Delta_{F,S}(c)$ resulting unique element of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta(c)$ such that $\Phi'_{F,S}(\Delta_{F,S}(c)) = \Delta'(c)$, we have:

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta(c) = \mathbf{C}\cdot\Delta_{F,S}(c) \tag{5.5.1.9}$$

and formation of $\Delta_{F,S}(c)$ is holomorphic as a function of $c \in \Omega(J_{F,S})$.

PROOF. (1) : One-dimensionality of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta(c)$ for every $c \in \Omega(J_{F,S})$ follows from Theorem 3.4.3.3, which also provides the canonical generators $\Delta'(c)$ and $\Delta_{F,S}(c)$.

Since, for every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, $\langle\Delta'(c), f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta,\mathcal{S}(\mathbf{A}_{F,S})}$ is a holomorphic function of $c \in \Omega(J_{F,S})$ by Theorem 3.4.3.3, then by Theorem C.5.2.6 and Theorem C.5.2.4 formation of Δ' is holomorphic over $\Omega(J_{F,S})$.

(2) : The adjoint operator $\Phi'_{F,S}$ of $\Phi_{F,S} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ is continuous by [**NB**, Thm.8.11.3] and injective by [**NB**, Thm.8.13.2]. Every element of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ is therefore in a unique manner an $\mathcal{O}^\times_{F,S}$-invariant tempered distribution on $\mathcal{S}(\mathbf{A}_{F,S})$.

Let T be an S-local eigendistribution of weight $c \in \Omega(J_{F,S})$. Then $T = A\cdot\Delta'(c)$, with Δ' defined in (5.5.1.5). Since, by (1), $\Delta' : \Omega(J_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S})'_\beta$ is a holomorphic function, its value at c is controlled by its values over non-empty open subsets of $\Omega(J_{F,S})$ by the Identity Theorem.

We then have, for every $f \in \ker(\Phi_{F,S})$:

$$L_S(c')\langle\Delta'(c'), f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta,\mathcal{S}(\mathbf{A}_{F,S})} = 0 \tag{5.5.1.10}$$

for $L_S(c')$ the S-local Hecke L-function, and for c' away from the polar set $\mathcal{U}_S \subset \Omega(J_{F,S})$ of the meromorphic function $\rho_S(c')$ (see (3.3.1.3)) serving as correcting factor in the S-local functional equation (see (3.3.1.2)),

because $L_S(c')\Delta'(c') = Z_S(f,c')$ by (3.4.3.13), and Lemma 5.1.2.2 applies. Since, over $\Omega(J_{F,S}) - \mathcal{U}_S$, $L_S(c')$ is holomorphic, and as such its set of zeros is discrete, we deduce that we have:

$$\langle \Delta'(c'), f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta, \mathcal{S}(\mathbf{A}_{F,S})} = 0 \tag{5.5.1.11}$$

as holomorphic scalar valued functions on the entire $\Omega(J_{F,S})$, and hence

$$\langle \Delta'(c), f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta, \mathcal{S}(\mathbf{A}_{F,S})} = 0. \tag{5.5.1.12}$$

This means that $\Delta'(c)$ factors through the quotient of $\mathcal{S}(\mathbf{A}_{F,S})$ by $\ker(\Phi_{F,S})$, i.e. $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$. Being the topology on the latter space the quotient topology inherited from $\mathcal{S}(\mathbf{A}_{F,S})$ under $\Phi_{F,S}$, we deduce that $\Delta'(c)$ is in the image of $\Phi'_{F,S}$, and hence so is T.

The continuous operator $\Phi'_{F,S}$ restricts to a linear bijection:

$$\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta(c) \to \mathcal{S}(\mathbf{A}_{F,S})'_\beta(c) \tag{5.5.1.13}$$

for every $c \in \Omega(J_{F,S})$. To see this, we observe that $\Phi'_{F,S}$ is a linear topological embedding: indeed, by [**NB**, Thm.8.13.2] it is injective and weakly relatively open, and by [**NB**, Thm.8.12.4(b)] applied to the quotient map $\Phi_{F,S}$, the assumptions of [**NB**, Thm.8.13.4] are satisfied, and $\Phi'_{F,S}$ is a linear topological embedding. By the foregoing its range contains the linear subspace of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ linearly generated by the S-local eigendistributions.

By Theorem 3.4.3.3, the target in (5.5.1.13) is 1-dimensional and contained in the range of $\Phi'_{F,S}$, and hence the continuous injective linear operator from (5.5.1.13) is a linear topological isomorphism, which proves (5.5.1.9).

By the foregoing, $c \mapsto \Delta'(c)$, a holomorphic function $\Omega(J_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S})'_\beta$, factors through $\Phi'_{F,S}$. By Lemma C.4.3.8 and injectivity of $\Phi'_{F,S}$ (which is a consequence of [**NB**, Thm.8.13.1]), we deduce from Theorem C.5.2.6 and Theorem C.5.2.4 that formation of $\Delta_{F,S}$ is holomorphic over $\Omega(J_{F,S})$. □

The same argument for Theorem 5.5.1.2, establishes the following global analog:

THEOREM 5.5.1.3. *Let F be a number field.*

(1) *For $\Delta'(c)$ the eigendistribution of weight c defined as:*

$$\Delta'(c) := \bigotimes_{v\in\Sigma_F} \Delta'_v(c) \tag{5.5.1.14}$$

for Δ'_v the local eigendistribution constructed in [**Weil-a**, §1]*, we have:*

$$\mathcal{S}(\mathbf{A}_F)'_\beta(c) = \mathbf{C}\cdot\Delta'(c), \tag{5.5.1.15}$$

Moreover, formation of $\Delta'(c)$ is holomorphic as a function of $c \in \Omega(C_F)$.

(2) *Let T be a fixed element of $\mathcal{S}(\mathbf{A}_F)'_\beta$. Assume T is a global eigendistribution of weight $c \in \Omega(C_F)$. There exists a unique $T_0 \in \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta$, such that the adjoint action ρ'_F of C_F on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta$ satisfies:*

$$\rho'_F(g, T_0) = c(g)T_0 \tag{5.5.1.16}$$

for all $g \in \mathrm{GL}_1(\mathbf{A}_F)$, and such that:

$$\langle T_0, \Phi_F(f)\rangle_{\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))'_\beta, \mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))} = \langle T, f\rangle_{\mathcal{S}(\mathbf{A}_F)'_\beta, \mathcal{S}(\mathbf{A}_F)}. \tag{5.5.1.17}$$

Moreover, upon denoting $\Delta_F(c)$ the resulting unique element of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta(c)$ such that $\Phi'_F(\Delta_F(c)) = \Delta'(c)$, we have:

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta(c) = \mathbf{C}\cdot\Delta_F(c) \tag{5.5.1.18}$$

and formation of $\Delta_F(c)$ is holomorphic as a function of $c \in \Omega(C_F)$.

PROOF. By [NB, Thm.8.13.2], Φ'_F is injective and weakly relatively open, and by [NB, Thm.8.12.4(b)] applied to the quotient map Φ_F, the assumptions of [NB, Thm.8.13.4] are satisfied, and Φ'_F is a linear topological embedding. The same argument shows that $\Phi'_{F,\omega}$ is a linear topological embedding, for every $\omega \in \widehat{C^1_F}$.

Since, by Theorem 5.2.1.1 and Theorem 2.1.4.1, respectively, Lemma C.1.6.1 applies to both $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))'_\beta$ and $\mathcal{S}(\mathbf{A}_F)'_\beta$, as these are strict countable locally convex inductive limits (directed, up to choosing a cofinal sequence of finite subsets of Σ_F containing all the Archimedean places of F) of the S-local spaces $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathcal{S}(\mathbf{A}_{F,S})$, all reflexive, being Montel, both (1) and (2) reduce to Theorem 5.5.1.2. □

REMARK 5.5.1.4. We note that Theorem 5.5.1.2(1) partially informs us about the structure of the linear image of $\Phi'_{F,S}$ within $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$. It contains the S-local eigendistributions, and hence the linear subspace they topologically (being $\Phi'_{F,S}$ a linear topological embedding) generate. In fact, a converse is also true: it turns out that the S-local eigendistributions are linear topological generators of the range of $\Phi'_{F,S}$, and hence of the strong dual of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$. This is the content of the Third Structure Theorem (see Theorem 5.5.3.2) and it rests on the Second Structure Theorem, describing the topology on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ (see Theorem 5.4.1.1), and on the Spectral Theorem for unbounded densely defined normal operators on Hilbert spaces (see [Cnw-a, Thm.4.11]). Similar considerations hold for Theorem 5.5.1.3(1), which says that the range of Φ'_F within $\mathcal{S}(\mathbf{A}_F)'_\beta$ contains the global eigendistributions, and hence the linear subspace they topologically (being Φ'_F a linear topological embedding) generate.

5.5.2. An example: global eigendistributions of weight $c = 1, |\cdot|_F$. We illustrate Theorems on an example.

Recall from (2.2.4.23) that we have defined:

$$\mathcal{S}_0(\mathbf{A}_F) := \ker(\delta_0) \cap \ker(\widehat{\delta_0}) \tag{5.5.2.1}$$

where $\delta_0 : \mathcal{S}(\mathbf{A}_F) \to \mathbf{C}$ is the strongly continuous functional $\delta_0(f) := f(0)$, and $\widehat{\delta_0}$ is its adelic Fourier transform, i.e. $\widehat{\delta_0}(f) := \widehat{f}(0)$.

We endow $\mathcal{S}_0(\mathbf{A}_F)$ with the subspace topology, promoting it to a closed subspace of $\mathcal{S}(\mathbf{A}_F)$, being δ_0 and $\widehat{\delta_0}$ continuous functionals on $\mathcal{S}(\mathbf{A}_F)$.

LEMMA 5.5.2.1. *Let F be a number field. The continuous functionals $\delta_0, \widehat{\delta_0} \in \mathcal{S}(\mathbf{A}_F)'_\beta$ descend uniquely to continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. They are homogeneous with respect to the adjoint of the continuous left-translation action of C_F on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and satisfy:*

$$\rho'_F(g, \delta_0) = \delta_0, \quad \rho'_F(g, \widehat{\delta_0}) = |g|_F \widehat{\delta_0} \tag{5.5.2.2}$$

for all $g \in C_F$, i.e. δ_0 is a global eigendistribution of weight $c = 1$, and $\widehat{\delta_0}$ is a global eigendistribution of weight $c = |\cdot|_F$. Moreover, for every $\omega \in \widehat{C^1_F}$, both δ and $\widehat{\delta_0}$ vanish on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ if and only if $\omega \neq 1$.

In the notation of Theorem 5.5.1.3, we have

$$\delta_0 = \Delta_F(1), \quad \widehat{\delta_0} = \Delta_F(|\cdot|_F). \tag{5.5.2.3}$$

PROOF. The value of the functionals $\delta_0, \widehat{\delta_0}$ on $f \in \mathcal{S}(\mathbf{A}_F)$ depend, respectively, only on $\Phi_F(f)$. Indeed, $\delta_0(f)$ is the residue of $Z(f, c)$ at $c = 1$ up to a nonzero constant. Similarly, $\widehat{\delta_0}(f) = \widehat{f}(0)$ is the residue of $Z(f, c)$ at $c = |\cdot|_F$, up to a nonzero constant (see Theorem 3.2.2.1). Assume $f_1, f_2 \in \mathcal{S}(\mathbf{A}_F)$ are such that $\Phi_F(f_1) = \Phi_F(f_2)$. Then by Lemma 5.1.2.1 we have $Z(f_1, c) = Z(f_2, c)$ as meromorphic functions on $\Omega(C_F)$, and their poles and residues are also equal.

Both functionals δ_0 and $\widehat{\delta_0}$ induce unique linear functionals on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, continuous with respect to the quotient topology. Such residues vanish over all the connected components of $\Omega(C_F)$ parametrized by non-trivial continuous characters of C^1_F. □

REMARK 5.5.2.2. We observe that while δ_0 and $\widehat{\delta_0}$ induce continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, which, throughout this remark, we shall denote the same way, this is not saying that $\delta_0(\Phi_F(f)) = \sum_{\xi\in F^\times} \delta_0(f(\xi^{-1}x))$: indeed, this series diverges.

The value of δ_0 at $\Phi_F(f)$, for some $f \in \mathcal{S}(\mathbf{A}_F)$, is $f(0)$: Lemma 5.5.2.1 informs us that this is well-defined modulo $\ker(\Phi_F)$, i.e. modulo $\ker(Z)$ (see Lemma 5.1.2.1). The same holds for $\widehat{\delta_0}$.

This can also be seen directly from Theorem 2.2.3.4(2): let $f \in \mathcal{S}(\mathbf{A}_F)$ be such that $\Phi_F(f) = 0$. Then by (2.2.3.5) from Theorem 2.2.3.4(2), we have, for every integer $N \geq 0$, that there exists a real number $C_N > 0$ such that:

$$|-|g|_F^{-1}\widehat{f}(0) + f(0)| \leq C_N|g|_F^N$$

for all $g \in C_F$ as $|g|_F \to 0^+$. This forces $\widehat{f}(0) = 0$, and hence also $f(0) = 0$. Therefore, if $f \in \mathcal{S}(\mathbf{A}_F)$ is arbitrary and $f_0 \in \ker(\Phi_F)$, δ_0 and $\widehat{\delta_0}$ satisfy $\delta_0(f) = \delta_0(f + f_0)$ and $\widehat{\delta_0}(f) = \widehat{\delta_0}(f + f_0)$.

5.5.3. Third Structure Theorem. The Third Structure Theorem (Theorem 5.5.3.2) places Weil's eigendistributions into their natural place to work organically, obtaining a precise result on the structure of the locally convex spaces of co-invariants of $\mathcal{S}(\mathbf{A}_F)$ and $\mathcal{S}(\mathbf{A}_{F,S})$ under the action of $\mathrm{GL}_1(\mathbf{A}_F)$ and its subgroups, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and its subgroups.

The following Proposition provides a local converse to Theorem 5.5.1.2(1) (i.e. local at a single place v of the number field F), and builds towards Theorem 5.5.3.2.

PROPOSITION 5.5.3.1. *Let F be a number field, v a place of F, $\omega \in \Omega(F_v^\times)$ a unitary character. Then $\mathcal{S}(F_v)'_\beta$ is linearly and topologically generated by the functionals $\{\Delta'_v(c), c \in \Omega(F_v^\times)\}$, and the closed subspace of $\mathcal{S}(F_v)'_\beta$ of the ω-homogeneous tempered distributions is linearly and topologically generated by the functionals $\{\Delta'_v(c), c \in \Omega(F_v^\times)(\omega)\}$.*

PROOF. We show that $\{\Delta'_v(c), c \in \Omega(F_v^\times)\}$ separates the points of $\mathcal{S}(F_v)$. Indeed, let $f \in \mathcal{S}(F_v)$ and assume $\Delta'_v(c)(f)$ for every $c \in \Omega(F_v^\times)$. Then $Z_v(f, c) = \Delta'_v(c)(f)L_v(c) = 0$ as a meromorphic function over $\Omega(F_v^\times)$. In particular, f vanishes identically over $F_v^\times$, and hence over F_v by continuity. We now call E the linear subspace of $\mathcal{S}(F_v)'_\beta$ generated by $\{\Delta'_v(c), c \in \Omega(F_v^\times)\}$. Since $\{\Delta'_v(c), c \in \Omega(F_v^\times)\}$ separates the points of $\mathcal{S}(F_v)$, then by [**NB**, Thm.7.7.3] E is dense with respect to the $\sigma(\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v))$-topology on $\mathcal{S}(F_v)'$, which is the strong topology by [**NB**, Thm.11.3.6] since $\mathcal{S}(F_v)'_\beta$ is barrelled, as $\mathcal{S}(F_v)$ is Montel, and hence $\sigma(\mathcal{S}(F_v)'_\beta, \mathcal{S}(F_v)) = \sigma(\mathcal{S}(F_v)'_\beta, (\mathcal{S}(F_v)'_\beta)')$ since $\mathcal{S}(F_v)$ is reflexive (recall that the bi-dual of a Hausdorff locally convex space is defined to be the dual of the strong dual; see [**Jar**, Part II,Ch.11,§11.2]).

Assume now that v is non-Archimedean.

Since the closed subspace of ω-homogeneous tempered distributions on $\mathcal{S}(F_v)$ is closed in $\mathcal{S}(F_v)'_\beta$, which is complemented. Indeed, integration over the compact Hausdorff abelian group $\mathcal{O}_{F_v}^\times$ with respect to the local multiplicative Tamagawa measure produces a linear form, for each $f \in \mathcal{S}(F_v)$:

$$P_v(f)(\varphi) := \int_{\mathcal{O}_{F_v}^\times} \langle \rho_v(g, f), \varphi\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'_\beta} \overline{\omega}(g)\, \mathrm{d}\mu_v(g)$$

for $\varphi \in \mathcal{S}(F_v)'$, and for each f, $P_v(f)$ is valued in $(\mathcal{S}(F_v)')^*$. Since $\mathcal{S}(F_v)$ is complete, hence quasi-complete, then by Theorem C.4.3.5 we see that $P_v(f)$ is contained in $\mathcal{S}(F_v)$, and since its formation is linear, it is a linear operator $P_v : \mathcal{S}(F_v) \to \mathcal{S}(F_v)$.

By Lemma C.4.3.12, P_v is continuous. Its range is the closed subspace of ω-homogeneous elements of $\mathcal{S}(F_v)$, and hence $\ker(P_v)^\perp$, which already contains all ω-homogeneous tempered distributions in $\mathcal{S}(F_v)'_\beta$, coincides with their closed subspace in $\mathcal{S}(F_v)'_\beta$. □

For F a number field and S a finite set of places of F containing all the Archimedean places, the S-local and global analogs of Proposition 5.5.3.1 rest upon the description of the topology of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ provided by the Second Structure Theorem (see Theorem 5.4.1.1), on the First Structure Theorem (see Theorem

5.2.1.1) to reduce the global assertion to the S-local case, and on the Spectral Theorem for unbounded densely defined normal operators on Hilbert spaces.

THEOREM 5.5.3.2 (S-local Third Structure Theorem). *Let F be a number field, S a finite set of places containing all Archimedean places.*

(1) *For any $\omega \in \widehat{J^1_{F,S}}$, the continuous linear operator $\Phi'_{F,S,\omega}$ is injective and maps $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ into $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ within the closed subspace of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ of the ω-homogeneous tempered distributions. The strong closure of its range is linearly and topologically generated by the image under $\Phi'_{F,S,\omega}$ of the functionals $\{\Delta_{F,S}(c), c \in \Omega(J_{F,S})(\omega)\}$.*

(2) *The continuous linear operator $\Phi'_{F,S}$ is injective and maps $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ into $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ within the closed subspace of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ of the $\mathcal{O}^\times_{F,S}$-invariant tempered distributions, and the strong closure of its range is linearly and topologically generated by the image under $\Phi'_{F,S}$ of the functionals $\{\Delta_{F,S}(c), c \in \Omega(J_{F,S})\}$.*

PROOF. We start by recalling the following general argument: let $A : X \to Y$ be a continuous linear operator between Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Assume $Y = \prod_{i\in I} H_i$ is a locally convex product of Hilbert spaces, and A is a linear topological embedding. Then the canonical linear bijection $P : \bigoplus_{i\in I} H'_i \to Y'$ provided by [**Jar**, Part II,Ch.8,§8.8,Prop.1] is continuous with respect to the locally convex direct sum topology on the source, and the strong topology on the target, and the composition

$$A' \circ P : \bigoplus_{i\in I} H'_i \to Y' \xrightarrow{A'} X' \tag{5.5.3.1}$$

is continuous, linear and surjective. To see this, we proceed as follows. From [**Jar**, Part II,Ch.8,§8.8,Prop.1], we know that P is a continuous linear bijection with respect to the stated topologies on source and target. It is therefore enough to show that $A' : Y' \to X'$ is continuous and surjective. Since A is injective and relatively open, being a linear topological embedding, this follows from [**NB**, Thm.8.13.1(c)].

(1) : We now call $X := \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, for an arbitrary choice $\omega \in \widehat{J^1_{F,S}}$. The Second Structure Theorem (Theorem 5.4.1.1) applied with $p = 2$, provides us with a linear topological embedding:

$$U_S : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \to \prod_{m\geq 0} \prod_{N\geq 0} (L^2(\mathbf{R}, \mathrm{d}x) \times L^2(\mathbf{R}, \mathrm{d}x))$$

defined as, upon choosing $\chi \in \widehat{J_{F,S}}$ such that $\chi|_{J^1_{F,S}} = \omega$:

$$U_S(f) := (s^N Z_S(f, \chi|\cdot|^s_{F,S}), s^N Z_S(\widehat{f}, \chi|\cdot|^s_{F,S}))_{s\in\mathbf{C}_{>0}, \mathrm{Re}(s)=a_m}$$

for $\{a_m\}_{m\geq 0} \subset \mathbf{Q}_{>0}$ a sequence of positive rational numbers such that 0 is an accumulation point, and $\limsup_{m\geq 0} a_m = +\infty$.

The linear topological embedding U_S induces a family $\mathcal{P}$ of seminorms on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ that generate the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$. We define, for each $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$:

$$p_{m,N}(\phi) := \|(a_m+ix)^N M(\phi)(a_m+ix)\|_{L^2(\mathbf{R},\mathrm{d}x)},\ \ \widehat{p}_{m,N}(\phi) := \|(a_m+ix)^N M(\mathcal{F}_S(\phi))(a_m+ix)\|_{L^2(\mathbf{R},\mathrm{d}x)},\ m, N \geq 0$$

where $M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(s) t^s \frac{\mathrm{d}t}{t}$ is the Mellin transform.

By Theorem 5.2.1.1(2), the family $\mathcal{P} = \{p_{m,N}, \widehat{p}_{m,N}, m, N \in \mathbf{N}\}$ is a separating countable family of seminorms on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ generating the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$.

The completion of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ with respect to any $p \in \mathcal{P}$ is a Hilbert space, which we denote H_p.

The linear map:

$$A : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \to \prod_{p\in\mathcal{P}} H_p \tag{5.5.3.2}$$

obtained from the completion operators $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \to H_p$, is a linear topological embedding, since $\mathcal{P}$ generates the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$.

In particular, the canonical linear operator

$$A' \circ P : \bigoplus_{p\in\mathcal{P}} H'_p \to (\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))' \tag{5.5.3.3}$$

from the foregoing (where P is the linear bijection $P : \bigoplus_{p\in\mathcal{P}} H'_p \to (\prod_{p\in\mathcal{P}} H_p)'$ provided by [**Jar**, Part II,Ch.8,§8.8,Cor.1]) is a surjective linear operator from the direct sum $\bigoplus_{p\in\mathcal{P}} H'_p$ onto $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'$. Upon endowing each H'_p with the strong topology (promoting them to Hilbert spaces, anti-isomorphic to H_p itself by [**Cnw-a**, Ch.I,§3,Thm.3.4]), their direct sum with the resulting Hausdorff locally convex topology, and $(\prod_{p\in\mathcal{P}} H_p)'$ with its strong topology, then P is continuous, and $A' \circ P$ is continuous with respect to the locally convex direct sum topology of the $(H_p)'_\beta$ on the source, and the strong topology of $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ on the target.

We choose a continuous homomorphic section τ to the S-idelic norm, and observe that each Hilbert space H_p is equipped with a strongly continuous one-parameter group of operators as follows induced by the $\mathbf{R}_{>0}$-action on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ through τ. Indeed, we define:

$$\Psi : \mathbf{R} \times H_p \to H_p \tag{5.5.3.4}$$

by, for $f \in \mathcal{S}(\mathbf{A}_{F,S})$ and $p = p_{m,N}$ with $m, N \geq 0$ integers:

$$\Psi(t', \Phi_{F,S,\omega}(f))(g) := \Phi_{F,S,\omega}(f)(\tau(t')^{-1}g) \tag{5.5.3.5}$$

the right side being a continuous function of $g \in J_{F,S}$, $\overline{\omega}$-homogeneous under the left-translation action of $J^1_{F,S}$.

To see this, we proceed as follows. We have, for the above choice of $p \in \mathcal{P}$:

$$p(\Psi(t', \Phi_{F,S,\omega}(f))(g)) = \left\| (a_m + ix)^N \int_{\mathbf{R}_{>0}} t^{a_m+ix} \left(\int_{J^1_{F,S}} \omega(g_0)\Phi_{F,S}(f)(\tau(t')^{-1}g_0\tau(t))\, \mathrm{d}\overline{\mu}_{S,1}(g_0) \right) \frac{\mathrm{d}t}{t} \right\|_{L^2(\mathbf{R},\mathrm{d}x)}$$

for unique integers $m, N \geq 0$.

We compute:

$$p(\Psi(t', \Phi_{F,S,\omega}(f))(g)) = \|(a_m + ix)^N (t')^{a_m+ix} M(\Phi_{F,S,\omega}(f)(\tau(t))(a_m + ix)\|_{L^2(\mathbf{R},\mathrm{d}x)} = (t')^{a_m} p(\Phi_{F,S,\omega}(f)).$$

We deduce that the new strongly continuous 1-parameter group of operator given by

$$t \mapsto t^{-a_m}\Psi(t, \cdot)$$

is unitary.

The infinitesimal generator Θ, which exists and is unbounded and densely defined on H_p by [**Cnw-a**, Thm.5.6], has real spectrum $\sigma(\Theta) \subset \mathbf{R}$, and by [**Cnw-a**, Thm.4.11], there exists a unique spectral measure $E : \mathcal{B}(\sigma(\mathbf{R})) \to \mathscr{B}_{\mathrm{SOT}}(H_p)$ valued in orthogonal projectors, such that the action $\rho_{F,S}$ of $J_{F,S}$ on H_p defined as the unique continuous extension of the action of $J_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, can be written as, for each $\phi \in H_p$:

$$\rho_{F,S}(g, \phi) = t^{a_m}\omega(g_0) \int_{\sigma(\mathbf{R})} t^{ix}\, \mathrm{d}E(x)\phi$$

upon writing $g \in J_{F,S}$ uniquely as $\tau(t)g_0$ for $(t, g_0) \in \mathbf{R}_{>0} \times J^1_{F,S}$.

From the construction of E from the proof of [**Cnw-a**, Thm.4.11], we may choose, for every real number $X \geq 0$, increasingly fine partitions $\{\Delta_{k,X}\}_{k\in\mathbf{N},X}$ of $[-X, X] \subset \mathbf{R}$ by Borel sets, and points $x_k \in \Delta_k$, ordering the intervals $[-X, X]$ by inclusion and the partitions by refinement, obtaining a net of continuous linear operators $\sum_{k\in\mathbf{N}} t^{ix_k} E(\Delta_{k,X})$ on H_p such that

$$\sum_{k\in\mathbf{N}} t^{ix_k} E(\Delta_{k,X}) \to \int_{\sigma(\mathbf{R})} t^{ix}\mathrm{d}E(x) \tag{5.5.3.6}$$

as $X \to +\infty$, and the partitions of $[-X, X]$ are increasingly finer, with respect to the strong operator topology on $\mathscr{B}(H_p)$ (the $\mathbf{C}$-algebra of bounded operators on H_p).

The range of $E(\Delta_{k,X})$ splits off as a linear topological direct summand of H_p, being $E(\Delta_{k,X})$ an orthogonal projector.

Since the convergence in (5.5.3.6) is with respect to the strong operator topology on $\mathscr{B}(H_p)$, then we deduce that any $\phi \in H_p$, can be approximated to an arbitrary degree of precision in the norm-topology on H_p, by a finite linear combination of elements ϕ_k of the form $\phi_k = E(\Delta_{X,k})\phi$, on which the continuous action of $J_{F,S}$ has the diagonal form:

$$\rho_{F,S}(\tau(t)g_0, \phi_k) = t^{a_m + ix_k}\omega(g_0)\phi_k.$$

Since $(H_p)'_\beta$ is anti-isometric to H_p compatibly with the strongly continuous action $t \mapsto t^{-a_m}\Psi(t, \cdot)$, which the anti-isometry carries over to the dual action, the same conclusion holds for the elements T of $(H_p)'_\beta$: they can be approximated to an arbitrary degree of precision with respect to the norm-topology (which is the strong topology on H'_p) by finite $\mathbf{C}$-linear combinations of norm-continuous functionals T_k on H_p on which the action $\rho'_{F,S}$ of $\mathbf{R}_{>0} \times J^1_{F,S}$ is of the diagonal form:

$$\rho'_{F,S}(\tau(t)g_0, T_k) = t^{-a_m - ix_k}\overline{\omega}(g_0)T_k.$$

Each T_k is in particular continuous as a functional on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, with respect to the finer topology of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$, and therefore induces an element of $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ (which is none other than the image of $T_k \in (H_p)'_\beta \subset \bigoplus_{p\in\mathcal{P}}(H_p)'_\beta$ under the continuous linear operator $A' \circ P$ of (5.5.3.3)).

We have just shown that the functionals of the form T_k, on which the (dual) continuous $J_{F,S}$-action is diagonal, of weight c given by the quasi-character corresponding to $(t^{a_m+ix_k}, \omega)$ under the bi-holomorphism from (3.1.2.5) induced by the continuous homomorphic section τ to the S-idelic norm, are norm-dense in $(H_p)'_\beta$. They therefore span a dense linear subspace and the direct sum within $\bigoplus_{p\in\mathcal{P}}(H_p)'_\beta$ is sent onto a dense linear subspace of $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ by the continuous linear surjective operator $A' \circ P$ from (5.5.3.3).

Recall now the purpose of this argument. We want to show that the closed $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ of those tempered distributions T in the closure of the range of $\Phi'_{F,S,\omega}$, is comprised of tempered distributions that are $\overline{\omega}$-homogeneous, i.e. such that $\rho'_{F,S}(g, T) = \overline{\omega}(g)T$ for all $g \in J^1_{F,S}$, and it is linearly and topologically generated by the eigendistributions $\{\Delta_{F,S}(c), c \in \Omega(J_{F,S})(\omega)\}$ from (5.5.1.6) of Theorem 5.5.1.2.

The desired homogeneity condition is a consequence of the design of the operator $\Phi_{F,S,\omega}$, which factors through the projector $P_{F,S,\omega}$ from (5.2.3.9). We have just shown that c-homogeneous continuous functionals in $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$, as c varies over all S-idelic quasi-characters such that $c|_{J^1_{F,S}} = \omega$, linearly topologically generate $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$. Their image in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ is itself a homogeneous tempered distribution with weight given by an S-idelic quasi-character in $\Omega(J_{F,S})(\omega)$, which is what we wanted to show.

(2) : We define the linear subspace H of $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ generated by the functionals $\{\Delta_{F,S}(c), c \in \Omega(J_{F,S})\}$. We compute $H^\perp$ with respect to the duality $\langle(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))\rangle$.

Let $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ be viewed as an element of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ under the linear topological identification of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ with $\mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))$, the closed subspace of ω-homogeneous elements in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$.

The condition

$$\langle T, \phi\rangle_{(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))} = 0$$

is valid, in particular, for T in the subspace of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ generated by the functionals $\{\Delta_{F,S}(c), c \in \Omega(J_{F,S})(\omega)\}$. By (1), this is the closure of the range under $\Phi'_{F,S,\omega}$ of $(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$. Which forces $\phi = 0$.

Since the algebraic direct sum $\bigoplus_{\omega\in\widehat{J^1_{F,S}}} \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ is dense in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, we deduce that we have $H^\perp = 0$, and hence H is weakly dense in $(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta$ by [**NB**, Thm.8.9.2(e)], noting

that $\sigma((\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S})))'_{\beta},\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))) = \sigma((\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S})))'_{\beta},((\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S})))'_{\beta})')$ since $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ is reflexive by Theorem 5.2.1.1(1), being Montel. The weak closure of H in $(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S})))'_{\beta}$ agrees with its strong closure by the Hahn-Banach Theorem, and so H is dense in $(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S})))'_{\beta}$ (see also [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.230]). □

REMARK 5.5.3.3. An alternative argument for Theorem 5.5.3.2 uses the Mellin transform L_S. Indeed, the eigendistributions $\{\Delta_{F,S}(c)\}$ are multiples of point-evaluation functionals on the meromorphic functions on $\Omega(J_{F,S})$ given by the linear image in $\mathscr{M}(\Omega(J_{F,S}))$ under L_S from (5.1.1.2).

The joint kernel of the $\Delta_{F,S}(c)$ for c varying over $\Omega(J_{F,S})$, or simply over a connected subset containing an interval with non-empty 1-dimensional interior, is therefore trivial in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ by the Identity Theorem.

It follows that the continuous functionals $\{\Delta_{F,S}(c)\}$ separate the points of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$, and they linearly generate a weakly dense subspace of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))'_{\sigma}$ by [**NB**, Thm.8.9.2(e)].

By [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.230], since $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ is Montel by Theorem 5.2.1.1(1), such subspace is also strongly dense.

Theorem 5.5.3.2 enables us to establish the following global analog:

THEOREM 5.5.3.4 (Global Third Structure Theorem). *Let F be a number field.*

(1) *For any $\omega \in \widehat{C_F^1}$, the continuous linear operator $\Phi'_{F,\omega}$ is a linear topological embedding of $\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F))'_{\beta}$ into $\mathcal{S}(\mathbf{A}_F)'_{\beta}$, whose range is contained within the closed subspace of $\mathcal{S}(\mathbf{A}_F)'_{\beta}$ of the ω-homogeneous tempered distributions, and it is linearly and topologically generated by the image under $\Phi'_{F,\omega}$ of the functionals $\{\Delta_F(c), c \in \Omega(C_F)(\omega)\}$.*

(2) *The continuous linear operator Φ'_F is a linear topological embedding of $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))'_{\beta}$ into $\mathcal{S}(\mathbf{A}_F)'_{\beta}$, whose range is contained within the closed subspace of $\mathcal{S}(\mathbf{A}_F)'_{\beta}$ of the $F^{\times}$-invariant tempered distributions, and it is linearly and topologically generated by the image under Φ'_F of the functionals $\{\Delta_F(c), c \in \Omega(C_F)\}$.*

PROOF. (1) follows from (2), because Lemma 4.3.3.1 applies to the continuous adjoint action of C_F^1 on $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))'_{\beta}$, this space being complete by Theorem 5.2.3.1, hence quasi-complete by [**Scha**, Ch.III,§4,Corollary,p.85], and the action of C_F^1 being uniformly continuous as a consequence of Lemma 5.1.3.1.

Since, by Theorem 5.2.1.1 and Theorem 2.1.4.1, respectively, Lemma C.1.6.1 applies to both $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))'_{\beta}$ and $\mathcal{S}(\mathbf{A}_F)'_{\beta}$, as these are strict countable locally convex inductive limits (directed, up to choosing a cofinal sequence of finite subsets of Σ_F containing all the Archimedean places of F) of the S-local spaces $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathcal{S}(\mathbf{A}_{F,S})$, all reflexive, being Montel, both (1) and (2) reduce to Theorem 5.5.1.2. □

CHAPTER 6

Local-Global structures

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6.1. Structure Theorem IV: local within global

In this section we establish one more Structure Theorem on $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$ and their isotypes, identifying the range of the transition operators $\overline{\pi}_{S,S'}$ and $\overline{\pi}_S$, and proving their eventual surjectivity on isotypes for all large enough S.

6.1.1. Description of $\overline{\pi}_{S,S'}$ and $\overline{\pi}_S$. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

We provide a description of the transition operators $\overline{\pi}_{S,S'}$ and $\overline{\pi}_S$ from Theorem 5.2.3.1, which will elucidate their relation to Euler products, with a view towards the later constructions of Chapter 5.

PROPOSITION 6.1.1.1. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Upon choosing a $\mathbf{Z}$-basis $\Xi_{S,S'} := \{\xi_1, \ldots, \xi_{|S'|-|S|}\}$ for $\mathcal{O}_{F,S'}^{\times}/\mathcal{O}_{F,S}^{\times}$, and upon denoting by $\theta_\xi : \mathcal{S}(\mathbf{A}_{F,S'}) \to \mathcal{S}(\mathbf{A}_{F,S'})$ the continuous linear operator $f \mapsto \rho_{F,S}(\xi, f) = f(\xi^{-1}\cdot)$, we have, as elements of

$$\mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^{\times}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

that the following identity holds:

$$\overline{\pi}_{S,S'} = \prod_{\xi \in \Xi_{S,S'}} (1 - \theta_\xi)^{-1}. \tag{6.1.1.1}$$

Upon choosing a $\mathbf{Z}$-basis $\Xi_S := \{\xi_k, k \geq 1\}$ for $F^{\times}/\mathcal{O}_{F,S}^{\times}$, and upon denoting by $\theta_\xi : \mathcal{S}(\mathbf{A}_F) \to \mathcal{S}(\mathbf{A}_F)$ the continuous linear operator $f \mapsto \rho_F(\xi, f) = f(\xi^{-1}\cdot)$, we have, as elements of

$$\mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F)))$$

that the following identity holds:

$$\overline{\pi}_S = \prod_{\xi \in \Xi_S} (1 - \theta_\xi)^{-1} \tag{6.1.1.2}$$

the convergence of the product being uniform on the bounded subsets of $\mathcal{S}(\mathbf{A}_{F,S})$.

PROOF. This is a consequence of the fact that every element ξ in $\mathcal{O}_{F,S'}^\times/\mathcal{O}_{F,S}^\times$ acts non-trivially by left multiplication only through the submonoid $\mathcal{O}_{F,S'} \cap \xi^{\mathbf{Z}}$, since for any $f \in \mathcal{S}(\mathbf{A}_{F,S})$, we have $\pi_{S,S'}(f) = f \otimes 1_K$ with $K = \prod_{v\in S'-S} 1_{\mathcal{O}_{F_v}}$ with $K = \prod_{v\in S'-S} \mathcal{O}_{F_v} \subset \widehat{\prod}_{v\in S'-S}(F_v, \mathcal{O}_{F_v})$. Injectivity of $\overline{\pi}_{S,S'}$ (see Theorem 5.2.3.1(1)) implies that $\overline{\pi}_{S,S'} = \sum_{m\geq 0} \theta_\xi^m$ and $(1-\theta_\xi)^{-1}$ are the same element in

$$\mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'}))).$$

Similarly, every $\xi \in F^\times/\mathcal{O}_{F,S}^\times$ is such that acts on $\pi_S(\mathcal{S}(\mathbf{A}_{F,S}))$ by left multiplication through the submonoid $\mathcal{O}_F \cap \xi^{\mathbf{Z}}$. The convergence assertion is a consequence of Lemma 2.2.2.1. □

6.1.2. Ranges of $\overline{\pi}_{S,S'}$ and $\overline{\pi}_S$, and subspaces of invariants. We recall that for S, S' finite sets of places of the number field F containing all Archimedean places, and such that $S \subset S'$, the Hausdorff locally compact abelian group $J_{F,S} = \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathcal{O}_{F,S}^\times$ from (3.1.2.1) acts continuously on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ by left translation, as does $J_{F,S'}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'}))$, and the compact Hausdorff abelian group $\prod_{v\in S'-S} \mathcal{O}_{F_v}^\times$ acts on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'}))$ through the insertion homomorphism:

$$\widehat{\mathcal{O}}_{F,S'-S}^\times := \prod_{v\in S'-S} \mathcal{O}_{F_v}^\times \to \{1\} \times \prod_{v\in S'-S} F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S}) \tag{6.1.2.1}$$

followed by the quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S}) \to J_{F,S}$.

We extend the notation $\widehat{\mathcal{O}}_{F,S'-S}^\times$ to the case S' is not finite, but coincides with the set Σ_F of all places of F.

We denote by $\widehat{\mathcal{O}}_{F,\Sigma_F-S}^\times$ the compact Hausdorff abelian group:

$$\widehat{\mathcal{O}}_{F,\Sigma_F-S}^\times := \prod_{v\notin S} \mathcal{O}_{F_v}^\times \to \{1\} \times \prod_{v\notin S} F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_F). \tag{6.1.2.2}$$

THEOREM 6.1.2.1 (Fourth Structure Theorem). *Let F be a number field, S, S' finite sets of places of F containing all Archimedean places such that $S \subset S'$. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

(1) *The range of the linear topological embedding $\overline{\pi}_{S,S'}$ from* (5.2.1.1) *is the closed subspace of elements of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'}))$ invariant under the action of the closed subgroup $\widehat{\mathcal{O}}_{F,S'-S}^\times$ of $\mathrm{GL}_1(\mathbf{A}_{F,S'})$.*
(2) *The range of the linear topological embedding $\overline{\pi}_S$ induced by the linear topological isomorphism $\overline{\pi}$ from* (5.2.1.2) *is the closed subspace of elements of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ invariant under the action of the closed subgroup $\widehat{\mathcal{O}}_{F,\Sigma_F-S}^\times$ of $\mathrm{GL}_1(\mathbf{A}_F)$.*

We need the following:

LEMMA 6.1.2.2. *Let F be a number field.*

(1) *Let v be a finite place of F. The $\mathbf{C}$-linear space:*

$$\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \tag{6.1.2.3}$$

is a free $\mathbf{C}[z^{\pm 1}]$-module of rank 1.
   (1) *Upon defining the action of $z \in \mathbf{C}[z^{\pm 1}]$ by left-translation on functions by $\pi_v^{\mathbf{Z}}$, for π_v any choice of uniformizer at v, then a $\mathbf{C}[z^{\pm 1}]$-module free generator is the function $1_{\mathcal{O}_{F_v}}$.*
   (2) *Upon endowing the subspace $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \subset \mathcal{S}(F_v)$ with the subspace topology, it is a closed subspace, and the $\mathbf{C}$-linear isomorphism with $\mathbf{C}[z^{\pm 1}]$ is a linear topological isomorphism upon endowing $\mathbf{C}[z^{\pm 1}]$ with the final locally convex topology.*
(2) *Let S, S' be finite sets of places containing all Archimedean places, and such that $S \subset S'$. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. *The $\mathbf{C}$-linear space:*

$$\mathrm{H}^0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{6.1.2.4}$$

satisfies the following properties:

(1) *Upon endowing the* $\mathbf{C}$*-linear space* $\mathbf{C}[z_v^{\pm 1}, v \in S' - S]$ *with the final locally convex topology, then the linear topological isomorphism* $\mathbf{C}[z^{\pm 1}] \to \mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ *for every finite place* $v \in S'$ *induces a linear topological isomorphism:*

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes_\pi \mathbf{C}[z_v^{\pm 1}, v \in S' - S] \to \mathrm{H}^0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{6.1.2.5}$$

(2) *The map from* (6.1.2.5) *is uniquely determined by the assignment explicitly obtained by sending, for any* $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$*, the element* $f_S \otimes (\prod_{v \in S'-S} z_v^{a_v})$*, for* $(a_v)_{v \in S'-S} \in \mathbf{Z}^{|S'|-|S|}$*, to the Schwartz-Bruhat function:*

$$f_S \otimes \prod_{v \in S'-S} 1_{\pi_v^{a_v} \mathcal{O}_{F_v}}. \tag{6.1.2.6}$$

Remark 6.1.2.3. For the sake of clarity, we recall that by "left-translation action" of a Hausdorff locally compact abelian group G on a function space $\mathscr{F}(X)$ on a Hausdorff topological space X on which G acts continuously, we mean the assignment $\rho_G : G \times \mathscr{F}(X) \to \mathscr{F}(X)$ defined by $\rho_G(g, f)(x) := f(g^{-1}x)$ as functions of $x \in X$. ρ_G preserves continuity of functions.

In particular, for $f \in \mathcal{S}(F_v)$ as in Lemma 6.1.2.2(1), $z.f$ is the function of $x \in F_v$ given by $f(\pi_v^{-1}x)$.

Proof. (1) : We define $u_m := 1_{\pi_v^m \mathcal{O}_{F_v}}$ for $m \in \mathbf{Z}$. The set $\{u_m\}_{m \in \mathbf{Z}} \subset \mathcal{S}(F_v)$ is $\mathbf{C}$-linearly independent. Indeed, let $\{c_m\}_{-N \leq m \leq N} \subset \mathbf{C}$ be such that $u := \sum_{-N \leq m \leq N} c_m u_m = 0$ in $\mathcal{S}(F_v)$.

Since $\pi_v^m \mathcal{O}_{F_v} \subset \pi_v^{m-1} \mathcal{O}_{F_v}$ for every $m \in \mathbf{Z}$, we let $x \in \pi_v^m \mathcal{O}_{F_v} - \pi_v^{m+1} \mathcal{O}_{F_v}$. Then $u(x) = c_m$, which forces $c_m = 0$ for every $-N \leq m \leq N$.

We now let $f \in \mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$. The function f is locally constant and compactly supported over F_v. There exists a large enough integer $N \geq 0$ such that f is $\pi_v^N \mathcal{O}_{F_v}$-invariant.

By compactness of the support of f, there exists an integer N' such that f vanishes over $F_v - \pi_v^{N'} \mathcal{O}_{F_v}$, and since $\pi_v^k \mathcal{O}_{F_v} \subset \pi_v^{N'} \mathcal{O}_{F_v}$ for every $k \geq N'$, the set of integers $k \in \mathbf{Z}$ such that f is nonzero over $\pi_v^k \mathcal{O}_{F_v}$ is bounded below. There therefore exists a minimal such $k \geq N'$, and we rename it as N'.

Let $c_{N'}$ be the value of f over the collar $\pi_v^{N'} \mathcal{O}_{F_v} - \pi_v^{N'+1} \mathcal{O}_{F_v}$. Then $f - c_m u_m$ vanishes over $\pi_v^{N'} \mathcal{O}_{F_v} - \pi_v^{N'+1} \mathcal{O}_{F_v}$, and hence its support, again compact, is contained in $\pi_v^{N'+1} \mathcal{O}_{F_v}$.

Upon iterating finitely many times, we obtain a subset $\{c_m, m \in \mathbf{Z}\} \subset \mathbf{C}$ such that c_m is nonzero for at most finitely many integers m, and such that $f - \sum_{m \in \mathbf{Z}} c_m u_m$ vanishes over $\pi_v^k \mathcal{O}_{F_v} - \pi_v^{k+1} \mathcal{O}_{F_v}$ for every $k \in \mathbf{Z}$. This means that $f - \sum_{m \in \mathbf{Z}} c_m u_m$ vanishes over $F_v^\times$, and by continuity it vanishes over F_v, forcing $f = \sum_{m \in \mathbf{Z}} c_m u_m$.

The set $\{1_{\pi_v^m \mathcal{O}_{F_v}}, m \in \mathbf{Z}\}$ is therefore an algebraic $\mathbf{C}$-linear basis for $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$.

Since $1_{\pi_v \mathcal{O}_{F_v}} = z \cdot 1_{\mathcal{O}_{F_v}}$, we see that we have $\mathbf{C}[z^{\pm 1}] \cdot 1_{\mathcal{O}_{F_v}} = \mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$.

The Hausdorff locally convex topological vector space $\mathcal{S}(F_v)$ is itself endowed with the final locally convex topology, since it is the strict locally convex inductive limit of finite dimensional $\mathbf{C}$-linear subspaces, spanned by locally constant functions whose compact support is contained in $\pi_v^N \mathcal{O}_{F_v}$, as N varies over $\mathbf{Z}$, and that are invariant under $\pi_v^{N'} \mathcal{O}_{F_v}$ for some $N' \leq N$. We call such finite dimensional subspace $W_{N,N'}$.

Such functions that are contained in $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ are spanned by $z^m \cdot 1_{\mathcal{O}_{F_v}}$ for $N' \leq m \leq N$, which implies that the linear left-inverse of the inclusion of $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ is continuous when restricted to the intersection with $W_{N,N'}$, since such intersection is finite dimensional and hence linearly homeomorphic to its image in $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ endowed with the subspace topology of the locally convex inductive limit topology induced by the Euclidean topologies on the finite dimensional subspaces $\bigoplus_{N' \leq m \leq N} \mathbf{C} z^m \cdot 1_{\mathcal{O}_{F_v}}$. Such inductive limit topology on $\mathbf{C}[z^{\pm 1}] \cdot 1_{\mathcal{O}_{F_v}}$ is the final locally convex topology by [**Tre**, Part I,Ch.13,Example 1] and [**Scha**, Ch.II,p.56,Example] (see also [**Scha**, Ch.II,p.69,Exercise 7]).

By [**Tre**, Part I,Ch.13,Thm.13.1], $\mathbf{C}[z^{\pm 1}] \cdot 1_{\mathcal{O}_{F_v}}$ is complete with respect to such inductive limit topology, and since such topology is the subspace topology inherited from $\mathcal{S}(F_v)$, $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$ is a complete, hence closed (by [**Bou-GT**, Ch.II,§3.4,Prop.8]), subspace.

(2) : Since, by Corollary 2.1.4.2, the continuous linear operator

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes_\pi \mathcal{S}(\prod_{v\in S'-S} F_v) \to \mathcal{S}(\mathbf{A}_{F,S'})$$

is a linear topological isomorphism, and $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'}))$ is endowed with the subspace topology inherited from the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{A}_{F,S'})$, in order to prove (2) it is enough to show that the following assertions hold:

(1) The linear operator

$$\mathbf{C}[z_v^{\pm 1}, v \in S'-S] \to \mathcal{S}(\prod_{v\in S'-S} F_v) \tag{6.1.2.7}$$

defined in (b) and uniquely determined by the assignment from (6.1.2.6), is a linear topological embedding onto $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v))$.

(2) The linear map:

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

is a linear isomorphism.

Indeed, by Proposition 2.1.4.1(1), $\mathcal{S}(\mathbf{A}_{F,S})$ is nuclear, and hence the continuous linear operator $\mathcal{S}(\mathbf{A}_{F,S})\otimes_\pi E_1 \to \mathcal{S}(\mathbf{A}_{F,S}) \otimes_\pi E_2$ induced by a linear topological embedding $E_1 \to E_2$ of Hausdorff locally convex topological $\mathbf{C}$-vector spaces is itself a linear topological embedding (see [**Jar**, Part III,Ch.16.2,Cor.3]).

Upon applying this to $E_1 = \mathbf{C}[z_v^{\pm 1}, v \in S'-S]$ and $E_2 = \mathcal{S}(\prod_{v\in S'-S} F_v)$, we obtain from (1) that the continuous linear operator from (6.1.2.5) is a linear topological embedding. By (2), we obtain that it is surjective, and hence a linear topological isomorphism.

(1) follows from compatibility of the linear operator (6.1.2.7) with the linear topological isomorphism $\mathcal{S}(\prod_{v\in S'-S} F_v) \simeq \bigotimes_{v\in S'-S} \mathcal{S}(F_v)$, the tensor product being projective and already complete, since the factors are endowed with the final locally convex topology, being $S'-S$ comprised of non-Archimedean places only. The linear topological embeddings $\mathbf{C}[z_v^{\pm 1}] \to \mathcal{S}(F_v)$ combine into the continuous linear operator in (6.1.2.7), itself a linear topological embedding by [**Jar**, Part III,Ch.16.2,Cor.3].

Since $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, the quotient homomorphism

$$C_F/\widehat{\mathcal{O}}^\times_{F,\Sigma_F-S} \to J_{F,S} \tag{6.1.2.8}$$

is an isomorphism of Hausdorff locally compact abelian groups. Since we have that the continuous homomorphism

$$(\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^\times_{F,\Sigma_F-S})/\mathcal{O}^\times_{F,S} \to \mathrm{GL}_1(\mathbf{A}_F)/F^\times = C_F \tag{6.1.2.9}$$

is an isomorphism, being its cokernel $\mathrm{Pic}(\mathcal{O}_{F,S})$, the kernel of quotient homomorphism $C_F \to J_{F,S}$ is isomorphic to both $\widehat{\mathcal{O}}^\times_{F,\Sigma_F-S}/\mathcal{O}^\times_{F,S}$ and $\widehat{\prod}_{v\notin S}(F_v^\times, \mathcal{O}^\times_{F_v})/F^\times$.

This in turn implies that the continuous homomorphism:

$$F^\times/\mathcal{O}^\times_{F,S} \to \widehat{\prod_{v\notin S}}(F_v^\times, \mathcal{O}^\times_{F_v})/\widehat{\mathcal{O}}^\times_{F,\Sigma_F-S} \tag{6.1.2.10}$$

is also an isomorphism.

Since $S \subset S'$, then $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$, and (6.1.2.10) being an isomorphism holds for S' in place of S too. By [**DE**, Lemma 13.3.1(b)] it follows that the continuous homomorphism:

$$\prod_{v\in S'-S} (F_v^\times/\mathcal{O}^\times_{F_v}) \to \mathcal{O}^\times_{F,S'}/\mathcal{O}^\times_{F,S} \tag{6.1.2.11}$$

is an isomorphism.

To show that the linear topological embedding $\mathbf{C}[z_v^{\pm 1}, v \in S' - S] \to \mathcal{S}(\prod_{v\in S'-S} F_v)$ is surjective onto $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v))$, it is enough to check that it is so upon further composing it with the injective $\mathbf{C}$-linear map $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, C_c^\infty(\prod_{v\in S'-S} F_v^\times))$, which, explicitly, is uniquely determined by sending the monomial $\prod_{v\in S'-S} z_v^{a_v}$, for $(a_v)_{v\in S'-S} \in \mathbf{Z}^{|S'|-|S|}$, to $\prod_{v\in S'-S} 1_{\pi_v^{a_v}\mathcal{O}^\times_{F_v}}$.

Since $\widehat{\mathcal{O}}^\times_{F,S'-S}$ is a compact Hausdorff abelian group, integration of smooth compactly supported functions over $\widehat{\mathcal{O}}^\times_{F,S'-S}$ with respect to the Radon product measure $\bigotimes_{v\in S'-S} \mu_v$, μ_v being the local Tamagawa measure from Definition A.3.2.2, identifies $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, C_c^\infty(\prod_{v\in S'-S} F_v^\times))$ with $C_c^\infty(\prod_{v\in S'-S} F_v^\times/\mathcal{O}^\times_{F_v}) = \mathbf{C}[\mathcal{O}^\times_{F,S'}/\mathcal{O}^\times_{F,S}]$, using the isomorphism from (6.1.2.11), and the surjectivity assertion follows. To see the identification:

$$\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, C_c^\infty(\prod_{v\in S'-S} F_v^\times)) \to C_c^\infty(\prod_{v\in S'-S} F_v^\times/\mathcal{O}^\times_{F_v}) \tag{6.1.2.12}$$

the linear operator being given by integration over $\widehat{\mathcal{O}}^\times_{F,S'-S}$ with respect to $\bigotimes_{v\in S'-S} \mu_v$, we note that it holds upon replacing $C_c^\infty(\prod_{v\in S'-S} F_v^\times)$ with $C_c(\prod_{v\in S'-S} F_v^\times)$: surjectivity holds by the proof [**DE**, Lemma 1.5.1] upon choosing the function ϕ to be $\widehat{\mathcal{O}}^\times_{F,S'-S}$-invariant, which we can do by compactness of $\widehat{\mathcal{O}}^\times_{F,S'-S}$.

Indeed, we fix $f \in C_c(\prod_{v\in S'-S} F_v^\times/\widehat{\mathcal{O}}^\times_{F,S'-S})$, whose support K is compact. There exists a compact subset $C \subset \prod_{v\in S'-S} F_v^\times$ whose image in the quotient $\prod_{v\in S'-S} F_v^\times/\widehat{\mathcal{O}}^\times_{F,S'-S}$ contains K. Upon replacing the compact subset C with $\widehat{\mathcal{O}}^\times_{F,S'-S}\cdot C$, also compact by [**DE**, Lemma 1.1.3(d)], the indicator function 1_C qualifies as such ϕ.

Injectivity holds since the volume of $\widehat{\mathcal{O}}^\times_{F,S'-S}$ with respect to $\bigotimes_{v\in S'-S} \mu_v$ is finite and nonzero.

It is therefore enough to show that the linear inclusion $C_c^\infty(\prod_{v\in S'-S} F_v^\times) \to C_c(\prod_{v\in S'-S} F_v^\times)$ induces an isomorphism on $\widehat{\mathcal{O}}^\times_{F,S'-S}$-invariants.

Let $f : \prod_{v\in S'-S} F_v^\times \to \mathbf{C}$ be a continuous compactly supported function, and assume that f is $\widehat{\mathcal{O}}^\times_{F,S'-S}$-invariant. Let $K \subset \prod_{v\in S'-S} F_v^\times$ be the support of f. It is contained into a finite union of compact subsets of the form $K_{(a_v)_{v\in S'-S}} := \prod_{v\in S'-S} \pi_v^{a_v}\mathcal{O}^\times_{F_v}$, for $(a_v)_{v\in S'-S} \in \mathbf{Z}^{|S'|-|S|}$, each of which is $\widehat{\mathcal{O}}^\times_{F,S'-S}$-stable. The restriction of f to $K_{(a_v)_{v\in S'-S}}$ is therefore constant, since each $K_{(a_v)_{v\in S'-S}}$ is the orbit under the continuous $\widehat{\mathcal{O}}^\times_{F,S'-S}$-action on $\prod_{v\in S'-S} F_v^\times$ of the single point $(\pi_v^{a_v})_{v\in S'-S}$. Since each $K_{(a_v)_{v\in S'-S}}$ is open, we deduce that f is locally constant over $\prod_{v\in S'-S} F_v^\times$, hence smooth. This concludes the proof of (1).

In order to finish the proof of part (2), we prove claim (2): we call $G := \widehat{\mathcal{O}}^\times_{F,S'-S}$, $M_1 := \mathcal{S}(\mathbf{A}_{F,S})$, $M_2 := \mathcal{S}(\prod_{v\in S'-S} F_v)$ as a $\mathbf{C}[G]$-module, we have the following short-exact sequence:

$$0 \to M_1 \otimes_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}, M_2) \to \mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}, M_1 \otimes_{\mathbf{C}} M_2) \to \mathrm{Tor}_1^{\mathbf{C}}(M_1, \mathrm{Ext}^1_{\mathbf{C}[G]}(\mathbf{C}, M_2)) \to 0$$

where $\mathrm{Tor}_1^{\mathbf{C}}(M_1, \mathrm{Ext}^1_{\mathbf{C}[G]}(\mathbf{C}, M_2)) = 0$ since M_1 is a $\mathbf{C}$-vector space, and the first map is the inclusion $\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathcal{S}(\prod_{v\in S'-S} F_v))$, which is therefore a linear isomorphism. The claim therefore follows from Corollary 2.1.4.2 and the fact that the linear space underlying the projective tensor product in Corollary 2.1.4.2 is the algebraic tensor product. follows from the fact that $\widehat{\mathcal{O}}^\times_{F,S'-S}$ acts trivially on $\mathcal{S}(\mathbf{A}_{F,S})$. □

We need the following:

LEMMA 6.1.2.4. *Let F be a number field, v a finite place of F. Let π_v be a uniformizer at v. Let S, S' be finite sets of places of F containing all Archimedean places, with $S \subset S'$.*

(1) *The linear subspace V of $\mathcal{S}(F_v)$ generated by the elements f_ξ, with $f_\xi(x) := f(x) - f(\xi^{-1}x)$ for $f \in \mathcal{S}(F_v)$, as $\xi \in \mathcal{O}^\times_{F_v}$, is closed, and $\mathrm{H}_0(\mathcal{O}^\times_{F_v}, \mathcal{S}(F_v)) = \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F_v}, \mathcal{S}(F_v))$.*

(2) *The linear operator*

$$\Phi_v : \mathcal{S}(F_v) \to C_c(F_v) \tag{6.1.2.13}$$

defined by sending $f \in \mathcal{S}(F_v)$ to the continuous function of $g \in F_v$:

$$\Phi_v(f)(g) := \int_{\mathcal{O}_{F_v}^\times} f(gh)\,\mathrm{d}\mu_v(h) \tag{6.1.2.14}$$

is continuous upon endowing $C_c(F_v)$ with the uniform norm, it is valued in $\mathcal{S}(F_v)$ and continuous with respect to the Hausdorff locally convex topology on $\mathcal{S}(F_v)$, and its kernel is V. The induced linear map:

$$N_v : \mathrm{H}_0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \to \mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) \tag{6.1.2.15}$$

obtained by factoring Φ_v through the quotient $\mathcal{S}(F_v)/V \simeq \mathrm{H}_0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$, is a linear topological isomorphism.

(3) *The linear subspace $\mathrm{H}^0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\prod_{v\in S'-S} F_v))$ of $\mathcal{S}(\prod_{v\in S'-S} F_v)$ is closed. The induced linear map:*

$$N_{S,S'} : \mathrm{H}_0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathrm{H}^0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\prod_{v\in S'-S} F_v)) \tag{6.1.2.16}$$

obtained by factoring $\prod_{v\in S'-S} \Phi_v$ through $\mathrm{H}_0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathcal{S}(\prod_{v\in S'-S} F_v))$, is a linear topological isomorphism.

Proof. (1) : Since v is finite, $\mathcal{S}(F_v)$ is endowed with the final locally convex topology. Every linear subspace is therefore closed, and hence the quotient $\mathcal{S}(F_v)/V$, endowed with the quotient topology, is Hausdorff and itself endowed with the final locally convex topology. It follows that we have $\mathrm{H}_0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v)) = \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$.

(2) : Since $f \in \mathcal{S}(F_v)$ is compactly supported, then upon calling K its support we obtain that $\Phi_v(f)$, as a function of $g \in F_v$, continuous by Dominated Convergence and compactness of $\mathcal{O}_{F_v}^\times$, is supported over $\mathcal{O}_{F_v}^\times \cdot K$, itself a compact subset of F_v by [**DE**, Lemma 1.3.1(d)].

The function $\Phi_v(f)$ equals $f * 1_{\mathcal{O}_{F_v}^\times}$, as can be checked upon reducing to the case $f = 1_K$ by $\mathbf{C}$-linearity of Φ_v, given that every element of $\mathcal{S}(F_v)$ is a finite $\mathbf{C}$-linear combination of indicator functions 1_K for K a compact subset of F_v. This shows continuity of Φ_v. It also shows that, since $1_{\mathcal{O}_{F_v}^\times}$ is itself a Schwartz-Bruhat function on F_v, and the dual group of F_v is isomorphic to F_v itself as a Hausdorff locally compact abelian group, Φ_v is valued in $\mathcal{S}(F_v)$, since the Fourier transform of a convolution of two Schwartz-Bruhat functions is the product of their Fourier transforms, hence itself a Schwartz-Bruhat function, and the Fourier transform is a linear topological automorphism of $\mathcal{S}(F_v)$.

Since $\Phi_v(f)$ is $\mathcal{O}_{F_v}^\times$-invariant, Φ_v is valued in $\mathrm{H}^0(\mathcal{O}_{F_v}^\times, \mathcal{S}(F_v))$.

We note that the Mellin transform $Z_v(f,c)$ of any $f \in \mathcal{S}(F_v)$, for $c \in \Omega(F_v^\times)$, is meromorphic, with poles located within the polar set of the function $\rho_v(c)$, and the operator L_v factors through Φ_v by Theorem D.1.2.1. This implies $\ker(\Phi_v) \subset \ker(L_v)$.

The reverse inclusion also holds, since if $Z_v(f,c) = 0$, then $|\cdot|_v \cdot f$ satisfies:

$$\int_{F_v^\times} f(g)|g|_v^{it}\,\mathrm{d}\mu_v(g) = \int_{F_v} f(g)|g|_v^{it}\,\mathrm{d}\mu_v(g) = Z_v(f, |\cdot|_v^{it}) = 0 \tag{6.1.2.17}$$

for every $t \in \mathbf{R}$. This implies that the function $|\cdot|_v \cdot f$, which is smooth and compactly supported over $F_v^\times$, has vanishing Fourier transform, and hence $|\cdot|_v \cdot f = 0$. Since $|\cdot|_v$ is continuous and nowhere vanishing over $F_v^\times$, we deduce that $f = 0$ over $F_v^\times$, and hence it is zero everywhere over F_v by continuity.

By Theorem 3.4.3.3, the local eigendistribution $\Delta_v'(c)$, as $c \in \Omega(F_v^\times)$ is an element of $V^\perp$, where the orthogonal is formed with respect to the duality $\langle \mathcal{S}(F_v), \mathcal{S}(F_v)'_\beta \rangle$.

Let $f \in \ker(\Phi_v)$. Then $Z_v(f,c) = \Delta_v'(c)(f) \cdot L_v(c)$, where $L_v(c)$ is the local Hecke L-function at v. It follows that we have $\Delta_v'(c)(f) = 0$ for every $c \in \Omega(F_v^\times)(1)$ (i.e. $c|_{\mathcal{O}_{F_v}^\times} = 1$), which places f in the closure of

V, which is V itself by (1). We deduce that f is annihilated by $V^{\perp}$, which, by Theorem 5.5.3.2, is linearly and topologically generated by the functionals $\{\Delta'_v(c), c \in \Omega(F_v^{\times})(1)\}$, and so we have $V = \ker(\Phi_v)$ (since f is then contained in the closure of V with respect to the $\sigma(\mathcal{S}(F_v), \mathcal{S}(F_v)'_{\beta})$-topology by [**NB**, Thm.8.9.2(e)], which is the intrinsic topology on $\mathcal{S}(F_v)$ by [**NB**, Thm.11.3.6], which applies since $\mathcal{S}(F_v)$ is barrelled, and since V is closed by (1) we deduce $V = \ker(\Phi_v)$). It follows that Φ_v induces the linear topological isomorphism N_v.

(3) : Since $S' - S$ is comprised of non-Archimedean places only, the linear topologies on the source and target of $N_{S,S'}$ are both the final locally convex topology. $N_{S,S'}$ is therefore a linear topological isomorphism so long as it is bijective. The operator $N_{S,S'}$ is the convolution operator $f \mapsto f * 1_{\widehat{\mathcal{O}}^{\times}_{F,S'-S}}$, which is therefore valued in $C_c(\prod_{v\in S'-S} F_v)$ and its range is contained in $\mathcal{S}(\prod_{v\in S'-S} F_v)$, since $\widehat{\mathcal{O}}^{\times}_{F,S'-S}$ is compact open in $\prod_{v\in S'-S} F_v$ and hence $1_{\widehat{\mathcal{O}}^{\times}_{F,S'-S}}$ is itself a Schwartz-Bruhat function on $\prod_{v\in S'-S} F_v$.

Its kernel coincides with the subspace $V_{S,S'}$ which is linearly generated by the elements $f_\xi(x) := f(x) - f(\xi^{-1}x)$ as functions of $x \in \prod_{v\in S'-S} F_v$, as f varies over the Schwartz-Bruhat functions on $\prod_{v\in S'-S} F_v$ and ξ varies over $\widehat{\mathcal{O}}^{\times}_{F,S'-S}$.

To see this, we use Theorem 5.5.3.2 to see that if $f \in \ker(\prod_{v\in S'-S} \Phi_v)$, then $(\bigotimes_{v\in S'-S} \Delta'_v(c))(f) = 0$ for all $c \in \Omega(\prod_{v\in S'-S} F_v)(1) = \coprod_{v\in S'-S} \Omega(F_v^{\times})(1)$ (i.e. such that $c|_{\widehat{\mathcal{O}}^{\times}_{F,S'-S}} = 1$, equivalently $c_v|_{\mathcal{O}^{\times}_{F_v}} = 1$ for every $v \in S' - S$), which implies that $V_{S,S'}^{\perp}$ annihilates f, and hence $f \in V_{S,S'}$ (since f is then in the closure of $V_{S,S'}$ with respect to the $\sigma(\mathcal{S}(\prod_{v\in S'-S} F_v), \mathcal{S}(\prod_{v\in S'-S} F_v)'_{\beta})$-topology by [**NB**, Thm.8.9.2(e)], and since $\mathcal{S}(\prod_{v\in S'-S} F_v)$ is barrelled, then the weakened topology on $\mathcal{S}(\prod_{v\in S'-S} F_v)$ is equivalent to the intrinsic topology on $\mathcal{S}(F_v)$ by [**NB**, Thm.11.3.6], and since $V_{S,S'}$ is closed we deduce $V_{S,S'} = \ker(\prod_{v\in S'-S} \Phi_v)$). □

Let $R \to R'$ be a flat ring map, M an R-module, M' an R'-module. We refer to the Grothendieck spectral sequence (applied to the composition of the right-exact functor $M' \otimes_{R'} (\cdot)$ on R'-modules, and the exact functor $R' \otimes_R (\cdot)$ from R-modules to R'-modules, to which [**Gro-d**, Théorème.2.4.1] applies to the opposites of the categories of R-modules and R'-modules, since, by R-flatness of R', we have $\mathrm{Tor}_m^{R'}(R' \otimes_R P, \cdot) = 0$ for every projective R-module P and every integer $m \geq 1$):

$$E^2_{p,q} := \mathrm{Tor}_p^{R'}(\mathrm{Tor}_q^R(M, R'), M') \Rightarrow \mathrm{Tor}^R_{p+q}(M, M')$$

as the *change-of-rings* spectral sequence for Tor-functors.

Proof of Theorem 6.1.2.1. We consider the $\mathbf{C}$-linear map:

$$\mathrm{Tor}_0^{\mathbf{C}[\mathcal{O}^{\times}_{F,S}]}(\mathbf{C}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{Tor}_0^{\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]}(\mathbf{C}, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{6.1.2.18}$$

The $\mathbf{C}$-algebra map $\mathbf{C}[\mathcal{O}^{\times}_{F,S}] \to \mathbf{C}[\mathcal{O}^{\times}_{F,S'}]$ is flat, and identifies, upon choosing a homomorphic section σ to the surjective group homomorphism $\mathcal{O}^{\times}_{F,S'} \to \mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}$, the $\mathbf{C}$-algebra $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]$ with the $\mathbf{C}$-algebra $\mathbf{C}[\mathcal{O}^{\times}_{F,S}] \otimes_{\mathbf{C}} \mathbf{C}[\mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}]$.

By Lemma 6.1.2.2(2), the $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]$-module $\mathcal{S}(\mathbf{A}_{F,S'})$ is isomorphic, as a $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]$-module, to

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathbf{C}[\mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}]$$

where the $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]$-module structure on $\mathcal{S}(\mathbf{A}_{F,S})$ is defined through the quotient $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}] \to \mathbf{C}[\mathcal{O}^{\times}_{F,S}]$ induced by the quotient by $\sigma(\mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S})$, and on $\mathbf{C}[\mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}]$ by the quotient $\mathcal{O}^{\times}_{F,S'} \to \mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}$.

By the change-of-rings spectral sequence for Tor-functors, we have a canonical $\mathbf{C}$-linear isomorphism:

$$\mathrm{Tor}_0^{\mathbf{C}[\mathcal{O}^{\times}_{F,S}]}(\mathbf{C}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{Tor}_0^{\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]}(\mathbf{C}, \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathbf{C}[\mathcal{O}^{\times}_{F,S'}/\mathcal{O}^{\times}_{F,S}])$$

which implies that the image of the linear operator from (6.1.2.18) is:

$$\mathrm{Tor}_0^{\mathbf{C}[\mathcal{O}^{\times}_{F,S'}]}(\mathbf{C}, \mathrm{H}^0(\widehat{\mathcal{O}}^{\times}_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'}))).$$

By Lemma 6.1.2.4(3) combined with Corollary 2.1.4.2, the linear isomorphism $N_{S,S'}$ induces a linear isomorphism, which we also denote $N_{S,S'}$:

$$N_{S,S'} : \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{6.1.2.19}$$

Indeed, the linear topological isomorphism from (6.1.2.16) of Lemma 6.1.2.4 induces a linear isomorphism:

$$\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \tag{6.1.2.20}$$

where the source is linearly isomorphic to $\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathcal{S}(\prod_{v\in S'-S} F_v))$ since formation of tensor products commutes with formation of quotients on $\mathbf{C}$-linear spaces.

We have, upon calling $G := \widehat{\mathcal{O}}^\times_{F,S'-S}$, $M_1 := \mathcal{S}(\mathbf{A}_{F,S})$, $M_2 := \mathcal{S}(\prod_{v\in S'-S} F_v)$ as a $\mathbf{C}[G]$-module, the short-exact sequence:

$$0 \to M_1 \otimes_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}, M_2) \to \mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}, M_1 \otimes_{\mathbf{C}} M_2) \to \mathrm{Tor}_1^{\mathbf{C}}(M_1, \mathrm{Ext}^1_{\mathbf{C}[G]}(\mathbf{C}, M_2)) \to 0$$

where $\mathrm{Tor}_1^{\mathbf{C}}(M_1, \mathrm{Ext}^1_{\mathbf{C}[G]}(\mathbf{C}, M_2)) = 0$ since M_1 is a $\mathbf{C}$-vector space, and the first map is the inclusion $\mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\prod_{v\in S'-S} F_v)) \to \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathcal{S}(\prod_{v\in S'-S} F_v))$. It follows that such map is a linear isomorphism, therefore identifying the target of (6.1.2.20) with $\mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S}) \otimes \mathcal{S}(\prod_{v\in S'-S} F_v))$, giving the linear isomorphism $N_{S,S'}$ from (6.1.2.19) upon applying Corollary 2.1.4.2.

Upon evaluating the functor $\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \cdot)$ on the linear surjective map:

$$\mathcal{S}(\mathbf{A}_{F,S'}) \to \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

we obtain a linear surjection:

$$\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_0(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S'}))) \tag{6.1.2.21}$$

which factors through $\mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'})))$, since for every element $\xi \in \mathcal{O}^\times_{F,S'}$ and every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S'})$, the function $f(x) - f(\xi^{-1}x)$ of $x \in \mathbf{A}_{F,S'}$ is annihilated by the linear map from (6.1.2.21).

We therefore obtain a linear surjective map:

$$\mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathcal{S}(\mathbf{A}_{F,S'}))) \to \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))) \tag{6.1.2.22}$$

Since

$$\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))) \to \mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

is surjective, since $\mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ is and $\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \cdot)$ is right-exact on $\mathbf{C}[\widehat{\mathcal{O}}^\times_{F,S'-S}]$-modules, we deduce that the range of $\overline{\pi}_{S,S'}$ is the image of

$$\mathrm{H}_0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ under the linear surjective operator N_G from Lemma 4.3.3.1, with $G = \widehat{\mathcal{O}}^\times_{F,S'-S}$, and $E = \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$, which is complete by Theorem 5.2.1.1, hence quasi-complete, implying that the Lemma applies. Note that to see that the action of $\widehat{\mathcal{O}}^\times_{F,S'-S}$ on $\mathcal{S}(\mathbf{A}_{F,S'})$, and hence on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$, is uniformly continuous, it is enough to show that it is so on $\mathcal{S}(\prod_{v\in S'-S} F_v)$. Since $\mathcal{S}(\prod_{v\in S'-S} F_v)$ is the strict locally convex inductive limit of a countable sequence of increasingly nested finite dimensional subspaces, and every convergent Cauchy net in $\mathcal{S}(\prod_{v\in S'-S} F_v)$ is eventually bounded [1] (i.e. for $\{f_i\}_{i\in I}$ such Cauchy net, there exists $i_0 \in I$ such that $\{f_i\}_{i\in I, i\geq i_0}$ is bounded. We may rename $I_0 := \{i \in I : i \geq i_0\}$ as I) by

[1] Return here.

[Scha, Ch.II,§6,6.5] such fixed convergent Cauchy net $\{f_i\}_{i\in I} \subset \mathcal{S}(\prod_{v\in S'-S} F_v)$ and its limit are the image of a convergent Cauchy net contained in a finite dimensional subspace of $\mathcal{S}(\prod_{v\in S'-S} F_v)$. [2]

Without loss of generality, we may assume $S' - S = \{v\}$, a single finite place, and that such finite dimensional subspace V is the finite dimensional subspace linearly generated in $\mathcal{S}(F_v)$ by the locally constant compactly supported functions that are invariant under $\pi_v^N \mathcal{O}_{F_v}$ for some integer N, and whose support is contained in $\pi_v^{N'} \mathcal{O}_{F_v}$, for $N' \leq N$. Since $\pi_v^{N'} \mathcal{O}_{F_v}$ is $\mathcal{O}_{F_v}^\times$-stable, and for any $\pi_v^N \mathcal{O}_{F_v}$-invariant locally constant function f on F_v, its translate $\rho_{\mathcal{O}_{F_v}^\times}(g, f)$ is also π_v^N-invariant for every $g \in \mathcal{O}_{F_v}^\times$, we see that the continuous action of $\mathcal{O}_{F_v}^\times$ preserves V, and hence the convergent Cauchy net $\{f_i\}_{i\in I} \subset V$ and its limit, is such that its $\mathcal{O}_{F_v}^\times$-orbit is also contained in V. Since V has its Euclidean topology and the action is continuous, then being $\mathcal{O}_{F_v}^\times$ profinite its action on V factors through a finite quotient, and it is therefore uniformly continuous. [3]

We further refer the reader to Lemma 2.1.2.10.

The assumptions of Lemma 4.3.3.1 are therefore met.

By Lemma 4.3.3.1, the range of N_G, and hence of $\overline{\pi}_{S,S'}$, is $\mathrm{H}^0(\widehat{\mathcal{O}}_{F,S'-S}^\times, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S'}^\times, \mathcal{S}(\mathbf{A}_{F,S'})))$, as desired. □

6.2. Structure Theorem V: isotypic exhaustiveness

We conclude this Chapter with the final Structure Theorem on the spaces

$$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)), \quad \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$$

and their isotypes.

It amounts to the statement that so long as the finite set of places S containing all Archimedean places, is large enough, so as to ensure $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ holds and, for each given continuous character ω of C_F^1, that ω is unramified outside S, then the transition operator:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$$

is a linear topological isomorphism. Recall from the First Structure Theorem (see Theorem 5.2.3.1) that $\overline{\pi}_S$ is a linear topological embedding. The assertion on $\overline{\pi}_S$ being a linear topological isomorphism therefore amounts to the surjectivity of $\overline{\pi}_S$, and it requires the homology calculations from the previous section.

6.2.1. Eventual surjectivity of $\overline{\pi}_{S,S'}$ and $\overline{\pi}_S$.

Theorem 6.2.1.1 (Fifth Structure Theorem). *Let F be a number field, S, S' finite sets of places of F containing all Archimedean places, such that $S \subset S'$. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

(1) *Let ω be a continuous character of $J_{F,S'}^1$, unramified outside S. Then $\overline{\pi}_{S,S'} : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S'}))$ from* (5.2.3.1) *is surjective, hence a linear topological isomorphism.*

(2) *Let ω be a continuous character of C_F^1, unramified outside S. Then $\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ induced by the linear topological isomorphism from* (5.2.3.2) *is surjective, hence a linear topological isomorphism.*

We record here the following Corollary:

Corollary 6.2.1.2. *Let F be a number field, $\omega \in \widehat{C_F^1}$ a continuous character. Then $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ is a nuclear Fréchet space.*

Proof. By Theorem 6.2.1.1, upon choosing $\chi \in \widehat{C_F}$ a continuous character such that $\chi|_{C_F^1} = \omega$, and S a large enough set of places of F such that χ is unramified outside S and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, $\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ is a linear topological isomorphism, with ω_S the unique continuous

[2] Return here.
[3] Return here.

character of $J^1_{F,S}$ descended from $\omega|_{C^1_{F,S}}$ by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance. By Theorem 5.4.1.1(2), $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ is a nuclear Fréchet space. □

Before we prove the Theorem, we need the following result about the type of continuity for the continuous action $\rho_{F,S}$ of $J_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$.

LEMMA 6.2.1.3. *Let F be a number field, S a finite set of places of F containing all Archimedean places.*

(1) *The action of $J^1_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ is uniformly continuous, i.e. for any convergent Cauchy net $\{f_i\}_{i\in I} \subset \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ with limit f, we have $\rho_{F,S}(g, f_i) \to \rho_{F,S}(g, f)$ uniformly as $g \in J^1_{F,S}$.*

(2) *The action of $J^1_{F,S}$ twisted by the continuous character $\omega \in \widehat{J^1_{F,S}}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ is uniformly continuous, i.e. for any convergent Cauchy net $\{f_i\}_{i\in I} \subset \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ with limit f, we have $\omega(g)\rho_{F,S}(g, f_i) \to \omega(g)\rho_{F,S}(g, f)$ uniformly as $g \in J^1_{F,S}$.*

PROOF. (1) follows from (2). Since ω is unitary, for any continuous seminorm p on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ we have $p(\omega(g)\rho_{F,S}(g, f)) = p(\rho_{F,S}(g, f))$ for any $f \in \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, and so it is enough to treat the case $\omega = 1$, i.e. (2) follows from (1) too.

We enow the linear image $\mathcal{L}_{F,S}$ of $Z_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathscr{M}(\Omega(J_{F,S}))$ with a Hausdorff locally convex topology making the operator L_S continuous.

We have:

$$Z_S(\rho_{F,S}(g, f_i), c) = c(g)Z_S(f_i, c)$$

for each $g \in J_{F,S}$, and similarly for $Z_S(\rho_{F,S}(g, f), c) = c(g)Z_S(f, c)$. If $g \in J^1_{F,S}$, then $c(g)$ is unimodular, and so for any continuous seminorm p on $\mathcal{L}_{F,S}$, we have

$$p(Z_S(\rho_{F,S}(g, f_i), c) - Z_S(\rho_{F,S}(g, f), c)) = p(Z_S(\rho_{F,S}(g, f_i) - \rho_{F,S}(g, f), c)) = p(Z_S(f_i - f, c))$$

and since $f_i \to f$ in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and p is continuous, we have

$$\sup_{g\in J^1_{F,S}} (Z_S(\rho_{F,S}(g, f_i) - \rho_{F,S}(g, f), c)) = p(Z_S(f_i - f, c)) \to 0$$

which means $Z_S(\rho_{F,S}(g, f_i), c) \to Z_S(\rho_{F,S}(g, f), c)$ uniformly over $J^1_{F,S}$ in any locally convex topology on $\mathcal{L}_{F,S}$ making $Z_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathcal{L}_{F,S}$ continuous.

In particular, we may endow $\mathcal{L}_{F,S}$ with the quotient topology induced by L_S, which is the finest making L_S continuous and a linear topological isomorphism. For any continuous seminorm p on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, there therefore exist, by [**NB**, Thm.5.7.3], a continuous seminorm q on $\mathcal{L}_{F,S}$ such that $p \le q \circ L_S$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$.

By the foregoing, we obtain

$$\sup_{g\in J^1_{F,S}} p(\rho_{F,S}(g, f_i) - \rho_{F,S}(g, f)) \to 0$$

and hence the action $\rho_{F,S}$ of $J^1_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ is uniformly continuous. □

PROOF OF THEOREM 6.2.1.1. (1) : We apply Lemma 4.3.3.1 to $E := \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ and $G := J^1_{F,S'}$. The Lemma indeed applies because G is compact Hausdorff and abelian, E is complete by Theorem 5.2.1.1(2), hence quasi-complete, and the action $\rho_G := \overline{\omega} \cdot \rho_{F,S'}$ is uniformly continuous on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ by Lemma 6.2.1.3.

The operator Φ_G therefore takes values in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ itself and it is continuous, with range exactly $\mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$, i.e. the closed subspace of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$ of the ω-homogeneous elements, equivalently the G-invariant elements, i.e. $\mathrm{H}^0(G, E)$.

The linear operator N_G from Lemma 4.3.3.1 is a linear topological isomorphism $N_{S'} : \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$. Similarly, upon replacing S' with S, we obtain a linear topological isomorphism $N_S : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}^0(\omega_S, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})))$. The operator $N_{S'} \circ \overline{\pi}_{S,S'} \circ N_S^{-1}$ is the natural inclusion linear map:

$$p_{S,S'} : \mathrm{H}^0(\omega_S, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))) \to \mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

given by sending an ω_S-homogeneous element ϕ in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, to $\overline{\pi}_{S,S'}(\phi)$, which is ω-homogeneous since ω is unramified outside S. Since ω is unramified outside S, every element of $\mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$ is already $\widehat{\mathcal{O}}^\times_{F,S'-S}$-invariant, and hence contained in $\mathrm{H}^0(\omega, \overline{\pi}_{S,S'}(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))))$. Upon showing this, then upon observing that this latter is the range of $p_{S,S'}$ by Theorem 6.1.2.1(1), it follows that $\overline{\pi}_{S,S'} : \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S'}))$ is surjective.

We now prove that every $\widehat{\mathcal{O}}^\times_{F,S'-S}$-invariant element of

$$\mathrm{H}^0(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

is contained in $\mathrm{H}^0(\omega, \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))))$.

Since $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, we have that the continuous quotient homomorphism:

$$\mathrm{GL}_1(\mathbf{A}_{F,S'})/(\mathcal{O}^\times_{F,S'} \cdot \widehat{\mathcal{O}}^\times_{F,S'-S}) \to J_{F,S}$$

is an isomorphism, compatible with S'-, respectively S-idelic norms. It induces a continuous isomorphism of Hausdorff locally compact abelian groups:

$$J^1_{F,S'}/\widehat{\mathcal{O}}^\times_{F,S'-S} \to J^1_{F,S}. \tag{6.2.1.1}$$

Since ω is a continuous character of $J^1_{F,S'}$, unramified outside S, it factors through the quotient homomorphism from (6.2.1.1). We call ω_S the induced character of $J^1_{F,S}$. The inflation linear map in degree-zero group cohomology:

$$\mathrm{Inf} : \mathrm{H}^0(G/H, \mathrm{H}^0(H, M)) \to \mathrm{H}^0(G, M)$$

for G an abelian group, H a subgroup of G, and M a G-module, is an isomorphism, and hence, upon choosing $G = J^1_{F,S'}$, acting through ω, $H = \widehat{\mathcal{O}}^\times_{F,S'-S}$, and $M = \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'}))$, with G/H acting through ω_S, the linear map:

$$\mathrm{Inf} : \mathrm{H}^0(\omega_S, \mathrm{H}^0(\widehat{\mathcal{O}}^\times_{F,S'-S}, \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))) \to \mathrm{H}^0(\omega, \mathrm{H}_0(\mathcal{O}^\times_{F,S'}, \mathcal{S}(\mathbf{A}_{F,S'})))$$

is an isomorphism, which proves the assertion.

(2) : In light of Theorem 5.2.1.1, surjectivity of $\overline{\pi}_S$ follows from that of $\overline{\pi}_{S,S'}$ for every $S \subset S'$, proved in (1). □

Remark 6.2.1.4. We mention a few complements on the spaces $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, and their isotypes.

(1) The $\mathbf{C}$-linear direct sums:

$$\bigoplus_{\omega \in \widehat{J^1_{F,S}}} \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))(\omega), \quad \bigoplus_{\omega \in \widehat{C^1_F}} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega) \tag{6.2.1.2}$$

equipped with their locally convex inductive topology, are Hausdorff, locally convex, complete, and inherit the properties of the summand spaces proved in the First Structure Theorem (see Theorem 5.2.3.1). They linearly and continuously inject into $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^\times_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, with dense image. The direct sum topology is, however, strictly finer than the subspace topology. To see this, it is enough to produce an example of Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ such that $\Phi_{F,S_\infty,\omega}(f)$ is nonvanishing for all $\omega \in \widehat{J^1_{F,S_\infty}}$. Indeed, such example would propagate to all the S-local cases and the global case too, through the transition operators $\overline{\pi}_{S_\infty,S}$ and $\overline{\pi}_{S_\infty}$, which would produce examples of

elements in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and in $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))$ whose projections into each isotype under everywhere unramified characters (and there is an infinitude of them) are nonvanishing, hence elements that are in $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and in $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))$, but not in the direct sum subspaces from (6.2.1.2).

The function $f:=\prod_{v\in S_\infty} f_v$ for f_v the self-dual exponentials from [**CF**, Ch.XV,pp.316,318], is such an example.

(2) The subspace topology on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))$ inherited from the product topology from Lemma 5.2.3.2(1), is strictly coarser than the intrinsic topology on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))$. This is due to the fact that the Fourier transform of the restriction of $\Phi_{F,S}(f)(g)$ to $J_{F,S}^1$ is Schwartz on $\widehat{J_{F,S}^1}$, whereas elements of the product spaces from (5.2.3.12) and (5.2.3.13) are meromorphic functions on $\Omega(J_{F,S})$ with no such decay condition along the discrete parameter space $\widehat{J_{F,S}^1}$. This observation leads to realizing that $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ and $\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))$ are not closed in the product spaces from (5.2.2.3), and since they are complete with respect to their intrinsic topology by the First Structure Theorem (Theorem 5.2.1.1), their intrinsic topologies are strictly finer than the subspace topology inherited from the products.

6.2.2. Computation of the isotypes of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$. We conclude this Chapter by refining Theorem 6.2.1.1(2) to describe the range of $\overline{\pi}_S$ for large enough S explicitly, in terms of the image under $\Phi_{F,S}$ of Schwartz-Bruhat functions on $\mathbf{A}_{F,S_\infty}$ extended to Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$ by tensoring with specific, explicit Schwartz-Bruhat functions on each of the non-Archimedean local fields F_v, for each finite place $v\in S$.

This finer analysis will be instrumental to perform calculations in Chapter 6.

THEOREM 6.2.2.1 (Description of the Isotypes by Standard Functions). *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume $\mathrm{Pic}(\mathcal{O}_{F,S})=1$. We write $S_f:=S\cap\Sigma_{F,f}$.*

Let $\omega\in\widehat{C_F^1}$ be a continuous character. Let $\omega_S\in\widehat{J_{F,S}^1}$ be the unique continuous character obtained by descending $\omega|_{C_{F,S}^1}$ to $J_{F,S}^1$ by $\widehat{\mathcal{O}}_{\Sigma_F-S}^{\times}$-invariance.

Let:

(1) *For each finite place $v\in S$ at which $\omega_v\neq 1$, we let $\rho_{v,\omega_S}:=\overline{\omega_S}\cdot 1_{\mathcal{O}_{F_v}^{\times}}$, a Schwartz-Bruhat function in $\mathcal{S}(F_v)$ supported on $\mathcal{O}_{F_v}^{\times}$ and such that $Z_v(\rho_{v,\omega_S},c_v)=\mu_v(\mathcal{O}_{F_v}^{\times})$ for $c_v\in\Omega(F_v^{\times})(\omega_v)_{>0}$, and $Z_v(\rho_{v,\omega_S},c_v)=0$ otherwise.*
(2) *For each finite place $v\in S$ at which $\omega_v=1$, we define $\rho_{v,\omega_S}:=1_{\mathcal{O}_{F_v}}$.*
(3) *We define $\rho_{\omega_S}\in\mathcal{S}(\prod_{v\in S_f}F_v)$ to be $\rho_{\omega_S}:=\bigotimes_{v\in S_f}\rho_{v,\omega_S}$.*

Then the composition:

$$\overline{\pi}_S\circ\Phi_{F,S,\omega_S}:\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes\rho_{\omega_S}\to\mathrm{H}_{0,\mathrm{cont}}(\omega,\mathcal{S}(\mathbf{A}_F)) \tag{6.2.2.1}$$

is a continuous, surjective, open linear operator.

We choose $\chi_S\in\widehat{\mathrm{GL}_1(\mathbf{A}_{F,S})}$ such that its restriction to $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ induces ω_S on the quotient $J_{F,S}^1$.

We choose elements $\{g_1=1,\dots,g_{h_F}\}\subset\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ representing the finitely many elements of $\mathrm{Pic}(\mathcal{O}_F)$.

We define the linear operators:

$$\Psi_{F,S,\chi_S}^{\infty}:\mathcal{S}(\mathbf{A}_{F,S_\infty})\to C(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})),\ \Psi_{F,S,\chi_S}^{v}:\mathcal{S}(F_v^{\times})\to C(F_v^{\times}) \tag{6.2.2.2}$$

for each $v\in S_f$, by:

$$\Psi_{F,S,\chi_S}^{\infty}(f)(x):=\sum_{1\leq k\leq h_F}\int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1}f(g_{k,\infty}^{-1}g^{-1}x)\chi_{S,\infty}(g_{k,\infty}g)\,\mathrm{d}\mu_{S_\infty,1}(g) \tag{6.2.2.3}$$

where, for each $1 \leq k \leq h_F$, $g_{k,\infty}$ *is the projection of* g_k *onto* $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, *and*

$$\Psi^v_{F,S,\chi_S}(f)(x) := \int_{F_v^\times} f(g^{-1}x)\chi_{S,v}(g)\,\mathrm{d}\mu_v(g) \tag{6.2.2.4}$$

with $\chi_{S,\infty}$ *and* $\chi_{S,v}$ *the restrictions to* $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, *respectively* $F_v^\times$.

Then:

(1) $\ker(\Psi^\infty_{F,S,\chi_S})$ *and* $\ker(\Psi^v_{F,S,\chi_S})$ *are closed subspaces, and the operator from* (6.2.2.1) *agrees with the unique* $\mathbf{C}$*-linear extension of* $\Psi^\infty_{F,S,\chi_S} \otimes \bigotimes_{v\in S_f} \Psi^v_{F,S,\chi_S}$ *to* $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f} \mathcal{S}(F_v^\times)$, *and it induces a linear topological isomorphism:*

$$(\mathcal{S}(\mathbf{A}_{F,S_\infty})/\ker(\Psi^\infty_{F,S,\chi_{S,\infty}})) \otimes \rho_{\omega_S} \to \mathrm{H}_{0,\mathrm{cont}}(\omega_S, \mathcal{S}(\mathbf{A}_{F,S})). \tag{6.2.2.5}$$

(2) *For any* $f \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$, $\Psi^\infty_{F,S,\chi_{S,\infty}}(f)$ *has semi-exponential decay over* $J_{F,S}/J^1_{F,S}$.

Example 6.2.2.2. For $F = \mathbf{Q}$ and $\omega = 1$, Theorem 6.2.2.1 informs us that $\mathrm{H}_{0,\mathrm{cont}}(1, \mathcal{S}(\mathbf{A}_\mathbf{Q}))$ is linearly and topologically a quotient of $\mathcal{S}(\mathbf{R}) \otimes 1_{\widehat{\mathbf{Z}}}$. This is already a consequence of Theorem 6.2.1.1(2).

Proof. Continuity and linearity of $\overline{\pi}_S \circ \Phi_{F,S}$ is a consequence of continuity and linearity of the factor operators, and of the linear inclusion $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \rho_\omega \subset \mathcal{S}(\mathbf{A}_{F,S})$.

By Proposition 2.1.4.1(1), $\mathcal{S}(\mathbf{A}_{F,S})$ is linearly and topologically isomorphic to $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f} \mathcal{S}(F_v)$, with $S_f := S \cap \Sigma_{F,f}$, and the projective tensor product is already complete (see Proposition 2.1.4.1(1) and (2)). Moreover, $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is nuclear Fréchet by Proposition 2.1.1.6, since $\mathbf{A}_{F,S_\infty}$ is isomorphic, as a Hausdorff locally compact abelian group, to a finite dimensional real Euclidean space.

The one-dimensional subspace $\mathbf{C} \cdot \rho_\omega \subset \bigotimes_{v\in S_f} \mathcal{S}(F_v)$ is linearly and topologically complemented by [**NB**, Thm.7.3.5] (a consequence of the Hahn-Banach Theorem), or by Proposition C.2.5.1(6), since $\bigotimes_{v\in S_f} \mathcal{S}(F_v)$ carries the finest locally convex topology.

By [**Jar**, Part III,Ch.15,§15.2,Prop.3(b),p.327], $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \rho_\omega$ is a complemented, hence closed, subspace of $\mathcal{S}(\mathbf{A}_{F,S})$, and it is a nuclear Fréchet space since $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is (note that we do not need to use nuclearity of $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ to show that the tensor product $\mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes \rho_\omega$, as a subspace of $\mathcal{S}(\mathbf{A}_{F,S})$, carries the projective tensor product topology, because $\mathbf{C} \cdot \rho_\omega$ is finite dimensional, and so [**Jar**, Part III,Ch.15,§15.2,Prop.3(b),p.327] suffices).

By [**Gro-f**, Ch.4,§5,Thm.2(1),p.148] (see also [**SN**, §2,Cor.3,p.8]), which applies because both source, by the foregoing, and target, by Theorem 5.2.1.1(2) and Theorem 5.4.1.1(2), are LF-spaces, in order to show openness and surjectivity of the operator in (6.2.2.1), it is enough to show surjectivity.

The operator Ψ^∞_{F,S,χ_S} is well defined because the innermost integral converges for all $x \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, which is shown by means of an argument closely following that for Lemma 2.2.2.1, since f is of rapid decay at infinity over the real Euclidean space $\mathbf{A}_{F,S_\infty}$, and $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ is an extension of a compact Hausdorff abelian group by the countable discrete subgroup $\mathcal{O}_F^\times$, whose action on $\mathbf{A}_{F,S_\infty}$ by left translation is free and proper, admittiing, by Proposition A.2.4.4 applies to $S = S_\infty$, a fundamental domain with non-empty interior, of the form $\tau(\mathbf{R}_{>0}) \cdot E_{S_\infty,0}$, for τ a choice of continuous homomorphic section to the S_∞-idelic norm.

Upon endowing $C(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1)$ with the topology of uniform convergence on compact subsets, Ψ^∞_{F,S,χ_S} is continuous, by an application of Dominated Convergence, using an analog of the estimate from (2.2.2.3) of Lemma 2.2.2.1 (see also (2.3.1.4) from the proof of Proposition 2.3.1.2), and hence its kernel is closed.

Note that the integral defining Ψ^v_{F,S,χ_S} is convergent since K_v is compact in $F_v^\times$, the open subgroup $\mathcal{O}^\times_{F_v}$ is compact Hausdorff, $\pi_v^{\mathbf{Z}} \subset F_v^\times$ is discrete, and $F_v^\times \simeq \pi_v^{\mathbf{Z}} \times \mathcal{O}^\times_{F_v}$ as Hausdorff locally compact abelian groups, for any choice $\pi_v \in F_v^\times$ of uniformizer.

$\mathcal{S}(F_v)$ carries the finest locally convex topology by Proposition 2.1.2.4(1), and hence Ψ^v_{F,S,χ_S} is continuous with respect to any complex linear topology on $C(F_v^\times)$ promoting $C(F_v^\times)$ to a topological vector space, so

long as one exists. Since $F_v^\times$ is Hausdorff locally compact, $C(F_v^\times)$ carries the topology of uniform convergence on compact subsets.

The closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \subset \mathrm{GL}_1(\mathbf{A}_{F,S})^1$ is equipped with the Haar measure $\mu_{S_\infty,1}$, where both $\mu_{S_\infty,1}$ and $\mu_{S,1}$ are induced by the choice of a continuous homomorphic section

$$\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$$

to the S-idelic norm whose image is in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and is, therefore, also a continuous homomorphic section to the S_∞-idelic norm.

The quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S}) \to \mathrm{GL}_1(\mathbf{A}_{F,S})/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ has a canonical section induced by the isomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S}) \simeq \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \prod_{v\in S_f} F_v^\times$ which in turn induces an isomorphism:

$$\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \simeq \prod_{v\in S_f} F_v^\times \tag{6.2.2.6}$$

The Haar measure induced as per Theorem D.1.2.1 on $\prod_{v\in S_f} F_v^\times$ by the choices $\mu_{S,1}$ and $\mu_{S_\infty,1}$ made in the foregoing, both induced by τ, is unique, and equals the Haar measure $\bigotimes_{v\in S_f} \mu_v$, where μ_v is the Tamagawa measure on $F_v^\times$ for each place $v \in S_f$ (see Definition A.3.2.2), by design of the measure μ_S for each S.

The arbitrary $f \in \mathcal{S}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v\in S_f} \mathcal{S}(F_v^\times)$ is a finite $\mathbf{C}$-linear combination of Schwarz-Bruhat functions of the form $f_\infty \otimes \bigotimes_{v\in S_f} 1_{K_v}$ for $K_v \subset F_v^\times$ a non-empty compact subset.

We assume f is of this form. Such f is, by Proposition 2.1.4.1(1) and since, for each $v \in S_f$, $\mathcal{S}(F_v^\times) \subset \mathcal{S}(F_v)$, a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$.

We denote by $\chi_{S,f}$ the restriction of χ_S to $\prod_{v\in S_f} F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$.

We compute:

$$\Phi_{F,S,\omega_S}(f)(x) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})^1} f(g^{-1}x)\chi_S(g)\,\mathrm{d}\mu_{S,1}(g) \tag{6.2.2.7}$$

$$= \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1} \int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1} \chi_S(g) f(g_f^{-1} g_\infty^{-1} x)\,\mathrm{d}\mu_{S_\infty,1}(g_\infty)\,\mathrm{d}(\bigotimes_{v\in S_f} \mu_v)(g_f). \tag{6.2.2.8}$$

We now analyze the iterated integral in (6.2.2.7). We observe that each $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$ can be written uniquely as $g_\infty \cdot \xi_{S-S_\infty} \cdot g_{f,1} \cdot g_k$ for a unique $1 \le k \le h_F$, with $g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, $\xi_{S-S_\infty} \in \mathcal{O}_{F,S}^\times/\mathcal{O}_F^\times$, a free group of rank $|S| - r_1 - r_2 + 1$ and a direct factor of $\mathcal{O}_{F,S}^\times$ (we implicitly choose a splitting to the quotient homomorphism $\mathcal{O}_{F,S}^\times \to \mathcal{O}_{F,S}^\times/\mathcal{O}_F^\times$), with $g_{f,1} \in \prod_{v\in S_f} \mathcal{O}_{F_v}^\times$.

We write $g_k = g_{k,\infty} \cdot g_{k,f}$, with $g_{k,\infty}$ the projection of g_k onto $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and $g_{k,f}$ the projection of g_k onto $\prod_{v\in S_f} F_v^\times$.

We now use the special form of $f = f_\infty \otimes \bigotimes_{v\in S_f} 1_{K_v}$, and rewrite (6.2.2.7) as:

$$\int_{\prod_{v\in S_f} F_v^\times} \chi_{S,f}(g_f) \bigotimes_{v\in S_f} 1_{K_v}(g_v^{-1}x_v)\,\mathrm{d}(\bigotimes_{v\in S_f} \mu_v(g_v)) \cdot \sum_{1\le k\le h_F} \int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1} \chi_{S,\infty}(g_\infty g_{k,\infty}) f_\infty(g_\infty^{-1} g_{k,\infty}^{-1} x)\,\mathrm{d}\mu_{S_\infty,1}(g_\infty) \tag{6.2.2.9}$$

where $g_f = (g_v)_{v\in S_f}$ (noting that for any $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, the image of $g = (g_v)_{v\in S}$, where the v-indexed coordinates are obtained by viewing g as an element of $\mathrm{GL}_1(\mathbf{A}_{F,S})$, into the quotient $\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ is sent to $(g_v)_{v\in S_f}$ by the isomorphism from (6.2.2.6)), and $g_f = (\xi_{S-S_\infty})_f \cdot g_{f,1} \cdot g_{k,f}$. We further note that $\bigcup_{1\le k\le h_F} (\mathcal{O}_{F,S}^\times/\mathcal{O}_F^\times)_f \cdot \prod_{v\in S_f} \mathcal{O}_{F_v}^\times \cdot g_{k,f} = \prod_{v\in S_f} F_v^\times$.

We now use the isomorphism $\widehat{\prod_{v\in S_f} F_v^\times} \simeq \prod_{v\in S_f} \widehat{F_v^\times}$ from [**DE**, Prop.3.6.1] through its Corollary, [**DE**, Cor.3.6.2], combined with functoriality of the formation of the dual group on Hausdorff locally compact abelian

groups, to write $\chi_{S,f} = \bigotimes_{v\in S_f}\chi_v$, where χ_v is obtained by restricting χ_S to the v-indexed coordinate of $\mathrm{GL}_1(\mathbf{A}_{F,S})$.

We choose a g_∞ denotes the arbitrary element of $g\in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ and g_f the image in $\prod_{v\in S_f}F_v^\times$ of the arbitrary element of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1/\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ through the isomorphism from (6.2.2.6).

We therefore write (6.2.2.9) as:

$$\Psi^\infty_{F,S,\chi_S}(f_\infty)(x_\infty)\cdot\prod_{v\in S_f}\Psi^v_{F,S,\chi_S}(1_{K_v})(x_v) \tag{6.2.2.10}$$

a function of $x=(x_v)_{v\in S}\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, with x also written as $(x_\infty,(x_v)_{v\in S_f})$ for $x_\infty\in\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

We now study the operator:

$$\Psi^v_{F,S,\chi_S}:\mathcal{S}(F_v^\times)\to C(F_v^\times) \tag{6.2.2.11}$$

defined in (6.2.2.4).

It is uniquely determined by the assignment $1_{K_v}\mapsto\int_{F_v^\times}\chi_{S,v}(g_v)1_{K_v}(g_v^{-1}x_v)\,\mathrm{d}\mu_v(g_v)$ as a function of $x_v\in F_v^\times$, for any non-empty compact open subset $K_v\subset F_v^\times$.

The function $\overline{\chi}_v\cdot\Psi_{F,S,\chi_v}(1_{K_v})$ is constant over $F_v^\times$. We denote by $c_v(1_{K_v})$ its value. We then have $\Psi_{F,S,\chi_v}(1_{K_v})=\chi_v\cdot c_v(1_{K_v})$.

Since formation of c_v is $\mathbf{C}$-linear, we deduce that the operator $\chi_v\cdot\Psi_{F,S,\chi_v}$ is a continuous linear functional on $\mathcal{S}(F_v^\times)$.

It is nonzero, since for $K_v=\mathcal{O}_{F_v}^\times$ and $f_v:=\overline{\chi}_v\cdot 1_{K_v}\in\mathcal{S}(F_v^\times)$, we have $\Psi_{F,S,\chi_v}(f_v)=\mu_v(\mathcal{O}_{F_v}^\times)\neq 0$.

It follows that $\overline{\chi}_{S,f}\cdot\bigotimes_{v\in S_f}\Psi_{F,S,\chi_v}$ is a continuous linear functional on $\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)\simeq\mathcal{S}(\prod_{v\in S_f}F_v^\times)$, and it is nonzero, since it is nonzero on the function $\rho'_{\omega_S}:=\bigotimes_{v\in S_f}\rho'_v$ where $\rho'_v=\overline{\chi}_v\cdot 1_{\mathcal{O}_{F_v}^\times}$ for each $v\in S_f$.

We have proved that Φ_{F,S,ω_S} agrees on $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)\subset\mathcal{S}(\mathbf{A}_{F,S})$, with $\Psi_{F,S,\chi_S}=\Psi^\infty_{F,S,\chi_S}\otimes\bigotimes_{v\in S_f}\Psi^v_{F,S,\chi_S}$.

Since, by Lemma 5.3.1.1, $\mathcal{S}(F_v^\times)=P_v\mathcal{S}(F_v)$, it follows that Φ_{F,S,ω_S} maps $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes\mathrm{PV}_{S,S_f}(\rho_{\omega_S})$ onto $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$, since, by Lemma 5.3.1.3, $\rho'_{\omega_S}=\mathrm{PV}_{S,S_f}(\rho_{\omega_S})$, where PV_{S,S_f} is Weil's Principal Value Operator from Definition 5.3.1.2 and Definition 5.3.1.4 (see also Lemma 5.3.1.3).

It follows that $\mathrm{PV}_{S,S_f}\circ\Phi_{F,S,\omega_S}$ maps $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes\rho_{\omega_S}$ onto

$$\mathrm{PV}_{S,S_f}\Phi_{F,S,\omega_S}(\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}E_v)$$

where $E_v=\mathcal{S}(F_v)$ if χ_S, equivalently ω_S, equivalently ω, is unramified at $v\in S_f$, and $E_v=\mathcal{S}(F_v^\times)$ otherwise.

This proves that Φ_{F,S,ω_S} maps $\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes\rho_{\omega_S}$ surjectively onto $\Phi_{F,S,\omega_S}(\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}E_v)$.

In order to show that we have

$$\Phi_{F,S,\omega_S}(\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}E_v)=\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$$

it is enough to show that for every Schwartz-Bruhat function $f\in\mathcal{S}(F_v)$ where $v\in S_f$ is a place at which χ_S is ramified, then upon calling χ_v the restriction of χ_S to $F_v^\times$, there exists a Schwartz-Bruhat function $f_0\in\mathcal{S}(F_v)$ which vanishes on a compact open neighborhood of $0\in F_v$, and such that $Z_v(f_0,\chi_v|\cdot|_v^s)=Z_v(f,\chi_v|\cdot|_v^s)$ as meromorphic (in fact entire) functions of $s\in\mathbf{C}$.

To see this, we note that since f is Schwartz-Bruhat over F_v, it is locally constant and compactly supported over F_v. For any choice $\pi_v\in F_v^\times$ of uniformizer, there exists a large enough integer $N\geq 0$ such that f is constant over $\pi_v^N\mathcal{O}_{F_v}$. Let A be its value over $\pi_v^N\mathcal{O}_{F_v}$.

We define $f_0:=f-A\cdot 1_{\pi_v^N\mathcal{O}_{F_v}}$, and note that f_0 is indeed a Schwartz-Bruhat function over F_v supported away from a compact open neighborhood of $0\in F_v$.

We have $Z_v(f_0,\chi_v|\cdot|_v^s)=Z_v(f,\chi_v|\cdot|_v^s)-A\cdot Z_v(1_{\pi_v^N\mathcal{O}_{F_v}},\chi_v|\cdot|_v^s)$ as holomorphic functions on the complex s-plane.

We now note that $\pi_v^N\mathcal{O}_{F_v}$ is a $\mathcal{O}_{F_v}^\times$-invariant compact open neighborhood of $0\in F_v$, and hence $1_{\pi_v^N\mathcal{O}_{F_v}}$ is a $\mathcal{O}_{F_v}^\times$-invariant function. It follows, since $\chi_v|_{\mathcal{O}_{F_v}^\times}$ is non-trivial, that we have $Z_v(1_{\pi_v^N\mathcal{O}_{F_v}},\chi_v|\cdot|_v^s)=0$.

The calculation from (6.2.2.7) also shows that the induced operator from (6.2.2.5) is a continuous linear bijection, and since both source and target are Fréchet spaces, since $\mathcal{S}(\mathbf{A}_{F,S_\infty})/\ker(\Psi^\infty_{F,S,\chi_S})$ is a separated quotient of $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ by continuity of Ψ^∞_{F,S,χ_S} and $\mathcal{S}(\mathbf{A}_{F,S_\infty})$ is a Fréchet space by Proposition 2.1.1.6, and $\mathrm{H}_{0,\mathrm{cont}}(\omega_S,\mathcal{S}(\mathbf{A}_{F,S}))$ is a Fréchet space by Theorem 5.4.1.1(2).

By [**NB**, Thm.14.4.5], it is a linear topological isomorphism. This proves all the assertions of the Theorem, with the exception of (2).

(2) : We proceed as for the proof of (2.4.2.1) from Theorem 2.4.2.1.

It is enough to replace

$$\int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1} f(g_{k,\infty}^{-1}g^{-1}x)\chi_{S,\infty}(g_{k,\infty}g)\,\mathrm{d}\mu_{S_\infty,1}(g)$$

with

$$\int_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1} |f(g_{k,\infty}^{-1}g^{-1}x)|\,\mathrm{d}\mu_{S_\infty,1}(g)$$

and since $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1/\mathcal{O}_F^\times$ is compact Hausdorff and $f(g_{k,\infty}^{-1}g^{-1}x)$, for each $1\leq k\leq h_F$, decays faster than any positive power of $|x|_{S_\infty}^{-1}$ as $|x|_{S_\infty}\to+\infty$ uniformly as $g\in E_{S_\infty,0}$, we may replace $|f|$ with the indicator function of

$$\{x\in\mathbf{A}_{F,S_\infty}:|x|_v\in[-A,A]\text{ for all }v\in S_\infty\}$$

for large enough fixed $A>0$. By Theorem A.4.1.1, we deduce that there exists a real number $C>0$ such that, as $|g|_{S_\infty}\to 0^+$, we have:

$$|\Psi^\infty_{F,S,\chi_{S,\infty}}(f)(g)|\leq C|\log|g|_{F,S_\infty}|^{r_1+r_2-1}. \tag{6.2.2.12}$$

The estimate in (6.2.2.12) and the rapid decay of f over $\mathbf{A}_{F,S_\infty}$ imply that $\Psi^\infty_{F,S,\chi_{S,\infty}}(f)$ is of semi-exponential decay over $J_{F,S}/J^1_{F,S}$. □

6.3. Local and Global Converse Theorems

In this section, we describe the isotypes of $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ in terms of ideals of meromorphic functions on the Riemann surface $\Omega(C_F)$.

The main Theorem in this section describes the image of the Mellin transform L from (5.2.3.14) on $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F))$ and can be viewed as an incarnation of the Converse Theorem for GL_1 (as per [**CPS**, Thm.1,p.165]).

6.3.1. Recollections on the Euler factors of Hecke L-functions. Let F be a number field.

We fix a holomorphic parametrization:

$$\chi|\cdot|_F^s,\ s\in\mathbf{C} \tag{6.3.1.1}$$

which will be in force throughout the entirety of this Chapter.

We will be denoting the global holomorphic coordinate s by z when the letter s is otherwise required: the choice of holomorphic parametrization will, however, be understood to remain the same as in (6.3.1.1).

Recall that this means choosing, for each continuous character $\omega\in\widehat{C_F^1}$, a lift $\chi\in\widehat{C_F}$, i.e. a continuous character $\chi\in\widehat{C_F}$ such that $\omega=\chi|_{C_F^1}$.

We then use χ as base-point for the unique connected component of $\Omega(C_F)$ parametrized by ω, and parametrize the connected component $\Omega(C_F)(\omega)$ by the global holomorphic chart $\chi|\cdot|_F^s$, $s\in\mathbf{C}$.

We denoted, in (3.2.1.2), by $L(s,\chi)$ the completed Hecke L-function associated to χ, i.e. $L(\chi|\cdot|_F^s)$ (see (3.2.1.2) from Lemma 3.2.1.1).

The Euler factors of $L(\chi|\cdot|_F^s)$ are, therefore, by Lemma 3.2.1.1, the following.

There exists, for each finite place v of F at which χ is unramified, a real number t_v, such that:

$$(\chi|\cdot|_F^s)_v = \chi_v|\cdot|_v^s = |\cdot|_v^{s+it_v}. \tag{6.3.1.2}$$

We have:

$$t_v = i\,\frac{\log\chi_v}{\log(q_v)} \tag{6.3.1.3}$$

where $q_v := N(\mathfrak{p}_v)$, with $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ associated to v, noting that, for any chosen branch of the logarithm, $\log\chi_v$ is a purely imaginary number.

t_v is unique modulo the additive translation action of the discrete group $(2\pi/\log(q_v))\,\mathbf{Z}$ on $\mathbf{R}$.

In particular, we may find a unique such t_v satisfying $t_v \in [0, 2\pi/\log(q_v))$.

Throughout this Chapter, the real number t_v will always be chosen as in the foregoing, so as to arrange for the identities in (6.3.1.2) to hold.

If v is Archimedean, then if v is real, $\chi_v|\cdot|_v^s = |\cdot|_v^{s+it_v}$ if χ_v is even, for a unique real number t_v, and $\chi_v|\cdot|_v^s = \mathrm{sgn}\,|\cdot|_v^{s+it_v}$ if χ_v is odd, for a unique real number t_v.

If v is complex, then $\chi_v|\cdot|_v^s = \chi_{n,s}|\cdot|_v^{it_v}$ for a unique real number t_v, and for the unique integer $n \in \mathbf{Z}$ such that $\chi_v|\cdot|_v^{it_v}(e^{i\theta}) = e^{in\theta}$ for every $\theta \in \mathbf{R}$.

We deduce that the Euler factor at v, for v a place of F, of the Hecke L-function $L(c) = L(\chi|\cdot|_F^s)$, is, on the holomorphic parametrization of $\Omega(C_F)$ selected in (6.3.1.1):

$$L_v(c_v) = L_v(\chi_v|\cdot|_v^s) = \frac{1}{1 - q_v^{-s-it_v}} \tag{6.3.1.4}$$

if v is non-Archimedean and χ is unramified at v, it is $L_v(c_v) = 1$ if v is non-Archimedean and χ is ramified at v, and it is given by (3.2.1.4) and (3.2.1.5) if v is Archimedean, for v real, respectively v complex.

Terminology 6.3.1.1 (S-local Hecke L-functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{C_F}$ be a continuous character. Assume $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ and χ is unramified outside S.

We define χ_S to be the unique continuous character of $J_{F,S}$ obtained by descending the restriction of χ to $C_{F,S}$ by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance.

We define:

$$L_S(\chi_S, s) := \prod_{v\in S} L_v(\chi_v|\cdot|_v^s) \tag{6.3.1.5}$$

a meromorphic function on the complex s-plane whose poles are located at the poles of $\rho_S(\chi|\cdot|_{F,S}^s)$ (see (3.3.1.3)), with matching orders of poles.

We refer to $L_S(\chi_S, s)$ as the *S-local Hecke L-function*.

We further define:

$$L_{\Sigma_F - S}(\chi, s) := L_S(\chi_S, s)^{-1} L(s,\chi). \tag{6.3.1.6}$$

6.3.2. Converse Theorems. Recall from (2.2.4.23) and (5.5.2.1) that we have defined:

$$\mathcal{S}_0(\mathbf{A}_F) := \ker(\delta_0) \cap \ker(\widehat{\delta_0}) \tag{6.3.2.1}$$

where $\delta_0 : \mathcal{S}(\mathbf{A}_F) \to \mathbf{C}$ is the strongly continuous functional $\delta_0(f) := f(0)$, and $\widehat{\delta_0}$ is its adelic Fourier transform, i.e. $\widehat{\delta_0}(f) := \widehat{f}(0)$.

We endow $\mathcal{S}_0(\mathbf{A}_F)$ with the subspace topology, promoting it to a closed subspace of $\mathcal{S}(\mathbf{A}_F)$, being δ_0 and $\widehat{\delta}_0$ continuous functionals on $\mathcal{S}(\mathbf{A}_F)$.

From Lemma 5.5.2.1, we know hat the continuous functionals $\delta_0, \widehat{\delta}_0 \in \mathcal{S}(\mathbf{A}_F)'_\beta$ descend uniquely to continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. They are homogeneous with respect to the adjoint of the continuous left-translation action of C_F on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and satisfy:

$$\rho'_F(g, \delta_0) = \delta_0, \quad \rho'_F(g, \widehat{\delta}_0) = |g|_F \widehat{\delta}_0 \tag{6.3.2.2}$$

for all $g \in C_F$, i.e. δ_0 is a global eigendistribution of weight $c = 1$, and $\widehat{\delta}_0$ is a global eigendistribution of weight $c = |\cdot|_F$. Moreover, for every $\omega \in \widehat{C^1_F}$, both δ and $\widehat{\delta}_0$ vanish on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F))$ if and only if $\omega \neq 1$.

In the language of Theorem 5.5.1.3 (see (5.5.2.3)), we have

$$\delta_0 = \Delta_F(1), \quad \widehat{\delta}_0 = \Delta_F(|\cdot|_F).$$

THEOREM 6.3.2.1 (Global Converse Theorem). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $A\mathcal{S}(\mathbf{C})$ be the Schwartz-Bergman space as per Definition D.3.1.3. We parametrize $\Omega(C_F)(\chi|_{C^1_F})$ by the global holomorphic parametrization*

$$\chi|\cdot|_F^s, \quad s \in \mathbf{C}.$$

(1) *If $\chi|_{C^1_F} \neq 1$, then the continuous linear map:*

$$Z : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F)) \to A\mathcal{S}(\mathbf{C}) \tag{6.3.2.3}$$

is a linear topological embedding with closed range. equal to $L(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \subset A\mathcal{S}(\mathbf{C})$. If $\chi|_{C^1_F} = 1$, then the continuous linear map:

$$Z \circ \Phi_{F,1} : \mathcal{S}_0(\mathbf{A}_F) \to A\mathcal{S}(\mathbf{C}) \tag{6.3.2.4}$$

is relatively open with closed range, equal to $((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C^1_F}}} L(s,\chi)\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C})$, with $\delta_{\chi|_{C^1_F}} = 0$ if $\chi|_{C^1_F} \neq 1$, and kernel equal to $\ker(\Phi_{F,1}) \cap \mathcal{S}_0(\mathbf{A}_F)$.

(2) *The following sequence of $\mathbf{C}$-linear operators:*

$$0 \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F)) \to \mathbf{C}^{\oplus 2} \to 0 \tag{6.3.2.5}$$

where:

(1) *$\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is the quotient of $\mathcal{S}_0(\mathbf{A}_F)$ by $\ker(\Phi_F) \cap \mathcal{S}_0(\mathbf{A}_F)$, endowed with the quotient topology.*

(2) *$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$ is the quotient of $\mathcal{S}_0(\mathbf{A}_F)$ by $\ker(\Phi_{F,\chi|_{C^1_F}}) \cap \mathcal{S}_0(\mathbf{A}_F)$, endowed with the quotient topology.*

(3) *The rightmost operator sends $\Phi_{F,\chi|_{C^1_F}}(f)$ to the pair of complex numbers:*

$$-\overline{\mu}_1(C^1_F)\, f(0)\, \delta_{\chi|_{C^1_F}}, \quad \overline{\mu}_1(C^1_F)\, \widehat{f}(0)\, \delta_{\chi|_{C^1_F}} \tag{6.3.2.6}$$

is strict exact, and under the Mellin transform $\phi \mapsto M(\phi)(s) := \int_{C_F} \phi(g)\chi(g)|g|_F^s \,\mathrm{d}\overline{\mu}(g)$ it is linearly and topologically isomorphic to the short exact sequence:

$$0 \to L(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to L(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to 0 \tag{6.3.2.7}$$

if $\chi|_{C^1_F} \neq 1$, the middle operator in (6.3.2.7) being the identity, and the topology on $L(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C})$ being the subspace topology inherited from $A\mathcal{S}(\mathbf{C})$, and to the short exact sequence:

$$0 \to (s-it_0)(1-s-it_0)\widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to \widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}) \to \mathbf{C} \oplus \mathbf{C} \to 0 \tag{6.3.2.8}$$

for a real number $t_0 \in \mathbf{R}$ uniquely determined by χ, where the rightmost operator sends an element of $\widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C})$ to the pair of residues at $s = -it_0, 1-it_0$, the topology on the leftmost space of meromorphic functions from (6.3.2.8) being the subspace topology inherited from $A\mathcal{S}(\mathbf{C})$, and the

topology on the middle space of meromorphic functions being the quotient topology induced by L from $\mathrm{H}_{0,\mathrm{cont}}(|\cdot|_F^{it_0}, \mathcal{S}(\mathbf{A}_F))$.

PROOF. We will use the First Structure Theorem (see Theorem 5.2.3.1), and specifically the assertion that the continuous linear operator $\overline{\pi}$ from (5.2.3.2), induced by the insertion operators:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F)) \tag{6.3.2.9}$$

for S any finite set of places of F containing all the Archimedean places, such that χ is unramified away from$S$, and with $(\chi|_{C_F^1})_S$ the character of $J_{F,S}^1$ obtained by restricting χ to $C_{F,S}^1$ and descending it to the quotient $J_{F,S}^1$, is a linear topological embedding.

By the Fifth Structure Theorem (see Theorem 6.2.1.1(2)), for all finite sets of places S of F containing all the Archimedean places, such that $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ and χ is unramified outside S, the linear topological embedding $\overline{\pi}_S$ from (6.3.2.9) is itself a linear topological isomorphism.

We choose S so large as to have $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, and χ unramified outside S.

We choose a continuous homomorphic section τ to the idelic norm $|\cdot|_F$, and define τ_S to be the induced continuous homomorphic section to the restriction to $C_{F,S}$ of the idelic norm (noting that the continuous open group homomorphism $C_{F,S} \to C_F$ is an isomorphism for this choice of S), descended to $J_{F,S}$ by composition with the quotient homomorphism $C_{F,S} \to J_{F,S}$.

The continuous linear operator

$$U_{S,\tau_S,(\chi|_{C_F^1})_S} : \mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S, \mathcal{S}(\mathbf{A}_{F,S})) \to A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) \tag{6.3.2.10}$$

defined in (5.4.1.2) by $U_{S,\tau_S,(\chi|_{C_F^1})_S}(f) := (Z_S(f, \chi|\cdot|_{F,S}^s), Z_S(\widehat{f}, \overline{\chi}|\cdot|_{F,S}^{1-s}))$ is, by the Second Structure Theorem (see Theorem 5.4.1.1(1)), a linear topological embedding. Its range is closed, since the source is complete by the First Structure Theorem (see Theorem 5.2.3.1(2)).

We will abbreviate the notation and write:

$$U_{S,\chi} := U_{S,\tau_S,(\chi|_{C_F^1})_S}.$$

We now consider the linear map:

$$T_{S,\chi} : A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) \xrightarrow{(\times(1-s-it_0)^{\delta_\chi} L_{\Sigma_F-S}(\chi,s), \times(s+it_0)^{\delta_\chi} L_{\Sigma_F-S}(\overline{\chi},1-s))} A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1}) \tag{6.3.2.11}$$

with $\delta_\chi = 0$ if $\chi|_{C_F^1} \neq 1$, and $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and with $t_0 \in \mathbf{R}$ uniquely determined by χ and τ (i.e. $t_0 \in \mathbf{R}$ is the unique real number such that $\chi(\tau(t)) = t^{it_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$) such that $s = 1 - it_0$ is the unique simple pole of $L_{\Sigma_F-S}(\chi, s)$ (if any) contained in $\mathbf{C}_{>0}$, sending a pair (ϕ, ψ) of holomorphic functions on $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$, each Schwartz uniformly on the vertical strips of bounded width of $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$, to the pair $((1-s-it_0)^{\delta_\chi} L_{\Sigma_F-S}(\chi, s)\phi(s), (s+it_0)^{\delta_\chi} L_{\Sigma_F-S}(\overline{\chi}, 1-s)\psi(s))$, where

$$L_{\Sigma_F-S}(\chi, s) := \prod_{v \in S} (L_v(\chi_v, s))^{-1} L(s, \chi)$$

is defined in (6.3.1.6) of Terminology 6.3.1.1.

$T_{S,\chi}$ is indeed valued in $A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{<1})$ and continuous, since $(1-s-it_0)^{\delta_\chi} L_{\Sigma_F-S}(\chi, s)$ is holomorphic over $\mathbf{C}_{>0} - \{1 - it_0\}$, and with at most a removable singularity at $s = 1 - it_0$ (see Proposition 3.2.1.3), and it grows at most polynomially, uniformly on the vertical strips of bounded width of $\mathbf{C}_{>0}$, as do its derivatives (see Remark B.1.1.4).

The kernel of the composition $T_{S,\chi} \circ U_{S,\chi}$ is trivial if $\chi|_{C_F^1} \neq 1$, and it is trivial when restricted to the subspace

$$\mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S, \mathcal{S}(\mathbf{A}_{F,S})) \cap \ker(\Delta_F(1) \circ \overline{\pi}_S) \cap \ker(\Delta_F(|\cdot|_F) \circ \overline{\pi}_S) \tag{6.3.2.12}$$

if $\chi|_{C_F^1}=1$. This subspace of $\mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S,\mathcal{S}(\mathbf{A}_{F,S}))$ is linearly and topologically isomorphic to $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ under $\overline{\pi}_S$.

To study the range of L from (6.3.2.3) and (6.3.2.4), we first study the range of $T_{S,\chi}\circ U_{S,\chi}$ within $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$.

We define the continuous linear map:

$$R_{S,\chi}: A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})\xrightarrow{(\phi,\psi)\mapsto(\phi(s)-\psi(s))|_{\mathbf{C}_{(0,1)}}} A\mathcal{S}(\mathbf{C}_{(0,1)}) \tag{6.3.2.13}$$

and observe that by the S-local functional equation (see (3.3.1.2) of Theorem 3.3.1.1) and the global functional equation (see (3.2.2.2) of Theorem 3.2.2.1), $T_{S,\chi}\circ U_{S,\chi}$ is valued in the kernel of $R_{S,\chi}$.

The operator $T_{S,\chi}$ is sequentially closed. Indeed, for $\{(\phi_m,\psi_m)\}_{m\geq 0}\subset A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$ a sequence such that $\{T_{S,\chi}(\phi_m,\psi_m)\}_{m\geq 0}\subset A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$ is a Cauchy sequence with limit $(\widetilde{\phi},\widetilde{\psi})\in A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$, the convergence of $(1-s-it_0)^{\delta_\chi}L_{\Sigma_F-S}(\chi,s)\phi_m(s)$ to $\widetilde{\phi}(s)$ and of $(s+it_0)^{\delta_\chi}L_{\Sigma_F-S}(\overline{\chi},1-s)\psi_m(s)$ to $\widetilde{\psi}_m(s)$ is uniform over the vertical strips of bounded width within $\mathbf{C}_{>0}$, respectively within $\mathbf{C}_{<1}$, by the polynomial growth of $(1-s-it_0)^{\delta_\chi}L_{\Sigma_F-S}(\chi,s)$ and $(s+it_0)^{\delta_\chi}L_{\Sigma_F-S}(\overline{\chi},1-s)$ and their derivatives of any order, uniformly on the vertical strips of bounded width within $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$.

In particular, $\widetilde{\phi}(s)$ is divisible by $(1-s-it_0)^{\delta_\chi}L_{\Sigma_F-S}(\chi,s)$ and $\widetilde{\psi}(s)$ is divisible by $(s+it_0)^{\delta_\chi}L_{\Sigma_F-S}(\overline{\chi},1-s)$ as meromorphic functions over $\mathbf{C}_{<1}$, i.e. there exist meromorphic functions $\widetilde{\phi}_0$ and $\widetilde{\psi}_0$ over $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$, holomorphic away from the zeros of $(1-s-it_0)^{\delta_\chi}L_{\Sigma_F-S}(\chi,s)$, respectively $(s+it_0)^{\delta_\chi}L_S(\widetilde{\chi},1-s)$, such that $\widetilde{\phi}(s)=(1-s-it_0)^{\delta_\chi}L_{\Sigma_F-S}(\chi,s)\widetilde{\phi}_0(s)$ over $\mathbf{C}_{>0}$, and $\widetilde{\psi}(s)=(s+it_0)^{\delta_\chi}L_{\Sigma_F-S}(\overline{\chi},1-s)\widetilde{\psi}_0(s)$ over $\mathbf{C}_{<0}$.

We now prove that in fact both $\widetilde{\phi}_0(s)$ and $\widetilde{\psi}_0(s)$ are holomorphic. To this end, we use the Asymptotic Residue Theorem (see Theorem B.3.2.1), and the polynomial growth of the reciprocal of $L_{\Sigma_F-S}(\chi,s)$ and its derivatives, on a sequence of horizontal segments with growing absolute ordinates, of a vertical strip of the form $\mathbf{C}_{[-\varepsilon,1+\varepsilon]}$ for any $0<\varepsilon<1$ (see Corollary B.6.1.8).

Since the growth of $L_{\Sigma_F-S}(\chi,s)$, respectively $L_{\Sigma_F-S}(\overline{\chi},1-s)$, together with their holomorphic derivatives of any order, is at most polynomial uniformly over the vertical strips of bounded width of $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$, then $\widetilde{\phi}_0(s)$ and $\widetilde{\psi}_0(s)$ are again elements of $A\mathcal{S}(\mathbf{C}_{>0})$, respectively $A\mathcal{S}(\mathbf{C}_{<1})$, as a consequence of the Asymptotic Residue Theorem (see Theorem B.3.2.1) and Lemma D.3.1.1.

We have shown that we have:

$$\lim_{m\to+\infty}T_{S,\chi}(\phi_m,\psi_m)=T_{S,\chi}(\widetilde{\phi}_0,\widetilde{\psi}_0)$$

in $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$, which shows that the range of $T_{S,\chi}$ is sequentially closed.

Since $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$ is a Fréchet space by Lemma D.3.1.2, the range of $T_{S,\chi}$ is closed, and being $T_{S,\chi}$ injective and continuous, it is a linear topological isomorphism onto its image.

Since $R_{S,\chi}$ is continuous, its kernel is closed in $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$, and hence the range of $T_{S,\chi}\circ U_{S,\chi}$ is closed in the kernel of $R_{S,\chi}$.

We will now analyze the kernel of $R_{S,\chi}$ as a subspace of $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$.

We define the linear map:

$$A: A\mathcal{S}(\mathbf{C})\to A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$$

assigned by sending $\phi\in A\mathcal{S}(\mathbf{C})$ to $(\phi(s),\phi(s))\in A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$.

A is continuous, injective, and with closed range since $A\mathcal{S}(\mathbf{C}_{>0})$ and $A\mathcal{S}(\mathbf{C}_{<1})$ are Hausdorff. It is therefore a linear topological embedding, and its range contains the kernel of $R_{S,\chi}$ (in fact, it agrees with the kernel of $R_{S,\chi}$). It follows that $T_{S,\chi}\circ U_{S,\chi}$ factors through A linearly and topologically, inducing a unique continuous linear operator from $\mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S,\mathcal{S}(\mathbf{A}_{F,S}))$, if $\chi|_{C_F^1}\neq 1$, and from the subspace of $\mathrm{H}_{0,\mathrm{cont}}((\chi|_{C_F^1})_S,\mathcal{S}(\mathbf{A}_{F,S}))$ from (6.3.2.12) if $\chi|_{C_F^1}=1$, to $A\mathcal{S}(\mathbf{C})$: such operator is $L\circ\overline{\pi}_S$.

Since $\overline{\pi}_S$ is a linear topological isomorphism onto $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F))$ if $\chi|_{C_F^1} \neq 1$, and onto $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ if $\chi|_{C_F^1} = 1$, we have shown that L is valued in $A\mathcal{S}(\mathbf{C})$ and it is continuous, injective and with closed range. In particular, it is a linear topological embedding, hence relatively open.

To conclude the proof of (1), we are to identify the range explicitly within $A\mathcal{S}(\mathbf{C})$. In order to do this, it is enough to compute the range of $L \circ \overline{\pi}_S$, for the choice of S made at the outset.

Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be given. Then $Z(\pi_S(f), \chi|\cdot|_F^s) = Z_S(f, \chi_S|\cdot|_{F,S}^s) L_{\Sigma_F - S}(\chi, s)$, where χ_S is the unique character of $J_{F,S}$ obtained by descending to $J_{F,S}$ the restriction of χ to $C_{F,S}$, being χ unramified outside S.

By combining Theorem 6.2.1.1 with its refinement, Theorem 6.2.2.1, the range of L is linearly generated by all entire functions ϕ on the complex s-plane satisfying the following three conditions:

(1) There exists $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ such that ϕ is of the following form:

$$\phi(s) = \psi(s)((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_f(s,\chi) \tag{6.3.2.14}$$

for $\psi \in \mathscr{O}(\mathbf{C})$ of the following form:

$$\psi(s) = \int_{\mathbf{R}_{>0}} t^{s+it_0}\, \Psi^\infty_{F,S,\omega_S}(f_\infty)(\tau(t))\, \frac{\mathrm{d}t}{t} \tag{6.3.2.15}$$

where $\Psi^\infty_{F,S,\omega_S}$ is the linear operator defined in (6.2.2.2) of the Theorem, for $\omega_S := (\chi_S)|_{J^1_{F,S}}$).

(2) ϕ is an element of $A\mathcal{S}(\mathbf{C})$.

(3) ϕ has zeros at the zeros of $L_f(s,\chi)$ contained in $\mathbf{C}_{>0}$, with matching multiplicities.

Note that by Theorem 6.2.2.1, the $\mathbf{C}$-linear subspace of finite linear combinations of entire functions of the form described in (1), (2) and (3), agrees with the range of L in $A\mathcal{S}(\mathbf{C})$, and it is not merely dense in it. This is a consequence of each $\mathcal{S}(F_v)$ carrying the finest locally convex topology, rendering the algebraic tensor products in Proposition 2.1.4.1 and Theorem 6.2.2.1 topologized with the projective tensor product topology, already complete.

Note also that since we restrict the source of L to the closed co-dimension 2 subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, and formation of this closed subspace is compatible with $\overline{\pi}_S$, i.e. $\overline{\pi}_S$ for every S sends $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))$ linearly and topologically isomorphically onto $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, for $\mathcal{S}_0(\mathbf{A}_{F,S}) := \ker(\delta_0) \cap \ker(\widehat{\delta_0})$ (the Fourier transform being the S-adelic Fourier transform, defined using the standard additive character), we can choose f_∞ to satisfy these two vanishing conditions in $\mathcal{S}(\mathbf{A}_{F,S_\infty})$, in the sole case they are necessary (i.e. $\chi = 1$).

The meromorphic function $\psi(s)$ therefore has only removable singularities over $\mathbf{C}_{>0}$, because the integral defining ψ in (6.3.2.15) is absolutely and locally uniformly convergent for $s \in \mathbf{C}_{>0}$ (by an analog of the estimate from Theorem 2.4.2.1 leading to Corollary 2.4.3.2, the only difference being that the character of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ being used to define the θ-function operator Ψ_{F,S,ω_S} is not $\mathcal{O}_F^\times$: it is only the restriction to $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ of the $\mathcal{O}_{F,S}^\times$-invariant character χ_S. With this caveat, the same argument for Theorem 2.4.2.1 leads to the analog of Corollary 2.4.3.2 that implies the regularity of ψ over $\mathbf{C}_{>0}$. We refer the reader to the proof of Theorem 6.2.2.1(2)).

Moreover, since the poles of the Archimedean local factors $\prod_{v\in S_\infty} L_v(\chi_v|\cdot|_v^s)$ are resolved by the trivial zeros of $L_f(s,\chi)$, the meromorphic function:

$$u(s) := \psi(s) \prod_{v\in S_\infty} L_v(\chi_v|\cdot|_v^s)^{-1}$$

also has only removable singularities over the complex s-plane, because we know that ϕ is entire, as is $((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L(s,\chi)$. Since this latter is nowhere vanishing over $\mathbf{C}_{\leq 0}$, the only singularities of $\psi(s)$ can be located over $\mathbf{C}_{\leq 0}$, and are therefore resolved by the zeros of $\prod_{v\in S_\infty} L_v(\chi_v|\cdot|_v^s)^{-1}$.

Its unique holomorphic continuation to the complex s-plane, which we again denote $u(s)$, satisfies:

$$\phi(s) = u(s)((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L(s,\chi)$$

which shows that indeed the range of L is contained within

$$((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L(s,\chi)\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}).$$

Conversely, let $\phi \in A\mathcal{S}(\mathbf{C})$ be of the form:

$$\phi(s) = u(s)((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L(s,\chi)$$

for some entire function u on the complex s-plane.

The inverse Fourier transform of $\phi(it)$ is therefore Schwartz-Bruhat over $\mathbf{R}_{>0}$ and with exponential decay (see Definition D.3.3.1 for $I = \mathbf{R}$, and Proposition D.3.2.5).

We now choose a finite set S of finite places of F, large enough so as to arrange for χ to be unramified outside of S, and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

By Proposition 5.3.4.4(3), there exists a Schwartz-Bruhat function $f_0 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ (see (5.3.4.1) from Definition 5.3.4.1 and recall that $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ is the closed subspace of $\mathcal{S}(\mathbf{A}_{F,S})$ of those Schwartz-Bruhat functions that are entirely supported over $\mathrm{GL}_1(\mathbf{A}_{F,S})$, vanishing smoothly towards the boundary hyperplanes over $\mathbf{A}_{F,S_\infty}$), such that $Z_S(f_0, \chi_S|\cdot|_{F,S}^s) = \phi(s)$.

We note that by Corollary B.6.1.8, the meromorphic function

$$\lambda_S(s) := \frac{1}{((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_{\Sigma_F - S}(\chi, s)}$$

has slow growth near its poles, and it is holomorphic over the open vertical strips of bounded width located within the complex s-plane away from $\mathbf{C}_{(0,1)}$.

It grows at most polymomially as $|s| \to +\infty$ over each such strip, as do its holomorphic derivatives of any order (see Remark B.1.1.4).

It follows that $\phi(s)\lambda_S(s)$ is meromorphic on the complex s-pane, it has poles located only at the poles of the local Hecke L-function $L_S(\chi_S, s)$:

$$L_S(\chi_S, s) := \prod_{v \in S} L_v(\chi_v|\cdot|_v^s)$$

with polar multiplicities bounded above by the orders of pole of the poles of $L_S(\chi_S, s)$.

We recall from Terminology 6.3.1.1 that we denote by χ_S the unique continuous character of $J_{F,S}$ descended by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance from the restriction of χ to $C_{F,S}$.

We define $\phi_S(s) := \phi(s)\lambda_S(s)$ and

$$\psi_S(s) := \frac{Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})}{((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_{\Sigma_F - S}(\chi, s)} = \lambda_S(s) Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})$$

a meromorphic function on the complex s-plane.

Since $Z_S(f_0, \chi_S|\cdot|_{F,S}^s)$ is entire and by (3.3.1.2) from Theorem 3.3.1.1 we have

$$Z_S(f_0, \chi_S|\cdot|_{F,S}^s) = \rho_S(\chi_S|\cdot|_{F,S}^s) Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})$$

then $Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})$ has zeros at the poles of $\rho_S(\chi_S \cdot |\cdot|_{F,S}^s)$, with multiplicity bounded below by the orders of pole, respectively.

Such poles are exactly the zeros of $((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_S(\chi_S, s)^{-1}$, the polar order matching the multiplicity of the zero, respectively.

Moreover, the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1 ensures that $Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})$ also vanishes at each non-trivial zero of $L_f(s,\chi)$, (since $\phi(s) = Z_S(f_0, \chi_S|\cdot|_{F,S}^s)$ does) equivalently at each zero of $L_{\Sigma_F - S}(\chi, s)$ contained within $\mathbf{C}_{(0,1)}$.

It follows that the ratio:

$$\frac{Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})}{((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_{\Sigma_F - S}(\chi, s)}$$

has only removable singularities over $\mathbf{C}_{<1}$.

Both ϕ_S and ψ_S are Schwartz uniformly over the vertical strips of bounded width of $\mathbf{C}_{>0}$, respectively $\mathbf{C}_{<1}$: to see this, we use the functional equation for the Hecke L-function $L_f(s,\chi)$ (see (B.1.1.9)) and the boundedness of $L_f(s,\chi)$ over $\mathbf{C}_{>0}$ and the polynomial growth of $W(s,\chi)$ and its holomorphic derivatives, as $|s| \to +\infty$ uniformly over each vertical strip of bounded width of the complex s-plane, together with Corollary B.6.1.8 on the polynomial growth of the reciprocal of $L_{\Sigma_F - S}(\chi, s)$ on a sequence of horizontal segments with growing absolute ordinates, of a vertical strip of the form $\mathbf{C}_{[-\varepsilon, 1+\varepsilon]}$ for any $0 < \varepsilon < 1$.

By the functional equation for $L_f(s,\chi)$ (see (B.1.1.9)) and the S-local functional equation from (3.3.1.2) from Theorem 3.3.1.1, we have:

$$\phi_S(s) = \frac{Z_S(f_0, \chi_S|\cdot|_{F,S}^s)}{((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_S(\chi_S, s)} \tag{6.3.2.16}$$

$$= \frac{\rho_S(\chi|\cdot|_{F,S}^s) Z_S(\widehat{f_0}, \overline{\chi}_S|\cdot|_{F,S}^{1-s})}{((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} L_S(\chi_S, s)} \tag{6.3.2.17}$$

$$= \rho_S(\chi_S|\cdot|_{F,S}^s)\psi_S(s). \tag{6.3.2.18}$$

It follows by Theorem 5.4.1.1 that $(\phi_S(s), \psi_S(s))$ is in the image of the operator $U_{S,\tau_S,\chi_S|_{J_{F,S}^1}}$ from (5.4.1.2), and hence there exists a Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$ such that $Z_S(f, \chi_S|\cdot|_{F,S}^s) = \phi_S(s)$.

In particular, we have:

$$Z_S(f, \chi_S|\cdot|_{F,S}^s) L_{\Sigma_F - S}(\chi, s)((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}} = \phi(s) = Z(\pi_S(f), \chi|\cdot|_F^s)((s+it_0)(1-s-it_0))^{\delta_{\chi|_{C_F^1}}}$$

as meromorphic functions on the complex s-plane, and hence everywhere on the complex s-plane by analytic continuation.

This completes the description of the range of L.

(2) : By (1) and by Lemma 5.5.2.1, the sequence of operators in (6.3.2.5) is strict exact, and hence so are the sequences in (6.3.2.7) and (6.3.2.8). □

The following is a version of Theorem 6.3.2.1, providing a description of the range of the Mellin transform L from (5.2.3.14), restricted to closed subspaces of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ defined to account for the removal of finitely many non-Archimedean "Euler factors" from $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$. It can be viewed as an incarnation of a "non primitive version" of the Converse Theorem for GL_1 (see [**CPS**, Thm.1,p.165]).

THEOREM 6.3.2.2 (Converse Theorem for GL_1 - non-principal case). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $A\mathcal{S}(\mathbf{C})$ be the Schwartz-Bergman space as per Definition D.3.1.3. We parametrize $\Omega(C_F)(\chi|_{C_F^1})$ by the global holomorphic parametrization*

$$\chi|\cdot|_F^s, \quad s \in \mathbf{C}.$$

Let S_0 be a finite set of finite places of F and denote by S_χ the finite set of places of F at which χ is ramified. Let PV_{S_0} the Weil's Global Principal Value operator on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$.

We define $q_v := N(\mathfrak{p}_v)$, with $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ corresponding to the finite place $v \in S_0$.

(1) *If* $\chi|_{C_F^1} \neq 1$, *then the continuous linear map:*

$$Z : \mathrm{PV}_{S_0} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F)) \to A\mathcal{S}(\mathbf{C}) \tag{6.3.2.19}$$

is a linear topological embedding with closed range equal to $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)\,\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C})\subset A\mathcal{S}(\mathbf{C})$.

If $\chi|_{C_F^1}=1$, *then the continuous linear map:*

$$L\circ\Phi_{F,1} : \mathrm{PV}_{S_0}\mathcal{S}_0(\mathbf{A}_F)\to A\mathcal{S}(\mathbf{C}) \tag{6.3.2.20}$$

is relatively open with closed range equal to $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})s(s-1)\widehat{\zeta}_F(s)\,\mathscr{O}(\mathbf{C})\subset A\mathcal{S}(\mathbf{C})$, *and kernel equal to* $\mathrm{PV}_{S_0}(\ker(\Phi_{F,1})\cap\mathcal{S}_0(\mathbf{A}_F))$.

(2) *The following sequence of* $\mathbf{C}$*-linear operators:*

$$0\to \mathrm{PV}_{S_0}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\to \mathrm{PV}_{S_0}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}(\mathbf{A}_F))\to \mathbf{C}^{\oplus 2}\to 0 \tag{6.3.2.21}$$

where:

(1) $\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}_0(\mathbf{A}_F))$ *is the quotient of* $\mathcal{S}_0(\mathbf{A}_F)$ *by* $\ker(\Phi_F)\cap\mathcal{S}_0(\mathbf{A}_F)$, *endowed with the quotient topology.*

(2) $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ *is the quotient of* $\mathcal{S}_0(\mathbf{A}_F)$ *by* $\ker(\Phi_{F,\chi|_{C_F^1}})\cap\mathcal{S}_0(\mathbf{A}_F)$, *endowed with the quotient topology.*

(3) *The rightmost operator sends* $\Phi_{F,\chi|_{C_F^1}}(\mathrm{PV}_{S_0}(f))$ *to the pair of complex numbers:*

$$-\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{it_0})\overline{\mu}_1(C_F^1)\,f(0)\,\delta_{\chi|_{C_F^1}},\quad \prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-1+it_0})\overline{\mu}_1(C_F^1)\,\widehat{f}(0)\,\delta_{\chi|_{C_F^1}} \tag{6.3.2.22}$$

with $\delta_{\chi|_{C_F^1}}=0$ *if* $\chi|_{C_F^1}\neq 1$, *and* $\delta_1=1$.

is strict exact, and under the Mellin transform $\phi\mapsto M(\phi)(s):=\int_{C_F}\phi(g)\chi(g)|g|_F^s\,\mathrm{d}\overline{\mu}(g)$ *it is linearly and topologically isomorphic to the short exact sequence:*

$$0\to\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)\,\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C})\to\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)\,\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C})\to 0 \tag{6.3.2.23}$$

if $\chi|_{C_F^1}\neq 1$, *the middle operator in* (6.3.2.23) *being the identity, and the topology on* $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)\,\mathscr{O}(\mathbf{C})$ *being the subspace topology inherited from* $A\mathcal{S}(\mathbf{C})$, *and to the short exact sequence:*

$$0\to(s-it_0)(1-s-it_0)\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})\widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C}) \tag{6.3.2.24}$$

$$\to\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})\widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C}) \tag{6.3.2.25}$$

$$\to\mathbf{C}\oplus\mathbf{C}\to 0 \tag{6.3.2.26}$$

for a real number $t_0\in\mathbf{R}$ *uniquely determined by* χ, *where the rightmost operator sends an element of* $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})\widehat{\zeta}_F(s+it_0)\,\mathscr{O}(\mathbf{C})$ *to the pair of residues at* $s=-it_0,1-it_0$, *the topology on the leftmost space of meromorphic functions from* (6.3.2.24) *being the subspace topology inherited from* $A\mathcal{S}(\mathbf{C})$, *and the topology on the middle space of meromorphic functions being the quotient topology induced by* L *from* $\mathrm{H}_{0,\mathrm{cont}}(|\cdot|_F^{it_0},\mathcal{S}(\mathbf{A}_F))$.

Proof. Both (1) and (2) follow from the combination of Lemma 5.3.1.5 and Theorem 6.3.2.1. □

CHAPTER 7

Schauder Bases of Idelic Zeta functions

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In this Chapter we construct Schauder bases on certain important subquotients of the separable nuclear Fréchet spaces:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})), \quad \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

with F a number field, χ a continuous character of $J_{F,S}$, respectively C_F.

The constructions are made in the S-local case, and then propagated globally using the slow growth of the reciprocal of the Hecke L-function $L_f(s,\chi)$ on the 1-line, and classical zero-free regions.

For the purposes of applications, the S-local Schauder bases will be the main instrument in use in later Chapters.

We recall from [**Jar**, Part II,Ch.14,§14.2,p.292] that for E a Hausdorff locally convex topological $\mathbf{C}$-vector space, $\{x_m\}_{m\geq 0} \subset E$ is a Schauder basis if and only if:

(1) For every $u \in E$, there exists a unique sequence $\{c_m(u)\}_{m\geq 0} \subset \mathbf{C}$ of complex numbers such that

$$u = \sum_{m\geq 0} c_m(u) x_m$$

the convergence of the sum being in E.

(2) Formation of each c_m is continuous on E.

Condition (1) identifies the concept of *topological basis* (see [**Jar**, Part II,Ch.14,§14.2,p.292]) while the datum $\{(x_m, f_m)\}_{m\geq 0}$ with $\{x_m\}_{m\geq 0} \subset E$, and $\{f_m\}_{m\geq 0} \subset E'$, such that the linear span of the x_m's is dense

in E, and the linear span of the f_m's separates the points of E (equivalently, is weakly dense in E') is termed a *Markushevich basis*; see [**Jar**, Part II,Ch.14,§14.1,p.289].

For $\{x_m\}_{m\geq 0} \subset E$ a Schauder basis, upon denoting $f_m := c_m$, then $\{(x_m, f_m)\}_{m\geq 0}$ is a Markushevich basis for the duality $\langle E, E'\rangle$ (see [**Jar**, Part II,Ch.14,§14.2,Prop.1,p.292]).

Hausdorff locally convex topological $\mathbf{C}$-vector spaces with a Schauder basis satisfy several additional properties. We refer the reader to [**Jar**, Part II,Ch.14] for more on Schauder bases, unconditional and absolute bases, and to [**Kal**] for more on properties of Hausdorff locally convex topological vector spaces in the presence of Schauder basic sequences within their closed subspaces, as well as some terminology.

We recall from [**Kal**, §3,p.95] that a Schauder basis $\{b_k\}_{k\geq 0}$ in a Hausdorff locally convex topological $\mathbf{C}$-vector space E is *simple* if and only if the finite rank projection operators $P_N := \sum_{0\leq k\leq N}\langle\varphi_k, \cdot\rangle_{E',E} b_k$, for $\{\varphi_k\}_{k\geq 0}$ the dual basic sequence of $\{b_k\}_{k\geq 0}$, are a bounded set in $\mathscr{L}_\beta(E)$, i.e. they are strongly bounded.

We further recall from [**Kal**, §1,Def.1.1,p.91] that a Schauder basis $\{b_k\}_{k\geq 0}$ of E is *regular* if and only if there exists a 0-neighborhood V such that $b_k \notin V$ for all $k \geq 0$.

Finally, from [**Kal**, §1,Def.1.2], a Schauder basis is *normalized* if and only if it is bounded and regular.

The example to keep in mind is that of $E = \mathcal{S}(\mathbf{T})$, for $\mathbf{T}$ the circle group $\mathbf{R}/\mathbf{Z}$, with $x_m = e^{2\pi i m x}$, as a function of $x \in \mathbf{R}$ descended to $\mathbf{R}$ by 1-periodicity, for each $m \in \mathbf{Z}$, and with f_m the mth Fourier coefficient functional. Then $\{x_m\}_{m\in\mathbf{Z}}$ is an absolute Schauder basis for E, and its dual basis $\{f_m\}_{m\in\mathbf{Z}}$ is an unconditional Schauder basis for E'_β (see Lemma 4.2.1.2, with $\gamma = e$, below, noting that the exponential isomorphism $x \mapsto e^x$ induces a linear topological isomorphism $\mathcal{S}(\mathbf{T}) \simeq \mathcal{S}(\mathbf{R}_{>0}/e^{\mathbf{Z}})$).

7.1. Polar Multiplicities

In this section we discuss the poles of the S-idelic Zeta functions.

This will serve the two-fold purpose of later selecting the Sobolev degree that suffices for a Sobolev space of the global idele class group to serve as ambient for the Trace Formula which is the main subject of study of this work, and it will serve the purpose, at the end of this Chapter, of discussing the existence of periodic orbits and fixed loci of the action of C_F on certain sub-loci of the global adeles.

This section will enable us to sharpen the analogy between our trace formula, and Selberg's Trace Formula (see [**Iwaniec**, Ch.10], by appealing to a paper of P. Sarnak on the uniform distribution of long horocycles on the space of elements of a Riemann surface, equipped with the Poincaré metric, for which we refer the reader to [**Sarnak**, §0,Thm.1,p.719].

7.1.1. Almost periodic functions. Pioneering work of Harald Bohr from [**Bohr1**, **Bohr2**, **Bohr3**] investigated self-approximation conditions on a large class of functions directly related to Dirichlet series: "almost periodic" functions (we do not review its definition since the present discussion is purely for motivation; what we will really need is "strongly recurrent" (Definition B.5.3.1) . These are functions that are holomorphic on a vertical strip $\mathbf{C}_I$ of bounded width of the complex s-plane, and, while not necessarily periodic, recur enough often over $\mathbf{C}_I$ to allow the study of their distribution of values. Examples of almost periodic functions are finite sums and finite products of vertically periodic functions with different, possibly $\mathbf{Q}$-linearly independent, vertical periods.

An example of such function is any finite partial sum of a Dirichlet series. Let F be a number field, S a finite set of places containing all the Archimedean places, and S_f the subset of S of all of the finite places of F contained in S. Then the function:

$$\prod_{v\in S_f}(1 - q_v^{-s-it_v}) \tag{7.1.1.1}$$

is, for $q_v := N(\mathfrak{p}_v)$ for $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ corresponding to the finite place v, almost periodic.

Each factor is periodic of period $2\pi i/\log(q_v)$. For finite places v, v' contained in S with different residue characteristics, the real numbers $\log(q_v)$ and $\log(q_{v'})$ are $\mathbf{Q}$-linearly independent, and the function in (7.1.1.1) is not periodic.

However, it recurs enough often for us to find a sequence of vertical segments within the vertical strips $\mathbf{C}_I$ (with I an open interval containing $0 \in \mathbf{R}$) the absolute value of whose ordinates tend to $+\infty$, and on which the function in (7.1.1.1) remains bounded below a positive threshold $N_S > 0$ fixed at the beginning. For instance, it has zeros at $\{2\pi i\,\mathbf{Z}/\log(q_v), v \in S_f\}$.

The reciprocal function

$$\prod_{v\in S_f} (1 - q_v^{-s-it_v})^{-1} \tag{7.1.1.2}$$

is, however, not holomorphic and is even unbounded over some $\mathbf{C}_I$. The question of whether there exists a sequence of ordinates whose absolute value tends to $+\infty$, and such that on the corresponding vertical segments contained in $\mathbf{C}_I$, (7.1.1.2) is uniformly bounded, is a non-trivial one.

We cannot use [**Bohr**, Ch.2,§46,p.35] applied to the restriction of the function (7.1.1.1) to $i\mathbf{R}$ since the function has zeros within $\mathbf{C}_I$ on $i\mathbf{R}$. For this reason, we use a weaker notion than almost periodicity: *strong recurrence*.

In §B.5.3.1 we review the notion of (strong) "recurrence" for a holomorphic function on a vertical strip $\mathbf{C}_I$, for $I \subset \mathbf{R}$ a non-empty open interval, the study of which was initiated in B. Bagchi's PhD thesis [**Ba1**, Ch.1,2]. Strong recurrence is a vertical self-approximation property of holomorphic functions on vertical strips. We refer the reader to [**Ba1**, Ch.1, §1.4] (respectively [**Ba1**, Ch.2, §2.3]) for further background of a general character (respectively specific to the holomorphic functions of interest to number theory), and to [**Ba2**, §3] for applications. It is indeed weaker than almost periodicity (for analytic functions on vertical strips of bounded width, as defined in [**Bohr4**, Appendix II,§104,p.101]), in light of [**Bohr**, Ch.2,§47,Corollary,p.37] and [**Ba1**, Ch.2,§2.4,Prop.2.4.2,p.46] (since, by [**Bohr4**, Appendix II,§106,p.104], an almost periodic analytic function on $\mathbf{C}_I$ has uniformly strongly recurrent vertical sections in the sense of [**Ba1**, Ch.2,§2.4,Def.2.4.1]). It can be thought of as a "local" analog of almost periodicity.

The following Lemma also follows from Proposition B.5.3.3, to which the reader is referred, whose proof is direct and rests upon the Kronecker-Weyl Theorem (see Theorem B.5.2.1).

Recall that for F a number field, v a finite place of F, we denote by q_v the absolute norm of the unique prime ideal $\mathfrak{p}_v \subset \mathcal{O}_F$ associated to v.

LEMMA 7.1.1.1. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. We write $S_f := S \cap \Sigma_{F,f}$. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let $\delta > 0$ be an arbitrary real number. Let $\chi \in \widehat{C_F}$ be a continuous caracter, unramified outside S. Let $t_v \log(q_v)$ be a choice of logarithm for $\chi(\mathfrak{p}_v)$, for each $v \in S_f$ at which χ is unramified.

There exists a real number $N_S > 0$ and a sequence $\{T_k\}_{k\in\mathbf{Z}} \subset \mathbf{R}$, such that:

(1) *$T_k \geq 0$ if $k \geq 0$ and $T_k \leq 0$ if $k \leq 0$, with $T_k \leq T_{k+1}$ for every $k \in \mathbf{Z}$.*
(2) $\lim_{|k|\to+\infty} |T_k| = +\infty$.
(3) *For all $k \in \mathbf{Z}$, we have:*

$$\prod_{v\in S_f} |1 - q_v^{-\sigma-iT_k+it_v}|^{-1} \leq N_S \tag{7.1.1.3}$$

for all $\sigma \in [-\delta, 1+\delta]$.

For $A \subset \mathbf{R}$ a subset, we recall that the *upper density* of A is defined to be

$$\overline{d}(A) := \limsup_{T\to+\infty} \frac{\lambda(A \cap [-T, T])}{2T} \tag{7.1.1.4}$$

where λ is the Lebesgue measure on $\mathbf{R}$, normalized so as to give the interval $[0,1]$ volume 1.

Let $I \subset \mathbf{R}$ be an open interval containing 0. The proof of Lemma 7.1.1.1 rests on the concept of "strongly recurrent" holomorphic function on $\mathbf{C}_I$: for ϕ a holomorphic function on $\mathbf{C}_I$, we say that ϕ is *strongly recurrent* over $\mathbf{C}_I$ if and only if for every compact subset $K \subset \mathbf{C}_I$, and for every real number $\varepsilon > 0$, the subset:

$$\{t \in \mathbf{R} : \sup_{s \in K} |\phi(s+it) - \phi(s)| < \varepsilon\}$$

has positive upper-density.

The idea to prove Lemma 7.1.1.1 is to use discreteness of the zeros of $\phi(s) := \prod_{v \in S_f}(1 - q_v^{-s-it_v})$ within $\mathbf{C}_I$, to find a compact subset $K \subset \mathbf{C}_I$ with non-empty interior, on which $|\phi(s)|$ is nowhere vanishing and bounded below by a positive constant M, for then choosing $\varepsilon > 0$ small enough compared to M, and access the strong recurrence of ϕ over $\mathbf{C}_I$, on the compact subset K and for such choice of ε, to keep the values of $1/\phi(s)$ bounded in absolute value over the vertical translates $K + it$, for t in a set of positive upper-density in $\mathbf{R}$.

Proof of Lemma 7.1.1.1. We consider the space $\mathscr{H}(\mathbf{C}_I)$ of holomorphic functions on the vertical strip $\mathbf{C}_I$, with $I \subset \mathbf{R}$ any open interval containing $0 \in \mathbf{R}$, upon topologizing $\mathscr{H}(\mathbf{C}_I)$ with the topology of uniform convergence on compact subsets as in [**Ba1**, Ch.2,§2.2.2]. In [**Ba1**, Ch.2,§2.2.3], $\mathscr{H}(\mathbf{C}_I)$ is equipped with the vertical-shift action of $\mathbf{R}$, a continuous flow.

The holomorphic function ϕ defined in (7.1.1.1) is strongly recurrent as a holomorphic function on $\mathbf{C}_I$ due to the vertical periodicity of its finitely many factors and [**Ba1**, Ch.2,Prop.2.3.3].

This means that for any compact subset $K \subset \mathbf{C}_I$ with non-empty interior, and any real number $\varepsilon > 0$, we have:

$$\overline{d}(\{t \in \mathbf{R} : \sup_{s \in K} |\phi(s+it) - \phi(s)| < \varepsilon\}) > 0. \tag{7.1.1.5}$$

Since the set of zeros of ϕ is discrete in $\mathbf{C}_I$ and entirely contained in $i\mathbf{R} \subset \mathbf{C}_I$, there exists a compact subset K_0 of $\mathbf{C}_I$ with non-empty interior, such that its interior has non-empty intersection with $i\mathbf{R}$, and such that $|\phi(s)|$, as a real valued function on K_0, is nowhere vanishing and bounded below by a positive constant $M > 0$.

We may replace K_0 with the union of K_0 and two large enough compact boxes with non-empty interior $B_1, B_2 \subset \mathbf{C}$ such that the intersection of $K_0 \cup B_1 \cup B_2$ with the boundary of $\mathbf{C}_{[-\delta,1+\delta]}$ ($\delta > 0$ being fixed in the statement of the Lemma) is the union of two vertical segments, one within the vertical line $-\delta + i\mathbf{R}$, the other within the vertical line $1 + \delta + i\mathbf{R}$. We further choose B_1 and B_2 such that $K_0 \cup B_1 \cup B_2$ has connected and simply connected interior, containing a horizontal segment of the form $[-\delta, 1+\delta] + iT$ for some $T \in \mathbf{R}$, such that $(-\delta, 1+\delta) + iT$ is entirely contained in the interior of $K_0 \cup B_1 \cup B_2$.

We now rename $K_0 \cup B_1 \cup B_2$ as K_0, and choose $I \subset \mathbf{R}$ strictly larger than $[-\delta, 1+\delta]$.

Since ϕ only has zeros on the line $i\mathbf{R}$, we may rename as M the new strictly positive lower bound for $|\phi(s)$ over K_0.

Upon choosing $K = K_0$ and $\varepsilon := M/2$ in (7.1.1.5), we deduce that there exists a subset $A \subset \mathbf{R}$ of positive upper-density, such that on each of the vertical translates $K_0 + it$ as $t \in A$, the function $|\phi(s)|$ satisfies:

$$|\phi(s)| \geq M - \varepsilon = \frac{M}{2} > 0. \tag{7.1.1.6}$$

Indeed, we have:

$$||\phi(s+it)| - |\phi(s)|| \leq |\phi(s+it) - \phi(s)| < \varepsilon$$

for all $s \in K_0$ and all $t \in A$, and hence:

$$-\varepsilon < |\phi(s+it)| - |\phi(s)| < \varepsilon \tag{7.1.1.7}$$

for all $s \in K_0$ and all $t \in A$. From (7.1.1.7), it follows that we have:

$$M - \varepsilon \leq |\phi(s)| - \varepsilon < |\phi(s+it)|$$

and hence (upon renaming $s+it$ as s) (7.1.1.6) holds for all $s \in K_0 + it$ and all $t \in A$. We deduce that we have:

$$|1/\phi(s)| \leq 1/(M-\varepsilon)$$

for all $s \in \bigcup_{t\in A}(K_0 + it)$. Upon choosing any point $s_0 \in K_0 \cap i\mathbf{R}$ in the interior of K_0 such that the horizontal segment $L := [-\delta, 1+\delta] + s_0$ is contained in K_0 and its interior is contained in the interior of K_0, we deduce that we have:

$$|1/\phi(s)| \leq 1/(M-\varepsilon)$$

for all $s \in \bigcup_{t\in A}\{L + it\}$.

Since A has positive upper-density $\delta_A > 0$ in $\mathbf{R}$, for $0 < \delta < \delta_A$ there exists a sequence $\{T'_k\}_{k\in\mathbf{N}} \subset \mathbf{R}_{>0}$ such that $T'_{k+1} \geq T'_k$ for all $k \geq 0$, $\lim_{k\to+\infty} T'_k = +\infty$, and such that each intersection $A \cap [-T'_k, T'_k]$ satisfies:

$$\lambda(A \cap [-T'_k, T'_k]) \geq 2\delta T'_k. \tag{7.1.1.8}$$

Since the lower bound is tending toward ∞ as k grows, for each integer $k \geq 0$ we therefore find an integer $m_k > k$ such that $(A \cap [-T'_{m_k}, T'_{m_k}]) - (A \cap [-T'_k, T'_k])$ is non-empty.

We denote, for each integer $k \geq 0$, by $m^{(N)}(k)$ the integer defined by the recursion $m^{(1)}(k) := m_k$, and $m^{(N+1)}(k) := m_{m^{(N)}(k)} > m^{(N)}(k)$, and call $T''_k := T'_{m^{(k)}(0)}$. This defines a subsequence of $\{T'_k\}_{k\geq 0}$, which therefore satisfies $T''_k \leq T''_{k+1}$, $\lim_{k\to+\infty} T''_k = +\infty$, and, for every integer $k \geq 1$, the set $(A \cap [-T''_{k+1}, T''_{k+1}]) - (A \cap [-T''_k, T''_k])$ is non-empty.

We choose, for each integer $k \geq 0$, $T'''_k \in (A \cap [-T''_{k+1}, T''_{k+1}]) - (A \cap [-T''_k, T''_k])$, and we obtain:

$$|1/\phi(s_0 + iT'''_k)| \leq 1/(M-\varepsilon)$$

for all $k \in \mathbf{N}$. We now define $T_k := \mathrm{Im}(s_0) + T'''_k$ for $k \geq 0$. We obtain that for every integer $k \geq 0$, we have

$$|1/\phi(\sigma + iT_k)| \leq 1/(M-\varepsilon)$$

for all $\sigma \in [-\delta, 1+\delta]$. We have arranged the T_k to satisfy $\lim_{k\to+\infty} |T_k| = +\infty$. We now observe that if $|1/\phi(\sigma + iT_k)| \leq 1/(M-\varepsilon)$, then we also have $|1/\phi(\sigma - iT_k + 2it_v)| \leq 1/(M-\varepsilon)$, since we have

$$|1/\phi(\sigma - iT_k + 2it_v)| = |\overline{1/(\sigma + iT_k)}|$$

for all $\sigma \in [-\delta, 1+\delta]$ and for each integer $k \geq 0$. We now consider the countable subset of $i\mathbf{R}$ given by

$$V = \{iT_k, k \in \mathbf{N}\} \cup \{-iT_k + 2it_v, k \in \mathbf{N}\}$$

and relabel it according to the following rule: we order the ordinates of $V \cap i[0, +\infty)$ increasingly and label them by the nonnegative integers. We order the ordinates of $V \cap i(-\infty, 0)$ increasingly (i.e. according to the total order inherited from the real line) and label them by the negative integers, themselves equipped with their natural total ordering.

We rename the elements of $-iV$ as T_k for $k \in \mathbf{Z}$. Upon choosing $N_S := M - \varepsilon > 0$, we see that $\{T_k\}_{k\in\mathbf{Z}}$ satisfies the requirements of the Lemma. □

The Lemma says that the reciprocals of the partial Euler products of the Hecke L-function $L(s, \chi)$ are meromorphic functions with slow growth near their poles, in the sense of Definition B.3.1.1. The Lemma provides access to the Asymptotic Residue Theorem (see Theorem B.3.2.1).

7.1.2. Asymptotic expansion of $\Phi_{F,S}(f)(g)$ as $|g|_{F,S} \to 0^+$. The following Theorem passes Proposition 2.4.1.1 and its general version of Theorem 2.4.2.1, through the lens of the Mellin transform. We further refer the reader to Theorem B.2.1.7.

We will be using Theorem 7.1.2.3 throughout this Chapter as a source of motivation for the search of certain Schauder bases for subquotients of the nuclear Fréchet spaces

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

as well as a means to explain the description of certain periodic orbits associated to the non-Archimedean places of a number field F, to be discussed at the end of this Chapter, exhibiting them as analogs of the periodic orbits that serve as the supports of the orbital measures in [**Sarnak**, §0,Thm.1].

The role of the periodic orbits found throughout our work is in fact the same as that of the periodic orbits of the horocycle flow on the space of elements of a Riemann surface, that features in Sarnak's paper [**Sarnak**], which the higher rank version of our work will recast as a special case of the GL_2-analog of Theorem 7.1.2.3 below.

Throughout this work we refer to "order of pole" at $s_0 \in \Omega$ of a meromorphic complex valued function $f : \Omega \to \mathbf{C}$ on a domain of the complex s-plane $\Omega \subset \mathbf{C}$ as the nonnegative integer $N \geq 0$ such that $a_m(f, s_0)$, the mth coefficient of the Laurent series expansion of f centered at $s = s_0$ for $m \in \mathbf{Z}$, vanishes for $m < -N$, minimal (among the nonnegative integers) with respect to this property, i.e. $a_{-N}(f, s_0) \neq 0$. We therefore say *f has a pole of order N at $s = s_0$*.

We require the following

Definition 7.1.2.1. Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function.

We define the *polar S-multiplicity of c* to be the order of pole

$$m_S(f, c) := -\mathrm{ord}_c\, Z_S(f, \cdot) \geq 1 \tag{7.1.2.1}$$

of the meromorphic function $Z_S(f, \cdot)$ on $\Omega(J_{F,S})$ at c when it has a pole there, and define $m_S(f, c) := 0$ if $Z_S(f, \cdot)$ has no pole at c. We define, for each integer $m \in \mathbf{Z}$:

$$a_m(f, c) \tag{7.1.2.2}$$

to be the mth coefficient of the Laurent series expansion centered at $c \in \Omega(J_{F,S})$.

Recall from §3.1.1 that the connected components of $\Omega(J_{F,S})$ are parametrized as $\chi|\cdot|^s_{F,S}$ for $\chi \in \widehat{J_{F,S}}$ a unitary base-point and $s \in \mathbf{C}$. Orders of poles and coefficients of principal parts are all defined relative to a parametrization, and are seen to be independent of the parametrization, which is why they are denoted as in (7.1.2.1) and (7.1.2.2) without reference to a choice of holomorphic parametrization of the connected components of $\Omega(J_{F,S})$.

We recall the definition of the meromorphic function ρ_S on $\Omega(J_{F,S})$ and the exact location of its poles. We have:

$$\rho_S(c) := \prod_{v \in S} \rho_v(c_v)$$

where c_v is the image of $c \in \Omega(J_{F,S})$ in $\Omega(F_v^\times)$, the Riemann surface of continuous quasi-characters of the local field F_v. The meromorphic function ρ_v is the explicit meromorphic function from [**CF**, Ch.XV,§§2.5], obtained from the local functional equations. We further refer the reader to (3.3.1.3) from Lemma 3.3.1.1.

Lemma 7.1.2.2. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character of $J_{F,S}$. Let S_χ be the finite set of finite places in S at which χ is ramified, and define $S_f := S \cap \Sigma_{F,f}$,*

Identify the connected component $\Omega(J_{F,S})(\chi)$ of χ with $\mathbf{C}$ via the parameterization $\chi|\cdot|^s_{F,S}$ (so s is well-defined up to an overall vertical shift under change of unitary base point χ).

The poles of ρ_S on $\Omega(J_{F,S})(\chi)$ are precisely points $c_{v,m,\chi}$ for $v \in S - S_\chi$ and certain integers m (depending on v) with $m \leq 0$ for Archimedean v, defined as follows:

(1) *for v Archimedean we write $\chi_v = \chi_0|\cdot|_v^{it_{\chi_v}}$ with a unique $t_{\chi_v} \in \mathbf{R}$ and a unitary χ_0 that is either trivial or the sign character of $F_v^\times = \mathbf{R}^\times$ for real v and $\chi_0 : z \mapsto (z/|z|)^k$ for a unique integer $k \in \mathbf{Z}$ for complex v, and define*

$$c_{v,m,\chi} = \chi|\cdot|_{F,S}^{m-it_{\chi_v}}$$

where m varies as follows depending on χ_0: $m \in -2\mathbf{N}$ when $\chi_0 = 1$ for real v, $m \in -1-2\mathbf{N}$ when $\chi_0 = \mathrm{sgn}$ for real v, and $m \in (|k|/2) - \mathbf{N}$ for complex v;

(2) *for v non-Archimedean (so χ is unramified at v: $v \notin S_\chi$) and q_v the size of the residue field at v, we write $\chi_v = |\cdot|_v^{it_{\chi_v}}$ with a unique $t_{\chi_v} \in [0, 2\pi/\log(q_v))$ and define*

$$c_{v,m,\chi} = \chi|\cdot|_{F,S}^{(2\pi i/\log(q_v))m-it_{\chi_v}}$$

for all $m \in \mathbf{Z}$.

Beware that for such poles c, the pole-order $m_S(c) = -\mathrm{ord}_c(\rho_S) \geq 1$ may exceed $-\mathrm{ord}_{c_v}(\rho_v)$ for the unique place $v \in S - S_\chi$ such that $c = c_{v,m,\chi}$. For example, there may be finite places $v, w \in S - S_\chi$ over the same rational prime p for which $F_v \simeq F_w$ and compatibly $\rho_v = \rho_w$. This matter will be revisited in §7.1.3.

PROOF. By definition, $\rho_S(c) = \prod_{v\in S}\rho_v(c_v)$, where c_v is the quasi-character of $F_v^\times$ obtained from c. The functions $\rho_v \in \Omega(F_v^\times)$ are tabulated explicitly on each connected component in [**CF**, Ch. XV, pp. 316-322]. Inspection shows that on the connected components of ramified quasi-characters for non-Archimedean v the function ρ_v is entire and non-vanishing (related to local root numbers of ramified characters), while in all other cases a zero of any of these never cancels a pole of another (not even the real parts ever coincide).

Thus, the set of poles of $\rho_S(\chi|\cdot|_{F,S}^s)$ is exactly the union of the sets of poles of the meromorphic functions ρ_v (composed with the analytic covering map $\Omega(J_{F,S}) \to \Omega(F_v^\times)$) for $v \in S - S_\chi$, and the pole order of ρ_S at such a pole agrees with that of the corresponding ρ_v for a uniquely determined v (depending on the pole under consideration). Since we have parameterized the connected component of χ in $\Omega(J_{F,S})$ via $\chi|\cdot|_{F,S}^s$, for such a point to be a pole of ρ_v with $v \in S - S_\chi$ we note that its restriction to $F_v^\times$ is $\chi_v|\cdot|_v^s$, so the possibilities for such s are exactly read off from Tate's formulas for the ρ_v's in all cases (Archimedean v on each connected component of $\Omega(F_v)$, and non-Archimedean v on the connected component of unramified quasi-characters in $\Omega(F_v)$). This yields exactly the list of points (in effect, the values for s) as claimed. □

The following, is the main Theorem of this section. We will be subsequently explain the connection with Sarnak's Theorem on the uniform distribution of long horocycles (see [**Sarnak**, §0,Thm.1]), a connection that will accompany us all the way through this Chapter until we find and describe the periodic orbits associated to the non-Archimedean places of the number field F, and show they serve a similar role as the periodic orbits of the horocycle flow from [**Sarnak**]. The higher rank version of the methods being presented in this work will generalize [**Sarnak**, §0,Thm.1], eventually rendering the analogy discussed at the end of this Chapter, sharper.

We let F be a number field, S a finite set of places of F containing all the Archimedean places. For a prime number p, we denote by S_p the subset of those finite places v of F contained in S such that the residue characteristic of the non-Archimedean local field F_v is p.

We denote by Π the set of all prime numbers, and by $\Pi(S) \subset \Pi$ the subset of the prime residue characteristics of the non-Archimedean local fields F_v for each $v \in S_f := \Sigma_{F,f} \cap S$.

THEOREM 7.1.2.3 (On Asymptotic Expansions as $|g|_{F,S} \to 0^+$). *Let F be a number field, S a finite set of places containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character. We have, for every real*

number $A > 0$:

(7.1.2.3)

$$\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(g) = \sum_{v\in S_\infty}\sum_{m\in\mathbf{N}_{>0},m<A}\sum_{1\leq k\leq m_S(f,c_{v,m,\chi})}\frac{(-1)^{k-1}\,a_{k-1}(f,c_{v,m,\chi})}{(k-1)!}\,\overline{\chi}(g)|g|_{F,S}^{m+it_v}\,(\log|g|_{F,S})^{k-1}$$
$$+\sum_{p\in\Pi(S)}\sum_{v\in S_p}\sum_{m\in\mathbf{Z}}\sum_{1\leq k\leq m_S(f,c_{v,m,\chi})}\frac{(-1)^{k-1}\,a_{k-1}(f,c_{v,m,\chi})}{(k-1)!}\,\overline{\chi}(g)|g|_{F,S}^{-\frac{2\pi im}{\log(q_v)}+it_v}\,(\log|g|_{F,S})^{k-1}$$
$$+O_A(|g|_{F,S}^A)$$

as $|g|_{F,S}\to 0^+$, where the series converges uniformly on the compact subsets of $J_{F,S}$, the implied constant depends only on the restriction of $Z_S(f,\cdot)$ to the relative vertical line $\sigma(c) = -A$, and where each $c_{v,m,\chi}$ is a pole of the meromorphic function ρ_S from (3.3.1.3) *(these poles are made explicit in Lemma 7.1.2.2).*

Before we take up the proof, we make an extended Remark to provide broader perspective of the meaning of the result.

REMARK 7.1.2.4 (On the poles of the non-Archimedean Euler factors of induced L-functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi\in\widehat{J_{F,S}}$ be a continuous character, and v a finite place of F contained in S at which χ is unramified.

We assume S is so large so as to arrange $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, so that we can think χ as a continuous character of C_F by composition of the inverse isomorphism of Hausdorff locally compact abelian groups $C_{F,S}\to C_F$, with the quotient $C_{F,S}\to J_{F,S}$. Upon doing this, we can refer to the meromorphic function $(1-\chi(\pi_v)|\pi_v|^s)^{-1}$ as the non-Archimedean local factor of the Hecke L-function $L(s,\chi)$ without ambiguity.

The real numbers t_v are obtained as follows. We choose a logarithm $\log\chi(\pi_v)$ of $\chi(\pi_v)$, where v is a finite place of F contained in S at which χ is unramified, and π_v is a local uniformizer for the local field F_v.

If the choice of branch of the logarithm is the principal one translated vertically upwards by $i\pi$, then t_v is obtained as $\frac{\log\chi(\pi_v)}{\log(q_v)}$ where q_v is $N(\mathfrak{p}_v) = |\pi_v|_v^{-1}$, with $\mathfrak{p}_v\subset\mathcal{O}_F$ the unique prime ideal of $\mathcal{O}_F$ associated to the finite place v.

The poles of the non-Archimedean local factor $(1-\chi(\pi_v)q_v^{-s})^{-1}$ of the Hecke L-function $L(s,\chi)$ are a torsor under $\mathbf{Z}/\frac{2\pi i}{\log(q_v)}$, and hence the general such pole is obtained by vertical translation under $\mathbf{Z}/\frac{2\pi i}{\log(q_v)}$, of the point $\chi|\cdot|_{F,S}^{-it_v}\in\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$.

We now choose an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. We note that if S_p is "saturated", i.e. if the prime number p is such that S contains all the finite places of F whose residue characteristic is p, then the collection of the points:

$$\chi|\cdot|_F^{-\frac{2\pi im}{\log(q_v)}+it_v},\ v\in S_p, m\in\mathbf{Z}$$

are the poles of the product of non-Archimedean local factors $\prod_{v\in S_p}(1-\chi(\pi_v)q_v^{-s})^{-1}$, which is in turn the Euler factor at the prime number p of the L-function $L(s,\rho)$, where ρ is the continuous finite dimensional complex representation of the Weil group $W_{\mathbf{Q}}$ (relative to the choice of algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ we have made) obtained by inducing the inflation of the character χ to $W_{F'/F}$, with F' the Galois closure of F.

This observation will be reprised in Chapter 8 (see the proof of Proposition 14.1.2.3).

We now look at Theorem 7.1.2.3 through the lens of Sarnak's paper on the uniform distribution of long horocycles (see [**Sarnak**, §0,Thm.1]).

We let M be a Riemann surface, and $T^1M\to M$ its unit tangent bundle. We equip M with its Poincaré metric, promoting it to a complete Riemannian manifold with constant negative curvature. We assume M is non-compact, with finite volume.

For $r\geq 0$ the finite number of inequivalent cusps, there exist r pairwise disjoint immersed submanifolds $P_k\subset T^1M$, $1\leq k\leq r$, whose union consists of all the periodic points of the horocyclic flow on T^1M.

For each cusp x_k, $1 \leq k \leq r$, and for each real number $t > 0$, $\nu^{(k)}(t)$ is the probability measure on the closed orbit of period t^{-1} of the horocyclic flow on T^1M assigning uniform mass with respect to arc-length to such closed orbit.

$\nu^{(k)}(t)$ is constructed in [**Sarnak**, §2,(2.1)] as an element of the strong dual of $\mathcal{D}(T^1M)$, by means of the Uniformization Theorem. Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ be a discrete subgroup acting properly discontinuously on the upper half-plane $\mathbf{H}$ equipped with the Poincaré metric, such that M is isometric to $\Gamma\backslash\mathbf{H}$.

Γ acts on $T^1\mathbf{H} \simeq \mathrm{PSL}_2(\mathbf{R})$ through tangent maps and we identify T^1M isometrically to $\Gamma\backslash T^1\mathbf{H}$. Since Γ acts on $\mathbf{H}$ and $T^1\mathbf{H}$ by isometries, the horocyclic flow on $T^1\mathbf{H}$ descends to the quotient T^1M, and in order to define $\nu^{(k)}(t)$ it is enough to do so for $k = 1$ on $T^1\mathbf{H}$ for then propagating the measure to the cusps other than the first, i.e. the one at $\infty \in \overline{\mathbf{H}}$, by push-forward along the Γ-action.

This is done in [**Sarnak**, §2,(2.1)] by integrating over a fundamental domain within the standard fundamental domain for the Γ-action on $T^1\mathbf{H}$, for the closed orbit of period $1/t$ on T^1M of the horocyclic flow, the generic automorphic function f on $T^1\mathbf{H}$.

The $J_{F,S}$-parametrized families of continuous functionals

$$a_m(\cdot,\chi|\cdot|_{F,S}^{\frac{2\pi i m}{\log(q_v)}-it_v})\,\overline{\chi}(g)|g|_{F,S}^{-\frac{2\pi i m}{\log(q_v)}+it_v}(\log|g|_{F,S})^k$$

are the direct analogs of the measures $\nu^{(k)}(t)$.

Throughout this Chapter, the Riemann surface M is replaced by the S-adeles $\mathbf{A}_{F,S}$ for all large enough S, the automorphic function is replaced by the arbitrary S-local θ-function $\Phi_{F,S}(f)$ for f a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$ (noting that the continuous functionals $a_m(\cdot,c)$ are $\mathcal{O}_{F,S}^\times$-invariant and hence are continuous on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))$; see Definition 5.5.1.1 and Theorem 5.5.1.2 of the previous Chapter, discussing S-local eigendistributions; see also Theorem 7.2.1.4 in this Chapter).

The parameter t in $\nu^{(k)}(t)$ is replaced by $\log|g|_{F,S}$.

Throughout [**Sarnak**, §2], Sarnak proves continuity on $\mathcal{D}(T^1M)$ of the Mellin transform of $\nu^{(k)}(t)$: continuity of $a_m(\cdot,c)$ and of formation of the S-idelic Mellin transform of the S-local θ-functions $\Phi_{F,S}(\cdot)$, is what we have discussed throughout Chapter 3 and Chapter 4 of this work.

Sarnak then studies the measures $\nu^{(k)}(t)$ for all small enough $t > 0$ by means of their Mellin transform, relating them to certain functional families of Eisenstein series and recasting them as residue functionals of the meromorphic vector valued functions $E^{(k)}(s)$ introduced throughout [**Sarnak**, §2] (see [**Sarnak**, §2,(2.2),p.724]) which, when evaluated on the generic automorphic function, are seen to be related to classical Eisenstein series in [**Sarnak**, §2,Prop.2.6,p.726], which gives access to Selberg's work informing us on their singularities and asymptotic behavior, as well as vector-valued functional equations (see [**Sarnak**, §2,Thm.2.7,p.726], [**Sarnak**, §2,Thm.2.11,p.729], and the family of functional equations in [**Sarnak**, §2,Thm.2.16,p.731], as well as the section on poles and residues in [**Sarnak**, §2.17]). We further refer the reader to [**Iwaniec**, Ch.6,§6.3,6.4] and [**Kubota**, Ch.IV,§4.4]. We further refer the reader to Zagier's [**Zagier**, §1].

This is what we have also done concerning the functionals

$$a_m(\cdot,c)$$

for $c \in \Omega(J_{F,S})$ and $m \in \mathbf{N}$, by analyzing the functional properties of the S-local θ-functions $\Phi_{F,S}(f)$ (the GL_1-analogs of the Eisenstein series arising in Sarnak's paper) through their S-idelic Mellin transforms $Z_S(f,c)$, from which $a_m(f,c)$ are obtained as coefficients of principal parts.

Following the dictionary being outlined above, Theorem 7.1.2.3 is then analogous to the main Theorem in Sarnak's paper, namely [**Sarnak**, §0,Thm.1], relating the measures $\nu^{(k)}(t)$ to the canonical Riemannian measure μ on T^1M, and to residue functionals $\mu_m^{(k)}$ (obtained in [**Sarnak**, §2] in a similar manner as our

S-local eigendistributions $a_m(\cdot, c)$), for all small enough $t > 0$, by the asymptotic expansion:

$$\nu^{(k)}(t) = \frac{\mu}{2\pi V} + \sum_{1\leq m\leq N} t^{1-s_m}\mu_m^{(k)} + o(t^{1/2}) \tag{7.1.2.4}$$

as $t \to 0^+$, for V the volume of M.

The meaning of the asymptotic expansion in (7.1.2.4) is that for each $f \in \mathcal{D}(T^1M)$, the evaluation on the element f of all the functionals featuring in each side of (7.1.2.4) obeys the asymptotic relation:

$$\langle \nu^{(k)}(t), f\rangle_{\mathcal{D}(T^1M),\mathcal{D}(T^1M)'} = \langle \frac{\mu}{2\pi V}, f\rangle_{\mathcal{D}(T^1M),\mathcal{D}(T^1M)'} + \sum_{1\leq m\leq N} t^{1-s_m}\langle \mu_m^{(k)}, f\rangle_{\mathcal{D}(T^1M),\mathcal{D}(T^1M)'} + o(t^{1/2}) \tag{7.1.2.5}$$

where the writing $\langle \mathcal{D}(T^1M), \mathcal{D}(T^1M)'\rangle$ refers to the duality of $\mathcal{D}(T^1M)$, the Hausdorff locally convex topological $\mathbf{C}$-vector space of test functions on T^1M, with its continuous dual. The $o(t^{1/2})$ (7.1.2.4) therefore refers to the rate of convergence to zero with respect to the topology of simple convergence on $\mathcal{D}(T^1M)$, i.e. the $\sigma(\mathcal{D}(T^1M)', \mathcal{D}(T^1M))$-topology.

Here the monomial functions t^{1-s_m} play the same role as $\overline{\chi}(g)|g|_{F,S}^{-\frac{2\pi im}{\log(q_v)}+it_v}(\log|g|_{F,S})^k$ in (7.1.2.3) of Theorem 7.1.2.3, and the asymptotic expansion for $\Phi_{F,S}(f)$ obtained in Theorem 7.1.2.3 is also as $|g|_{F,S} \to 0^+$, as is the case in [**Sarnak**, §0,Thm.1].

The measures $\mu_m^{(k)}$ in Sarnak's work are supported on the periodic loci P_k, and the geometric meaning of [**Sarnak**, §0,Thm.1] from (7.1.2.4) is that long horocycles (i.e. the periodic orbits contained within the immersed cylinders P_k, as their period $1/t$ tends to $+\infty$) tend to distribute themselves uniformly over T^1M with respect to μ (obtained upon endowing Γ with the counting measure and $T^1\mathbf{H}$ with the push-forward of the normalized Haar measure $\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta}{y^2}$ on $\mathrm{PSL}_2(\mathbf{R})$ by uniformization).

The uniform distribution of long horocycles bears on neighboring questions of arithmetic significance. As is often the case for continuous-time uniform distribution results, a discrete analog of Sarnak's Theorem exists. We refer the reader to [**AtChe**], where the authors describe a Poincaré section to the horocyclic flow and establish discrete-time versions of [**Sarnak**, §0,Thm.1] (see [**AtChe**, Thm.1.3]) where the horocyclic flow is discretized into the discrete flow generated by a self-map constructed by Boca-Cobeli-Zaharescu in [**BCZ**], in analogy with C. Series' work on a Poincaré section of the geodesic flow on the unit tangent bundle of the modular surface, as opposed to the horocyclic flow, discretized into the discrete flow generated by the Gauss' map (see the map T on [**Series**, §2,p.76] and [**Series**, §2,Thm.C], which is ergodic with respect to the Gauss' measure $A \mapsto \frac{1}{\log(2)}\int_A \frac{\mathrm{d}x}{1+x}$ on $(0,1)$, in analogy with [**AtChe**, Thm.1.2]) with applications to continued fractions.

It is possible to look at the asymptotic expansion from Theorem 7.1.2.3 too through the lens of ergodic theory (using Dirichlet's S-Unit Theorem and the lattice point-counting techniques underlying the proof of Theorem 2.4.2.1).

What we are most interested in, however, is the question of whether an analog of the periodic orbits and periodic loci P_k in [**Sarnak**, §2] exists in the context of our Theorem 7.1.2.3 too.

By the end of this Chapter, this question will be answered in the affirmative, exhibiting orbital quotient stacks arising as topological quotient stacks of certain closed loci within $\mathbf{A}_{F,S}$ which are responsible for the existence of poles of the S-idelic Zeta functions $Z_S(f,c)$, and which are in correspondence with divisors on the Riemann surface $\Omega(J_{F,S})$ under L_S from (5.1.1.2).

This leads to the finding of periodic orbits and fixed loci that realize the structures predicted by C. Deninger in his expectations on the existence of a functional approach to the problem of a Trace-Formula theoretic interpretation to Weil's Explicit Formulas.

They will indeed be underlying indirectly the trace calculation which is the main subject of study of this work.

At the end of this Chapter, we will be describing the periodic orbits alluded to above, and we will be explaining how they indeed serve the same purpose of the periodic loci P_k within T^1M in [**Sarnak**]: in a precise sense they serve, similarly as the P_k's do, as the *supports* of the continuous functionals $a_m(\cdot, \chi|\cdot|_{F,S}^{\frac{2\pi i m}{\log(q_v)} - it_v})$, whenever v is a non-Archimedean place contained in S.

We will now prove Theorem 7.1.2.3. The proof mirrors to some meaningful extent the proof of [**Sarnak**, §0,Thm.1], and rests on the S-local functional equation of (3.3.1.3) from Lemma 3.3.1.1, which plays the same role as the vector-valued functional equations of [**Sarnak**, §2,Thm.2.7,p.726] and [**Sarnak**, §2,Thm.2.16,p.731], as well as Mellin inversion and the Asymptotic Residue Theorem (see Theorem B.3.2.1 and Theorem B.2.1.7).

The main difference between Theorem 7.1.2.3 and [**Sarnak**, §0,Thm.1] is that countably many poles are involved in the statement and in (7.1.2.3), and hence applying the Asymptotic Residue Theorem requires knowledge of the existence of horizontal segments on which partial Euler products are kept under control.

This knowledge is provided to us by either Lemma 7.1.1.1, or, resting on the Kronecker-Weyl Theorem (see Theorem B.5.2.1), by Proposition B.5.3.3.

We now prove Theorem 7.1.2.3.

Proof of Theorem 7.1.2.3. We first choose $A \in (-1, 0)$.

We define the meromorphic function on the complex s-plane:

$$\Phi(s) := Z_S(f, \chi|\cdot|_{F,S}^s) \prod_{v \in S_f} (1 - q_v^{-s-it_v}) \prod_{v \in S_\infty} (s - s_v)^{m_v}$$

for $q_v := N(\mathfrak{p}_v)$ with $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$, corresponding to $v \in S_f$, and for s_v the unique pole, if any, of the Archimedean local factor $L_v(\chi_v|\cdot|_v^s)$ at v, located on the vertical line $i\mathbf{R}$, where $m_v \in \{0, 1\}$ accordingly (for example, if $\chi = 1$, then $\prod_{v \in S_\infty}(s - s_v)^{m_v} = s^{r_1+r_2}$. Recall that $s = 0$ is a pole of $Z_S(f, |\cdot|_{F,S}^s)$ of order $r_1 + r_2$, since the Γ-function has a simple pole at $s = 0$, and each such pole of the $r_1 + r_2$ Archimedean local L-functions contributes one order of pole to the quasi-character $|\cdot|_{F,S}^{s=0} = 1 \in \Omega(J_{F,S})(1)$).

We further define the meromorphic function on the complex s-plane:

$$\phi(s) := \prod_{v \in S_f} (1 - q_v^{-s-it_v})^{-1} \prod_{v \in S_\infty} (s - s_v)^{-m_v}.$$

The meromorphic function $Z_S(f, \chi|\cdot|_{F,S}^s)$ on the complex s-plane is the product:

$$Z_S(f, \chi|\cdot|_{F,S}^s) = \Phi(s)\phi(s).$$

The function $\Phi(s)$ has only removable singularities over $\mathbf{C}_{(-1/2,1/2)}$, and decays, uniformly over $\mathbf{C}_{(-1/2,1/2)}$, faster than any positive power of $|s|^{-1}$ as $|s| \to +\infty$ from within the vertical strip $\mathbf{C}_{(-1/2,1/2)}$. (Explicitly, recall $|\Gamma(\sigma + it)| = O(e^{-(\pi/2)|t|}|t|^{1/2})$ for $|\sigma| \leq 1/2$ as $|t| \to +\infty$.)

The function $\phi(s)$ is meromorphic over $\mathbf{C}_{(-1/2,1/2)}$ and has slow growth near its poles, in the sense of Definition B.3.1.1, because $\prod_{v \in S_\infty}(s - s_v)^{-m_v}$ does, being a rational function, and $\prod_{v \in S_f}(1 - q_v^{-s-it_v})^{-1}$ also does, by Lemma 7.1.1.1 applied to the choice $I = (-1/2, 1/2)$.[1]

Lemma 7.1.2.2 provides the location of the poles of $Z_S(f, |\cdot|_{F,S}^s)$ over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$ holomorphically parametrized by $\chi|\cdot|_{F,S}^s$ (for example, for $\chi = 1$, they are all contained within the subset:

$$\left\{ |\cdot|_{F,S}^s, s \in -\mathbf{N} \cup \bigcup_{\{q \in N(\mathfrak{p}_v) : v \in S_f\}} \frac{2\pi i\, \mathbf{Z}}{\log(q)} \right\}$$

of the connected component $\Omega(J_{F,S})(1)$).

[1] check if Definition B.3.1.1 is actually applicable since no control on growth of $|T_k|$'s in Lemma 7.1.1.1, or if it needs to be adjusted while retaining Asymptotic Residue Theorem

By the Asymptotic Residue Theorem (see Theorem B.3.2.1), applied to $\Phi(s) := \Phi_0(s)$ and $f(s)$ in loc.cit, chosen to be $\phi(s)$, (7.1.2.3) follows, upon performing the following calculation for $t > 0$:

$$\mathrm{Res}_{s=a}\left(\frac{t^{-s}}{(s-a)^m}\right) = \frac{(-1)^{m-1}}{(m-1)!}t^a(\log(t))^{m-1}$$

and choosing a continuous homomorphic section τ to the S-idelic norm, so as to rewrite $Z_S(f, \chi|\cdot|_{F,S}^s)$ over $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$ as the Mellin transform $M(\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)\circ\tau)(s+ix_0)$, for $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions on $\mathbf{R}_{>0}$, of the smooth function of $t \in \mathbf{R}_{>0}$ given by $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(\tau(t))$.

The convergence of the series in (7.1.2.3) is uniform on the compact subsets of $\mathbf{R}_{>0}$ because the asymptotic vanishing of the line integrals arising in the Asymptotic Residue Theorem, of $\Phi(s)\phi(s)$ evaluated on the horizontal segments $\mathrm{Im}(s) = T_k$ intersected with $\mathbf{C}_{(-1/2,1/2)}$, $\{T_k\}_{k\in\mathbf{Z}}$ being the sequence provided by Lemma 7.1.1.1, is uniform with respect to $t \in \mathbf{R}_{>0}$, since t^{-s} is bounded for $\mathrm{Re}(s) \in (-1/2, 1/2)$, and as t is constrained to vary over a compact subset of $\mathbf{R}_{>0}$.

To obtain the general case of (7.1.2.3), i.e. for any $A > 0$, it suffices to modify the function $\Phi(s)$ by multiplying by $\prod_{v\in S_\infty}(s - s_{v,m})^{m_S(f,\chi|\cdot|_{F,S}^{s_{v,m}})}$, and multiplying $\phi(s)$ by the reciprocal, where $s_{v,m}$ is the unique pole of the Archimedean local factor $L_v(\chi_v|\cdot|_v^s)$ contained within the interior of the vertical strip $\mathbf{C}_{[-m-1/2,-m+1/2]}$, for the finitely many integers $m \in [0, A] \cap \mathbf{N}$.

The new $\Phi(s)$ and $\phi(s)$ again satisfy the assumptions of the Asymptotic Residue Theorem (see Theorem B.3.2.1), and more terms of the first series in (7.1.2.3) (a finite sum, in fact) occur, the error term becoming $O_A(|g|_{F,S}^A)$ for the new $A > 0$.

The implied constant is indeed dependent only on $Z_S(f, \chi|\cdot|_{F,S}^s)$ restricted to the vertical line $\mathrm{Re}(s) = -A$ within $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$, holomorphically parametrized as $\{\chi|\cdot|_{F,S}^s, s \in \mathbf{C}\}$, because the error term is provided by $t^{-(-A)}$ multiplied by the L^1-norm of $Z_S(f, \chi|\cdot|_{F,S}^s)$ over $\mathrm{Re}(s) = -A$ with respect to the Lebesgue measure on such vertical line. □

The fact that the sums in (7.1.2.3) converge, and the fact that the functionals $a_m(\cdot, c)$ on $\mathcal{S}(\mathbf{A}_{F,S})$ (and indeed, as is to be discussed shortly, on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, see Theorem 7.2.1.4), is a first indication that the continuous functionals $a_m(\cdot, c)$ may serve as the coefficient functionals of a Schauder basis on a quotient of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$.

Theorem 7.1.2.3 serves as a first source of motivation for the search for such bases.

It is indeed possible to use Theorem 2.4.2.1 and Proposition 2.3.1.4 to obtain the stronger result that for all $g \in J_{F,S}$ such that $|g|_{F,S} < 1$, the right side of (7.1.2.3) converges uniformly with respect to $f \in \mathcal{S}(\mathbf{A}_{F,S})$ varying over a fixed bounded subset $\mathcal{B}$, in which case the implied constant in $O_A(|g|_{F,S}^A)$ depends on $\mathcal{B}$ and not on the specific $f \in \mathcal{B}$.

The asymptotic expansion therefore reads

$$\begin{aligned}\Phi_{F,S,\chi|_{J^1_{F,S}}}(\cdot)(g) = &\sum_{v\in S_\infty}\sum_{m\in\mathbf{N}_{>0}, m<A}\sum_{1\le k\le m_S(\cdot,c_{v,m,\chi})}\frac{(-1)^{k-1}\,a_{k-1}(\cdot,c_{v,m,\chi})}{(k-1)!}\,\overline{\chi}(g)|g|_{F,S}^{m+it_v}\,(\log|g|_{F,S})^{k-1}\\ &+\sum_{p\in\Pi(S)}\sum_{v\in S_p}\sum_{m\in\mathbf{Z}}\sum_{1\le k\le m_S(\cdot,c_{v,m,\chi})}\frac{(-1)^{k-1}\,a_{k-1}(\cdot,c_{v,m,\chi})}{(k-1)!}\,\overline{\chi}(g)|g|_{F,S}^{-\frac{2\pi im}{\log(q_v)}+it_v}\,(\log|g|_{F,S})^{k-1}\\ &+O_{A,\mathcal{B}}(|g|_{F,S}^A)\end{aligned}$$

the mode of convergence of the infinite sums being pointwise as $g \in J_{F,S}$ satisfies $|g|_{F,S} < 1$, and uniform over the fixed bounded subset $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$. Recall that $\Pi(S)$ denotes the finite set of prime numbers p such that there exists a finite place v in S whose residue characteristic is p.

The notation $O_{A,\mathcal{B}}(|g|_{F,S}^{A})$ means that the function of $g \in J_{F,S}$ defined as the difference of $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(g)$ and the finite and infinite sums indexed by the places in S, is, in absolute value, bounded above by $C_{\mathcal{B},A}|g|_{F,S}^{A}$ for all $g \in J_{F,S}$ with $|g|_{F,S} < 1$ and uniformly with respect to f varying over the fixed bounded subset $\mathcal{B} \subset \mathcal{S}(\mathbf{A}_{F,S})$.

This uniform form of Theorem 7.1.2.3 is closer to Sarnak's [**Sarnak**, §0,Thm.1], insofar as it highlights the functional nature of the result. In other words, it concerns the vector-valued Mellin anti-transform of $Z_S(\cdot, c)$, the S-idelic zeta function as a vector valued meromorphic function, i.e. we think $Z_S(\cdot, c)$ as a meromorphic function over $\Omega(J_{F,S})$ valued in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ (i.e. valued in tempered distributions on $\mathbf{A}_{F,S}$).

The coefficients $a_m(\cdot, c)$ are thus obtained as the vector-valued Laurent coefficients provided by the formulas in Grothendieck's Theorem C.5.3.2, as contour integrals in the sense of Pettis (see (C.5.3.3)).

Namely, upon fixing a holomorphic parametrization $\{\chi|\cdot|_{F,S}^{s}\}$ as in §3.1.1 for the connected component of χ in $\Omega(J_{F,S})$, for every integer $k \geq 1$ the Laurent coefficient operator $a_k(\cdot, c)$ satisfies

$$a_{k-1}(\cdot, c) = \frac{1}{2\pi i}\int_{\Gamma} \frac{Z_S(\cdot, \chi|\cdot|_{F,S}^{s})}{(s - s_c)^k}\, \mathrm{d}s$$

as continuous operators on $\mathcal{S}(\mathbf{A}_{F,S})$, for s_c corresponding to c under the chosen parametrization for any continuously differentiable simple closed curve $\Gamma \subset \mathbf{C}$ enclosing s_c.

7.1.3. Polar Multiplicities. The following Theorem discusses the multiplicities of the poles of the local Euler factors of the Hecke L-function $L(c)$, across the Riemann surface $\Omega(C_F)$.

We refer to quasi-characters of the form

$$c = \chi_0|\cdot|_F^{s}$$

for some $s \in \mathbf{C}$ and for χ_0 of finite order, as quasi-characters of *Galois type*. If $s \in i\mathbf{R}$, we say $\chi_0|\cdot|_F^{it}$ is a character of Galois type.

Remark 7.1.3.1. Let F be a number field, and fix S_0 a finite set of finite places of F, χ a continuous character of C_F, and S_χ the finite set of finite places of F at which χ is ramified.

Let ε_p denote the arbitrary zero of the holomorphic function on the complex s-plane:

$$\prod_{v|p, v\in S_0-(S_0\cap S_\chi)} (1 - q_v^{-s-it_v})$$

where $q_v := N(\mathfrak{p}_v)$ is the norm of the prime ideal $\mathfrak{p}_v \subset \mathcal{O}_F$ corresponding to v. Since $v \mid p$, q_v is a power of p.

In particular, ε_p is always of the form:

$$\frac{2\pi i k}{\log(q)} - it_v$$

for q a power of p occurring as q_v for some $v \mid p$ in S_0, $k \in \mathbf{Z}$, and $t_v \in \mathbf{R}$ a unique real number such that $t_v \in [0, 2\pi/\log(q))$, and $\chi_v = |\cdot|_v^{it_v}$ as continuous unitary characters of $F_v^\times$.

The highest possible multiplicity of a zero of $\prod_{v|p, v\in S_0-(S_0\cap S_\chi)}(1 - q_v^{-s-it_v})$ is $[F : \mathbf{Q}]$, the degree of the number field over the field of the rational numbers.

This multiplicity is attained if p splits completely in F, all finite places v of F over p are contained in S_0, χ is unramified at each such v, and $\chi_v(\pi_v) = 1$ for each $v \mid p$, in which case

$$\prod_{v|p, v\in S_0} (1 - q_v^{-s-it_v}) = (1 - p^{-s})^{[F:\mathbf{Q}]}. \tag{7.1.3.1}$$

For any continuous character $\chi \in \widehat{C_F}$, there exist infinitely many primes p such that p splits completely in F and such that for each $v \mid p$, χ is unramified at v.

By Chebotarev, the density of the set of prime numbers p such that p splits completely in F, is positive. For χ of the form $\chi_0|\cdot|_F^{ix}$, for some $x \in \mathbf{R}$, with χ_0 of finite order (i.e. for χ of Galois type), there exist infinitely many prime numbers such that (7.1.3.1) holds for all large enough S_0.

We begin to note that the only pole that reciprocal non-Archimedean Euler factors of Hecke L-functions are allowed to have in common, is the "origin" of the connected component of $\Omega(C_F)$ parametrized by $\chi|\cdot|_F^s$, at least when χ is of finite order. Similarly, this remains the case if χ is of Galois type, i.e. of the form $|\cdot|_F^{ix_0}\chi_0$ for χ_0 of finite order.

This phenomenon is specific to number fields.

We further refer the reader to §14.2, for the differences with the case of global function fields of positive characteristic, where the analog of Theorem 7.1.3.2 below does not hold.

THEOREM 7.1.3.2 (On Polar Multiplicities). *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

Let $\chi \in \widehat{C_F}$ be a continuous character, unramified outside S. Denote by χ also the unique continuous character of $J_{F,S}$ obtained from the restriction of χ to $C_{F,S}$, by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance.

The polar multiplicities of ρ_S on the connected component of χ in $\Omega(J_{F,S})$ from Lemma 7.1.2.2 satisfy:

(1) *For $v_1, v_2 \in S$ two finite places with distinct residue characteristics at which χ is unramified, the local Euler factors $L(\chi_{v_1}|\cdot|_{v_1}^s)$ and $L(\chi_{v_2}|\cdot|_{v_2}^s)$ of the Hecke L-function have at most one pole s in common.*
(2) *For v Archimedean, $m_S(c_{v,0,\chi}) \leq |S|$, $m_S(c_{v,m,\chi}) \leq r_1 + r_2$ if m is a negative even integer, $m_S(c_{v,m,\chi}) \leq r_2$ if m is a negative odd integer. (In particular, $m_S(c_{v,m,\chi}) \leq [F:\mathbf{Q}]$ for $m < 0$ and Archimedean v.)*
(3) *Let $S_0 = S_f - S_\chi$ be the set of finite places in S at which χ is unramified. If $v \in S_0$ then $m_S(c_{v,m,\chi}) \leq |S|$ for all m, and as we vary across all such points $c = c_{v,m,\chi} \in \Omega(J_{F,S})(\chi)$ we have $m_S(c) \leq [F:\mathbf{Q}]$ with at most $r_1 + r_2 + |S_0|(|S_0|-1)/2$ exceptions.*
   (a) *If $\chi = \chi_0|\cdot|_F^{ix}$ for $x \in \mathbf{R}$ and χ_0 of finite order (i.e., χ is of Galois type) then the only pole with order exceeding $[F:\mathbf{Q}]$ is $s = -ix$.*
   (b) *If $\chi = \chi_0|\cdot|_F^{s_0}$ for $s_0 \in \mathbf{C}$ and χ_0 algebraic then assuming Schanuel's Conjecture the only points in $\Omega(J_{F,S})(\chi)$ where ρ_S can have a pole order exceeding $[F:\mathbf{Q}]$ are the imaginary axis points on the horizontal lines of Archimedean poles and at $s = -\mathrm{Im}(s_0)i$. Unconditionally for such χ, the only point on $\Omega(J_{F,S})(\chi)$ where the pole order can exceed $2[F:\mathbf{Q}]$ is $s = -\mathrm{Im}(s_0)i$.*

PROOF. (1): If s is a common pole then $\chi_{v_1}(\pi_{v_1})q_{v_1}^{-s} = 1 = \chi_{v_2}(\pi_{v_2})q_{v_2}^{-s}$, so $q_{v_1}^s = \chi_1(\pi_{v_1})$ and $q_{v_2}^s = \chi_2(\pi_{v_2})$ for uniformizers π_{v_k} at v_k. The condition $q_{v_k}^s = \chi_k(\pi_{v_k})$ determines s up to adding an element of $(2\pi i/\log(q_{v_k}))\mathbf{Z}$. Since $\log(q_{v_k})$ is a positive integer multiple of $\log(p_k)$, so the $\log(q_{v_k})$'s are $\mathbf{Q}$-linearly independent, the two equations $q_{v_k}^s = \chi_{v_k}(\pi_{v_k})$ determine s uniquely (if it exists at all).

(2) Since ρ_S is a product of $|S|$ meromorphic functions whose poles are all simple, it has pole-order at most $|S|$ at any of its poles. In particular $m_S(c_{v,0,\chi}) \leq |S|$. The non-Archimedean factors ρ_v have all poles on the imaginary axis in $\Omega(J_{F,S})(\chi)$ (since if $q_v^s = \chi_v(\pi_v) \in U(1)$ then $\mathrm{Re}(s) = 0$), whereas the Archimedean factors ρ_v have poles in the half-plane to the left of that axis.

Consider an Archimedean $v \in S$. If m is a negative even integer then $c_{v,m,\chi}$ has polar multiplicity at most multiplicity the number of Archimedean local factors of ρ_S, which is the number $r_1 + r_2$ of Archimedean places of F. For m odd, $c_{v,m,\chi}$ can only occur as a pole of ρ_v for complex v, so its multiplicity as a pole of ρ_S is bounded above by the number r_2 of complex places of F.

(3). Suppose for some $c_{v,m,\chi}$ with non-Archimedean $v \in S_0 = S_f - S_\chi$, the pole-order of ρ_S exceeds $[F:\mathbf{Q}]$. Since there are at most $[F:\mathbf{Q}]$ places of F with the same residue characteristic p as v, for such excess pole order to occur it must also be a pole of ρ_w for some Archimedean w or some non-Archimedean w with residue characteristic different from p. If there is a common pole with ρ_w for an Archimedean w then the pole is the rightmost of the Archimedean poles in $\Omega(J_{F,S})(\chi)$. There are at most $r_1 + r_2$ horizontal lines on which

the Archimedean local factors have their poles (the number of such distinct lines depends on the infinity-type of χ). Each such line has only its rightmost polar point on the imaginary axis, so setting aside those points (which have meaning independently of v), we aim to show at most $|S_0|(|S_0|-1)/2$ such points $c_{v,m,\chi}$ can arise.

There are at most $|S_0|(|S_0|-1)/2$ pairs of places v, w in S_0 with distinct residue characteristics, so it suffices to show that for any such pair there is at most one common pole for ρ_v and ρ_w. The relation of ρ_v to $L(\chi_v|\cdot|_v^s)$ reduces this "at most one" assertion to (1).

(3.a): We may and do reparameterize the connected component by a pure-imaginary translation so the unitary base point is $\chi = \chi_0$ (i.e., $x = 0$). Suppose ρ_S has pole-order exceeding $[F : \mathbf{Q}]$, so by (2) it occurs as a pole of ρ_v and ρ_w for some non-Archimedean v and some place $w \neq v$ that either is Archimedean or is non-Archimedean with a residue characteristic distinct from v. In the case of Archimedean w, the pole would have to be at $s = 0$ (as desired) since this is the only pole of an Archimedean local factor which occurs on the imaginary axis (where the non-Archimedean local factors have their poles): the infinity type for finite-order χ_0 puts all Archimedean poles on the same horizontal line.

Thus, we can assume v, w are as in (1), in which case the common pole s satisfies $q_v^s = \chi_0(\pi_v)$ and $q_w^s = \chi_0(\pi_w)$. These character values are roots of unity, so for a sufficiently divisible integer $N > 0$ we have $q_v^{Ns} = 1 = q_w^{Ns}$, forcing $Ns = 0$ (and so $s = 0$, as desired) since v and w have distinct residue characteristics.

(3.b): We may and do reparameterize the connected component by a pure-imaginary translation so the unitary base point is $\chi = \chi_0|\cdot|_F^{s_0}$ with $s_0 \in \mathbf{R}$. If there is a point c at which ρ_S has pole-order exceeding $[F : \mathbf{Q}]$, then as in the argument for (3.a) it occurs as a pole of ρ_v and ρ_w for some non-Archimedean v and some place $w \neq v$ that either is Archimedean or is non-Archimedean with a residue characteristic distinct from v. We'll use Schanuel's Conjecture to rule out the possibility of such v and w both non-Archimedean, so then such a common pole would have to be one of the points where the imaginary axis meets the finitely many lines of Archimedean poles.

The condition on v and w implies $q_v^{-s-s_0}\chi_0(\pi_v) = 1 = q_w^{-s-s_0}\chi_0(\pi_w)$ with $s \in i\mathbf{R}$. Since $s_0 \in \mathbf{R}$, if $s \neq 0$ then $\{1, s + s_0\}$ is $\mathbf{Q}$-linearly independent. Since $\log(q_v)$ and $\log(q_w)$ are $\mathbf{Q}$-linearly independent, the Four Exponentials Theorem (a consequence of Schanuel's Conjecture) would imply that one of the numbers q_v, q_w, $q_v^{s+s_0} = \chi_0(\pi_v)$, or $q_w^{s+s_0} = \chi_0(\pi_w)$ are transcendental. But $\chi_0(\pi_v)$ and $\chi_0(\pi_w)$ are algebraic since χ_0 is an algebraic Hecke character, so this is a contradiction. Thus, necessarily $s = 0$.

Suppose instead there is a pole $c \in \Omega(J_{F,S})(\chi)$ for ρ_S at which the pole-order exceeds $2[F : \mathbf{Q}]$. If this isn't the unique Archimedean pole on the imaginary axis, we can get an unconditional contradiction via the Six Exponentials Theorem (which is unconditional): there would have to be three non-Archimedean places v, v', v'' with pairwise distinct residue characteristics such that $q_v^{s+s_0} = \chi_0(\pi_v)$, $q_{v'}^{s+s_0} = \chi_0(\pi_{v'})$, and $q_{v''}^{s+s_0} = \chi_0(\pi_{v''})$ with $s \in i\mathbf{R}$. If $s \neq 0$ then once again $\{1, s + s_0\}$ is $\mathbf{Q}$-linearly independent, as are $\{\log(q_v), \log(q_{v'}), \log(q_{v''})\}$, so now by the Six Exponentials Theorem at least one among $q_v, q_{v'}, q_{v''}, q_v^{s+s_0}, q_{v'}^{s+s_0}, q_{v''}^{s+s_0}$ is transcendental. This is again a contradiction, so necessarily $s = 0$. □

From Theorem 7.1.3.2 and Theorem 7.1.2.3, we deduce the following:

COROLLARY 7.1.3.3. *Let F be a number field, S a finite set of places of F containing the unique Archimedean place. Let $\chi \in \widehat{J_{F,S}}$, and $f \in \mathcal{S}(\mathbf{A}_{F,S})$. There exists a real number $C > 0$ depending on S, χ and f, such that:*

$$|\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(g)| \leq C_{S,\chi,f}\,|\log|g|_{F,S}|^{|S|-1} + O_{S,\chi,f}(|\log|g|_{F,S}|^{m-1}) \tag{7.1.3.2}$$

for some integer $m < |S|$, and for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ as $|g|_{F,S} \to 0^+$, where the implied constant in (7.1.3.2) *depends on S, χ and f, and on the poles of the meromorphic function ρ_S from* (3.3.1.3) *contained within $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$.*

The integer $m \geq 0$ equals the highest order of pole of $Z_S(f, \chi|\cdot|_{F,S}^s)$ that is strictly less than the highest order of pole of $Z_S(f, \chi|\cdot|_{F,S}^s)$.

PROOF. From Theorem 7.1.2.3 we may choose $A > 0$, and $m = 0$ in (7.1.2.3), calling:

$$C := \sup_{v \in S} \max_{1 \leq m \leq |S|} \left| \frac{(-1)^{m-1} a_{m-1}(f, \chi|\cdot|_{F,S}^{s_0})}{(m-1)!} \right|$$

where s_0 is the pole of highest order of $Z_S(f, \chi|\cdot|_{F,S}^s)$.

The estimate in (7.1.3.2) follows from (7.1.2.3) upon calling $m_0 \in \mathbf{N}$ the smallest integer such that

$$m_S(f, c) \leq m_0$$

for every pole c of $Z_S(f, c)$ of order less than the order of pole of $\chi|\cdot|_{F,S}^{s_0}$, which, in turn, satisfies $m_0 \leq |S|$. □

We refer the reader to Theorem 2.4.2.1, where a version of the estimate from (7.1.3.2), with implied constants that are uniform on bounded subsets, is discussed.

We further refer the reader to [Co, §VII,(6),p.30], where this type of estimates is alluded to, and a lattice-point argument in a similar vein as the proof of Theorem 2.4.2.1 is suggested.

7.1.4. Spectral bi-orthogonal sequences. Using Theorem 6.3.2.1, we can now describe the orthogonal of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$ within $A\mathcal{S}(\mathbf{C})$ for any $\omega \in \widehat{C_F^1}$.

We denote by $\mathscr{O}(\mathbf{C})$ the $\mathbf{C}$-vector space of all holomorphic functions on the complex s-plane, equipped with the Hausdorff locally convex topology of uniform convergence on compact subsets, promoting it to a nuclear Fréchet space.

SITUATION 7.1.4.1. We choose a continuous homomorphic section τ to the idelic norm.

We recall that the Mellin transform $Z : \mathcal{S}_0(\mathbf{A}_F) \to A\mathcal{S}(\mathbf{C})$ defined as $Z(f, \chi|\cdot|_F^s)$, for χ varying over a choice of representative continuous characters of C_F for the continuous characters of the compact Hausdorff abelian closed subgroup $C_F^1 \subset C_F$, factors through the continuous linear map:

$$Z : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \to A\mathcal{S}(\mathbf{C}) \tag{7.1.4.1}$$

sending $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ to $Z(\phi, s) := \int_{\mathbf{R}_{>0}} t^{s+ix_0} \phi(\tau(t)) \frac{\mathrm{d}t}{t}$, i.e. $Z(\phi, s) = Z(f, \chi|\cdot|_F^s)$ for $\phi = \Phi_{F,\chi|_{C_F^1}}(f)$, and $x_0 \in \mathbf{R}$ is the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$.

THEOREM 7.1.4.2 (Spectral bi-orthogonal sequences). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. We adopt the notation and choices from Situation 7.1.4.1.*

(1) *The orthogonal Z_χ of the ideal $(s + ix_0)^{\delta_\chi}(1 - s - ix_0)^{\delta_\chi} L(s, \chi)\, \mathscr{O}(\mathbf{C}) \subset \mathscr{O}(\mathbf{C})$ with respect to the topology of uniform convergence on compact subsets of the complex s-plane, with $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and $\delta_\chi = 0$ otherwise, is linearly and topologically isomorphic to the locally convex product:*

$$\prod_{\rho : L(\rho, \chi) = 0} J_\rho \tag{7.1.4.2}$$

where J_ρ is the finite dimensional $\mathbf{C}$-vector space:

$$J_\rho := \bigoplus_{0 \leq m \leq N(\rho) - 1} \mathbf{C}\, \delta_\rho^{(m)} \tag{7.1.4.3}$$

where $N(\rho)$ is the multiplicity of ρ as a zero of $L(s, \chi)$, and $\delta_\rho^{(m)} \in \mathscr{O}(\mathbf{C})'_\beta$ is the continuous functional:

$$\delta_\rho^{(m)}(\phi) := (\partial_s^m \phi)(\rho). \tag{7.1.4.4}$$

(2) *The continuous linear inclusion operator $A\mathcal{S}(\mathbf{C}) \to \mathscr{O}(\mathbf{C})$ induces a continuous linear map:*

$$U : Z_\chi \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp \tag{7.1.4.5}$$

where the orthogonal is formed with respect to the duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot\circ L,\cdot\rangle_{ZH,A\mathcal{S}(\mathbf{C})}$, *where* ZH *denotes the image in* $A\mathcal{S}(\mathbf{C})$ *under the operator* Z *from* (7.1.4.1) *of* $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$. *The continuous linear operator* U *from* (7.1.4.5) *is injective with dense image, and for every zero of* $L(s,\chi)$ *and every integer* $0\leq m\leq N(\rho)-1$, *we have:*

$$U(\delta_\rho^{(m)})(\phi)=\frac{1}{2\pi i}\int_{\mathbf{R}_{>0}}\left(\int_{-i\infty}^{+i\infty}t^{-s}\phi(s)\,\mathrm{d}s\right)t^{\rho+ix_0}(\log(t))^m\,\frac{\mathrm{d}t}{t}. \tag{7.1.4.6}$$

(3) *Any enumeration of the set of continuous functionals* $\{U(\delta_\rho^{(m)})\}$, *indexed by the pairs* (ρ,m) *as* ρ *varies over the zeros of* $L(s,\chi)$ *and, for each* ρ, m *varies over the integers* $0\leq m\leq N(\rho)-1$, *admits a bi-orthogonal sequence in*

$$\mathscr{H}_F(\chi|_{C_F^1}) := \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}.\mathcal{S}_0(\mathbf{A}_F)) \tag{7.1.4.7}$$

In particular, it is topologically free.

(4) *Let:*

$$H_\chi := \left\{\frac{1}{m!}U(\delta_\rho^{(m)})\right\}_{(\rho,m)}\subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))^\perp \tag{7.1.4.8}$$

be the set of continuous functionals from (7.1.4.6) *of Theorem 7.1.4.2, where* $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))^\perp$ *is endowed with the strong topology on the continuous dual of the quotient of* $\mathcal{S}(C_F)(\chi|_{C_F^1})$ *by* $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$.

Then H_χ *is equicontinuous and bounded.*

Proof. (1) : By [**Bar**, Thm.2], we have, for every $s\in\mathbf{C}$ and upon defining

$$\Lambda(s,\chi) := \left(\frac{1}{2}(s+ix_0)(s-1-ix_0)\right)^{\delta_\chi}L(s,\chi)$$

where $\delta_\chi=0$ if $\chi|_{C_F^1}$ is non-trivial, and $\delta_\chi=1$ if $\chi|_{C_F^1}=1$:

$$\Lambda(s,\chi)=\Lambda(0,\chi)\prod_{\rho\in\mathbf{C}:L(\rho,\chi)=0}{}'\left(1-\frac{s}{\rho}\right) \tag{7.1.4.9}$$

where the convergence of the product on the right side ($\prod'$ meaning that the factors are indexed by the pairs of conjugate zeros, counted with multiplicity, whose imaginary part has absolute value $\leq T$, for $T\in\mathbf{R}_{>0}$, and the product is the limit of the resulting finite products as $T\to+\infty$) is uniform on the compact subsets of the complex s-plane (note that in [**Bar**, Thm.2] the exponent δ_χ vanishes if and only if χ is the principal character: any unitary Hecke character χ such that $\chi|_{C_F^1}=1$ is of the form $\chi=|\cdot|_F^{ix_0}$, and hence $\chi|\cdot|_F^{-ix_0}$ is trivial. We are using [**Bar**, Thm.2] upon replacing χ with $\chi|\cdot|_F^{-ix_0}$).

The quotient $\mathbf{C}$-linear space:

$$\mathscr{O}(\mathbf{C})/\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}) \tag{7.1.4.10}$$

is Hausdorff, since $\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C})$ is closed in $\mathscr{O}(\mathbf{C})$, and it is complete, since $\mathscr{O}(\mathbf{C})$ is a Fréchet space, and Hausdorff quotients of Fréchet spaces are complete by [**Scha**, Ch.I,§6,6.3,p.31].

Its continuous dual is linearly and topologically isomorphic to Z_χ by [**NB**, Thm.8.12.1(a)] and [**NB**, Thm.8.12.2(b)].

We define:

$$K_\rho := \mathbf{C}[s]/((1-s/\rho)^{N(\rho)}(1-s/\overline{\rho})^{N(\rho)}) \tag{7.1.4.11}$$

equipped with the Euclidean topology.

K_ρ is linearly and topologically isomorphic to $\mathscr{O}(\mathbf{C})/((1-s/\rho)^{N(\rho)}(1-s/\overline{\rho})^{N(\rho)})\,\mathscr{O}(\mathbf{C})$. Indeed, calling I_ρ the ideal of holomorphic functions on the complex s-plane generated by $((1-s/\rho)^{N(\rho)}(1-s/\overline{\rho})^{N(\rho)})$, we have:

$$I_\rho = (((1-s/\rho)^{N(\rho)}(1-s/\overline{\rho})^{N(\rho)})\mathbf{C}[s])\otimes_{\mathbf{C}[s]}\mathscr{O}(\mathbf{C})$$

since $\mathscr{O}(\mathbf{C})$ is $\mathbf{C}[s]$-flat as an algebra over $\mathbf{C}[s]$ (in fact, faithfully flat). This gives injectivity of $K_\rho \to \mathscr{O}(\mathbf{C})/I_\rho$.

Since $\mathbf{C}[s]$ is dense in $\mathscr{O}(\mathbf{C})$ by [**Cnw-b**, Ch.VIII,Cor.1.15,p.200], then K_ρ is dense in $\mathscr{O}(\mathbf{C})/I_\rho$, being the quotient maps $\mathbf{C}[s]\to K_\rho$ and $\mathscr{O}(\mathbf{C})\to\mathscr{O}(\mathbf{C})/I_\rho$ open. Since K_ρ is finite dimensional, it carries the subspace topology inherited from $\mathscr{O}(\mathbf{C})/I_\rho$, which is the Euclidean topology, being K_ρ finite dimensional. Being K_ρ complete, the map $K_\rho\to\mathscr{O}(\mathbf{C})/I_\rho$ is surjective.

As a $\mathbf{C}$-vector space, the quotient $\mathscr{O}(\mathbf{C})/\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C})$ is linearly isomorphic to the countable product of the K_ρ, i.e. to $\prod_{\rho\in\mathbf{C}:L(\rho,\chi)=0}K_\rho$.

To see this, we note that since $\mathbf{C}$ is Stein, then upon denoting by $\mathscr{J}\subset\mathscr{O}$ the ideal sheaf globally generated by $\Lambda(s,\chi)$, we have

$$\check{\mathrm{H}}(\mathbf{C},\mathscr{O}/\mathscr{J}) = \mathrm{H}^1(\mathbf{C},\mathscr{O}/\mathscr{J}) = 0$$

and hence upon choosing a locally finite countable open cover $\{B_m\}_{m\geq 0}$ of the complex s-plane by open balls each of which contains at most one of the zeros ρ of $L(s,\chi)$, then, upon denoting by $B_{\underline{k}}$ the intersection $\bigcap_{m\in\underline{k}}B_m$ for $\underline{k}$ varying over all non-empty finite subsets of the nonnegative integer, we deduce that $\mathrm{H}^0(B_{\underline{k}},\mathscr{J})$ is trivial if $\underline{k}$ contains at least two elements, and it is either trivial or K_ρ if $\underline{k}$ contains one element exactly. This computes

$$\mathscr{O}(\mathbf{C})/\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}) = \check{\mathrm{H}}^0(\mathbf{C},\mathscr{O}/\mathscr{J})$$

to be $\prod_{\rho:L(\rho,\chi)=0}K_\rho$. The quotient topology $\mathscr{O}(\mathbf{C})/\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C})$ is the product topology, since any compact subset $K\subset\mathbf{C}$ is contained in the union of at most finitely many of the B_m, and hence the linear bijection $\mathscr{O}(\mathbf{C})/\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C})\to\prod_{\rho:L(\rho,\chi)=0}K_\rho$ is continuous. Being source and target Fréchet spaces, it is a linear topological isomorphism by [**NB**, Thm.14.4.5(b)].

To compute the orthogonal of $\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C})$ it is therefore enough to compute the continuous dual of $\prod_{\rho\in\mathbf{C}:L(\rho,\chi)=0}\mathbf{C}[s]/(s-\rho)^{N(\rho)}$ with respect to the product topology: by [**Tre**, Part II,Ch.22,Thm.22.1(b)], this is exactly the locally convex direct sum of the finite dimensional $\mathbf{C}$-vector spaces:

$$H_\rho := \bigoplus_{0\leq m\leq N(\rho)-1}\mathbf{C}\,D_\rho^m$$

where D_ρ^m is the composition of the formal mth derivative on polynomials in $\mathbf{C}[s]$ followed by the evaluation-at-ρ of the resulting polynomial function of $s\in\mathbf{C}$.

Each such functional D_ρ^m is the image of $\delta_\rho^{(m)}$ under the linear isomorphism $Z_\chi = (\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp\to(\prod_{\rho\in\mathbf{C}:L(\rho,\chi)=0}\mathbf{C}[s]/(s-\rho)^{N(\rho)})'$, which provides the desired description of Z_χ as a $\mathbf{C}$-vector space.

The subspace topology on $(\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp$ inherited from the strong topology on $\mathscr{O}(\mathbf{C})'_\beta$ promotes the continuous linear bijection $(\Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp\to(\prod_{\rho\in\mathbf{C}:L(\rho,\chi)=0}\mathbf{C}[s]/(s-\rho)^{N(\rho)})'_\beta$ to a linear topological isomorphism.

(2) : Since every element $\phi\in AS(\mathbf{C})$ is the Mellin transform of some Schwartz-Bruhat function $f\in\mathcal{S}(\mathbf{R}_{>0})$ with exponential decay (i.e. such that f decays more rapidly than any positive power of $e^{-|\log(t)|}$ as $|\log(t)|\to+\infty$, see Definition B.2.1.1), and we have, by Fourier Inversion:

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}t^{-s}\phi(s)\,\mathrm{d}s$$

then we compute:

$$\begin{aligned}\delta_\rho^{(m)}(\phi) &= (\partial_s^m\phi)(\rho)\\ &= \left(\partial_s^m\int_{\mathbf{R}_{>0}} f(t)t^s\,\frac{\mathrm{d}t}{t}\right)|_{s=\rho}\\ &= \left(\int_{\mathbf{R}_{>0}} f(t)t^s(\log(t))^m\,\frac{\mathrm{d}t}{t}\right)|_{s=\rho}\\ &= \frac{1}{2\pi i}\int_{\mathbf{R}_{>0}}\left(\int_{-i\infty}^{+i\infty} t^{-s}\phi(s)\,\mathrm{d}s\right)t^\rho(\log(t))^m\,\frac{\mathrm{d}t}{t}\end{aligned}$$

and hence (7.1.4.6) holds. Denseness of the range of U follows from the observation that, by Theorem 6.3.2.1(1), the functionals $U(\delta_\rho^{(m)})$ separate the points of $A\mathcal{S}(\mathbf{C})/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$.

(3) : We consider, for ρ a zero of $\Lambda(s,\chi)$, of multiplicity $N(\rho)$, and for m an integer such that $0\leq m\leq N(\rho)-1$, the meromorphic function on the complex s-plane:

$$\phi_{\rho,m}(s) := \frac{1}{(s-\rho)^{N(\rho)-m}}\Lambda(s,\chi). \tag{7.1.4.12}$$

Since $0\leq m\leq N(\rho)-1$, then $\phi_{\rho,m}$ has only removable singularities. We let $\widetilde{\phi}_{\rho,m}$ be the unique holomorphic extension of $\phi_{\rho,m}$ to the complex s-plane, and rename it as $\phi_{\rho,m}$.

Since ρ can be located within the vertical strip $\mathbf{C}_{[0,1]}$ only, we may fix an arbitrary real number $\varepsilon>0$ and an integer $N\geq 0$, and consider the two line integrals, for $s\in\mathbf{C}_{(0,1)}$:

$$\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\frac{z^N\phi_{\rho,m}(z)}{z-s}\,\mathrm{d}z,\quad \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{z^N\phi_{\rho,m}(z)}{z-s}\,\mathrm{d}z. \tag{7.1.4.13}$$

Since $\phi_{\rho,m}$ is entire, by the Asymptotic Residue Theorem (see Theorem B.3.2.1), which applies since $z^N(z-\rho)^{m-N(\rho)}/(z-s)$ is meromorphic and with slow growth near its poles (see Definition B.3.1.1), and $\Lambda(s,\chi)$ decays faster than any positive power of $|s|^{-1}$ uniformly over the vertical strips of bounded width of the complex s-plane, we obtain:

$$|s^N\phi_{\rho,m}(s)| \leq \frac{1}{2\pi}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\left|\frac{z^N\phi_{\rho,m}(z)}{z-s}\right||\mathrm{d}z| + \frac{1}{2\pi}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left|\frac{z^N\phi_{\rho,m}(z)}{z-s}\right||\mathrm{d}z| \tag{7.1.4.14}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|-\varepsilon+iy|^N|\phi_{\rho,m}(-\varepsilon+iy)|}{|-\varepsilon+iy-s|}\,\mathrm{d}y + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\varepsilon+iy|^N|\phi_{\rho,m}(\varepsilon+iy)|}{|\varepsilon+iy-s|}\,\mathrm{d}y \tag{7.1.4.15}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|-\varepsilon+iy|^N|\Lambda(-\varepsilon+iy,\chi)|}{|-\varepsilon+iy-s|\,|-\varepsilon+iy-\rho|^{N(\rho)-m}}\,\mathrm{d}y \tag{7.1.4.16}$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\varepsilon+iy|^N|\Lambda(\varepsilon+iy,\chi)|}{|\varepsilon+iy-s|\,|\varepsilon+iy-\rho|^{N(\rho)-m}}\,\mathrm{d}y \tag{7.1.4.17}$$

and since both s and ρ are within $\mathbf{C}_{[0,1]}$, and $\Lambda(s,\chi)$ decays faster than any positive power of $|s|^{-1}$ uniformly over the vertical strip $\mathbf{C}_{[-\varepsilon,1+\varepsilon]}$, and hence in particular on the vertical lines $\mathrm{Re}(s)=-\varepsilon$ and $\mathrm{Re}(s)=1+\varepsilon$, both integrals in (7.1.4.17) converge and their value is bounded above by a positive real number that depends on $N\geq 0$ but not on s (since $\varepsilon>0$ and s is constrained to vary over $\mathbf{C}_{[0,1]}$).

The same argument shows that the holomorphic derivatives of $\phi_{\rho,m}$ of any order also decay faster than any positive power of $|s|^{-1}$ as $|s|\to+\infty$ uniformly over the vertical strips of bounded width of the complex s-plane.

In particular, $\phi_{\rho,m}$ is an element of $A\mathcal{S}(\mathbf{C})$ for every (ρ,m), with ρ a zero of $L(s,\chi)$, and m an integer such that $0\leq m\leq N(\rho)-1$.

They satisfy the following vanishing conditions:

$$U(\delta_\rho^m)(\phi_{(\rho',m')}) = 0 \tag{7.1.4.18}$$

if $\rho' \neq \rho$ and m' is an integer such that $0 \leq m' \leq N(\rho') - 1$, or if $\rho' = \rho$ and m' is an integer such that $0 \leq m \leq m' - 1$, and

$$U(\delta_\rho^m)(\phi_{(\rho',m')}) \neq 0 \tag{7.1.4.19}$$

if $\rho' = \rho$ and $m' \leq m \leq N(\rho) - 1$.

We know from (1) and (2) that the functionals $\{U(\delta_\rho^{(m)})\}_{(\rho,m)}$ are linear topological generators of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$.

Indeed, the continuous linear injection $A\mathcal{S}(\mathbf{C}) \to \mathscr{O}(\mathbf{C})$ has dense range, because both $A\mathcal{S}(\mathbf{C})$ and $\mathscr{O}(\mathbf{C})$ are reflexive, and hence [**NB**, Thm.8.10.6(e)] applies to its adjoint.

Let $\{c_{(\rho,m)}\}_{(\rho,m)}$ and $\{c'_{(\rho,m)}\}_{(\rho,m)}$ be two countable nets of complex numbers such that the following two series converge in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$ with respect to the topology of uniform convergence on bounded subsets, with same sum:

$$\sum_{(\rho',m')} c_{(\rho',m')} U(\delta_{\rho'}^{(m')}) = \sum_{(\rho',m')} c'_{(\rho',m')} U(\delta_{\rho'}^{(m')}). \tag{7.1.4.20}$$

Since the convergence is with respect to the strong topology, evaluation at $\phi_{\rho,m}$ is a continuous functional on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$ equipped with this topology, and we may therefore evaluate each series in (7.1.4.20) term by term, to obtain the finite sums:

$$\sum_{(\rho',m'):\rho'=\rho, m\leq m'\leq N(\rho)-1} c_{(\rho',m')} U(\delta_{\rho'}^{(m')}) = \sum_{(\rho',m'):\rho'=\rho, m\leq m'\leq N(\rho)-1} c'_{(\rho',m')} U(\delta_{\rho'}^{(m')})$$

and since the $U(\delta_{\rho'}^{(m')})$ are $\mathbf{C}$-linearly independent, we obtain $c_{(\rho,m)} = c'_{(\rho,m)}$ for all pairs (ρ, m).

This implies that for any enumeration of the pairs (ρ, m), the resulting sequence of functionals $\{U(\delta_\rho^{(m)})\}$ is topologically free within $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$ (since it admits a bi-orthogonal sequence, which can, in light of the foregoing, be constructed by replacing each $\phi_{(\rho,m)}$ with a finite linear combination of the finitely many elements $\phi_{(\rho,m')}$ as m' varies over the integers $0 \leq m' \leq N(\rho) - 1$; see [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.290]).

(4) : Let $\phi \in \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ be an arbitrary element. Since ϕ has exponential decay (see Definition D.3.3.1), its Mellin transform:

$$\Phi(s) := \int_{C_F} \phi(g)\chi(g)|g|_F^s \, \mathrm{d}\overline{\mu}(g)$$

is entire and Schwartz uniformly over each vertical strip of bounded width of the complex s-plane. In particular, the set of values of Φ over the vertical strip $\mathbf{C}_{(-\varepsilon,1+\varepsilon)}$, for any fixed real number $\varepsilon > 0$, is bounded with respect to the Euclidean topology.

We have, by the Asymptotic Residue Theorem (see Theorem B.3.2.1):

$$-\frac{k!}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\Phi(z)}{(z-s)^{k+1}}\,\mathrm{d}z + \frac{k!}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\Phi(z)}{(z-s)^{k+1}}\,\mathrm{d}z = \Phi^{(k)}(s) \tag{7.1.4.21}$$

for every $s \in \mathbf{C}_{[0,1]}$ and every $\varepsilon > 0$. From (7.1.4.21) we obtain the estimate:

$$|\Phi^{(k)}(s)| \leq \frac{k!}{2\pi\varepsilon^{k+1}} \left\{ \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} |\Phi(z)|\,|\mathrm{d}z| + \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} |\Phi(z)|\,|\mathrm{d}z| \right\} \tag{7.1.4.22}$$

which implies that, for all $s \in \mathbf{C}_{[0,1]}$, we have, upon choosing $\varepsilon = 1$:

$$\left|\frac{\Phi^{(k)}(s)}{k!}\right| \leq \frac{1}{2\pi} \max\{\|\Phi\|_{L^1(-1+i\mathbf{R},\mathrm{d}x)}, \|\Phi\|_{L^1(2+i\mathbf{R},\mathrm{d}x)}\}.$$

This implies that the subset $H_\chi \subset \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})'_\beta$ is simply bounded, hence bounded by [**Scha**, Ch.III,§3,Corollary,p.82], since $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})'_\beta$ is complete.

By [**Scha**, Ch.III,§4,Thm.4.2,p.83], since $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ is barrelled, being a Fréchet space, it is equicontinuous. □

The following extension of Theorem 7.1.4.2 incorporates the inversion of finitely many Euler factors of the Hecke L-function.

THEOREM 7.1.4.3 (Global Orbital bi-orthogonal sequences). *Let F be a number field, $\chi \in \widehat{C_F}$ a unitary character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S_0 be a finite set of finite places of F.*

(1) *The orthogonal $Z_{S_0,\chi}$ of the ideal*

$$\prod_{v\in S_0-(S_0\cap S_\chi)} (1-q_v^{-s-it_v})(s+it_0)^{\delta_\chi}(1-s-it_0)^{\delta_\chi}L(s,\chi)\,\mathscr{O}(\mathbf{C}) \subset \mathscr{O}(\mathbf{C})$$

with respect to the topology of uniform convergence on compact subsets of the complex s-plane, with $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and $\delta_\chi = 0$ otherwise, is linearly and topologically isomorphic to the locally convex product:

$$\prod_{p:(p,S_0)\neq 1} J_{S_0,p} \times \prod_{\rho:L(\rho,\chi)=0} J_\rho \tag{7.1.4.23}$$

where J_ρ is the finite dimensional $\mathbf{C}$-vector space from (7.1.4.3), where $(p,S_0)\neq 1$ denotes the condition on the prime number p, of some $v\in S_0$ existing such that $v \mid p$, and J_p is the finite dimensional $\mathbf{C}$-vector space:

$$J_{S_0,p} := \bigoplus_{0\le m\le N(\varepsilon_p)-1} \delta_{\varepsilon_p}^{(m)} \tag{7.1.4.24}$$

where ε_p varies over the zeros of $\prod_{v|p,v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})$, $N(\varepsilon_p)$ is the zero-multiplicity of ε_p, and $\delta_{\varepsilon_p}^{(m)} \in \mathscr{O}(\mathbf{C})'_\beta$ is the continuous functional:

$$\delta_{\varepsilon_p}^{(m)}(\phi) := (\partial_s^m\phi)(\varepsilon_p). \tag{7.1.4.25}$$

(2) *The continuous linear operator $A\mathcal{S}(\mathbf{C}) \to \mathscr{O}(\mathbf{C})$ induces a continuous linear map:*

$$U : Z_{S_0,\chi} \to (\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)))^\perp \tag{7.1.4.26}$$

where the orthogonal is formed with respect to the duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot\circ L,\cdot\rangle_{ZH,A\mathcal{S}(\mathbf{C})}$, where ZH denotes the image in $A\mathcal{S}(\mathbf{C})$ under Z of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

The continuous linear operator U from (7.1.4.26) is injective with dense image, and for every zero z of either $L(s,\chi)$ or $\prod_{v|p,v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})$, and every integer $0\le m\le N(z)-1$, we have:

$$U(\delta_z^{(m)})(\phi) = \frac{1}{2\pi i}\int_{\mathbf{R}_{>0}} \left(\int_{-i\infty}^{+i\infty} t^{-s}\phi(s)\,\mathrm{d}s\right) t^z(\log(t))^m\,\frac{\mathrm{d}t}{t}. \tag{7.1.4.27}$$

(3) *Any enumeration of the countable subset $\{U(\delta_z^{(m)})\}_{(z,m)}$ in $(\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)))^\perp$ equipped with strong topology of the quotient topology on the quotient of $\mathcal{S}(C_F)(\chi|_{C_F^1})$ by $\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ as z varies over the zeros of $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)$, and, for each such z, m is an integer satisfying $0\le m\le N(z)-1$, admits a bi-orthogonal sequence in*

$$\mathscr{E}_{F,S_0}(\chi|_{C_F^1}) := \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)). \tag{7.1.4.28}$$

In particular, it is a topologically free set.

PROOF. The argument for both (1) and (2) is formally analogous to that for (1) and (2) from Theorem 7.1.4.2, with only one additional input: the non-vanishing of $L(s,\chi)$ on the vertical lines $\mathrm{Re}(s)=0,1$, which is required to ensure that no zero of $L(s,\chi)$ is also a zero of $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})$. This property of $L(s,\chi)$ is logically equivalent to the Prime Number Theorem (in its general form for Hecke L-functions) and is proved in [**CF**, Ch.VIII,Thm.2].

(3) : The argument is closely analogous to that for (3) from Theorem 7.1.4.2, upon replacing $L(s,\chi)$ with $\prod_{v\in S_0-(S_0\cap S_\chi)}(1-q_v^{-s-it_v})L(s,\chi)$, and upon constructing additional functions $\phi_{\varepsilon_p,m}$ for ε_p varying among the zeros of $\prod_{v\in S_0-(S_0\cap S_\chi),v|p}(1-q_v^{-s-it_v})$ for each rational prime p in a manner similar to that of $\phi_{\rho,m}$ from (7.1.4.12), to test any topological linear dependence relation among the functionals $U(\delta_z^{(m)})$ and deduce its triviality. By [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289], any enumeration of the countable set $\{U(\delta_z^{(m)})\}$ is topologically free. □

7.2. Poles of the S-idelic Zeta functions

Let X be a Riemann surface, $D\subset X$ a divisor. The following is a short exact sequence of sheaves of $\mathscr{O}_X$-modules, the sheaf of holomorphic functions on X:

$$0\to\mathscr{O}_X(-D)\to\mathscr{O}_X\to\mathscr{O}_D\to 0 \tag{7.2.0.1}$$

and if X is Stein, then $\mathrm{H}^m(X,\mathscr{F})=0$ for all integers $m>0$ and for any coherent analytic sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, which leads to the exact sequence of global sections:

$$0\to\mathscr{O}_X(-D)\to\mathscr{O}_X\to\mathscr{O}_D\to 0.$$

Twisting by $\mathscr{O}_X(D)$, we further obtain the short exact sequence of sheaves of $\mathscr{O}_X$-modules:

$$0\to\mathscr{O}_X\to\mathscr{O}_X(D)\to\mathscr{P}_{X,D}\to 0 \tag{7.2.0.2}$$

where $\mathscr{P}_{X,D}$ is the sheaf of principal parts.

The $\mathbf{C}$-linear spaces of global sections involved can be topologized so as to render the above short exact sequences strict exact. We refer to (7.2.0.1) and to (7.2.0.2) as the *divisor exact sequence* and the *principal parts exact sequence*, respectively.

We refer the reader to work of Grosse-Erdman on topologizing spaces of meromorphic functions, focusing on the Hausdorff locally convex topologies on spaces of meromorphic functions, and comparing the topology induced by combining the topology of uniform convergence on compact subsets for holomorphic functions, with the product topology on Euclidean lines generated by principal parts, termed Mittag-Leffler topology, with other Hausdorff locally convex topologies on spaces of meromorphic functions on complex domains: the standard topology inherited from the topology of uniform convergence on compact subsets for holomorphic functions on complex domains (an inductive topology by nature), and the Holdgründ topology, a projective topology obtained by writing spaces of meromorphic functions on complex domains as projective limits of spaces of holomorphic functions up to rational functions obtained as principal parts.

We refer the reader to [**GE**, §3.2], as well as to [**Jor-a**, §3], for vector-valued analogs, and particularly to [**Jor-a**, Thm.3.6].

7.2.1. Principal parts. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

We recall from Theorem 3.3.1.1 that the S-local Mellin transform operator:

$$Z_S:\mathcal{S}(\mathbf{A}_{F,S})\to\mathscr{M}(\Omega(J_{F,S}))$$

from (5.1.1.2), factors through $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times}, \mathcal{S}(\mathbf{A}_{F,S}))$, and for any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the meromorphic function $Z_S(f,c)$ of $c \in \Omega(J_{F,S})$ satisfies the functional equation:

$$Z_S(f,c) = \rho_S(c) Z_S(\widehat{f}, \widehat{c}) \tag{7.2.1.1}$$

for which we refer the reader to (3.3.1.2).

Recall that $\mathscr{M}(\Omega(J_{F,S}))$ denotes the $\mathbf{C}$-linear space of meromorphic functions on the Riemann surface $\Omega(J_{F,S})$ with poles located within the subset $\mathcal{U}_S$ of $\Omega(J_{F,S})$ of the poles of $\rho_S(c)$ from (3.3.1.3).

Recall that $\widehat{f}$ is the S-adelic Fourier transform, which we defined using the standard additive character of $\mathbf{A}_{F,S}$, and that $\rho_S(c)$ is the meromorphic function on $\Omega(J_{F,S})$ defined in (3.3.1.3) as:

$$\rho_S(c) := \prod_{v \in S} \rho_v(c_v) \tag{7.2.1.2}$$

where ρ_v is the meromorphic function on $\Omega(F_v^{\times})$ defined in [**CF**, Ch.XV,§2.5] for each $v \in S$.

We further recall that $Z_S(f,c)$ is holomorphic over the relative half-surface $\Omega(J_{F,S})_{>0}$, as is $Z(\widehat{f},c)$, and so the functional equation (7.2.1.1) extends $Z_S(f,c)$ to a meromorphic function over $\Omega(J_{F,S})$ whose only poles are located at the poles of $\rho_S(c)$ (possibly with lower order of pole).

The location and order of the poles of $\rho_S(c)$ was described in Lemma 7.1.2.2 and Theorem 7.1.3.2.

We established, using input from the theory of almost periodic functions and the Asymptotic Residue Theorem (see Theorem B.3.2.1), that for any $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the continuous function $\Phi_{F,S,\omega}(f)$ on $J_{F,S}$ satisfies the following asymptotic expansion.

For any continuous character $\chi \in \widehat{J_{F,S}}$, we have, for any real number $A > 0$ and upon denoting by S_χ the subset of S of those places in S at which χ is ramified:

$$\begin{aligned}
\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)(g) = & \sum_{v \in S_\infty} \sum_{m \in \mathbf{N}_{>0}, m<A} \sum_{1 \leq k \leq m_S(f, c_{v,m,\chi})} \frac{(-1)^{k-1} a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} \overline{\chi}(g) |g|_{F,S}^{m+it_v} (\log|g|_{F,S})^{k-1} \\
& + \sum_{p \in \Pi(S)} \sum_{v \in S_p} \sum_{m \in \mathbf{Z}} \sum_{1 \leq k \leq m_S(f, c_{v,m,\chi})} \frac{(-1)^{k-1} a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} \overline{\chi}(g) |g|_{F,S}^{-\frac{2\pi i m}{\log(q_v)} + it_v} (\log|g|_{F,S})^{k-1} \qquad (7.2.1.4) \\
& + O(|g|_{F,S}^{A}) \qquad (7.2.1.5)
\end{aligned}$$

as $|g|_{F,S} \to 0^+$, where q_v is the norm of the prime ideal $\mathfrak{p}_v$ corresponding to the finite place v of F, the series converges uniformly on the compact subsets of $J_{F,S}$, the implied constant depends only on the restriction of the meromorphic function $Z_S(f,c)$ on the vertical line $\chi|\cdot|_{F,S}^{-A+i\mathbf{R}}$ within $\Omega(J_{F,S})(\omega)$, and where $c_{v,m,\chi}$ is a pole of $\rho_S(c)$, as described in Lemma 7.1.2.2 according to the chosen parametrization of $\Omega(J_{F,S})(\omega)$.

In this subsection we reprise the study of the Mellin transform of $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$, by discussing continuity properties of the coefficients $a_k(f,c)$ for all integers $k \geq 0$, as c varies over all the poles of $\rho_S(c)$, and as f varies over all Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$.

They will turn out to be S-local eigendistributions (see Definition 5.5.1.1, and Theorem 5.5.1.2 for their classification), which is the content of Theorem 7.2.1.1 below.

In the last subsection, we will use the eigendistributions $a_k(\cdot,c)$ to describe the orthogonals within the strong dual $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ of certain closed subspaces of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, and the functionals $\{a_k(\cdot,c)\}$ will provide Schauder bases for such orthogonals, equipped with the strong topology on the continuous dual of the quotient of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ by the such closed subspace (we refer the reader to Lemma 4.2.1.3 on strong adjoints to quotient maps).

THEOREM 7.2.1.1 (Coefficients of Principal Parts). *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $m \geq 0$ be an integer.*

(1) *Formation of $a_m(f,c)$ is continuous with respect to $f \in \mathcal{S}(\mathbf{A}_{F,S})$.*
(2) *The tempered distribution $a_m(\cdot,c) \in \mathcal{S}(\mathbf{A}_{F,S})'_\beta$ factors through $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, through its ω-isotype for $\omega = c|_{J^1_{F,S}}$. It is an S-local eigendistribution of weight $c \in \Omega(J_{F,S})$.*
(3) *If $a_m(\cdot,c)$ is nonzero, then $c \in \mathcal{U}_S$ and $m \leq m_S(f,c)$ for all $f \in \mathcal{S}(\mathbf{A}_{F,S})$.*
(4) *Let $\omega \in \widehat{J^1_{F,S}}$ be a continuous character. Let $\mathcal{U} \subset \mathcal{U}_S$ be a subset, $m : \mathcal{U}_S \to \mathbf{N}$ a function supported on $\mathcal{U}$. Let $\mathcal{U}(\omega)$, respectively $\mathcal{U}_S(\omega)$, denote the intersections with the connected component $\Omega(J_{F,S})(\omega)$.*

The linear subspace:

$$H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \subset \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \tag{7.2.1.6}$$

linearly generated by those elements $\Phi_{F,S,\omega}(f)$ such that $Z_S(f,c)$ has poles with order of pole $m_S(f,c) \leq m(c)$, is closed, and the orthogonal $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))^\perp$ with respect to the duality

$$\langle \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta \rangle$$

is linearly and topologically generated by the S-local eigendistributions:

$$\{a_k(\cdot,c) : c \in \mathcal{U}_S(\omega), k > m(c)\}. \tag{7.2.1.7}$$

Proof. (1) : By the S-local functional equation (see (3.3.1.2) from Theorem 3.3.1.1), we have:

$$Z_S(f,c) = \rho_S(c) Z_S(\widehat{f}, \widehat{c})$$

for all $f \in \mathcal{S}(\mathbf{A}_{F,S})$, and formation of both sides is continuous with respect to the Schwartz-Bruhat topology on $\mathcal{S}(\mathbf{A}_{F,S})$ by Lemma 5.1.3.1 for each fixed c, and also upon endowing the linear space $\mathscr{M}(\Omega(J_{F,S}))$ with the topology of uniform convergence on the compact subsets of $\Omega(J_{F,S}) - \mathcal{U}_S$.

We now choose a holomorphic parametrization $\chi|\cdot|^s_{F,S}$, $s \in \mathbf{C}$, for $\Omega(J_{F,S})$, i.e. a set of base points, equivalently representatives, $\chi \in \widehat{J_{F,S}}$ for the continuous characters $\omega \in \widehat{J^1_{F,S}}$, which parametrize the connected components of $\Omega(J_{F,S})$.

Let $c_0 \in \mathcal{U}_S$ be a point, and we parametrize it as $c_0 = \chi_0|\cdot|^{s_0}_{F,S}$ for a unique base point χ_0 and a unique point $s_0 \in \mathbf{C}$.

Formation of $a_m(f,c)$ is therefore continuous with respect to the Schwartz-Bruhat topology on $\mathcal{S}(\mathbf{A}_{F,S})$, since we may write, by Cauchy's Integral Formula:

$$a_m(f,c_0) = \frac{1}{2\pi i} \int_\Gamma \frac{(s-s_0)^m Z_S(f, \chi|\cdot|^s_{F,S})}{s-s_0} \, \mathrm{d}s \tag{7.2.1.8}$$

for any closed rectifiable curve $\Gamma \subset \mathbf{C}$ containing s_0 within the unique bounded domain of $\mathbf{C}$ whose boundary is Γ, and formation of the integral in the right side of (7.2.1.8) is continuous.

(2) : Since $a_m(f,c_0)$, in light of (7.2.1.8), depends on f only through $Z_S(f,c)$ as a meromorphic function of $c \in \Omega(J_{F,S})$, then by Lemma 5.1.2.2, then $a_m(\cdot,c_0)$ annihilates the kernel of Φ_{F,S,ω_0}, for $\omega_0 = c_0|_{J^1_{F,S}} = \chi_0|_{J^1_{F,S}}$. Since this holds for all $c_0 \in \mathcal{U}_S$, we deduce that $a_m(\cdot,c)$ factors through $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, and in fact through its quotient $\mathrm{H}_{0,\mathrm{cont}}(c|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$. c-homogeneity of $a_m(\cdot,c)$ follows from c-homogeneity of $Z_S(\cdot,c)$, which is given by (5.1.3.2) from Lemma 5.1.3.1, and from (7.2.1.8). $a_m(f,c)$ is therefore an S-local eigendistribution of weight c.

(3) : If $a_m(\cdot,c_0)$ is nonzero, then there exists $f \in \mathcal{S}(\mathbf{A}_{F,S})$ such that $a_m(f,c_0) \neq 0$, which means that c_0 is a pole of $Z_S(f,c)$, and the polar multiplicity $m_S(f,c_0)$ of c_0 is at least m. It follows that we have $c_0 \in \mathcal{U}_S$, and $m \leq m_S(f,c_0)$, this latter inequality holding for all $f \in \mathcal{S}(\mathbf{A}_{F,S})$.

(4) : By design of $a_m(\cdot,c)$, for each $\omega \in \widehat{J^1_{F,S}}$, $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ is the intersection:

$$H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S})) = \bigcap_{(c,k): c \in \mathcal{U}_S, k > m(c)} \ker(a_k(\cdot,c)) \tag{7.2.1.9}$$

and hence by (1), $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ is closed, and the eigendistributions $\{a_k(\cdot, c)\}$ as (c,k) varies over the set of pairs $I := \{(c,k) : c \in \mathcal{U}, 1 \leq k \leq m(c)\}$ generate $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))^\perp$ within $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta$ linearly and topologically. Indeed, $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))^\perp$ is closed in $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'$ with respect to the $\sigma(\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))', \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})))$-topology by [**NB**, Thm.8.9.2(d)], and hence it is closed with respect to the strong topology on $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'$, since the strong topology is finer than the topology of simple convergence.

It is therefore enough to show that $\{a_k(\cdot, c)\}_{(c,k)\in I}$, linearly and topologically generates $H_m(\omega, \mathcal{S}(\mathbf{A}_{F,S}))^\perp$ with respect to the topology of simple convergence. By (7.2.1.9), this follows from [**NB**, Thm.8.9.2(a)]. □

Remark 7.2.1.2. Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\omega \in \widehat{J^1_{F,S}}$ be a continuous character.

The short algebraically-exact sequence:

$$0 \to H_{m_{S_0}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \xrightarrow{\prod_{(c,k)} a_k(\cdot,c)} \prod_{(c,k)} \mathbf{C}$$

as (c,k) ranges over the countable set of pairs:

$$\{(c,k) : c \in \mathcal{U}_{S_0}, 1 \leq k \leq m_{S_0}(c)\}$$

is turned by the S-local Mellin transform L_S into the short algebraically-exact sequence:

$$0 \to \mathscr{O}_{\Omega(J_{F,S})}(\mathcal{U}_S - \mathcal{U}_{S_0}) \to \mathscr{O}_{\Omega(J_{F,S})}(\mathcal{U}_S) \to \mathscr{P}_{\Omega(J_{F,S}),\mathcal{U}_S - \mathcal{U}_{S_0}}$$

up to identifying the $m_{S_0}(c)$-dimensional linear space generated by the principal part of $Z_S(f,c)$ at $c \in \mathcal{U}_{S_0}$ with $\prod_{1\leq k\leq m_{S_0}(c)} \mathbf{C}$, the principal part operator being turned, by such identification, into $\prod_{1\leq k\leq m_{S_0}} a_k(\cdot, c)$.

Remark 7.2.1.3. Let F be a number field, S a finite set of places of F containing all the Archimedean places. We may choose $S_0 = S$ in Theorem 7.3.1.2. In this case, the closed subspace $H_{m_S}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ is, for each continuous character $\omega \in \widehat{J^1_{F,S}}$, the closure of the linear image under $\Phi_{F,S,\omega}$ of the closed subspace:

$$\mathcal{S}_{00}(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S}) \tag{7.2.1.10}$$

which agrees with the closure of the linear subspace of those Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$ that vanish over the union $\bigcup_{v\in S} H_v$, for $H_v \subset \mathbf{A}_{F,S}$ the closed subspace defined by the equation $x_v = 0$, and vanish with their real, holomorphic and anti-holomorphic directional derivatives of arbitrary order along $F_v \subset \mathbf{A}_{F,S}$ at each H_v for v ranging over all the Archimedean places.

We further note that $\Phi_{F,S,\omega}$ maps $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ onto a closed subspace of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))$ which is linearly and topologically isomorphic to $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\omega)$, the ω-isotype of the Hausdorff locally convex topological $\mathbf{C}$-vector space of exponentially decaying Schwartz-Bruhat functions on the modulated group $J_{F,S}$, compatibly with the continuous left-translation actions of $J_{F,S}$.

This is a consequence of Proposition 5.3.4.4.

The next Theorem describes the structure of the tempered distributions $a_m(\cdot, c)$ more closely. Recall that the set $\mathcal{U}_S$ of the poles of $\rho_S(c)$, counted with multiplicity, is the union of the sets:

$$\mathcal{U}_{S,v} := \psi_v^{-1}(\mathcal{U}_v) \tag{7.2.1.11}$$

for $\mathcal{U}_v \subset \Omega(F_v^\times)$ the set of the poles of the meromorphic function $\rho_v(c_v)$ on the Riemann surface $\Omega(F_v^\times)$ of local quasi-characters, counted with multiplicity, in light of the definition of $\rho_S(c)$ from (7.2.1.2).

Recall that each Riemann surface $\Omega(J_{F,S})$ admits holomorphic submersions

$$\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^\times)$$

to the Riemann surfaces of local quasi-characters for each $v \in S$, defined by restricting a quasi-character $c \in \Omega(J_{F,S})$ to $F_v^\times$ along the closed injective homomorphism of Hausdorff locally compact abelian groups $F_v^\times \subset J_{F,S}$ induced by the insertion homomorphism $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$ into the v-indexed "coordinate".

We recall that for such S, the S-local Hecke L-function $L_S(c)$ is defined for $c \in \Omega(J_{F,S})_{>0}$ by the assignment:

$$L_S(c) := \prod_{v\in S} L_v(\psi_v(c)) = \prod_{v\in S} L_v(c_v)$$

where $L_v(c_v) := Z_v(f_v, c_v)$, for $f_v \in \mathcal{S}(F_v)$ the basic Schwartz-Bruhat functions on the local fields F_v, for each $v \in S$, defined in [**CF**, Ch.XV,§2.5] (see also [**RV**, Ch.7,§7.1(iii),pp.249-259]).

$L_S(c)$ is explicitly computed in [**CF**, Ch.XV,§2.5] (see also [**RV**, Ch.7,§7.1(iii),pp.249-259]) and it is seen to be meromorphic on $\Omega(J_{F,S})$, with poles located exactly at the points of $\mathcal{U}_S$, counted with multiplicity.

For $c \in \Omega(J_{F,S})_{>0}$, we let $\widehat{L}_S(c)$ be defined as

$$\widehat{L}_S(c) := \prod_{v\in S} Z_v(\widehat{f}_v, \psi_v(\widehat{c})) = \prod_{v\in S} Z_v(\widehat{f}_v, \widehat{c}_v). \tag{7.2.1.12}$$

The meromorphic function $1/\widehat{L}_S(c)$ has only removable singularities over $\Omega(J_{F,S})$, and the meromorphic function $\rho_S(c)$, which is defined as $L_S(c)/\widehat{L}_S(c)$, and computed explicitly in [**CF**, Ch.XV,§2.5] (see also [**RV**, Ch.7,§7.1(iii),pp.249-259]), therefore extends to a meromorphic function on $\Omega(J_{F,S})$, with poles located at exactly the points of $\mathcal{U}_S$, i.e. at the poles of $L_S(c)$, counted with multiplicity.

We further refer the reader to Theorem 3.3.1.1.

The following Theorem provides formulas for the functionals $a_k(\cdot, c)$.

Before we state it and prove it, we recall the following. For F a number field, S a finite set of places of F containing all the Archimedean places, we fix a continuous character $\chi \in \widehat{J_{F,S}}$ for each parameter $\omega \in \widehat{J^1_{F,S}}$, i.e. such that $\chi|_{J^1_{F,S}} = \omega$, and use it as a base point for the connected component $\Omega(J_{F,S})(\omega)$, to parametrize the connected component by the global holomorphic chart $\chi|\cdot|^s_{F,S}$.

For ϕ a meromorphic function on $\Omega(J_{F,S})$, we can therefore consider the holomorphic derivative:

$$\partial_s^k \phi(\chi|\cdot|^s_{F,S})$$

for each integer $k \geq 0$. If we change the choice of χ for each $\omega \in \widehat{J^1_{F,S}}$, then the holomorphic parametrization of $\Omega(J_{F,S})$ changes into a new one, $\chi'|\cdot|^z_{F,S}$, under a bi-holomorphism, which is given, explicitly, by the formula $z = -it_0 + s$, where $t_0 \in \mathbf{R}$ is the unique real number such that $\chi' = |\cdot|^{it_0}_{F,S}$.

In particular, we have:

$$\partial_z^k(\phi(\chi'|\cdot|^z_{F,S})) = \partial_z^k(\phi(\chi|\cdot|^{z+it_0}_{F,S})) = (\partial_s^k\phi)(\chi|\cdot|^{z+it_0}_{F,S}) = (\partial_s^k\phi)(\chi|\cdot|^s_{F,S})$$

and hence the holomorphic derivative of order k is independent of the choice of global holomorphic parametrization $\chi|\cdot|^s_{F,S}$ of $\Omega(J_{F,S})$. We will be denoting it by ∂^k.

THEOREM 7.2.1.4 (Coefficients of Principal Parts Formula). *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $v_1, \ldots, v_N$ be the places of F contained in S such that $c \in \mathcal{U}_{S,v_1} \cap \ldots \cap \mathcal{U}_{S,v_N}$. Let $m := m_S(c)$ be the multiplicity of c in $\mathcal{U}_S$.*

Then $m = N$. Moreover:

(1) *The image $\Delta'(c)$ in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$ from (3.4.3.14) of the S-local eigendistribution $\Delta_{F,S}(c)$ from Theorem 5.5.1.2, satisfies:*

$$a_m(\cdot, c) = \widehat{\Delta}'(\widehat{c}) \prod_{v\in\{v_1,\ldots,v_m\}} \mathrm{Res}_{c'_v=\psi_v(c)} L_v(c'_v) \prod_{v\in S-\{v_1,\ldots,v_m\}} L_v(\psi_v(c)) \tag{7.2.1.13}$$

where $L_v(c_v)$ is the local factor at v of the Hecke L-function $L(c)$, and $\widehat{\Delta}'(c)$ is the S-adelic Fourier transform of the eigendistribution $\Delta'(\widehat{c})$, i.e. $\widehat{\Delta}'(f)(\widehat{c}) = \Delta'(\widehat{f})(\widehat{c})$.

In particular, $a_m(\cdot, c)$ is an S-local eigendistribution of weight c.

(2) *For any integer $0 \leq k \leq m-1$, we have:*

$$a_k(f,c) = \sum_{j=0}^{m-k} a_{m-j}(\rho_S(c'),c)\, \frac{\partial^{m-k-j} Z_S(\widehat{f},\widehat{c'})|_{c'=c}}{(m-k-j)!} \tag{7.2.1.14}$$

where $a_{m-j}(\rho_S(c'),c)$ is the principal part of order $m-j$ of the meromorphic function $\rho_S(c')$ from (3.3.1.3) *at $c=c'$, i.e. the coefficient of $(s-s_0)^{-m+j}$ of the Laurent series expansion of $\rho_S(\chi_0|\cdot|_{F,S}^s)$ at $s=s_0$, for $\{\chi_0|\cdot|_{F,S}^s, s\in\mathbf{C}\}$ a choice of holomorphic parametrization of $\Omega(J_{F,S})(c|_{J^1_{F,S}})$, and for c being parametrized by $s=s_0$.*

We recall that the residues of the local L-functions $Z_v(f_v,c_v)$ for $f_v \in \mathcal{S}(F_v)$ are described in Lemma 3.3.2.1, for v Archimedean, and Lemma 3.3.2.2, for v non-Archimedean.

Proof. (1) : By (3.4.3.13) from Theorem 3.4.3.3, we have the identity, as meromorphic functions over $\Omega(F_v^\times)$ for each place $v \in S$:

$$\widehat{\Delta}'_v(\widehat{c}_v)(f) = \frac{Z_v(\widehat{f},\widehat{c})}{\widehat{L}_v(c)} \tag{7.2.1.15}$$

for every $f \in \mathcal{S}(F_v)$. By design of Δ' (see (3.4.3.11) from Theorem 3.4.3.3), we deduce the identity as S-local eigendistributions of weight c:

$$\widehat{\Delta}'(\widehat{c}) = \frac{Z_S(\mathcal{F}_S(\cdot),\widehat{c})}{\widehat{L}_S(c)}. \tag{7.2.1.16}$$

From the S-local functional equation (see (3.3.1.2) from Theorem 3.3.1.1), we have, upon choosing a holomorphic parametrization $\chi|\cdot|_{F,S}^s$, where $\{\chi\} \subset \widehat{J_{F,S}}$ is a set of representatives for the continuous characters of $J^1_{F,S}$, serving as base-points for each connected components of $\Omega(J_{F,S})$:

$$Z_S(f,\chi|\cdot|_{F,S}^s) = \rho_S(\chi|\cdot|_{F,S}^s)\, Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1-s}) = \epsilon_S(\chi|\cdot|_{F,S}^s)\frac{L_S(\chi|\cdot|_{F,S}^s)}{\widehat{L}_S(\chi|\cdot|_{F,S}^s)}\, Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1-s}) \tag{7.2.1.17}$$

for any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$.

The identity of meromorphic functions over $\mathbf{C}$ from (7.2.1.17) therefore reads:

$$Z_S(f,\chi|\cdot|_{F,S}^s) = L_S(\chi|\cdot|_{F,S}^s)\, \widehat{\Delta}'(\overline{\chi}|\cdot|_{F,S}^{1-s})(f) \tag{7.2.1.18}$$

for each $f \in \mathcal{S}(\mathbf{A}_{F,S})$, the identity in (7.2.1.18) being in the sense of meromorphic functions on the complex s-plane, using Theorem 3.4.3.3 on Δ' (and hence its Fourier transform) being a holomorphic vector valued function valued in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$.

Let $c \in \bigcap_{1\leq k\leq N} \mathcal{U}_{S,v_k}$ be as in the statement of the Theorem. We parametrize c as $\chi_0|\cdot|_{F,S}^{s_0}$ according to the chosen holomorphic parametrization of $\Omega(J_{F,S})$.

We therefore have, for each $f \in \mathcal{S}(\mathbf{A}_{F,S})$:

$$a_m(f,c) = \lim_{s\to s_0}(s-s_0)^m Z_S(f,\chi_0|\cdot|_{F,S}^s) = \left\{\lim_{s\to s_0}(s-s_0)^m L_S(\chi_0|\cdot|_{F,S}^s)\right\} \lim_{s\to s_0}\left\{\widehat{\Delta}'(\overline{\chi_0}|\cdot|_{F,S}^{1-s})(f)\right\}. \tag{7.2.1.19}$$

By Theorem 7.2.1.18, Δ', and hence its Fourier transform, is an $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$-valued holomorphic function over $\Omega(J_{F,S})$, and hence we have:

$$\lim_{s\to s_0}\left\{\widehat{\Delta}'(\overline{\chi_0}|\cdot|_{F,S}^{1-s})(f)\right\} = \widehat{\Delta}'(\overline{\chi}_0|\cdot|_{F,S}^{1-s_0})(f) = \widehat{\Delta}'(\widehat{c})(f). \tag{7.2.1.20}$$

We are reduced to computing the complex limit:

$$\lim_{s\to s_0}(s-s_0)^m L_S(\chi_0|\cdot|_{F,S}^s). \tag{7.2.1.21}$$

Assume first that we have $m > N$. Since $c \in \bigcap_{1\leq k\leq N}\mathcal{U}_{S,v_k}$, then $\psi_v(c)$ is a pole of $L_v(c_v)$ for $v \in \{v_1,\dots,v_N\}$ only, and hence the limit in (7.2.1.21) is zero. We may therefore assume $m \leq N$, and, up to relabelling, assume that $\psi_v(c)$ is a pole of $L_v(c_v)$ exactly for $v \in \{v_1,\dots,v_m\}$, since, by Lemma 3.3.2.1 and Lemma 3.3.2.2, $L_v(c_v)$ only has at most a simple pole over $\Omega(F_v^\times)$.

We deduce the identity:

$$\lim_{s\to s_0}\prod_{v\in\{v_1,\dots,v_m\}}(s-s_0)L_v(\psi_v(\chi_0|\cdot|_{F,S}^s))\prod_{v\in\{v_{m+1},\dots,v_N\}}L_v(\psi_v(\chi_0|\cdot|_{F,S}^s)) \tag{7.2.1.22}$$

$$=\prod_{1\leq k\leq m}\mathrm{Res}_{c'_{v_k}=\psi_{v_k}(c)}L_{v_k}(\psi_{v_k}(c')\prod_{m+1\leq k\leq N}L_{v_k}(\psi_{v_k}(c)). \tag{7.2.1.23}$$

From (7.2.1.22), we deduce that (7.2.1.13) holds, for each Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S})$, and since both sides in (7.2.1.13) are continuous functionals on $\mathcal{S}(\mathbf{A}_{F,S})$, it holds in the sense of tempered distributions on $\mathbf{A}_{F,S}$, homogeneous of weight c.

(2) : We parametrize $c \in \bigcap_{1\leq k\leq N}\mathcal{U}_{S,v_k}$ as $\chi_0|\cdot|_{F,S}^{s_0}$. By (7.2.1.19), we have:

(7.2.1.24)

$$\lim_{s\to s_0}\partial_s^{m-k}(s-s_0)^m Z_S(f,\chi_0|\cdot|_{F,S}^s)=\lim_{s\to s_0}\partial_s^{m-k}\left\{(s-s_0)^m\rho_S(\chi_0|\cdot|_{F,S}^s)Z_S(\widehat{f},\overline{\chi}_0|\cdot|_{F,S}^{1-s})\right\}$$

$$=\lim_{s\to s_0}\sum_{j=0}^{m-k}\binom{m-k}{j}\partial_s^j\left\{(s-s_0)^m\rho_S(\chi_0|\cdot|_{F,S}^s)\right\}\partial_s^{m-k-j}Z_S(\widehat{f},\overline{\chi}_0|\cdot|_{F,S}^{1-s}) \tag{7.2.1.25}$$

$$=\sum_{j=0}^{m-k}\binom{m-k}{j}j!\,a_{m-j}(\rho_S(\chi_0|\cdot|_{F,S}^s),s_0)\,\partial_s^{m-k-j}Z_S(\widehat{f},\overline{\chi}_0|\cdot|_{F,S}^{1-s})|_{s=s_0} \tag{7.2.1.26}$$

where $a_{m-j}(\rho_S(\chi_0|\cdot|_{F,S}^s,s_0)$ is the principal part of order $m-j$ at $s=s_0$, of the meromorphic function $\rho_S(\chi_0|\cdot|_{F,S}^s)$, i.e. the coefficient of $(s-s_0)^{-m+j}$ in the Laurent series expansion of $\rho_S(\chi_0|\cdot|_{F,S}^s)$ around $s=s_0$.

The last sum in (7.2.1.26) reads:

$$a_k(f,\chi_0|\cdot|_{F,S}^{s_0})=\sum_{j=0}^{m-k}a_{m-j}(\rho_S(\chi_0|\cdot|_{F,S}^s),s_0)\,\frac{\partial_s^{m-k-j}Z_S(\widehat{f},\overline{\chi}_0|\cdot|_{F,S}^{1-s})|_{s=s_0}}{(m-k-j)!} \tag{7.2.1.27}$$

and formation of both sides of (7.2.1.27) is $\mathbf{C}$-linear with respect to f.

We deduce that (7.2.1.14) holds. □

7.2.2. Local poles and global zeros. The following Proposition provides a first dictionary between the poles of the S-idelic Zeta functions and the zeros of the global idelic Zeta functions, using the transition operators:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times,\mathcal{S}(\mathbf{A}_{F,S}))\to\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}(\mathbf{A}_F)).$$

Following (7.2.1.11), we denote by

$$\mathcal{U}_{S_0}\subset\mathcal{U}_S$$

the subset of the set of poles $\mathcal{U}_S$ of $\rho_S(c)$ on $\Omega(J_{F,S})$ given by

$$\mathcal{U}_{S_0}:=\bigcup_{v\in S_0}\psi_v^{-1}(\mathcal{U}_v)=\bigcup_{v\in S_0}\mathcal{U}_{S,v} \tag{7.2.2.1}$$

each point being counted with multiplicity, defined, for $c \in \mathcal{U}_{S_0}$, as the sum of the orders of poles of ρ_v at $\psi_v(c)$ as v ranges over S_0. Since, by Lemma 3.3.2.1 and Lemma 3.3.2.2, each such order is 1, the multiplicity of c in $\mathcal{U}_{S_0}$ is the number of places $v \in S_0$ such that $\psi_v(c)$ is a pole of ρ_v over $\Omega(F_v^\times)$.

We define the function:

$$m_{S_0} : \mathcal{U}_{S_0} \to \mathbf{N} \tag{7.2.2.2}$$

to be the function assigning to $c \in \mathcal{U}_{S_0}$ its multiplicity in $\mathcal{U}_{S_0}$ as defined in the foregoing.

PROPOSITION 7.2.2.1. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. Let $S_0 \subset S$ be a non-empty subset of finite places.*

Then for any continuous character $\chi \in \widehat{C_F}$ unramified outside S_0 (and hence outside S), and for χ_S the unique continuous character of $J_{F,S}$ obtained by descending to $J_{F,S}$ the restriction of χ to $C_{F,S} \subset C_F$, by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance, given the transition operator:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

induced by the transition operators $\overline{\pi}_{S,S'}$ from (5.2.3.1) *of Theorem 5.2.3.1, the following conditions on $f \in \mathcal{S}(\mathbf{A}_{F,S})$ are equivalent:*

(1) *$a_k(f,c) = 0$ for all $c \in \mathcal{U}_{S_0} \cap \Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})$ and all integers $m_S(c) - m_{S_0}(c) + 1 \leq k \leq m_S(c)$.*
(2) *$Z(\pi_S(f), c) = Z(\overline{\pi}_S(\Phi_{F,S}(f)), c)$ has a zero at c of order $m_{S_0}(c)$, for all $c \in \psi^{-1}_{F,S}(\mathcal{U}_{S_0}) \cap \Omega(C_F)(\chi|_{C^1_F})$, with the sole exception of $c = 1$, in which case its order is $m_{S_0}(c) - 1$ if and only if $c = 1$ is a pole of at least one of the Euler factors at $v \in S_0$ of the Hecke L-function.*
(3) *$f \in \mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.*
(4) *$\pi_S(f) \in \mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$.*

PROOF. (1) implies (2) : By Theorem 7.2.1.4(2), $Z_S(\widehat{f}, \widehat{c'})$ has a zero of order $m_{S_0}(c)$ at $c' = c \in \mathcal{U}_{S_0}$. It follows that, since χ is unramified outside S, we have:

$$Z_S(\widehat{f}, \widehat{c'})\widehat{L}_S(c') = Z(\widehat{\pi_S(f)}, \widehat{c'}) = Z(\pi_S(f), c')$$

where we used the global functional equation from (3.2.2.2) of Theorem 3.2.2.1.

It follows that $Z(\pi_S(f), c')$ has a zero of order $m_{S_0}(c)$ at $c' = c$, unless $L_S(c')$ has a pole at $c' = c$, in which case such pole can only be simple and occurs at $c' = 1$. This proves (2).

(2) implies (4) : By (2), upon fixing a holomorphic parametrization $\chi|\cdot|_F^s$ for the Riemann surface $\Omega(C_F)$, the meromorphic function:

$$\phi(s) := \prod_{v \in S_0} \frac{1}{1 - \chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}} Z(\pi_S(f), \chi|\cdot|_F^s)$$

has only removable singularities, and it is an element of $A\mathcal{S}(\mathbf{C})$ (a consequence of the Asymptotic Residue Theorem, Theorem B.3.2.1, applied to the vertical lines bounding any vertical strip of bounded width containing the line $\chi|\cdot|_F^{i\mathbf{R}}$ in its interior, from which we deduce that not only ϕ but also its holomorphic derivatives decay faster than any positive power of $|s|^{-1}$ as $|s| \to +\infty$ uniformly over any such strip).

It follows from Theorem 6.3.2.1 that ϕ is in the range of L, and hence $\pi_S(f)$ is in the image of PV_{S_0} on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$.

(4) implies (3) : By (4), $\pi_S(f)$ is in the range of PV_{S_0} on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$.

By Lemma 5.3.1.5, f is in the range of PV_{S,S_0} on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

(3) implies (1) : By Theorem 7.2.1.1(4), if $Z_S(f, c')$ has a pole of order $k > m_S(c) - m_{S_0}(c)$ at $c' = c$, then f cannot be in the range of PV_{S,S_0}, since all elements of $\mathcal{S}(\mathbf{A}_{F,S})$ in the range of PV_{S,S_0} have S-idelic Mellin transforms with poles of order at most $m_S(c) - m_{S_0}(c)$. □

COROLLARY 7.2.2.2. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. *Let $S_0 \subset S$ be a non-empty subset of finite places.*

Then for any continuous character $\chi \in \widehat{C_F}$ unramified outside S_0 (and hence outside S), and for χ_S the unique continuous character of $J_{F,S}$ obtained by descending to $J_{F,S}$ the restriction of χ to $C_{F,S} \subset C_F$, by $\widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}$-invariance, given the transition operator:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$$

induced by the transition operators $\overline{\pi}_{S,S'}$ from (5.2.3.1) *of Theorem 5.2.3.1, with:*

$$\mathcal{S}_0(\mathbf{A}_{F,S}) = \ker(\delta_0) \cap \ker(\widehat{\delta}_0) \subset \mathcal{S}(\mathbf{A}_{F,S}) \tag{7.2.2.3}$$

then the following conditions on $f \in \mathcal{S}_0(\mathbf{A}_{F,S})$ are equivalent:

(1) *$a_k(f,c) = 0$ for all $c \in \mathcal{U}_{S_0} \cap \Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})$ and all integers $m_S(c) - m_{S_0}(c) + 1 \leq k \leq m_S(c)$, with the sole possible exception of $c = 1$, in which case $a_k(f,c) = 0$ for all integers $m_S(c) - m_{S_0}(c) \leq k \leq m_S(c) - 1$.*
(2) *$Z(\pi_S(f), c) = Z(\overline{\pi}_S(\Phi_{F,S}(f)), c)$ has a zero at c of order $m_{S_0}(c)$, for all $c \in \psi^{-1}_{F,S}(\mathcal{U}_{S_0}) \cap \Omega(C_F)(\chi|_{C^1_F})$.*
(3) *$f \in \mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))$.*
(4) *$\pi_S(f) \in \mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$.*

PROOF. This is a direct consequence of Proposition 7.2.2.1, noting that the linear conditions $\delta_0(f) = \widehat{\delta}_0(f) = 0$ resolve one order of pole at $c = 1$ in (1), and increase the multiplicity of the zero at $c = 1$ in (2) of the Proposition. □

7.3. Paley-Wiener Levels and Schauder Bases of Principal Parts

We now discuss an S-local analog of the Theorem 7.1.4.2, describing certain S-local divisor exact sequences encoding the poles of the S-idelic Zeta functions, and describing orthogonals by means of Schauder bases comprised of S-local eigendistributions.

We will also be analyzing principal parts at the poles of the Archimedean local Euler factors, and relate them to point evaluation functionals on global idelic Zeta functions supported at the trivial zeros.

7.3.1. Orbital Schauder Bases. The following Theorem describes the orthogonals of the closed subspaces $\mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ as $S_0 \subset S$ is a fixed subset of finite places.

SITUATION 7.3.1.1 (Saturated sets of places). Let F be a number field, S be a finite set of places of F containing all the Archimedean places. Let S_0 be an arbitrary finite set of places of F.

(1) We will term S_0 *saturated* if and only if for every prime number p, if a place v of F whose residue characteristic is p is contained in S_0, then all places of F whose residue characteristic is p are also contained in S_0.
(2) Let p be a prime number such that at least one place of F whose residue characteristic is p, is contained in S. Assume S is saturated.
(3) We choose, as in Situation 4.4.2.3 applied to the $\mathbf{R}_{>0}$-moduated group $G = J_{F,S}$ with $|\cdot|_G = |\cdot|_{F,S}$, a set of representatives $\widetilde{J^1_{F,S}}$ of the continuous characters of the compact Hausdorff abelian group $J^1_{F,S}$, equivalently a holomorphic parametrization for $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$ of the form

$$\chi|\cdot|^s_{F,S}, \quad s \in \mathbf{C}$$

for $\chi \in \widetilde{J^1_{F,S}}$.

We begin by constructing bi-orthogonal sets to the sets of continuous functionals given by the coefficients of the principal parts of the S-idelic Zeta functions (see Definition 7.1.2.1 for the definition of the functionals $a_m(\cdot, c)$, as $c \in \Omega(J_{F,S})$).

THEOREM 7.3.1.2 (Orbital bi-orthogonal sets). *Let F be a number field, S a saturated finite set of places containing all the Archimedean places. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let $S_0 \subset S$ be a non-empty subset containing all the Archimedean places.

Assume the notation and choices in Situation 7.3.1.1 are in force.

Let $\chi \in \widehat{J^1_{F,S}}$ be a continuous character.

Then:

(1) *The closed subspace* $\mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ *is the intersection of* $\ker(a_k(\cdot,c))$ *as c ranges over $\mathcal{U}_S$, and as k ranges over the integers $m_S(c) - m_{S_0}(c) < k \leq m_S(c)$.*

(2) *The orthogonal*

$$\mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))^{\perp} \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta} \tag{7.3.1.1}$$

with respect to the duality $\langle \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))' \rangle$ is linearly and topologically generated by the eigendistributions $\{a_k(\cdot,c)\}$ as c ranges over the points in $\mathcal{U}_S$ and, for each $c \in \mathcal{U}_S$, k ranges over the integers $m_S(c) - m_{S_0}(c) < k \leq m_S(c)$. The set $\{a_k(\cdot,c)\}$ admits a bi-orthogonal set in

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})).$$

In particular, it is topologically free.

PROOF. (1) follows from Theorem 7.2.1.1(4) applied to $\mathcal{U} = \mathcal{U}_{S_0}$, to $m := m_S - m_{S_0}$.

(2) : By (1) and Theorem 7.2.1.1(4), the set of S-local eigendistributions $\{a_k(\cdot,c)\}$ as c ranges over the points in $\mathcal{U}_S$ and, for each $c \in \mathcal{U}_S$, k ranges over the integers $m_S(c) - m_{S_0}(c) < k \leq m_S(c)$ linearly and topologically generates the orthogonal of $\mathrm{PV}_{S,S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ with respect to the duality $\langle \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta} \rangle$.

We let $\omega \in \widehat{J^1_{F,S}}$ be a continuous character, and we denote by $I_{S_0,\omega}$ the countable set of pairs:

$$I_{S_0,\omega} := \{(c,k) \in (\mathcal{U}_S \cap \Omega(J_{F,S})(\omega)) \times \mathbf{N} : m_{S_0}(c) < k \leq m_S(c)\}$$

and let $\{C_{(c,k)}\}_{(c,k)\in I_{S_0,\omega}}$ and $\{C'_{(c,k)}\}_{(c,k)\in I_{S_0,\omega}}$ be two sets of complex numbers such that for any finite subset $J \subset I_{S_0,\omega}$, the assignments $J \mapsto \sum_{(c,k)\in J} C_{(c,k)} a_k(\cdot,c)$, respectively $J \mapsto \sum_{(c,k)\in J} C'_{(c,k)} a_k(\cdot,c)$, is a convergent net in $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta}$.

We denote by $\sum_{(c,k)\in I_{S_0,\omega}} C_{(c,k)} a_k(\cdot,c)$, respectively $\sum_{(c,k)\in I_{S_0,\omega}} C'_{(c,k)} a_k(\cdot,c)$, their respective limits.

Assume we have, as elements of $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta}$:

$$\sum_{(c,k)\in I_{S_0,\omega}} C_{(c,k)} a_k(\cdot,c) = \sum_{(c,k)\in I_{S_0,\omega}} C'_{(c,k)} a_k(\cdot,c). \tag{7.3.1.2}$$

We select a single pair $(c_0, m_0) \in I_{S_0,\omega}$, and let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be a Schwartz-Bruhat function such that $Z_S(f,c)$ has a pole at $c = c_0$ of order $m_0 > m_{S_0}(c_0)$, and is holomorphic over $\Omega(J_{F,S}) - \{c_0\}$.

By continuity of both sides of (7.3.1.2), and the mode of convergence of the sums in (7.3.1.2) being uniform on bounded subsets, we deduce that the nets of complex numbers $J \mapsto \sum_{(c,k)\in J} C_{(c,k)} a_k(f,c)$, respectively $J \mapsto \sum_{(c,k)\in J} C'_{(c,k)} a_k(f,c)$, both converge, and their limits are $\langle \sum_{(c,k)\in I_{S_0,\omega}} C_{(c,k)} a_k(\cdot,c), f \rangle$, respectively $\langle \sum_{(c,k)\in I_{S_0,\omega}} C'_{(c,k)} a_k(\cdot,c), f \rangle$.

Since $a_k(f,c) = 0$ if either $c \neq c_0$ or if $k > m_0$, we deduce that the identity:

$$\sum_{m_{S_0}(c_0)<k\leq m_0} C_{(c_0,k)} a_k(f,c_0) = \sum_{m_{S_0}(c_0)<k\leq m_0} C'_{(c_0,k)} a_k(f,c_0) \tag{7.3.1.3}$$

Upon repeating the reasoning with f replaced by a Schwartz-Bruhat function $f' \in \mathcal{S}(\mathbf{A}_{F,S})$ such that $Z_S(f', c)$ has only one pole at c_0 of order $m_0 - 1$, we further obtain the identity:

$$\sum_{m_{S_0}(c_0)<k\leq m_0-1} C_{(c_0,k)} a_k(f', c_0) = \sum_{m_{S_0}(c_0)<k\leq m_0-1} C'_{(c_0,k)} a_k(f', c_0) \tag{7.3.1.4}$$

and upon iterating the reasoning m_0 times, we obtain a non-degenerate system of m_0 linear equations relating the $C_{(c_0,k)}$ to the $C'_{(c_0,k)}$, deducing the identities $C_{(c_0,k)} = C'_{(c_0,k)}$.

Since we selected $(c_0, m_0) \in I_{S_0,\omega}$ arbitrarily, we deduce the identities $C_{(c,k)} = C'_{(c,k)}$ for all $(c, k) \in I_{S_0,\omega}$.

By [**Jar**, Part II,Ch.14,§14.1,Prop.1.p.289], this implies that the countable set $\{a_k(\cdot, c)\}_{(c,k)\in I_{S_0,\omega}}$, and hence, for any countable subset $I \subset I_{S_0,\omega}$, the resulting countable subset $\{a_k(\cdot, c)\}_{(c,k)\in I}$, is topologically free.

We are reduced to showing that for any choice $(c_0, m_0) \in I_{S_0,\omega}$, there exists $f \in \mathcal{S}(\mathbf{A}_{F,S})$ such that $Z_S(f, c)$ is holomorphic over $\Omega(J_{F,S}) - \{c_0\}$, and the order of pole of $Z_S(f, c)$ at $c = c_0$ is exactly m_0.

We choose a holomorphic parametrization $\chi|\cdot|^s_{F,S}$ for the Riemann surface $\Omega(J_{F,S})$, i.e. we choose a continuous character $\chi \in \widehat{J_{F,S}}$ for each $\omega \in \widehat{J^1_{F,S}}$, and parametrize the connected component $\Omega(J_{F,S})(\omega)$ as $\chi|\cdot|^s_{F,S}$, with $s \in \mathbf{C}$ a global holomorphic coordinate.

The point $c_0 \in \Omega(J_{F,S})$ is therefore parametrized as $\chi_0|\cdot|^{s_0}_{F,S}$, for a unique χ_0, the base point with respect to the chosen holomorphic parametrization of $\Omega(J_{F,S})$, of the connected component of $\Omega(J_{F,S})$ containing c_0.

We let $\varphi \in \mathcal{D}(\mathbf{R}_{>0})$ such that the entire function:

$$\psi(s) := \int_{\mathbf{R}_{>0}} t^{s+ix_0}\varphi(t)\,\frac{\mathrm{d}t}{t}$$

satisfies $\psi(s_0) \neq 0$, for $x_0 \in \mathbf{R}$ the unique real number such that $\chi_0(t) = t^{ix_0}$ as smooth functions on $t \in \mathbf{R}_{>0}$.

We let $S_{c_0} := \{v_1, \ldots, v_{m_S(c_0)}\} \subset S$ be the subset of those places such that $c_0 \in \mathcal{U}_{S,v_1} \cap \ldots \cap \mathcal{U}_{S,v_{m_S(c_0)}}$, i.e. such that $\psi_{v_i}(c_0)$ is a pole of the local factor $L_{v_i}(c_{v_i})$ at v_i of the Hecke L-function $L(c)$.

There therefore exists a unique integer $N_{v_i} \in \mathbf{N}$ such that $c_0 = \chi|\cdot|^{\frac{2\pi i N_{v_i}}{\log(q_{v_i})} - it_{v_i}}_{F,S}$.

Recall that we have $m_0 > m_{S_0}(c_0)$.

The function:

$$u(s) := \prod_{1\leq i\leq m_0 : v_i\in S_{c_0}} \left\{ \frac{1}{s - \frac{2\pi i N_{v_i}}{\log(q_{v_i})} - it_{v_i}} (1 - q_{v_i}^{-s-it_{v_i}}) \right\} \psi(s)$$

has only removable singularities and we rename as u its unique holomorphic extension to the complex s-plane.

The entire function $u(s)$ has simple zeros at all the points $\{\chi|\cdot|_F^{\frac{2\pi i m}{\log(q_{v_i})} - it_{v_i}}\}_{m\in\mathbf{Z}}$ for all $1 \leq i \leq m_0$, where, recall, v_i is a non-Archimedean place contained in S_0 such that $\psi_{v_i}(c_0)$ is a pole of local-at-v_i L-functions, with the exception of the unique point $\psi^{-1}_{F,S}(c_0)$, which does not feature among such zeros.

The inverse Mellin transform ϕ of u is a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ with exponential decay. By Lemma 5.3.4.3, there exists a Schwartz-Bruhat function $f_0 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$ such that $\Phi_{F,S,1}(f_0) = \phi$.

We deduce that the function $f_1 := \chi^{-1} \cdot f_0$, also a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$ and an element of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ by Lemma 5.3.4.2(3), satisfies $\Phi_{F,S,\omega}(f_1) = \phi$.

Note that since $f_1 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$, f_1 is in the range of Weil's Principal Value operator PV_{S,S_0}, which is a linear topological embedding with closed range (see Lemma 5.3.1.5).

The Schwartz-Bruhat function $f := \mathrm{PV}^{-1}_{S,S_0}(f_1)$ satisfies the desired properties: $Z_S(f, c)$ has, over the connected component $\Omega(J_{F,S})(\omega)$, exactly one pole, c_0, with order of pole exactly m_0, and it is holomorphic away from c_0. □

We now start the study of the Fréchet spaces

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

through the study of certain subquotients which we will now introduce.

DEFINITION 7.3.1.3 (p-primary Orbital Quotients). Let F be a number field, S a finite set of places of F containing all the Archimedean places.

Assume the notation and choices in Situation 7.3.1.1 are in force.

We denote by:

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) := \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \tag{7.3.1.5}$$

endowed with the quotient topology.

We refer to $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$ as the *p-primary Orbital Quotient*.

We now return to the multiplicity function m_{S_0} defined in (7.2.2.2) and associated to a subset $S_0 \subset S$.

Recall that for $\mathcal{U}_{S_0}$ the subset defined as the union $\bigcup_{v\in S_0}\mathcal{U}_{S,v}$ (see (7.2.2.1)), where $\mathcal{U}_{S,v} := \psi_v^{-1}(\mathcal{U}_v)$, with $\psi_v : \Omega(J_{F,S}) \to \Omega(F_v^\times)$ the quotient surjective local analytic isomorphism defined by restriction of a continuous quasi-character c to $F_v^\times \subset J_{F,S}$, and where $\mathcal{U}_v$ is the set of poles (each simple) over $\Omega(F_v^\times)$ of the meromorphic function $\rho_v(c)$ from [**CF**, Ch.XV,§2.5,p.322] (i.e. the correcting meromorphic factor from the local-at-v functional equation; see Proposition 3.3.1.5), m_{S_0} counts the cumulative multiplicity of $c \in \mathcal{U}_{S_0}$ as a member of each $\psi_v^{-1}(\mathcal{U}_v)$, each counted with multipicity 1.

DEFINITION 7.3.1.4 (Primary Multiplicity Functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Assume the notation and choices in Situation 7.3.1.1 are in force.

We define the multiplicity function:

$$N_{S_p} : \mathcal{U}_{S_p} \to \mathbf{N} \tag{7.3.1.6}$$

to be $N_{S_p} := m_{S_p}$, the integer-valued function assigning to $c \in \mathcal{U}_{S_p}$ the value:

$$N_{S_p}(c) := \sum_{v\in S_p} \mathrm{ord}_{c'=\psi_v(c)}\rho_v(c'). \tag{7.3.1.7}$$

We refer to N_{S_p} as the *p-primary multiplicity function*.

The following Theorem provides absolute Schauder bases for the Fréchet spaces $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$, and unconditional bases for its strong dual.

We will be using the following Lemma:

LEMMA 7.3.1.5. *Let E_1, E_2 be nuclear Fréchet spaces, $H_k \subset E_k$ closed subspaces for $k = 1, 2$. Assume:*

(i) *there exists a continuous injective linear operator $A : E_1 \to E_2$ such that $A^{-1}(H_2) = H_1$.*
(ii) *The induced continuous injective linear operator $A_0 : E_1/H_1 \to E_2/H_2$ is surjective.*
(iii) *There exists a sequence $\{x_m\}_{m\geq 0} \subset E_2$ whose image in E_2/H_2 is an absolute Schauder basis.*
(iv) *For any continuous seminorm q on E_2/H_2, there exists a real-valued nonnegative function $F_q : \mathbf{N} \to \mathbf{R}_{\geq 0}$ such that:*

$$q(\pi_2(x_m)) \ll_q F_q(m) \tag{7.3.1.8}$$

as $m \to +\infty$.

Then:

(1) *For any choice of sequence $\{y_m\}_{m\geq 0} \subset E_1$ such that $A(y_m) = x_m$ for each integer $m \geq 0$, the image sequence $\{\pi_1(y_m)\}_{m\geq 0}$, with $\pi_1 : E_1 \to E_1/H_1$ the quotient map, is an absolute Schauder basis for E_1/H_1. Its bi-orthogonal sequence in $(E_1/H_1)'_\beta$ is an unconditional Schauder basis.*

(2) *The sequence* $\{\pi_1(y_m)\}_{m\geq 0} \subset E_1/H_1$ *satisfies the following condition: for any continuous seminorm* q *on* E_1/H_1*, there exists a real-valued nonnegative function* $F_q : \mathbf{N} \to \mathbf{R}_{\geq 0}$ *such that:*

$$q(\pi_1(y_m)) \ll_q F_{q\circ A_0^{-1}}(m) \tag{7.3.1.9}$$

as $m \to +\infty$*.*

Proof. We note that by Lemma 4.1.2.4, the bi-orthogonal sequence to $\{\pi_2(x_m)\}_{m\geq 0} \subset E_2/H_2$ is an unconditional Schauder basis. The Lemma applies since E_2, and hence E_2/H_2, being $H_2 \subset E_2$ is a Fréchet space, hence complete and barrelled, nuclear by [**Scha**, Ch.III,§7,Thm.7.4,p.103].

(1) : By (i) and [**NB**, Thm.14.4.5(b)], A_0 is a linear topological isomorphism. The convergence condition from [**Jar**, Part II,Ch.14,§14.7,p.314] defining absolute Schauder bases is preserved under linear topological isomorphisms, which also preserve topological bases in both directions, and hence Schauder bases, by [**Jar**, Part II,Ch.14,§14.2,Thm.5,p.293].

(2) : We let q be a continuous seminorm on E_1/H_1. Then $q \circ A_0^{-1}$ is a continuous seminorm on E_2/H_2 since A_0 is a linear topological isomorphism. The estimate in (7.3.1.9) folows from (iv). By Lemma 4.1.2.4, the bi-orthogonal sequence to $\{\pi_1(y_m)\}_{m\geq 0}$ is an unconditional Schauder basis in $(E_1/H_1)'_\beta$. □

We first recall the description of the poles of $\rho_S(c)$ from Lemma 7.1.2.2, in accordance with the holomorphic parametrization $\{\chi|\cdot|_{F,S}^s\}$ of $\Omega(J_{F,S})$ chosen in Situation 7.3.1.1.

For each $\chi \in \widetilde{J^1_{F,S}}$ and for each $v \in S_p$, there exists a unique real number $t_v \in \mathbf{R}$ contained within $[0, 2\pi/\log(q_v))$, for $q_v := |\pi_v|_v^{-1}$ with $\pi_v \in F_v^\times$ a choice of local uniformizer, such that:

$$\chi|\cdot|_F^{-it_v} \in (\pi_v^{\mathbf{Z}})^\perp \subset \widehat{J_{F,S}} \tag{7.3.1.10}$$

and for each $c \in \mathcal{U}_{S_p}$, we let $S_{p,c} \subset S_p$ be the subset $\{v_1, \ldots, v_{N_p(c)}\} \subset S_p$ of those finite places $v \in S_p$ such that $c \in \mathcal{U}_{S,v} = \psi_v^{-1}(\mathcal{U}_v)$.

For each $v \in S_{p,c}$, there therefore exists a unique integer $m_{c,v} \in \mathbf{Z}$ such that:

$$c = \chi|\cdot|_{F,S}^{\frac{2\pi i m_{c,v}}{\log(q_v)} - it_v}. \tag{7.3.1.11}$$

Note that for $v, w \in S_{p,c}$, the two real numbers t_v and t_w are related by a unique translation by an element of the discrete subgroup $\frac{2\pi\mathbf{Z}}{\log(p)} \subset \mathbf{R}$.

Theorem 7.3.1.6 (Orbital Schauder Bases - I). *Let* F *be a number field,* S *a saturated finite set of places containing all the Archimedean places. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Assume the notation and choices in Situation 7.3.1.1 are in force.

We denote by N_p *is the number of finite places of* F *whose residue characteristic is* p*.*

Let $\chi \in \widetilde{J^1_{F,S}}$ *be a continuous character.*

We define the two subsets:

$$P_{S,S_p}(\chi|_{J^1_{F,S}}) := \{a_k(\cdot, c) : m_S(c) - N_p(c) < k \leq m_S(c)\} \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta \tag{7.3.1.12}$$

and:

$$E_{S,S_p}(\chi|_{J^1_{F,S}}) := \{\widehat{\delta}_c^{(k)} : 0 \leq k \leq N_p(c) - 1\} \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta \tag{7.3.1.13}$$

where $\widehat{\delta}_c^{(k)}$ *is the continuous functional obtained as the composition of the Fourier transform*

$$\mathcal{F}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

followed by the Mellin transform L_S*, and by the* k*th derivative of the point-evaluation functional at* $\widehat{c} \in \Omega(J_{F,S})$*, i.e.* $\widehat{\delta}_c^{(k)}$ *is uniquely determined by the assignment, on the arbitrary* $f \in \mathcal{S}(\mathbf{A}_{F,S})$*:*

$$\widehat{\delta}_c^{(k)}(f) := \partial_s^k Z_S(\widehat{f}, \overline{\chi}|\cdot|_{F,S}^{1-s})|_{s=s_c} \tag{7.3.1.14}$$

with $s_c = \frac{2\pi i m_{c,v}}{\log(q_v)} - it_v$ *for a, equivalently any, choice of* $v \in S_p$ *such that* $c \in \mathcal{U}_{S,v}$.

Then the sets in (7.3.1.12)*, respectively* (7.3.1.13)*, satisfy the following properties:*

(1) $P_{S,S_p}(\chi|_{J^1_{F,S}})$ *and* $E_{S,S_p}(\chi|_{J^1_{F,S}})$ *are contained within the range of the continuous linear inclusion*

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})'_\beta \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_\beta.$$

(2) *Any enumeration of either countable set* $P_{S,S_p}(\chi|_{J^1_{F,S}})$ *and* $E_{S,S_p}(\chi|_{J^1_{F,S}})$ *is an unconditional Schauder basis for*

$$(\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})))'_\beta \tag{7.3.1.15}$$

equipped with the strong topology.

(3) *Their bi-orthogonal sets:*

$$P^\vee_{S,S_p}(\chi|_{J^1_{F,S}}), E^\vee_{S,S_p}(\chi|_{J^1_{F,S}}) \subset \mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) \tag{7.3.1.16}$$

equipped with the induced enumeration, are absolute Schauder bases.

(4) *The elements* $\psi_{(c,k)}$ *of the set* $E^\vee_{S,S_p}(\chi|_{J^1_{F,S}})$ *further satisfy the following property. For any continuous seminorm* q *on* $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$*, and for each* $v \in S_p$*, upon parametrizing the points of* $\mathcal{U}_{S_p,v} = \psi_v^{-1}(\mathcal{U}_v)$ *as:*

$$c = \chi| \cdot |_{F,S}^{\frac{2\pi i m_{c,v}}{\log(q_v)} - it_v}, m_{c,v} \in \mathbf{Z}$$

there exists a polynomial $P \in \mathbf{C}[X]$ *depending on* v *and such that, as* c *varies over* $\mathcal{U}_{S_p,v}$ *as a function of* $m_{c,v}$*, we have:*

$$q(\psi_{(c,k)}) \ll_{q,v} P(m_{c,v}) \tag{7.3.1.17}$$

uniformly with respect to $0 \leq k \leq N_p(c) - 1$*, as* $|m_{c,v}| \to +\infty$*, where the implied constant depends on the chosen enumeration.*

(5) *For every* $f \in \mathcal{S}(\mathbf{A}_{F,S})$*, and for each pair* (c,k) *as* c *varies over* $\mathcal{U}_{S_p,v}$*, then the sequence of complex numbers* $\{\widehat{\delta}_c^{(k)}(\Phi_{F,S,\chi|_{J^1_{F,S}}}(f))\}_{m_{c,v}\in\mathbf{Z}}$ *is of rapid decay, i.e. an element of* $\mathcal{S}(\mathbf{Z})$ *as a function of* $m_{c,v} \in \mathbf{Z}$.

Proof. (1) : We first treat the case $E_{S,S_p}(\chi|_{J^1_{F,S}})$.

We choose a continuous homomorphic section τ to the S-idelic norm. We call $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions on $\mathbf{R}_{>0}$.

We define $I := (0, +\infty) \subset \mathbf{R}$.

By Proposition 2.4.3.1, for every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, we have that $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$ is, as a smooth function on $J_{F,S}$, is contained within $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$. In particular, upon replacing f with $\widehat{f}$, so is $\Phi_{F,S,\overline{\chi}|_{J^1_{F,S}}}(\widehat{f})$.

We define $V_{S_p,\chi} \subset \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ the linear subspace linearly generated by the elements $|\cdot|_{F,S}\Phi_{F,S,\overline{\chi}|_{J^1_{F,S}}}(\widehat{f})$, i.e. by the functions defined as the product of the continuous quasi-character $|\cdot|_{F,S}$ and $\Phi_{F,S,\overline{\chi}|_{J^1_{F,S}}}(\widehat{f})$.

We endow $V_{S_p,\chi}$ with the quotient topology inherited from $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

We note that $V_{S_p,\chi}$ is sequentially dense within $\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ by Lemma D.3.6.1.

As discussed in Remark D.3.3.6, $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ is stable under multiplication by continuous quasi-characters of $J_{F,S}$ of arbitrary exponent.

By Proposition 5.3.4.4 combined with surjectivity of $\Phi_{F,S,\overline{\chi}|_{J^1_{F,S}}}$ as a continuous linear map:

$$\Phi_{F,S,\overline{\chi}|_{J^1_{F,S}}} : \mathcal{S}_{00}(\mathbf{A}_{F,S}) \to \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \tag{7.3.1.18}$$

where we refer the reader to (5.3.4.1) from Definition 5.3.4.1 for the construction of the closed subspace $\mathcal{S}_{00}(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$.

The surjectivity assertion for (7.3.1.18) follows from the combination of Lemma 5.3.4.3 together with continuity and continuous invertibility of the multiplication operator on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ by a continuous quasi-character of arbitrary exponent, which holds by Lemma 5.3.4.2(3).

We refer the reader to the proof of Proposition 5.3.4.4 for a similar argument.

$\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ is, as a linear subspace of $\mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, therefore contained within the linear image of $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi},\mathcal{S}(\mathbf{A}_{F,S}))$ into $\mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, and hence so is $W := |\cdot|^{-1}_{F,S}\cdot\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \subset V_{S_p,\chi}$ by another application of Lemma 5.3.4.2(3) and Remark D.3.3.6.

Since the linear map $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})) \to \mathcal{S}_I(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ is injective, we identify W with its pre-image in $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$ and rename such linear subspace of $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$ as W.

We further denote by $\widehat{W}$ the pre-image of W in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$ under the Fourier transform

$$\mathcal{F}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$$

from (5.2.4.1) of Theorem 5.2.4.1, a linear topological isomorphism by Theorem 5.2.4.1.

We are not topologizing W nor $\widehat{W}$.

The idea is to now use $\widehat{W}$ to produce within the subquotient:

$$(|\cdot|_{F,S}\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))/(|\cdot|_{F,S}\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))$$

a copy of $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})/P_X\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, where X denotes, following the notation of Situation 4.6.1.1 (see item (2)) from the previous subsection and from Theorem 4.6.1.5, the vector $X = (\pi_v, v \in S_p)$, for π_v a choice of local uniformizer in $F_v^\times$ for each $v \in S_p$, and for any choice of ordering on the finitely many elements of S_p, and P_X is the operator from (4.6.1.3) from Lemma 4.6.1.3.

Indeed, upon doing so, we will be deducing from Corollary 4.6.3.1 that the continuous linear map:

(7.3.1.19)

$$(|\cdot|_{F,S}\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))/(|\cdot|_{F,S}\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))) \to \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})/P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$$

is a linear topological isomorphism.

This entails showing that the intersection of $P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ with

$$(|\cdot|_{F,S}\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))$$

is indeed the closed subspace

$$(|\cdot|_{F,S}\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))$$

which amounts to using the Mellin transform L_S from (5.1.1.2) and the very definition of Weil's Principal Value operators PV_{S,S_p} from (5.3.1.6) of Definition 5.3.1.4, which L_S turns into a multiplication operator by a product of reciprocal Euler factors, and shows injectivity of (7.3.1.19).

Surjectivity is indeed given by the existence of a copy of $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ within $(|\cdot|_{F,S}\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})))$, constructed using $\widehat{W}$, for then using the surjectivity assertion in Corollary 4.6.3.1 applied to the $\mathbf{R}_{>0}$-modulated group $G = J_{F,S}$ and to the continuous character $\overline{\chi}$ of $J_{F,S}$.

Since source and target of (7.3.1.19) are Fréchet spaces, the continuous linear bijection in (7.3.1.19) is therefore a linear topological isomorphism by [**NB**, Thm.14.4.5(a)].

Theorem 4.6.1.5 and Lemma 7.3.1.5 (which applies by nuclearity of the source, by the First Structure Theorem, Theorem 5.2.3.1(2), and the target, by Theorem D.3.3.7(1)) combined, then provide us with the desired absolute basis property for the set $E_{S,S_p}(\chi|_{J^1_{F,S}})$ from (7.3.1.13), and with the estimates from (7.3.1.17) (see (4.6.1.17)).

We note that indeed the continuous functionals comprising the set $E_{S,S_p}(\chi|_{J^1_{F,S}})$ in (7.3.1.13) are the images under the continuous adjoint of the quotient operator $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) \to \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})/P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$,

of the continuous functionals defined in (4.6.1.14), which were shown to be an unconditional Schauder basis in Theorem 4.6.1.5, with its dual basis being absolute. We further refer the reader to the general Lemma 4.1.2.4, which applies since $E = \mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$ is a nuclear Fréchet space, hence Montel, hence complete barrelled and reflexive.

We now complete the argument by discussing the details of the above steps.

We inspect the following diagram of linear operators:

(7.3.1.20)

$$\begin{array}{ccccc}
\mathrm{PV}_{S,S_p}(\widehat{W}) & \xrightarrow{(1)} & \widehat{W} & \xrightarrow{(2)} & \widehat{W}/\mathrm{PV}_{S,S_p}(\widehat{W}) \\
\downarrow{\scriptstyle (a)} & & \downarrow{\scriptstyle (b)} & & \downarrow{\scriptstyle (c)} \\
\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{(3)} & \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{(4)} & \mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) \\
\downarrow{\scriptstyle (d)} & & \downarrow{\scriptstyle (e)} & & \downarrow{\scriptstyle (f)} \\
\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{(5)} & \mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \xrightarrow{(6)} & \mathcal{F}_S\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}) \\
\downarrow{\scriptstyle (g)} & & \downarrow{\scriptstyle (h)} & & \downarrow{\scriptstyle (i)} \\
P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \times P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) & \xrightarrow{(7)} & \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \times \mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}) & \xrightarrow{(8)} & \mathcal{S}_{X,\overline{\chi}}(J_{F,S}) \\
\downarrow{\scriptstyle (j)} & & \downarrow{\scriptstyle (k)} & & \downarrow{\scriptstyle (l)} \\
P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) & \xrightarrow[(9)]{} & \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) & \xrightarrow[(10)]{} & \mathcal{S}_{X,\overline{\chi}}(J_{F,S})
\end{array}$$

where:

$$\mathcal{S}_{X,\overline{\chi}}(J_{F,S}) := \mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})/P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \tag{7.3.1.21}$$

where the horizontal arrows are:

(1) is the inclusion linear operator induced by the inclusion of the closed subspace $\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ into $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

(2) is the quotient map resulting from the inclusion in (1).

(3) is the inclusion linear operator of the closed subspace $\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ into $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

(4) is the quotient map resulting from (3).

(5) is the inclusion linear operator of the closed subspace $\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ into

$$\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

obtained upon applying the linear topological isomorphism $\mathcal{F}_S$ from (5.2.4.1), inherited from the S-adelic Fourier transform $\mathcal{F}_S$ on $\mathcal{S}(\mathbf{A}_{F,S})$, onto the continuous inclusion with closed range from (3).

(6) is the quotient map resulting from the inclusion with closed range from (5).

(7) is the product of the inclusions of the closed subspaces $P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, respectively $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$, into $\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, respectively $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$.

(8) is the quotient induced by the inclusion with closed range from (7).

(9) is the inclusion of the closed subspace $P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ into $\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$.

(10) is the quotient map induced by the inclusion with closed range from (9).

and the vertical arrows are:

(a) is the inclusion induced by the inclusion of $\widehat{W}$ into $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, which we constructed throughout the discussion preceding the diagram in (7.3.1.20), resting on Proposition 5.3.4.4 and Lemma 5.3.4.2(3).

(b) is the inclusion of $\widehat{W}$ into $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, which we constructed throughout the discussion preceding the diagram in (7.3.1.20), resting on Proposition 5.3.4.4 and Lemma 5.3.4.2(3).

(c) is the linear inclusion induced by the top leftmost square being cartesian as a diagram of $\mathbf{C}$-linear maps among $\mathbf{C}$-vector spaces. To see this, we proceed as follows: we let $\phi \in \widehat{W}$ be of the form:

$$\phi = \mathrm{PV}_{S,S_p}(\psi) \tag{7.3.1.22}$$

for some $\psi \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$. We compute:

$$Z_S(\mathcal{F}_S(\phi), \overline{\chi}|\cdot|^{1-s}_{F,S}) = Z_S(\mathcal{F}_S(\mathrm{PV}_{S,S_p}(\psi)), \overline{\chi}|\cdot|^{1-s}_{F,S}) = \prod_{v\in S_p}(1-\chi(\mathfrak{p}_v)q_v^{-s})Z_S(\mathcal{F}_S(\psi), \overline{\chi}|\cdot|^{1-s}_{F,S}) \tag{7.3.1.23}$$

where we used the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1, and Weil's Identity from (3.4.1.6) of Theorem 3.4.1.1, to deduce the second of the two identities in (7.3.1.23).

We now use the assumption $\phi \in \widehat{W}$. We deduce that $|\cdot|_{F,S}\mathcal{F}_S(\phi) \in \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ (where the linear space $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ is thought as a linear subspace of $\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$). In particular, $(1-\chi(\mathfrak{p}_v)q_v^{-s})Z_S(\mathcal{F}_S(\psi), \overline{\chi}|\cdot|^{1-s}_{F,S})$ is entire, and since $Z_S(\mathcal{F}_S(\psi), \overline{\chi}|\cdot|^{1-s}_{F,S})$ can only have poles located over

$$\widehat{\mathcal{U}}_S = \{\widehat{c}, c \in \mathcal{U}_S\}$$

by Theorem 3.3.1.1(1), where $(1-\chi(\mathfrak{p}_v)q_v^{-s})$ has no zeros for every $v \in S_p$, we deduce that $Z_S(\mathcal{F}_S(\psi), \overline{\chi}|\cdot|^{1-s}_{F,S})$ is itself entire.

It follows that we have $|\cdot|_{F,S}\mathcal{F}_S(\psi) \in \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ by Proposition 5.3.4.4, and hence $\psi \in \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ by Remark D.3.3.6 and Lemma 5.3.4.2(3) (as earlier in the argument, $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ is identified with its linear copy within $|\cdot|_{F,S}\mathcal{F}_S\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$).

In particular, we have $\mathcal{F}_S(\psi) \in |\cdot|^{-1}_{F,S} \cdot \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ by another application of Remark D.3.3.6 combined with Lemma 5.3.4.2(3).

It follows that we have $\psi \in \widehat{W}$, and hence $\phi \in \mathrm{PV}_{S,S_p}(\widehat{W})$, which was the assertion to be shown.

(d) is the Fourier transform $\mathcal{F}_S$, restricted to the closed subspace $\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

(e) is the Fourier transform $\mathcal{F}_S$.

(f) is induced by the Fourier transform $\mathcal{F}_S$ from (d) and (e).

(g) is the linear topological embedding induced by the linear topological embedding in (h): we need to show that the linear topological embedding in (h) (see below) indeed maps $\mathcal{F}_S\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ into $P_X\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \times P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ and promotes the square within (7.3.1.20) whose edges are (5),(h),(g) and (7), to a cartesian diagram of $\mathbf{C}$-linear maps among $\mathbf{C}$-vector spaces.

To see this, we proceed as follows: by Theorem 5.4.1.1, (h) is a linear topological embedding with closed range.

Under Mellin transform (the operator L_S from (5.1.1.2), therefore incorporating in its definition the vertical translation on the complex s-plane by ix_0, for $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$, for τ the given choice of continuous homomorphic section to the S-idelic norm), it is isomorphic to the operator sending $\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)$ to

$$(Z_S(\widehat{f}, \overline{\chi}|\cdot|^{1+s}_{F,S}), Z_S(f, \chi|\cdot|^{s}_{F,S})) \tag{7.3.1.24}$$

and if the pair in (7.3.1.24) is of the form:

$$(\prod_{v\in S}(1-\chi(\mathfrak{p}_v)q_v^{-s})\phi(s), \prod_{v\in S}(1-\chi(\mathfrak{p}_v)q_v^{-s})\psi(s)) \tag{7.3.1.25}$$

then $Z_S(f, \chi|\cdot|^s_{F,S})$ has no poles over $\mathcal{U}_{S_p}$ by the S-local functional equation of Theorem 3.3.1.1 (since all its poles are located at the poles of $\rho_S(\chi|\cdot|^s_{F,S})$, and multiplication by $\prod_{v\in S}(1-\chi(\mathfrak{p}_v)q_v^{-s})$ removes exactly

the poles located over $\mathcal{U}_{S_p}$, if any, by introducing zeros of $Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1+s})$, and hence of $Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1-s})$, that remove the singularities of $\rho_S(\chi|\cdot|_{F,S}^{s})$ located over $\mathcal{U}_{S_p}$. Here we are using the assumption $\prod_{v\in S}(1-\chi(\mathfrak{p}_v)q_v^{-s})\phi(s)=Z_S(\widehat{f},\overline{\chi}|\cdot|_{F,S}^{1+s})$, where $\psi\in A\mathcal{S}(\mathbf{C}_{>0})$).

By Theorem 7.3.1.2(2) applied to $S_0=S_p$, we have

$$\Phi_{F,S,\chi|_{J^1_{F,S}}}(f)\in \mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$$

which shows that the square within (7.3.1.20) whose edges are (5),(h),(g) and (7), is cartesian, as a diagram of $\mathbf{C}$-linear maps among $\mathbf{C}$-vector spaces, which is the assertion to be shown.

(h) is the inclusion of the closed space on the source, into the target, obtained as follows: by Theorem 5.4.1.1, upon choosing a continuous homomorphic section τ to the S-idelic norm, and calling $\omega:=\chi|_{J^1_{F,S}}$, the operator $U_{S,\tau}(\Phi_{F,S,\omega}(f)):=(Z_S(f,c),Z_S(\widehat{f},\widehat{c}))$ is a linear topological embedding of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$ into $A\mathcal{S}(\mathbf{C}_{>0})\times A\mathcal{S}(\mathbf{C}_{<1})$. The operator $U_{S,\tau}$ is isomorphic, under conjugation by the Mellin transform (L_S from (5.1.1.2), i.e. incorporating the vertical translation by ix_0 on the complex s-plane, for $x_0\in\mathbf{R}$ the unique real number such that $\chi(\tau(t))=t^{ix_0}$ as smooth functions of $t\in\mathbf{R}_{>0}$) to the operator sending $\Phi_{F,S,\omega}(f)$ to the pair of functions $(\Phi_{F,S,\omega}(f),|\cdot|_{F,S}J\Phi_{F,S,\omega}(\widehat{f}))$, for $J:\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})\to\mathcal{S}_{-I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ the inversion linear topological automorphism (see (5.4.2.6)). This is, by Theorem 5.4.2.2, itself a linear topological embedding.

Finally, the operator in (h) sends the element $\Phi_{F,S,\omega}(\widehat{f})$ to the pair $(|\cdot|_{F,S}\Phi_{F,S,\omega}(\widehat{f}),\Phi_{F,S,\omega}(f))$. We deduce from the foregoing that this assignment is itself a linear topological embedding.

(i) is an injective continuous linear operator. Injectivity follows from the diagram comprised by the arrows (g),(7),(5) and (h) being cartesian as a diagram of $\mathbf{C}$-linear maps among $\mathbf{C}$-vector spaces, which was shown in point (g) above.

(j) is the projection onto the first factor.

(k) is the projection onto the first factor.

(l) is a linear topological isomorphism, noting that the quotient map in (8) annihilates the second factor in the product $\mathcal{S}_{-1+I}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})\times\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$, because $0\notin I$, and so $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})=\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ (because the zeros of $\prod_{v\in S_p}L_v(\chi_v|\cdot|_v^s)$ are all located on the vertical line $i\mathbf{R}$, and so the orthogonal of $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})\subset\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ with respect to the duality

$$\langle\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}),\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})'\rangle$$

is trivial. It follows that $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ is $\sigma(\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}}),\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})')$-dense in $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$. Since, by Theorem D.3.3.7(1), $\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ is Montel, $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$ is strongly dense by [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.230], and being a closed subspace by Lemma 4.6.1.3(1) applies to $G=J_{F,S}$, $|\cdot|_G=|\cdot|_{F,S}$, $S_0=X$, we indeed have $P_X\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})=\mathcal{S}_I(J_{F,S})(\chi|_{J^1_{F,S}})$.

In light of the foregoing, we know that the continuous linear map:

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})\to\mathcal{S}_{X,\overline{\chi}}(J_{F,S}) \tag{7.3.1.26}$$

is a continuous injective linear operator.

Since $\widehat{W}$ is mapped by it onto a copy of $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ within $|\cdot|_{F,S}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$, such that the restriction to it agrees with the surjective (by Corollary 4.6.3.1) linear operator $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})\to\mathcal{S}_{X,\overline{\chi}}(J_{F,S})$, we deduce that the composition of (c),(f),(i) and (l) is surjective, and hence so is (7.3.1.26). This finishes the proof of the Theorem in the case $E_{S,S_p}(\chi|_{J^1_{F,S}})$.

The case of $P_{S,S_p}(\chi|_{J^1_{F,S}})$, the set of coefficients of principal parts from (7.3.1.12), follows from the previous case by Theorem 7.2.1.4, which relates the functionals $\widehat{\delta}_c^{(k)}$ to the functionals $a_k(\cdot,c)$.

Note that the correcting coefficients involved in passage from a set of functionals to the other do not preserve the growth and decay conditions asserted in part (4).

To see this, we compute them: from Theorem 7.2.1.4, they are either residues or special values of the local factors of the Hecke L-function corresponding to the finite or Archimedean places in S, or special values of the reciprocal Euler factors of the Hecke L-function corresponding to the finite or Archimedean places in S (through $\widehat{L}_S(\widehat{c})$, defined in (7.2.1.12)). Since the logarithms of prime numbers are $\mathbf{Q}$-linearly independent, the special values of $1/L_v(\chi_v|\cdot|_v^s)$ at the points $s = \frac{2\pi im}{\log(q_w)} - it_w$, for $v \in S - S_p$ and $w \in S_p$, and for t_w the unique real number $t_w \in [0, 2\pi/\log(q_w))$, with $q_w = N(\mathfrak{p}_w)$ for $\mathfrak{p}_w \subset \mathcal{O}_F$ the unique prime ideal corresponding to the finite place $w \in S$, such that $\chi|\cdot|_{F,S}^{\frac{2\pi im}{\log(q_w)} - it_w} \in \pi_v^{\mathbf{Z}}$, oscillate too much, since q_w is a prime power coprime to the residue characteristic of the non-Archimedean local field F_v: a consequence of Proposition B.5.3.3. □

Definition 7.3.1.7 (Orbital Schauder Bases - I). Let F be a number field, S a saturated finite set of places of F containing all the Archimedean places. Assume the notation and choices in Situation 7.3.1.1 are in force.

Let $\chi \in \widetilde{J^1_{F,S}}$ be a choice of continuous character from the chosen set of representatives $\widetilde{J^1_{F,S}}$ for $\widehat{J^1_{F,S}}$. Let p be a prime number such that at least one finite place of F whose residue characteristic is p, is contained in S.

We refer to the sets $P_{S,S_p}(\chi|_{J^1_{F,S}})$ and their bi-orthogonals $E_{S,S_p}(\chi|_{J^1_{F,S}})$ from Theorem 7.3.1.6, as the:

(1) *Polar Orbital Schauder System*, for $P_{S,S_p}(\chi|_{J^1_{F,S}})$ the set from (7.3.1.12) and $P^{\vee}_{S,S_p}(\chi|_{J^1_{F,S}})$ is its bi-orthogonal.
(2) *Orbital Fourier Schauder System* is $E_{S,S_p}(\chi|_{J^1_{F,S}})$ is the set from (7.3.1.13) and $E^{\vee}_{S,S_p}(\chi|_{J^1_{F,S}})$ is its bi-orthogonal.

In light of the topological properties of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathrm{H}_{0,\mathrm{cont}}(F^{\times}, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^{\times}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S}))$ and their isotypes, we deduce the following result on the local and global Spectral and Orbital Schauder Bases.

Finer information on the Orbital Schauder bases will be shown in the first two subsections of the next section, where the main results of this Chapter are discussed.

Corollary 7.3.1.8. *Let F be a number field, S a finite set of places containing all the Archimedean places, $\chi \in \widehat{C_F}$ a continuous character, unramified outside S. Let χ_S be the unique continuous character of $J_{F,S}$ obtained as the descent of $\chi|_{C_{F,S}}$ by $\widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}$-invariance.*

Then the Orbital Schauder Systems from Theorem 7.3.1.2(3) are such that for any choice of enumeration, the resulting Schauder bases satisfy the following conditions:

(1) *They are equicontinuous, shrinking and boundedly complete bases.*
(2) *Their dual bases are equicontinuous, shrinking and boundedly complete bases.*

Proof. We observe that [**Jar**, Part II,Ch.14,§14.3,Cor.3,p.296] applies since, by Theorem 5.4.1.1 and by Theorem 6.2.1.1 (see also Corollary 6.2.1.2), $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, are barrelled by [**NB**, Thm.11.7.2], since they are Fréchet spaces. Their strong duals are Montel by Theorem 5.2.3.1(2). It follows that said bases and their duals are equicontinuous Schauder bases. Being $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$, $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ and their strong duals Montel by Theorem 5.2.3.1(2), they are reflexive, and by [**Jar**, Part II,Ch.14,§14.5,Thm.1(a)], all said bases are shrinking and boundedly complete. □

7.3.2. Paley-Wiener Levels. We let F be a number field, S a finite set of places of F containing the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character.

In this subsection we construct a one-parameter family of closed subspaces of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, incorporating a support condition on the Fourier transform.

Their aim is to enable an approximation method to perform the final trace calculation of this work in Chapter 7.

The one-parameter family of closed subspaces of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ we are about to construct, collects "most" of the continuous left-translation action of $J_{F,S}$, and it is where most of the trace will be "concentrated".

We will begin explaining this at the end of this Chapter, when we discuss the existence and role of the periodic orbits and fixed points that correspond to the Schauder bases of point-evaluation functionals we are in the process of identifying (see Theorem 7.3.1.6).

Definition 7.3.2.1 (Absolute Paley-Wiener spaces of level X). Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let S_0 be a saturated finite set of finite places of F disjoint from S, and define $S' := S \cup S_0$.

Let $\chi_{S'}$ be the unique continuous character of $J_{F,S'}$ obtained by descending by $\widehat{\mathcal{O}}^\times_{F,\Sigma_F - S'}$-invariance the restriction of χ to $C_{F,S'}$.

Let $X > 0$ be a real number.

We define:

(1) The linear subspace:

$$\mathcal{S}(\mathbf{A}_{F,S'})_X \subset \mathcal{S}(\mathbf{A}_{F,S'}) \tag{7.3.2.1}$$

of those Schwartz-Bruhat functions over $\mathbf{A}_{F,S'}$ of the following form:

$$\pi_{S,S'}(f) = f \otimes \bigotimes_{v \in S_0} 1_{\mathcal{O}^\times_{F_v}} \tag{7.3.2.2}$$

where $\pi_{S,S'}$ is the transition operator $\mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S'})$ from (2.1.4.2) of Proposition 2.1.4.1, and $f \in \mathcal{S}(\mathbf{A}_{F,S})$ is a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$ whose Fourier transform is compactly supported, with support contained within the hypercube:

$$Q_X := \prod_{v \in S} \{x_v \in F_v : |x_v|_v \in [-X, X]\} \subset \mathbf{A}_{F,S}. \tag{7.3.2.3}$$

We further define

$$\mathcal{S}_0(\mathbf{A}_{F,S'})_X \subset \mathcal{S}(\mathbf{A}_{F,S'})_X \tag{7.3.2.4}$$

to be the codimension-2 subspace obtained by intersecting $\mathcal{S}(\mathbf{A}_{F,S'})_X$ with $\ker(\delta_0)$ and $\ker(\widehat{\delta_0})$ within $\mathcal{S}(\mathbf{A}_{F,S'})$.

(2) Let ρ_{S_0} be the Schwartz-Bruhat function on $\widehat{\prod}_{v \in S_0}(F_v, \mathcal{O}_{F_v})$:

$$\rho_S := \bigotimes_{v \notin S_0} 1_{\mathcal{O}_{F_v}} \in \mathcal{S}\left(\widehat{\prod}_{v \in S_0}(F_v, \mathcal{O}_{F_v})\right). \tag{7.3.2.5}$$

We define the linear subspaces:

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.2.6}$$

and

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \tag{7.3.2.7}$$

to be the closure of the linear images under $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}$ of $\mathcal{S}(\mathbf{A}_{F,S'})_X$, respectively of $\mathcal{S}_0(\mathbf{A}_{F,S'})_X$. We refer to $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'}))$, as the *absolute Paley-Wiener subspaces of level* X in $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

We abbreviate the terminology and refer to the absolute Paley-Wiener levels as the *absolute PW-levels*.

We begin by describing the functions $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f)$ on $J_{F,S'}$ as f varies over $\mathcal{S}(\mathbf{A}_{F,S'})_X$ and $\mathcal{S}_0(\mathbf{A}_{F,S'})_X$.

PROPOSITION 7.3.2.2 (Semigroup $J_{F,S'}^{\geq 1}$-action on the absolute PW-levels). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.*

Let S_0 be a saturated finite set of finite places of F disjoint from S, and define $S' := S \cup S_0$.

Let $\chi_{S'}$ be the unique continuous character of $J_{F,S'}$ obtained by descending by $\widehat{\mathcal{O}}_{F,\Sigma_F - S'}^{\times}$-invariance the restriction of χ to $C_{F,S'}$.

Let $X > 0$ be a real number.

(1) *If $Y > X$, then we have:*

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{PW}_Y(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.2.8}$$

and:

$$\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \subset \mathrm{PW}_{Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{7.3.2.9}$$

The directed unions $\mathrm{PW}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{PW}_0(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, of the linear subspaces $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, as $X \to +\infty$, is sequentially dense in $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'}))$.

(2) *There exists a real number $\lambda(X) > 0$ depending on X and χ, such that*

$$\overline{\pi}_{S'}(\Phi_{F,\chi_{S'}|_{J^1_{F,S'}}}(f)) = \Phi_{F,\chi_{S'}|_{J^1_{F,S'}}}(f \otimes \rho_{\Sigma_F - S'})$$

for $\rho_{\Sigma_F - S'} := \bigotimes_{v \notin S'} 1_{\mathcal{O}_{F_v}}$, as a function of $g \in C_F$, is supported over $C_F^{\geq 1/\lambda(X)}$, for every $f \in \mathcal{S}_0(\mathbf{A}_{F,S'})_X$.

(3) *There exists a real number $\lambda(X) > 0$ depending on X and χ, such that $\Phi_{F,\chi_{S'}|_{J^1_{F,S'}}}(\widehat{f})$, as a function of $g \in J_{F,S'}$, is supported over $J_{F,S'}^{\leq \lambda(X)}$.*

(4) *The linear subspaces $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, are stable under the continuous left-translation action of the semigroup $J_{F,S'}^{\geq 1}$.*

PROOF. (1) : Let $f \in \mathcal{S}_0(\mathbf{A}_{F,S'})_X$. Then $f = f_0 \otimes \rho_{S_0}$, and $\widehat{f_0}$ is compactly supported and its support is contained in the hypercube $Q_X \subset \mathbf{A}_{F,S}$.

If $Y > X$, then we have $Q_X \subset Q_Y$, and hence we have $f \in \mathcal{S}_{Y,0}(\mathbf{A}_{F,S'})$, which gives $\Phi_{F,\chi_{S'}|_{J^1_{F,S'}}}(f_0 \otimes \rho_{S_0}) \in \mathrm{PW}_{Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

Upon showing that the directed union of the $\mathcal{S}_0(\mathbf{A}_{F,S'})_X$ within $\ker(\delta_0) \cap \ker(\widehat{\delta_0}) \subset \mathcal{S}(\mathbf{A}_{F,S}) \otimes \rho_S$ is sequentially dense, then so is the directed union of the $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ within $\overline{\pi}_{S,S'}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))$ (where $\overline{\pi}_{S,S'}$ is the transition operator, a linear topological embedding, induced by $\pi_{S,S'}$ compatibly with $\Phi_{F,S}$, respectively $\Phi_{F,S'}$, as per the First Structure Theorem, Theorem 5.2.1.1), since $\Phi_{F,S,\chi_S|_{J^1_{F,S}}}$ is open.

The directed union of the $\mathcal{S}_0(\mathbf{A}_{F,S'})_X$ within $\ker(\delta_0) \cap \ker(\widehat{\delta_0}) \subset \mathcal{S}(\mathbf{A}_{F,S}) \otimes \rho_{S_0}$ contains the linear subspace

$$\mathcal{F}_{S'}^{-1}((\mathcal{D}(\mathbf{A}_{F,S}) \otimes \rho_{S_0}) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0})) = \mathcal{F}_{S'}^{-1}((\mathcal{D}(\mathbf{A}_{F,S}) \otimes \rho_{S_0})) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0}).$$

Since the Fourier transform $\mathcal{F}_{S'}$ is a linear topological automorphism of $\mathcal{S}(\mathbf{A}_{F,S'})$, it is enough to show that the linear subspace:

$$V_0 := \mathcal{D}(\mathbf{A}_{F,S}) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0}) \subset \mathcal{S}(\mathbf{A}_{F,S}) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0})$$

is sequentially dense.

Let $f_1, f_2 \in \mathcal{S}(\mathbf{A}_{F,S})$ be such that $f_1(0) = 0$ and $\widehat{f_1}(0) \neq 0$, $f_2(0) \neq 0$ and $\widehat{f_2}(0) = 0$.

We choose f_1 and f_2 to be compactly supported.

We have:

$$(\mathcal{S}(\mathbf{A}_{F,S}) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0})) \oplus \mathbf{C} \cdot f_1 \oplus \mathbf{C} \cdot f_2 = \mathcal{S}(\mathbf{A}_{F,S}) \tag{7.3.2.10}$$

linearly and topologically.

By [**Sch-a**, Ch.VII,§3,Thm.III,p.237], $\mathcal{D}(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ is sequentially dense.

The closure in the direct sum within the left side of (7.3.2.10) of $X_0 \oplus \mathbf{C} \cdot f_1 \oplus \mathbf{C} \cdot f_2$ contains $\mathcal{D}(\mathbf{A}_{F,S})$ by [**Jar**, Part I,Ch.4,§4.5,Cor.4,p.82], and hence it cannot be a proper closed subspace. It follows that V_0 is sequentially dense in $\ker(\delta_0) \cap \ker(\widehat{\delta_0})$.

The case of $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, without incorporating the two linear conditions $f(0) = \widehat{f}(0) = 0$, is seen by means of the same argument.

(2) and (3) : By Theorem 2.2.3.2, we have:

$$\Phi_{F,\chi|_{C^1_F}}(f \otimes \rho_{\Sigma_F - S'})(g) = |g|_F^{-1} \Phi_{F,\overline{\chi}|_{C^1_F}}(\widehat{f} \otimes \rho_{\Sigma_F - S'})(g^{-1})$$

as continuous functions on C_F. It is therefore enough to show that there exists a real number $\lambda(X) > 0$ such that $\Phi_{F,\overline{\chi}|_{C^1_F}}(\widehat{f} \otimes \rho_{\Sigma_F - S'})$ vanishes outside of $C_F^{\leq \lambda(X)}$.

We let $f = f_0 \otimes \rho_{S_0}$ for some $f_0 \in \mathcal{S}(\mathbf{A}_{F,S})$ whose Fourier transform is compactly supported, with support contained within $Q_X \subset \mathbf{A}_{F,S}$ (see (7.3.2.3)), and $f_0(0) = \widehat{f_0}(0) = 0$.

We call $f_1 := \widehat{f_0}$ and note that f_1 is supported over $Q_X \subset \mathbf{A}_{F,S}$. Moreover, we have $\widehat{f} = f_1 \otimes \rho_{S_0}$ since χ is unramified outside S.

We let τ be a continuous homomorphic section to the idelic norm with image contained within $F_{v_0}^{\times} \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_F)$ for a choice $v_0 \in S_\infty$ of Archimedean place.

The condition on $(\xi, g_0, t) \in F^{\times} \cdot E_F^1 \times \mathbf{R}_{>0}$, for E_F^1 the global idelic fundamental domain from (A.2.3.14) of Definition A.2.3.6:

$$\widehat{f}(\xi g_0 \tau(t)) \neq 0 \tag{7.3.2.11}$$

reads $|\xi(g_0)_v|_v \leq 1$ for all $v \in \Sigma_{F,f} - S_\chi$, $|\xi(g_0)_v|_v = 1$ for all $v \in S_\chi$, and $|\xi(g_0)_v|_v \leq X$ for all $v \in S_\infty - \{v_0\}$, and $|\xi(g_0)_{v_0} \tau(t)|_{v_0} \leq X$.

We then obtain that ξ must vary over the nonzero elements of a fractional ideal $I_\chi \subset F^{\times}$, whose image into $\mathbf{A}_{F,S}$ is a discrete subset $I_{S,\chi} \subset \mathbf{A}_{F,S}$.

Since E_F^1, and hence its image in $\mathrm{GL}_1(\mathbf{A}_{F,S})$ under the continuous quotient $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S})$, is relatively compact, we deduce that for (7.3.2.11) to hold, we must have $|\xi|_v \leq X_v$ for each $v \in S - \{v_0\}$ and for real numbers $X_v \geq X$. There exists a real number $X_{v_0} \geq X$ such that $|\xi(g_0)_{v_0} \tau(t)|_{v_0} \leq X$ for every $g_0 \in E_F^1$, implies that $|\xi \tau(t)|_{v_0} \leq X_{v_0}$ also holds.

Since, for any fixed $Y > 0$, $|\xi|_{v_0} \leq Y$ holds for at most finitely many $\xi \in I_{S,\chi}$, in light of discreteness of $I_{S,\chi}$ and the fact that the boundedness conditions $|\xi|_v \leq X_v$ for all Archimedean places $v \neq v_0$ already all hold, we deduce that the condition $t = |\xi \tau(t)|_{v_0} \leq X_{v_0}$ holds only for $t \in (0, \lambda(X))$ for some $\lambda(X) > 0$.

Formation of $I_{S,\chi}$ does not depend on f_1, but only on χ.

Alternatively, we observe that we have $|I_{S,\chi}|_{F,S} = \prod_{v \in S} |I_{S,\chi}|_v \subset [a, +\infty)$ for some real number $a > 0$.

Moreover, we have:

$$Q_X \cap \mathrm{GL}_1(\mathbf{A}_{F,S}) \subset B_X := \{x \in \mathrm{GL}_1(\mathbf{A}_{F,S}) : |x|_{F,S} \leq X^{|S|}\}$$

and since $I_{S,\chi}^{-1} \cdot B_X \subset a^{-1} \cdot B_X$, and since the image of E_F^1 in $\mathrm{GL}_1(\mathbf{A}_{F,S})$ is relatively compact, hence contained in a large enough compact subset $K \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$, we deduce that (7.3.2.11) holds so long as $\tau(t) \in K^{-1} \cdot a^{-1} \cdot B_X$, a compact subset of $\mathbf{A}_{F,S}$.

The subset $(K^{-1} \cdot a^{-1} \cdot B_X) \times \prod_{v \in S'} \mathcal{O}_{F_v}$ is compact within $\mathbf{A}_{F,S'}$.

There therefore exists $\lambda(X) > 0$ such that $\tau([\lambda(X), +\infty)) \subset \mathrm{GL}_1(\mathbf{A}_{F,S'})$ does not intersect $((K^{-1} \cdot a^{-1} \cdot B_X) \times \prod_{v \in S'} \mathcal{O}_{F_v}) \cap \mathrm{GL}_1(\mathbf{A}_{F,S'})$ within $\mathrm{GL}_1(\mathbf{A}_{F,S'})$. This proves (3).

(4) : To see this, we choose a continuous homomorphic section τ to the idelic norm, whose image is contained within $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_F)$.

Let $g_0 \in \mathrm{GL}_1(\mathbf{A}_{F,S'})^1$ be an arbitrary element. Then if $f_0 \in \mathcal{S}(\mathbf{A}_{F,S})$ is such that $\widehat{f_0}$ is supported over $Q_X \subset \mathbf{A}_{F,S}$ and $f_0(0) = \widehat{f_0}(0) = 0$, the element $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_0 \otimes \rho_{S_0})(g_0^{-1}x)$ as a continuous function of $x \in J_{F,S'}$, equals $\overline{\chi}_{S'}(g_0)\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_0 \otimes \rho_{S_0})(x)$, i.e. a nonzero scalar multiple of $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_0 \otimes \rho_{S_0})$, and it is therefore again an element of $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

We let $t \in \mathbf{R}_{>0}$ be an arbitrary positive real number. Then as functions of $x \in \mathbf{A}_{F,S'}$, both $f_0 \otimes \rho_{S_0}(x)$ and $f_0 \otimes \rho_{S_0}(\tau(t)^{-1}x)$ are in the intersection of the two hyperplanes $\ker(\delta_0)$ and $\ker(\widehat{\delta_0})$.

Since $\widehat{f_0}$ is compactly supported over the hypercube $Q_X \subset \mathbf{A}_{F,S}$, then $\mathcal{F}_{S'}(f_0 \otimes \rho_{S_0}(\tau(t)^{-1}x)) = t \cdot \widehat{f_0}(\tau(t)x)\rho_{S_0}(x_{S'})$ (where $x_{S'}$ is the projection of x onto the S'-indexed coordinates) is nonzero only if $(\tau(t)x)_S = \tau(t)x_S \in Q_X \subset \mathbf{A}_{F,S}$, which reads $x_S \in \tau(t)^{-1} \cdot Q_X \subset Q_X$.

This shows that the effect of the continuous left translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S'})^{\geq 1}$ on $f_0 \otimes \rho_{S_0}$ preserves the defining conditions of $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

We deduce that we have:

$$\rho_{F,S'}(J^{\geq 1}_{F,S'}, \mathrm{PW}_{S,X,0}(\chi|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))) \subset \mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

and hence $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ is a continuous sub-representation of the abelian semigroup $J^{\geq 1}_{F,S'}$, upon endowing $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ with the subspace topology inherited from $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

The same argument also applies to show the same assertion for $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ □

The reason why we termed the closed subspaces

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

and its closed codimension-2 subspace $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *absolute* Paley-Wiener Levels, is that the defining constraint on the Fourier transform of the elements of $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ is assigned by requiring that the S-local tensor factor of the arbitrary representative

$$\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_S \otimes \rho_{S_0})$$

for ρ_{S_0} as defined in (7.3.2.5) and $f_S \in \mathcal{S}_0(\mathbf{A}_{F,S})$ (such representative always existing by the Fifth Structure Theorem.

See Theorem 6.2.1.1) to have Fourier transform $\widehat{f_S}$ that is supported over the compact hypercube $Q_X \subset \mathbf{A}_{F,S}$ of (7.3.2.3), as opposed to being supported over the closed $\mathcal{O}^{\times}_{F,S}$-invariant subset

$$\widetilde{Q}_Y := \{x \in \mathbf{A}_{F,S} : \prod_{v \in S} |x|_v \leq Y\}.$$

Note that for $Y = X^{|S|}$, the latter contains the former.

We will now discuss and study a variant of the absolute Paley-Wiener Levels, by using $\widetilde{Q}_Y$ as opposed to Q_X: we will be terming them *invariant Paley-Wiener Levels*, and the absolute Paley-Wiener levels will be used to provide examples of elements of the invariant Paley-Wiener Levels.

Definition 7.3.2.3 (Invariant Paley-Wiener spaces of level Y). Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let S_0 be a saturated finite set of finite places of F disjoint from S, and define $S' := S \cup S_0$.

Let $\chi_{S'}$ be the unique continuous character of $J_{F,S'}$ obtained by descending by $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F - S'}$-invariance the restriction of χ to $C_{F,S'}$.

Let $Y > 0$ be a real number.

We define:

(1) The linear subspace:

$$\widetilde{\mathcal{S}}(\mathbf{A}_{F,S'})_Y \subset \mathcal{S}(\mathbf{A}_{F,S'}) \tag{7.3.2.12}$$

of those Schwartz-Bruhat functions over $\mathbf{A}_{F,S'}$ of the following form:

$$\pi_{S,S'}(f) = f \otimes \bigotimes_{v \in S_0} 1_{\mathcal{O}_{F_v}^\times} \tag{7.3.2.13}$$

where $\pi_{S,S'}$ is the transition operator $\mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S'})$ from (2.1.4.2) of Proposition 2.1.4.1, and $f \in \mathcal{S}(\mathbf{A}_{F,S})$ is a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$ whose Fourier transform is compactly supported, with support contained within the hypercube:

$$\widetilde{Q}_Y := \{x \in \mathbf{A}_{F,S} : \prod_{v \in S} |x|_v \leq Y\} \subset \mathbf{A}_{F,S}. \tag{7.3.2.14}$$

We further define:

$$\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y \subset \widetilde{\mathcal{S}}(\mathbf{A}_{F,S'})_Y \tag{7.3.2.15}$$

to be the codimension-2 subspace obtained by intersecting $\widetilde{\mathcal{S}}(\mathbf{A}_{F,S'})_Y$ with $\ker(\delta_0)$ and $\ker(\widehat{\delta_0})$ within $\mathcal{S}(\mathbf{A}_{F,S'})$.

(2) The linear subspaces:

$$\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.2.16}$$

and

$$\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \tag{7.3.2.17}$$

to be the closures of the linear images under $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}$ of $\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y$, respectively under $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}$ of $\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y$.

We refer to $\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, as the *invariant Paley-Wiener subspace of level* Y in $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'}))$.

We abbreviate the terminology and refer to the invariant Paley-Wiener levels as the *invariant PW-levels*.

The following is an analog of Proposition 7.3.2.2:

Proposition 7.3.2.4 (Semigroup $J^{\geq 1}_{F,S'}$-action on the invariant PW-levels). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.*

Let S_0 be a saturated finite set of finite places of F disjoint from S, and define $S' := S \cup S_0$.

Let $\chi_{S'}$ be the unique continuous character of $J_{F,S'}$ obtained by descending by $\widehat{\mathcal{O}}^\times_{F,\Sigma_F - S'}$-invariance the restriction of χ to $C_{F,S'}$.

Let $Y > 0$ be a real number.

(1) *If $Y' > Y$, then we have:*

$$\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \widetilde{\mathrm{PW}}_{S,Y',0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.2.18}$$

and:

$$\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \widetilde{\mathrm{PW}}_{S,Y'}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{7.3.2.19}$$

The directed unions $\widetilde{\mathrm{PW}}_0(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *respectively* $\widetilde{\mathrm{PW}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *of the linear subspaces* $\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *respectively* $\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *as* $Y \to +\infty$, *is sequentially dense in* $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'}))$, *respectively* $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

(2) *There exists a real number* $\lambda(Y) > 0$ *depending on* Y *and* χ, *such that* $\Phi_{F,\chi|_{C^1_F}}(f \otimes \rho_{\Sigma_F - S'})$, *as a function of* $g \in C_F$, *is supported over* $C_F^{\geq 1/\lambda(Y)}$, *for every* $f \in \widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y$.

(3) *The linear subspaces* $\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *respectively* $\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, *are stable under the continuous left-translation action of the semigroup* $J^{\geq 1}_{F,S'}$.

(4) *The following inclusions hold:*

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \widetilde{\mathrm{PW}}_{S,X^{|S|}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.2.20}$$

and

$$\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset \widetilde{\mathrm{PW}}_{S,X^{|S|},0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})). \tag{7.3.2.21}$$

Proof. (1) : Let $f_0 \in \mathcal{S}_0(\mathbf{A}_{F,S})$ and assume that $\widehat{f_0}$ is supported over $\widetilde{Q}_Y$, which is contained in $\widetilde{Q}_{Y'}$, and hence the containments in (7.3.2.18) and (7.3.2.19) both hold. The sequential denseness assertion follows from (4), to be proved below, and Proposition 7.3.2.2(1).

(2) and (3) : By Theorem 2.2.3.2, for any $f \in \widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y$, in order to show that there exists $\lambda(Y) > 0$ such that $\Phi_{F,\chi|_{C^1_F}}(f \otimes \rho_{\Sigma_F - S'})$ is supported over $C_F^{\geq 1/\lambda(Y)}$, it is enough to show that there exists $\lambda(Y) > 0$ such that $\Phi_{F,\overline{\chi}|_{C^1_F}}(\widehat{f} \otimes \rho_{\Sigma_F - S'})$ is supported over $(0, \lambda(Y)]$.

We replace f with $\widehat{f}$, and write $f = f_0 \otimes \rho_{\Sigma_F - S'}$ for some $f_0 \in \mathcal{S}_0(\mathbf{A}_{F,S})$, noting that we have $\widehat{\rho}_{\Sigma_F - S'} = \rho_{\Sigma_F - S'}$, supported over $\widetilde{Q}_Y$.

We choose a continuous homomorphic section τ to the idelic norm, whose image is contained in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_F)$. We let $g_0 \in E^1_F$ be arbitrary, where E^1_F is the idelic fundamental domain from (A.2.3.14), and let $t \in \mathbf{R}_{>0}$ be arbitrary.

The condition on $\xi \in F^\times$:

$$f(\xi g_0 \tau(t)) \neq 0 \tag{7.3.2.22}$$

implies that the following inequalities all hold: $|\xi|_v \leq 1$ for all $v \notin S$, since $|g_0|_v \leq 1$ for all such v, since $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

We choose a splitting of the quotient homomorphism $F^\times \to F^\times/\mathcal{O}^\times_{F,S}$ (which exists, its target being a free abelian group), and write $F^\times = \mathcal{O}^\times_{F,S} \cdot \Gamma$, where Γ is a free abelian subgroup of $F^\times$ mapped isomorphically onto $F^\times/\mathcal{O}^\times_{F,S}$, such that $\Gamma \cap \mathcal{O}^\times_{F,S} = 1$.

Since $|\Gamma|_{F,S} \subset [1, +\infty)$, we have that, for each $\gamma \in \Gamma$, the condition on $g_{0,S}\tau(t)$:

$$f_0(\gamma g_{0,S} \tau(t)) \neq 0$$

for $g_{0,S}$ the projection of g_0 onto $\mathrm{GL}_1(\mathbf{A}_{F,S})$, implies $g_{0,S}\tau(t) \in \gamma^{-1} \cdot \widetilde{Q}_Y \subset \widetilde{Q}_Y$, and so $\tau(t) \in K_S^{-1} \cdot \widetilde{Q}_Y$, where K_S is the relatively compact image of E^1_F onto $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ under the quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

It follows that we have $t = |\tau(t)|_{F,S} \in |K_S^{-1} \cdot \widetilde{Q}_Y|_{F,S} \subset \mathbf{R}_{\geq 0}$, which is relatively compact, since $|\cdot|_{F,S}$ is continuous. It follows that there exists $\lambda(Y) := \sup |K_S^{-1} \cdot \widetilde{Q}_Y|_{F,S} < +\infty$, such that $\Phi_{F,\overline{\chi}|_{C^1_F}}(f)$ is supported over $(0, \lambda(Y)]$.

The Mellin transform L_S from (5.1.1.2), respectively L from (5.1.1.1), turns the action by left-translation of $g_0 \in J^1_{F,S}$, respectively $g_0 \in C^1_F$, into multiplication by $\overline{\chi}_S(g_0)$, respectively $\overline{\chi}(g_0)$, and therefore preserves the linear subspaces $\widetilde{\mathrm{PW}}_{S,Y}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

Upon reprising the argument from the foregoing, for any $t \in [1, +\infty)$ we have:

$$\mathcal{F}_{S'}(f_0(\tau(t)^{-1}x_S)\rho_{S_0}(x_{S_0})) = t\widehat{f_0}(\tau(t)x_S)\rho_{S_0}(x_{S_0}) \neq 0$$

which implies that $\tau(t)x \in \widetilde{Q}_Y$, which reads $x \in \tau(t)^{-1} \cdot \widetilde{Q}_Y \subset \widetilde{Q}_Y$.

It follows that the continuous left-translation action of $J^{\geq 1}_{F,S'}$ preserves the linear subspaces $\widetilde{\mathrm{PW}}_Y(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, respectively $\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

(4) : Both inclusions in (7.3.2.20) are seen to hold upon observing that we have $Q_X \subset \widetilde{Q}_{X^{|S|}}$ within $\mathbf{A}_{F,S}$. □

7.3.3. Orbital Schauder Bases on the Paley-Wiener Levels. In this subsection, we extend Weil's Principal Value operators PV_{S,S_0}, introduced in Definition 5.3.1.4 of §5.3.1, for S_0 a finite set of finite places satisfying certain saturation properties, to the invariant and the asbolute Paley-Wiener spaces introduced in the previous subsection.

This will enable us to construct Schauder bases for the orthogonals to their range, even on the Paley-Wiener spaces, building on the work that was done in §6.3.2 on the Orbital Schauder Bases.

The Orbital Schauder Bases on the Paley-Wiener levels will then enable us to deconstruct the global geometric kernel $k^{\mathrm{geom}}_{\varphi,a,s}$ from 10.4.3.27 of Chapter 6, "lifted" to a kernel on $\mathrm{GL}_1(\mathbf{A}_F) \times C_F$ using the methods from §12.2.1, step by step, into an S-local kernel whose integral along the diagonal can be computed explicitly with a volume calculation.

We recall from Situation 7.3.1.1 the following:

Situation 7.3.3.1. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

Let S_0 be a finite set of finite places of F satisfying the following assumptions:

(1) $S_0 \cap S = \varnothing$.
(2) We assume S_0 is saturated in the sense of Situation 7.3.1.1(1): if $v \in S_0$ and p is the residue characteristic of v, then all places w of F of residue characteristic p are also contained in S_0. We denote by $\{p_1, \ldots, p_N\}$ the finitely many residue characteristics of the finite places contained in S_0, and call S_{p_k}, for $1 \leq k \leq N$, the finite set of those finite places of F whose residue characteristic is p_k. In particular, we have $S_0 = \bigcup_{1 \leq k \leq N} S_{p_k}$.
(3) We assume S is so large so as to ensure $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

We define $S' := S \cup S_0$.

Proposition 7.3.3.2 (Weil's Principal Values on the Paley-Wiener Levels). *Let F be a number field. Let S be a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{C_F}$ be a continuous character, unramified outside S. Let S_0 be a saturated finite set of finite places disjoint from S, and define $S' := S \cup S_0$.*

Assume the choices and assumptions from Situation 7.3.3.1 are in force.

Let $X, Y > 0$ be real numbers.

For any saturated finite subset $S_1 \subset S_0$, Weil's Principal Value Operator:

$$\mathrm{PV}_{S',S_1} : \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

from (5.3.1.9)*, extends by continuity to the operators* $\mathrm{PV}^{\mathrm{ABS}}_{S',S_1}$ *and* $\widetilde{\mathrm{PV}}_{S',S_1}$ *below, promoting the following to a commutative diagram of continuous linear operators:*

(7.3.3.1)

$$\begin{array}{ccccc}
\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}_0(\mathbf{A}_{F,S'})) \\
\downarrow{\scriptstyle \mathrm{PV}^{\mathrm{ABS}}_{S',S_1}} & & \downarrow{\scriptstyle \widetilde{\mathrm{PV}}_{S',S_1}} & & \downarrow{\scriptstyle \mathrm{PV}_{S',S_1}} \\
\mathrm{PW}_{S,X',0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \widetilde{\mathrm{PW}}_{S,Y',0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}_0(\mathbf{A}_{F,S'}))
\end{array}$$

compatible with the structures of continuous representations of the closed abelian semigroup $J^{\geq 1}_{F,S'} \subset J_{F,S'}$*, where* X' *and* Y' *are related to* X *and* Y *by the following formulas:*

$$Y \geq X^{|S|}, \quad X' = q_{S_1}X, \quad Y' = q_{S_1}Y \tag{7.3.3.2}$$

with $q_{S_1} := \prod_{v\in S_1} N(\mathfrak{p}_v)$*, where, for each* $v \in S_1$*,* $\mathfrak{p}_v \subset \mathcal{O}_F$ *is the unique prime ideal associated to the finite place* $v \in S_1$*.*

Proof. Since PV_{S',S_1} is already defined on the level of the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(\mathbf{A}_{F,S'})$ of S'-local Schwartz-Bruhat functions on the S'-adeles $\mathbf{A}_{F,S'}$, to prove the existence of $\mathrm{PV}^{\mathrm{ABS}}_{S',S_1}$ and $\widetilde{\mathrm{PV}}_{S',S_1}$ it is sufficient to show that $\mathrm{PV}_{S',S_1} : \mathcal{S}(\mathbf{A}_{F,S'}) \to \mathcal{S}(\mathbf{A}_{F,S'})$ sends the closed subspace $\mathcal{S}_0(\mathbf{A}_{F,S'})_X$, respectively $\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_Y$, up to averaging under the continuous left-translation action of $\mathcal{O}^{\times}_{F,S'}$ on $\mathcal{S}(\mathbf{A}_{F,S'})$, into $\mathcal{S}_0(\mathbf{A}_{F,S'})_{X'}$, respectively $\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S'})_{Y'}$, for some X', Y' related to X, Y.

Since $\mathrm{PV}_{S',S_1} = \prod_{v\in S_1} \mathrm{PV}_v$, the product denoting composition of continuous linear endomorphisms of $\mathcal{S}(\mathbf{A}_{F,S'})$, we show that the same assertions hold for the operator PV_v for a single finite place v of F contained in S_1.

We denote by $\mathfrak{p}_v \subset \mathcal{O}_F$ the unique prime ideal of $\mathcal{O}_F$ associated to the finite place $v \in S_1$, and by q_v its norm $N(\mathfrak{p}_v)$.

We let $\pi_v \in \mathcal{O}_{F_v}$ be a choice of uniformizer at v.

Then for any $f \in \mathcal{S}(\mathbf{A}_{F,S'})$, $\mathrm{PV}_v(f) = f - \rho_{F,S'}(\pi_v, f)$, where $\rho_{F,S'}$ denotes the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S'})$ on $\mathcal{S}(\mathbf{A}_{F,S'})$, i.e. $\rho_{F,S'}(g,f)(x) := f(g^{-1}x)$ as functions of $x \in \mathbf{A}_{F,S'}$.

We let $f_0 \in \mathcal{S}_0(\mathbf{A}_{F,S})$ be such that $\widehat{f_0}$ is compactly supported over Q_X, respectively supported (not necessarily compactly) over $\widetilde{Q}_Y$.

Then $\mathrm{PV}_v(f_0 \otimes \rho_{S_0})$ is the Schwartz-Bruhat function, up to a linear topological isomorphism induced by reordering of the finite places of F:

$$\mathrm{PV}_v(f_0 \otimes \rho_{S_0}) = f_0 \otimes \rho_{S_0^v} \otimes 1_{\mathcal{O}^{\times}_{F_v}} = f_0 \otimes \rho_{S_0^v} \otimes 1_{\mathcal{O}_{F_v}} - f_0 \otimes \rho_{S_0^v} \otimes 1_{\pi_v\mathcal{O}_{F_v}}$$

where $\rho_{S_0^v} := \bigotimes_{w\in S_0, w\neq v} 1_{\mathcal{O}_{F_w}} \in \mathcal{S}(\prod_{w\in S_0, w\neq v}(F_w, \mathcal{O}_{F_w}))$.

We now show that there exist real numbers X', Y', such that there exists $f_1 \in \mathcal{S}_0(\mathbf{A}_{F,S})$ such that $\widehat{f_1}$ is compactly supported with support contained within $Q_{X'}$, respectively supported (not necessarily compactly) over $\widetilde{Q}_{Y'}$, satisfying the following property:

$$\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(\mathrm{PV}_v(f_0 \otimes \rho_{S_0})) = \Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_1 \otimes \rho_{S_0}).$$

To this end, it suffices to find $f_2 \in \mathcal{S}_0(\mathbf{A}_{F,S})$ such that $\widehat{f_2}$ is compactly supported with support contained within $Q_{X'} \subset \mathbf{A}_{F,S}$, respectively supported (not necessarily compactly) over $\widetilde{Q}_{Y'}$, such that:

$$\Phi_{F,S',\chi|_{J^1_{F,S'}}}(f_0 \otimes \rho_{S_0^v} \otimes 1_{\pi_v\mathcal{O}_{F_v}}) = \Phi_{F,\chi_{S'}|_{J^1_{F,S'}}}(f_2 \otimes \rho_{S_0}) \tag{7.3.3.3}$$

in which case we define $f_1 := f_0 - f_2$. Since $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, we can arrange π_v to be the image in F_v of an S-integer ξ_0, satisfying $|\xi_0|_w = 1$ for all $w \notin S$, $w \neq v$. We call $\xi_{0,S}$ the image of ξ_0 in $\mathrm{GL}_1(\mathbf{A}_{F,S})$ under the composition $F^\times \to \mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S})$, the second map being the continuous homomorphic projection from the global ideles to the S-ideles.

We define $f_2 := \rho_{F,S}(\xi_{0,S}^{-1}, f_0)$, i.e. $f_2(x) := f_0(\xi_{0,S}x)$ as a function of $x \in \mathbf{A}_{F,S}$. Up to left-translation of ρ_{S_0} by an element of $\prod_{w\in S_0} \mathcal{O}_{F_w}^\times$, which does not change ρ_{S_0} and $\Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f_2 \otimes \rho_{S_0})$ since χ is unramified outside S, we have that (7.3.3.3) holds.

Since $f_0(0) = \widehat{f_0}(0) = 0$, the same vanishing conditions are satisfied by f_2.

Moreover, the Fourier transform $\mathcal{F}_S(f_2)$ is supported over $\xi_{0,S} \cdot Q_X$, respectively $\xi_{0,S} \cdot \widetilde{Q}_Y$ (noting that the Fourier transform conjugates the left-translation action by $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathcal{S}(\mathbf{A}_{F,S})$ over to the right-translation action by $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathcal{S}(\widehat{\mathbf{A}}_{F,S}) \simeq \mathcal{S}(\mathbf{A}_{F,S})$).

We have:

$$\xi_{0,S} \cdot Q_X \subset Q_{q_v X},\ \xi_{0,S} \cdot \widetilde{Q}_Y = \widetilde{Q}_{q_v Y}. \tag{7.3.3.4}$$

We define $X' := q_v X$ and $Y' := q_v Y$ and deduce that indeed $\mathrm{PV}^{\mathrm{ABS}}_{S',S_1}$ and $\widetilde{\mathrm{PV}}_{S',S_1}$ exist, for $X' := q_{S_0} X$ and $Y' := q_{S_0} Y$.

By Proposition 7.3.2.4, the condition $Y \geq X^{|S|}$ ensures that the leftmost top and bottom continuous inclusion operators in (7.3.3.1) exist.

It remains to show that the sub-squares of the diagram in (7.3.3.1) are cartesian as diagrams of $\mathbf{C}$-linear maps among $\mathbf{C}$-vector spaces. □

Before we state and prove the main Theorem of this section, we introduce some terminology and notation:

Definition 7.3.3.3 (p-primary PW Orbital Quotients). Let F be a number field, S a saturated finite set of places of F containing all the Archimedean places. Let S_0 be a saturated finite set of finite places of F disjoint from S. We define $S' := S \cup S_0$.

Let p be a prime number such that $S_p \subset S_0$.

Assume the notation and choices in Situation 7.3.3.1 are in force.

Let $X, Y > 0$ be real numbers. Denote by q_{S_p} the integer $q_{S_p} := \prod_{v\in S_p} q_v$, with $q_v := N(\mathfrak{p}_v)$, for $\mathfrak{p}_v \subset \mathcal{O}_F$ the unique prime ideal of $\mathcal{O}_F$ corresponding to the finite place v.

We define:

(1) The quotient:

$$\mathscr{O}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S'}}) := \mathrm{PW}_{S,q_{S_p}^{-1}X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))/\mathrm{PV}^{\mathrm{ABS}}_{S',S_p}\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.3.5}$$

endowed with the quotient topology, where $\mathrm{PV}^{\mathrm{ABS}}_{S',S_p}$ is the continuous linear operator with closed range induced by Weil's Principal Value Operators as per Proposition 7.3.3.2.

We refer to $\mathscr{O}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S}})$ as the *p-primary absolute PW Orbital Quotient.*

(2) The quotient:

$$\widetilde{\mathscr{O}}_{S',S_p,Y}(\chi|_{J^1_{F,S'}}) := \widetilde{\mathrm{PW}}_{S,q_{S_p}^{-1}Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))/\widetilde{\mathrm{PV}}_{S',S_p}\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.3.6}$$

endowed with the quotient topology, where $\widetilde{\mathrm{PV}}_{S',S_p}$ is the continuous linear operator with closed range induced by Weil's Principal Value Operators as per Proposition 7.3.3.2.

We refer to $\widetilde{\mathscr{O}}_{S',S_p,X}(\chi|_{J^1_{F,S}})$ as the *p-primary invariant PW Orbital Quotient.*

The following is the main Theorem of this section. It says that so long as we choose the constraint on the support X, Y on the Fourier transform, which defines the Paley-Wiener levels of $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$, large enough, i.e. $X, Y \gg 0$, depending on the finitely many places in S_p for a fixed prime p, with $S_p \subset S_0$,

then the Paley-Wiener levels inherit the absolute Schauder bases and their dual bases, which we have already identified on the p-primary Orbital Quotients

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$$

and their strong duals (see Definition 7.3.1.3 and Theorem 7.3.1.6).

THEOREM 7.3.3.4 (Orbital Schauder Bases II - Bases on the PW-levels). *Let F be a number field. Let S be a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{C_F}$ be a continuous character, unramified outside S. Let S_0 be a saturated finite set of finite places disjoint from S, and define $S' := S \cup S_0$.*

Assume the choices and assumptions from Situation 7.3.3.1 are in force.

We denote by N_p is the number of finite places of F whose residue characteristic is p. Assume $S_p \subset S_0$.

Let $\chi \in \widehat{C^1_F}$ be a continuous character.

We define the two subsets:

$$P_{S',S_p}(\chi|_{J^1_{F,S}}) := \{a_k(\cdot,c) : m_S(c) - N_p(c) < k \leq m_S(c)\} \subset \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))'_\beta \tag{7.3.3.7}$$

and:

$$E_{S',S_p}(\chi|_{J^1_{F,S'}}) := \{\widehat{\delta}^{(k)}_c : 0 \leq k \leq N_p(c) - 1\} \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))'_\beta \tag{7.3.3.8}$$

where $\widehat{\delta}^{(k)}_c$ is the continuous functional obtained as the composition of the Fourier transform

$$\mathcal{F}_{S'} : \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

followed by the Mellin transform $L_{S'}$, and by the kth derivative of the point-evaluation functional at $\widehat{c} \in \Omega(J_{F,S'})$, i.e. $\widehat{\delta}^{(k)}_c$ is uniquely determined by the assignment, on the arbitrary $f \in \mathcal{S}(\mathbf{A}_{F,S'})$:

$$\widehat{\delta}^{(k)}_c(f) := \partial^k_s L_{S'}(\widehat{f}, \overline{\chi}_{S'}|\cdot|^{1-s}_{F,S'})|_{s=s_c} \tag{7.3.3.9}$$

with $s_c = \frac{2\pi i m_{c,v}}{\log(q_v)} - it_v$ for a, equivalently any, choice of $v \in S_p$ such that $c \in \mathcal{U}_{S',v}$.

Then the sets in (7.3.3.7), respectively (7.3.3.8), satisfy the following properties:

(1) *Any enumeration of the sets:*

$$P^{\mathrm{ABS}}_{S',S_p}(\chi|_{J^1_{F,S'}}),\ E^{\mathrm{ABS}}_{S',S_p}(\chi|_{J^1_{F,S'}}),\ \widetilde{P}_{S',S_p}(\chi|_{J^1_{F,S'}}),\ \widetilde{E}_{S',S_p}(\chi|_{J^1_{F,S'}}) \tag{7.3.3.10}$$

of continuous functionals obtained as the image of either countable set $P_{S',S_p}(\chi|_{J^1_{F,S'}})$ and $E_{S',S_p}(\chi|_{J^1_{F,S'}})$ under the continuous linear map:

$$\mathscr{E}_{S',S_p}(\chi_{S'}|_{J^1_{F,S'}})'_\beta \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))'_\beta \to \widetilde{\mathrm{PW}}_{S',Y,0}(\chi_{S'}|_{J^1_{F,S'}})'_\beta \to \mathrm{PW}_{S',X,0}(\chi_{S'}|_{J^1_{F,S'}})'_\beta \tag{7.3.3.11}$$

for $Y \geq X^{|S|}$, where $\widetilde{\mathrm{PW}}_{S',Y,0}(\chi_{S'}|_{J^1_{F,S'}})$, respectively $\mathrm{PW}_{S',X,0}(\chi_{S'}|_{J^1_{F,S'}})$, are the invariant, respectively absolute, PW-levels, is an unconditional Schauder basis for:

$$\mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S}})'_\beta \tag{7.3.3.12}$$

and for:

$$\widetilde{\mathscr{E}}_{S',S_p,Y}(\chi|_{J^1_{F,S}})'_\beta \tag{7.3.3.13}$$

both equipped with the strong topology, for all large enough X, respectively Y.

(2) *Their bi-orthogonal sets:*

$$P^\vee_{S',S_p}(\chi|_{J^1_{F,S'}}), E^\vee_{S',S_p}(\chi|_{J^1_{F,S'}}) \subset \mathscr{E}_{S',S_p}(\chi|_{J^1_{F,S'}}) \tag{7.3.3.14}$$

equipped with the induced enumeration, are contained in the range of the continuous linear operators induced by the inclusions of each Paley-Wiener level into $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ provided by Proposition 7.3.3.2,

and they are an absolute Schauder basis in each of the two subquotients $\mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S}})$*, respectively* $\widetilde{\mathscr{E}}_{S',S_p,Y}(\chi|_{J^1_{F,S}})$*.*

(3) *The absolute Schauder basis induced by the pre-image of* $E_{S',S_p}(\chi_{S'}|_{J^1_{F,S'}})$ *in the subquotients* $\mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S}})$*, respectively* $\widetilde{\mathscr{E}}_{S',S_p,Y}(\chi|_{J^1_{F,S}})$*, and its dual basis in each strong dual, satisfies the growth and decay estimates from Theorem 7.3.1.6*(4) *and* (5)*, with same terminology and notation.*

The proof of the Theorem proceeds in the same manner as the proof of Theorem 7.3.1.6, with one important difference. Instead of using Theorem 4.6.1.5 to source the sought for Schauder bases, we use its "compactly supported" version, Theorem 4.6.2.1, exercising caution in keeping track of how the support condition on the Fourier transforms, defining the Paley-Wiener levels, is adjusted under the effect of Weil's Principal Value Operators: this, we can do thanks to Proposition 7.3.3.2.

Proof. We define $I := (0, +\infty) \subset \mathbf{R}$.

We inspect the following diagram of linear operators:

(7.3.3.15)

$$\begin{array}{ccccc}
\mathrm{PV}_{S',S_p}(\widehat{W}) & \xrightarrow{(1)} & \widehat{W} & \xrightarrow{(2)} & \widehat{W}/\mathrm{PV}_{S',S_p}(\widehat{W}) \\
\downarrow{\scriptstyle (a)} & & \downarrow{\scriptstyle (b)} & & \downarrow{\scriptstyle (c)} \\
E_{S',S_p} & \xrightarrow{(3)} & E & \xrightarrow{(4)} & E/E_{S',S_p} \\
\downarrow{\scriptstyle (d)} & & \downarrow{\scriptstyle (e)} & & \downarrow{\scriptstyle (f)} \\
\mathcal{F}_{S'}E_{S',S_p} & \xrightarrow{(5)} & \mathcal{F}_{S'}E & \xrightarrow{(6)} & (\mathcal{F}_{S'}E)/(\mathcal{F}_{S'}E_{S',S_p}) \\
\downarrow{\scriptstyle (g)} & & \downarrow{\scriptstyle (h)} & & \downarrow{\scriptstyle (i)} \\
P_U\mathcal{S}_{-1+I}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}}) \times P_U\mathcal{S}_I(J_{F,S'})(\chi_{S'}|_{J^1_{F,S'}}) & \xrightarrow{(7)} & \mathcal{S}_{-1+I}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}}) \times \mathcal{S}_I(J_{F,S'})(\chi_{S'}|_{J^1_{F,S'}}) & \xrightarrow{(8)} & \mathcal{S}_{U,\overline{\chi}_{S'}}(J_{F,S'}) \\
\downarrow{\scriptstyle (j)} & & \downarrow{\scriptstyle (k)} & & \downarrow{\scriptstyle (l)} \\
P_U\mathcal{S}_{-1+I}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}}) & \xrightarrow[(9)]{} & \mathcal{S}_{-1+I}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}}) & \xrightarrow[(10)]{} & \mathcal{S}_{U,\overline{\chi}_{S'}}(J_{F,S'})
\end{array}$$

where, we follow the notation from Situation 4.6.1.1 and (4.5.3.3) for the operator $P_U : \mathcal{S}_J(G)(\xi|_{G^1}) \to \mathcal{S}_J(G)(\xi|_{G^1})$ for G an $\mathbf{R}_{>0}$-modulated group, $\xi \in \widehat{G}$ a continuous character, $J \subset \mathbf{R}$ a non-empty connected interval, $U = (\gamma_1, \ldots, \gamma_N)$ a vector of N elements of G such that $|\gamma_k|_G > 1$ for each $1 \leq k \leq |S_p|$, we set $G = J_{F,S'}$, $\xi = \overline{\chi}_{S'}$, $U = (\pi_v^{-1})_{v \in \in S_p}$, $J = I$.

The $\mathbf{C}$-vector space E is defined to be either:

$$\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.3.16}$$

or:

$$\widetilde{\mathrm{PW}}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \tag{7.3.3.17}$$

respectively, and E_{S',S_p} is either:

$$\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \cap \mathrm{PV}_{S',S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \tag{7.3.3.18}$$

or:

$$\widetilde{\mathrm{PW}}_{S,Y,0}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \cap \mathrm{PV}_{S',S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_0(\mathbf{A}_{F,S'})) \tag{7.3.3.19}$$

respectively, where $\mathcal{F}_{S'}$ is the Fourier transform:

$$\mathcal{F}_{S'} : \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) \to \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

from (5.2.4.1) of Theorem 5.2.4.1, and $\mathcal{F}_{S'}E$, respectively $\mathcal{F}_{S'}E_{S',S_p}$, denotes the linear image of E, respectively E_{S',S_p}, under $\mathcal{F}_{S'}$, a linear topological isomorphism onto its range.

The $\mathbf{C}$-vector space $\widehat{W}$ is defined as the pre-image under $\mathcal{F}_{S'}$ of a $\mathbf{C}$-vector subspace $W \subset \mathcal{F}_{S'}E$ to be constructed. We will proceed by means of analogy with the proof of Theorem 7.3.1.6, where W was constructed as a linear copy of $|\cdot|_{F,S}^{-1} \cdot \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, providing access to Theorem 4.6.1.5 to source the absolute Schauder basis on $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$.

The existence of such linear copy of $|\cdot|_{F,S}^{-1} \cdot \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ was, in turn, provided by the combination Proposition 5.3.4.4 together with Lemma 5.3.4.2 and Remark D.3.3.6 (on the stability of $\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$ and $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ under pointwise multiplication by $|\cdot|_{F,S}^a$ for $a \in \mathbf{R}$), and the Second Structure Theorem (see Theorem 5.4.1.1) was then used to prove cartesianity properties of some of the sub-squares in the diagram from (7.3.1.20), which (7.3.3.15) mirrors.

We choose a continuous homomorphic section τ to the S-idelic norm.

We consider K_S a compact subset of the $\mathbf{R}_{>0}$-modulated group $J_{F,S}$, of the following form:

$$K_S = \tau([1/a^2, 1/a]) \cdot J^1_{F,S} \subset J_{F,S} \tag{7.3.3.20}$$

for $a > 1$ a real number. We assume that a satisfies:

$$a > q_{S_0} := \prod_{v \in S_0} q_v = \prod_{v \in S_0} N(\mathfrak{p}_v). \tag{7.3.3.21}$$

For any $\varphi \in \mathcal{D}_K(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, we then consider the smooth compactly supported function obtained from φ as follows: we enumerate $S_0 = \{v_1, \ldots, v_N\}$, and we define $\varphi_0 := \varphi$, and, for each $1 \leq k \leq N$:

$$\varphi_k(x) := \varphi_{k-1}(x) - c_{k-1} \cdot \varphi_{k-1}(\tau(q_{v_k})x) \tag{7.3.3.22}$$

as smooth compactly supported functions on $J_{F,S}$, where $c_{k-1} \in \mathrm{U}(1) \subset \mathbf{C}^\times$ is the complex number:

$$c_{k-1} := \overline{\chi}(\mathfrak{p}_{v_k}) q_{v_k}^{-ix_0} \tag{7.3.3.23}$$

for $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$.

Since $\varphi \in \mathcal{D}_{K_S}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, then $\varphi_N \in \mathcal{D}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}}) \subset \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}|_{J^1_{F,S}})$, which, by Proposition 5.3.4.4(3), implies $\varphi_N = \Phi_{F,S',\chi_{S'}|_{J^1_{F,S'}}}(f)$ for some choice of $f = f(\varphi_N) \in \mathcal{S}(\mathbf{A}_{F,S})$.

We note that the function:

$$f(\varphi_N) \otimes \bigotimes_{v \in S_0} 1_{\mathcal{O}_{F_v}} \tag{7.3.3.24}$$

is a Schwartz-Bruhat function over $\mathbf{A}_{F,S'}$, with $f(\varphi_N)$ a Schwartz-Bruhat function over $\mathbf{A}_{F,S}$ whose image under $\Phi_{F,S,\overline{\chi}_S|_{J^1_{F,S}}}$ is contained within $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_S, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$.

We refer the reader to (5.3.4.1) from Definition 5.3.4.1, and to Proposition 5.3.4.4 for more on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$: this is a closed subspace of $\mathcal{S}(\mathbf{A}_{F,S})$ providing elements $\Phi_{F,S,\overline{\chi}_S|_{J^1_{F,S}}}(f)$ in $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ have entire Mellin transform $Z_S(f,c)$ for $c \in \Omega(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$, i.e. over the connected component of the Riemann surface $\Omega(J_{F,S})$ parametrized by $\overline{\chi}_S|_{J^1_{F,S}}$.

If $\chi_S = |\cdot|_{F,S}^{ix_0}$ and we assume φ satisfies the vanishing condition $M(\varphi)(s)|_{s=1-ix_0} = 0$, then so does φ_N, since this is a linear condition, and φ_N is a finite linear combination of dilations of φ, each of which has, therefore, itself Mellin transform that vanishes at $s = 1 - ix_0$, since φ does.

In this case, φ_N is contained within $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}) \cap \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, i.e. $f(\varphi_N)$ can be chosen to satisfy the following properties: $f(\varphi_N)$ is compactly supported, with support entirely contained within $\mathrm{GL}_1(\mathbf{A}_{F,S}) \subset \mathbf{A}_{F,S}$ (in particular, we have $f(\varphi_N)(0) = 0$), and it satisfies $\mathcal{F}_S(f(\varphi_N))(0) = 0$.

We note that if $\chi_S|_{J^1_{F,S}} \neq 1$, then

$$\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S''}|_{J^1_{F,S''}}, V) = \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S''}|_{J^1_{F,S''}}, \mathcal{S}(\mathbf{A}_{F,S''})) \tag{7.3.3.25}$$

for $V = \ker(\delta_0)$, respectively $V = \ker(\widehat{\delta}_0)$, and hence for $V = \mathcal{S}_0(\mathbf{A}_{F,S''})$, for any finite set of places of F containing S. Indeed, if the inclusion of the left side into the right side from (7.3.3.25) was strict, then in the case $V = \ker(\delta_0)$, δ_0 would descend to the quotient $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S''}|_{J^1_{F,S''}}, \mathcal{S}(\mathbf{A}_{F,S''})$ inducing a continuous functional, which cannot be since δ_0 is homogeneous of weight zero, and hence it cannot be $\chi_{S''}|_{J^1_{F,S''}}$-homogeneous, since $\chi_{S''}|_{J^1_{F,S''}}$ is non-trivial. The same argument applies to $\widehat{\delta}_0$, which is homogeneous of weight 1, and hence cannot be $\chi_{S''}|_{J^1_{F,S''}}$-homogeneous (as all continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}_{S''}|_{J^1_{F,S''}}, \mathcal{S}(\mathbf{A}_{F,S''})$, instead, are).

We have, as meromorphic functions over $\Omega(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}})$:

$$L_{S'}(f(\varphi_N) \otimes \bigotimes_{v\in S_0} 1_{\mathcal{O}_{F_v}}, \overline{\chi}_{S'}|\cdot|^{1+s}_{F,S'}) = Z_S(f(\varphi_N), \overline{\chi}_S|\cdot|^{1+s}_{F,S}) \prod_{v\in S_0} \frac{1}{1-\overline{\chi}(\mathfrak{p}_v)q_v^{-1-s}} \tag{7.3.3.26}$$

and we further have, as meromorphic (in fact, entire) functions over $\Omega(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$:

$$Z_S(f(\varphi_N), \overline{\chi}_S|\cdot|^{1+s}_{F,S}) = M(\varphi_N)(1+s-ix_0) \tag{7.3.3.27}$$

$$= M(\varphi)(1+s-ix_0) \prod_{1\leq k\leq N} (1-\overline{\chi}(\mathfrak{p}_{v_k})q_{v_k}^{-ix_0}q_{v_k}^{-1-s+ix_0}) \tag{7.3.3.28}$$

$$= M(\varphi)(1+s-ix_0) \prod_{1\leq k\leq N} (1-\overline{\chi}(\mathfrak{p}_{v_k})q_{v_k}^{-1-s}). \tag{7.3.3.29}$$

We define W to be the linear image under $\Phi_{F,S',\overline{\chi}_{S'}|_{J^1_{F,S'}}}$ of the $\mathbf{C}$-linear subspace of $\mathcal{S}(\mathbf{A}_{F,S'})$ linearly generated by the elements $f((|\cdot|^{-1}_{F,S}\cdot\varphi)_N) \otimes \bigotimes_{v\in S_0} 1_{\mathcal{O}_{F_v}}$ (noting that for any smooth compactly supported function φ on $J_{F,S}$, $|\cdot|^a_{F,S}\cdot\varphi$ is also smooth compactly supported for any $a \in \mathbf{R}$. We further refer the reader to Remark D.3.3.6).

With this choice of W, up to enlarging $X > 0$, respectively $Y > 0$, once, to ensure that as φ varies over $\mathcal{D}_{K_S}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$, then the image of $(|\cdot|^{-1}_{F,S}\cdot\varphi)_N$ under the linear inclusion

$$\mathcal{D}(J_{F,S})(\overline{\chi}_S|_{J_{F,S}}) \to \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}}) \to \mathcal{S}_I(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$$

is contained in the image of

$$\mathcal{F}_S(\mathcal{S}_0(\mathbf{A}_{F,S})_X), \quad \mathcal{F}_S(\widetilde{\mathcal{S}}_0(\mathbf{A}_{F,S})_Y)$$

into $\mathcal{S}_I(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$ under $\Phi_{F,S,\overline{\chi}_S|_{J^1_{F,S}}}$.

The argument provided in the proof of Theorem 7.3.1.6 then applies, and by the choice made for K_S, by Theorem 4.6.2.1 and Corollary 4.6.3.1 we obtain that the composition of the four rightmost vertical arrows is surjective, which implies that the composition of (f) and (i) is surjective, hence open by [**NB**, Thm.14.4.5], being source and target Fréchet spaces.

The vertical arrows (c), (f) and (i) are injective by design of the closed subspaces E_{S',S_p} and $\mathcal{F}_{S'}E_{S',S_p}$ from (7.3.3.18), respectively (7.3.3.19).

Indeed, since $0 \in -1+I$, the composition of the vertical arrows (b), (e), (h) and (k), for the definition of which we refer the reader to the proof of Theorem 7.3.1.6, the linear subspace $\widehat{W} \subset E$ is sent onto the linear subspace

$$\mathcal{D}_{K_S}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}}) \subset \mathcal{S}_{\mathrm{E}}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}})$$

where the inclusion is induced by the inclusion $\mathcal{D}_{K_S}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}}) \subset \mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}})$, followed by the linear topological isomorphism:

$$\mathcal{S}_{\mathrm{E}}(J_{F,S})(\overline{\chi}_S|_{J^1_{F,S}}) \to \mathcal{S}_{\mathrm{E}}(J_{F,S'})(\overline{\chi}_{S'}|_{J^1_{F,S'}}) \tag{7.3.3.30}$$

induced by the continuous linear inclusion $\mathcal{S}_{\mathrm{E}}(J_{F,S}) \to \mathcal{S}_{\mathrm{E}}(J_{F,S'})$ of the closed subspace of Γ-invariant Schwartz-Bruhat functions of exponential decay on $J_{F,S'}$, for $\Gamma \subset J^1_{F,S'}$ the closed subgroup defined as the image of the closed subgroup $\prod_{v\in S_0} \mathcal{O}^{\times}_{F_v} \subset \mathrm{GL}_1(\mathbf{A}_{F,S'})$.

The linear inclusion in (7.3.3.30) is indeed surjective since χ is unramified outside S.

By Lemma 7.3.1.5, the remaining assertions follow from Theorem 4.6.1.5. □

7.3.4. Poles of the Γ-factors. We discuss an analog of Theorem 7.3.1.2 isolating the coefficients of the principal parts at the poles of the Γ-factors only.

NOTATION 7.3.4.1 (Filtration by orders of poles - Archimedean local factors). Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character.

We denote by $S_{\infty,r}$, respectively $S_{\infty,c}$, the sets of real places, respectively complex places of F. We choose an enumeration $S_{\infty,r} = \{v_1, \ldots, v_{r_1}\}$ and $S_{\infty,c} = \{w_1, \ldots, w_{r_2}\}$.

We denote by $(\underline{n}, \underline{p})$ the arbitrary element of $\mathbf{N}^{r_1} \times \mathbf{N}^{r_2}$, lexicographically according to the chosen enumerations (i.e. $(\underline{n}, \underline{p}) \leq (\underline{n}', \underline{p}')$ if and only if for every $\underline{n} \leq \underline{n}'$ under the lexicographical ordering on $\mathbf{N}^{r_1}$, and $\underline{p} \leq \underline{p}'$ under the lexicographical ordering on $\mathbf{N}^{r_2}$).

We choose an integer tuple $(\underline{n}, \underline{p}) \in \mathbf{N}^{r_1} \times \mathbf{N}^{r_2}$ and define:

$$\mathcal{U}_{\leq(\underline{n},\underline{p})} := \bigcup_{1\leq k\leq r_1} \psi_{v_k}^{-1}(\mathcal{U}_{v_k}^{\leq n_{v_k}}) \cup \bigcup_{1\leq k\leq r_2} \psi_{w_k}^{-1}(\mathcal{U}_{w_k}^{\leq p_{v_k}}) \tag{7.3.4.1}$$

where $\mathcal{U}_{v_k}^{\leq n_{v_k}}$ denotes the set of poles of the Euler factor $L_v(c_v)$ of the Hecke L-function $L(c)$ in $\Omega(F_v^{\times})(\omega_v)$ whose ordinate $\sigma(c_{v_k})$ satisfies $\sigma(c_{v_k}) \geq -2n_{v_k} - \epsilon(c_{v_k})$ for $\epsilon(c_{v_k}) = 0$ if ω_{v_k} is trivial, and $\epsilon(c_{v_k}) = 1$ if ω_{v_k} is the sign character, and where $\mathcal{U}_{w_k}^{\leq p_{w_k}}$ denotes the set of poles of the Euler factor $L_v(c_v)$ of the Hecke L-function $L(c)$ in $\Omega(F_v^{\times})(\omega_v)$ whose ordinate $\sigma(c_{w_k})$ satisfies $\sigma(c_{w_k}) \geq -p_{w_k} - \epsilon(c_{w_k})$ where $\epsilon(c_{w_k}) = |n|/2$ if $\omega_{w_k} = c_n \in \widehat{\mathrm{U}(1)}$.

We define the filtration:

$$F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}}) \subset \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \tag{7.3.4.2}$$

of those elements $\Phi_{F,S,\omega}(f) \in \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ whose S-idelic Mellin transform $Z_S(f,c)$ (which, recall from Lemma 5.1.2.2(2), only depends on f through $\Phi_{F,S,\omega}(f)$) has no poles over $\mathcal{U}_{\leq(\underline{n},\underline{p})}$. We endow the linear subspace $F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}})$ with the subspace topology, promoting it to a Hausdorff locally convex topological $\mathbf{C}$-vector subspace.

We refer to the filtration $F_S^{(\bullet,\bullet)}(\chi|_{J^1_{F,S}})$ as the *filtration by orders of pole* on $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

We denote by:

$$\mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}}) := \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S})) \tag{7.3.4.3}$$

the quotient of the closed subspace $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, by the closed subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, equipped with the quotient topology, promoting it to a nuclear Fréchet space.

THEOREM 7.3.4.1 (S-local Polar Orbital Schauder Basis - II). *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $\chi \in \widehat{J_{F,S}}$ be a continuous character.*

(1) *The subspace $F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}})$ of $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ is closed.*

(2) *The orthogonal $F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}})^{\perp}$ with respect to the duality*

$$\langle \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})), \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta} \rangle$$

endowed with the strong topology on the continuous dual of the quotient

$$\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))/F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}})$$

is linearly and topologically generated by the finitely many $\mathbf{C}$*-linearly independent continuous functionals:*

$$\{a_k(\cdot,c), c\in\mathcal{U}_{\leq(\underline{n},\underline{p})}, 1\leq k\leq m_{S_\infty}(c)\}. \tag{7.3.4.4}$$

(3) *We have, as closed subspaces of* $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$*:*

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_{00}(\mathbf{A}_{F,S})) = \bigcap_{(\underline{n},\underline{p})} F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}}). \tag{7.3.4.5}$$

(4) *The quotient topology on the linear quotient:*

$$\mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}}) := \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_{00}(\mathbf{A}_{F,S})) \tag{7.3.4.6}$$

is equivalent to the projective topology induced by the countable family of linear quotient maps:

(7.3.4.7)

$$q_{(\underline{n},\underline{p})}:\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_{00}(\mathbf{A}_{F,S}))\to\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})/F_S^{(\underline{n},\underline{p})}(\chi|_{J^1_{F,S}}).$$

The quotient in (7.3.4.6) *is linearly and topologically isomorphic to the linear space* $\prod_{(\underline{n},\underline{p})}\mathbf{C}$ *topologized with the product topology. Its strong dual is linearly and topologically isomorphic to the orthogonal* $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_{00}(\mathbf{A}_{F,S})^\perp$ *topologized with the strong topology on the continuous dual of the quotient, in turn linearly and topologically isomorphic to* $\bigoplus_{(\underline{n},\underline{p})}\mathbf{C}$ *topologized with the finest locally convex topology.*

(5) *Any enumeration of the continuous functionals*

$$\{a_k(\cdot,c), 1\leq k\leq m_{S_\infty}(c), c\in\mathcal{U}_{S_\infty}\} \tag{7.3.4.8}$$

is a Schauder basis for the orthogonal $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_{00}(\mathbf{A}_{F,S})^\perp$ *topologized with the strong topology on the continuous dual of the quotient.*

Proof. (1), (2), (3) and (4) of the Theorem were proved throughout the proof of Theorem 5.4.1.1, to which we refer the reader. We will recall here how to construct dual basis elements for each of the sets of finitely many functionals in (7.3.4.4).

We exhibit Schwartz-Bruhat functions in $\mathcal{S}(\mathbf{A}_{F,S})$ whose image under $\Phi_{F,S,\omega}$ has prescribed poles and orders of poles at the points of each $\mathcal{U}^{\leq(\underline{n},\underline{p})}$.

Each local character $\omega_v = \chi_v|_{\{u\in F_v^\times:|u|_v=1\}}$ for each Archimedean place v, uniquely determines a Schwartz-Bruhat function f_v on F_v whose local L-function is the Γ-factor at v. We refer the reader to [**CF**, Ch.XV,§2.5,pp.316-319] and [**RV**, Ch.7,§7.1,pp.249-253] for the explicit definition of such functions, satisfying the self-duality properties in [**CF**, Ch.XV,§2.5,pp.316-319] with respect to the real and complex Fourier transforms.

We consider a tuple of nonnegative integers $(\underline{n},\underline{p}) = ((n_v)_{v\in\Sigma_{F,r}},(p_v)_{v\in S_{\infty,c}})$ where $S_{\infty,r}$ denotes the real places of F, and $S_{\infty,c}$ denotes the complex places, and define the linear map:

$$M_{(\underline{n},\underline{p})} := \bigotimes_{v\in S_{\infty,r}} M_{x_v^{2n_v}}\otimes\bigotimes_{v\in S_{\infty,c}} M_{z_v^{p_v}\overline{z}_v^{p_v}} : \mathcal{S}(\mathbf{A}_{F,S_\infty})\to\mathcal{S}(\mathbf{A}_{F,S_\infty}). \tag{7.3.4.9}$$

We further define

$$M^S_{(\underline{n},\underline{p})} := M_{(\underline{n},\underline{p})}\otimes\bigotimes_{v\in S_f-S_\chi} 1_{\mathcal{O}_{F_v}^\times}\otimes\bigotimes_{v\in S_\chi}\omega_v^{-1}1_{\mathcal{O}_{F_v}^\times} : \mathcal{S}(\mathbf{A}_{F,S_\infty})\to\mathcal{S}(\mathbf{A}_{F,S_\infty})\otimes\bigotimes_{v\in S_f}\mathcal{S}(F_v^\times)$$

and, upon calling $f_\infty := \bigotimes_{v\in S_\infty} f_v$ for $f_v\in\mathcal{S}(F_v)$ the above choice of "self-dual" Schwartz-Bruhat function, the Schwartz-Bruhat functions on $\mathbf{A}_{F,S}$ obtained as $M^S_{(\underline{n}',\underline{p}')}(f_\infty)$ for each $(\underline{n}',\underline{p}')\leq(\underline{n},\underline{p})$, satisfy the desired spanning properties (see Proposition 5.3.3.4).

We now show that any enumeration of the countable set of continuous functionals from (7.3.4.8) indeed is a Schauder basis for the orthogonal $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S})^{\perp} \subset \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))'_{\beta}$.

Since the orthogonal carries the finest locally convex topology, any countable algebraically linearly independent set of topological generators is a topological basis. The countable set in (7.3.4.8) is algebraically linearly independent by (2), and it is a set of topological generators by (2) and (3).

By [**Jar**, Part II,Ch.14,§14.2,Thm.5,p.293], it is a Schauder basis. □

7.4. Nuclearity

Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $a \in \mathbf{R}$ be a fixed real number.

For p a fixed prime number, we denote by S_p the finite set of places of F dividing p.

In this subsection we discuss nuclearity of the regularized action of C_F on

$$\mathscr{H}_{F,a}(\chi|_{C^1_F}) := \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})/|\cdot|_F^a \,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \tag{7.4.0.1}$$

and on the subquotients:

$$\mathscr{E}_{S_p}(\chi|_{C^1_F}) := \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)). \tag{7.4.0.2}$$

7.4.1. Spectral Nuclearity. The following Theorem will never be used throughout this work. Rather, Theorem 10.3.1.1 will, which is a weaker yet sufficient analog.

THEOREM 7.4.1.1 (Strong Topologies and Nuclearity - I). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.*

Let $N \geq 1$ be an integer.

1. *The linear subspace $|\cdot|_F^a \,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \subset \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})$ is a closed $\mathcal{D}(C_F)$-submodule.*
2. *For each $\varphi \in \mathcal{D}(C_F)$, upon denoting by $A_{\mathrm{spec},a,\chi,N}(\varphi)$ the operator induced on the closed subspace of $\mathscr{H}_{F,a}$ linearly and topologically generated by the subset of the bi-orthogonal set $\{\psi_{\chi,t,m}\}$ to the functionals $\{U(\delta_\rho^m)\}$ provided by Theorem 7.1.4.2, for which $0 \leq m \leq N-1$, by $A(\varphi)$ on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})$, then $A_{\mathrm{spec},a,\chi,N}(\varphi)$ is a strongly nuclear operator whose nuclear trace is computed by the formula:*

$$\mathrm{Trace}(A_{\mathrm{spec},a,\chi,N}(\varphi)) = \sum_{\rho: L(-a+\rho,\chi)=0} N_0(\chi|\cdot|_F^{-a+\rho}) \int_{C_F} \varphi(g)\chi(g)|g|_F^{-a+\rho}\, \mathrm{d}\overline{\mu}(g) \tag{7.4.1.1}$$

where the sum is over the complex numbers ρ such that $\chi|\cdot|_F^{-a+\rho}$ is a zero of the Hecke L-function $L(c)$ contained in the connected component $\Omega(C_F)(\chi|_{C^1_F})$, and $N_0(\chi|\cdot|_F^{-a+\rho})$ is the minimum between the multiplicity of $\chi|\cdot|_F^{-a+\rho}$ as a zero of $L(c)$, and N.

PROOF. Since multiplication by $|\cdot|_F^{-a}$ on $\mathcal{S}_{\mathrm{E}}(C_F)$ is a bounded and boundedly invertible linear endomorphism, we may and do assume $a = 0$ in (1).

By Theorem 7.1.4.2, to show strong nuclearity of the operator $A_{\mathrm{spec},\chi,N}(\varphi)$ (i.e. that it is an operator of order 0, for the discussion of which we refer the reader to [**Gro-a**, Ch.II,§1,No.2,Def.2,p.6]) it is enough to show that the sequence $\{\Phi^{(m)}(\rho)\}_{(\rho,m)}$ where $\Phi(s) := \int_{C_F} \varphi(g)\chi(g)|g|_F^s\, \mathrm{d}\overline{\mu}(g)$ and the labelling of the pairs $\{(\rho,m)\}$ with $0 \leq m \leq N_0(\chi|\cdot|_F^{\rho}) - 1$ is prescribed by the chosen enumeration of the set of functionals provided by Theorem 7.1.4.2(3), is a sequence with of rapid decay (see [**Gro-a**, Ch.II,§1,No.1,(1),p.4] and the discussion on [**Gro-a**, Ch.II,§1,No.1,p.5]).

To see this, we proceed as follows.

We prove that the expression:

$$A_{\mathrm{spec},a,\chi,N}(\varphi) = \sum_{\rho} \sum_{0 \leq m_\rho \leq N_0(\rho)-1} \langle \cdot, A(\varphi)'U(\delta_\rho^{(m_\rho)})\rangle_{\mathscr{H}_{F,0},\mathscr{H}'_{F,0}}\, \psi_{(\rho,m_\rho)} \tag{7.4.1.2}$$

where $\{\psi_{(\rho,m_\rho)}\}$ is the bi-orthogonal set in $\mathscr{H}_{F,0}$ associated to the set $\{U(\delta_\rho^{(m)})\}$ (the subset of the full set of continuous functionals $U(\delta_\rho^{(m)})$ from (7.1.4.6), for which $0 \leq m \leq N-1$) exhibits $A(\varphi)'$ as being of the form $\sum_{i\geq 0}\lambda_i\langle\cdot,\varphi_i\rangle_{\mathscr{H}_{F,0},\mathscr{H}'_{F,0}}u_i$ for $\{u_i\}_{i\geq 0}\subset\mathscr{H}_{F,a}$ a bounded subset (given by the elements of the bi-orthogonal sequence to $\{U(\delta_\rho^{(m)}\}_{(\rho,m_\rho)}$ as $0\leq m_\rho\leq N_0(\rho)-1$, suitably renormalized by a real number $c_{(\rho,m)}>0$), and $\{\varphi_i\}_{i\geq 0}\subset(\mathscr{H}_{F,0})'_\beta$ an equicontinuous subset (given by the set of continuous functionals $\{U(\delta_\rho^{(m)}\}_{(\rho,m_\rho)}$ as $0\leq m_\rho\leq N_0(\rho)-1$, suitably renormalized by a real number $c'_{(\rho,m)}>0$), with $\{\lambda_i\}_{i\geq 0}$ a sequence of complex numbers of rapid decay (obtained from the values of $\Phi^{(m)}(s)$ at the zeros ρ and as m varies over $0\leq m\leq N_0(\rho)-1$, from sums of length at most $N_0(\rho)$ of such values and with uniformly bounded (by $2^{N_0(\rho)}$) coefficients, with the index set i being the chosen enumeration of the set of functionals indexed by the pairs (ρ,m), listing ψ_{ρ,m_ρ} with multiplicity if necessary, and multiplied by the reciprocal of $c_{(\rho,m)}\cdot c'_{(\rho,m)}$). Compare with the proof of Theorem 10.3.1.1(1).

Since the zeros of $L(s,\chi)$ are all located within the vertical strip of bounded width $\mathbf{C}_{[0,1]}$ of the complex s-plane, and they are a discrete subset, the rapid decay of the λ_i's then follows from $\Phi(s)$ and all of its holomorphic derivatives being Schwartz uniformly on any vertical strip of bounded width of the complex s-plane, only finitely many of which (at most N) occur in (7.4.1.2).

The real numbers $c_{(\rho,m)}, c'_{(\rho,m)}$ that qualify for the intended purpose, can be chosen to be equal, for each pair (ρ,m), and to be $1/N(\rho)!$, where we note that $N(\rho)!$ is $O(|\mathrm{Im}(\rho)|)$: multiplying a sequence of complex numbers depending on (ρ,m) that decays rapidly as $|\mathrm{Im}(\rho)|\to+\infty$, by $N(\rho)!$, therefore preserves its rapid decay condition, providing the coefficients λ_i satisfying the required decay property.

The nuclear trace of $A_{\mathrm{spec},a,\chi,N}(\varphi)$ therefore agrees with its matrix trace by [**Lit-II**, §4,Cor.2,p.61], which is computed by the formula in (7.4.1.1). □

7.4.2. Orbital Nuclearity - non-Archimedean places.

THEOREM 7.4.2.1 (Strong Topologies and Nuclearity - II). *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let p be a prime number.*

(1) *The linear subspace*
$$\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$$
is a closed $\mathcal{D}(C_F)$-submodule.

(2) *For each $\varphi\in\mathcal{D}(C_F)$, upon denoting by $A_{\mathrm{geom},p,\chi}(\varphi)$ the operator induced on $\mathscr{E}_{S_p}$ by $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$, then $A_{\mathrm{geom},p,\chi}(\varphi)$ is strongly nuclear:*

$$\mathrm{Trace}(A_{\mathrm{geom},p,\chi}(\varphi))=\sum_{\varepsilon_p}N(\chi|\cdot|_F^{\varepsilon_p})\int_{C_F}\varphi(g)\chi(g)|g|_F^{\varepsilon_p}\,\mathrm{d}\overline{\mu}(g)\tag{7.4.2.1}$$

where the sum is over the complex numbers ε_p such that $\chi|\cdot|_F^{\varepsilon_p}$ is a zero of the entire function $\prod_{v\in S_p}L_v(\chi|\cdot|_F^s)^{-1}$ of multiplicity $N(\chi|\cdot|_F^{\varepsilon_p})$, contained in the connected component $\Omega(C_F)(\chi|_{C_F^1})$, where $L_v(\chi|\cdot|_F^s)$ is the local Euler factor at v of the Hecke L-function $L(s,\chi)$.

PROOF. The same argument as for Theorem 7.4.1.1, upon using Theorem 7.1.4.3 in place of Theorem 7.1.4.2 and Proposition 7.1.4.3 in place of Theorem 7.1.4.2, shows that $A_{\mathrm{geom},p,\chi}(\varphi)$ is strongly nuclear (noting that, following the argument in Proposition 7.1.4.3, $[F:\mathbf{Q}]$ bounds from above the multiplicity function $N(\chi|\cdot|_F^{\varepsilon_p})$). It follows that its trace agrees with its matrix trace by [**Lit-II**, §4,Cor.2,p.61], which is computed by the formula in (7.4.2.1) upon repeating the calculation from (11.2.2.12) of Theorem 11.2.2.1. □

7.4.3. Orbital Nuclearity - Archimedean places. The following is an analog of Theorem 7.1.4.3 treating the Archimedean local factors of the Hecke L-function $L(s,\chi)$, as opposed to the non-Archimedean local factors as discussed in Theorem 7.1.4.3.

It uses Theorem 7.3.4.1 as an input, relating the poles of the Archimedean local factors and the resulting polar Schauder basis whose existence is discussed in Theorem 7.3.4.1, to the trivial zeros of the finite Hecke L-function.

THEOREM 7.4.3.1 (Global Orbital Schauder Basis - infinite places). *Let F be a number field. Let $\chi \in \widehat{C_F}$ be a continuous character. If $\chi|_{C_F^1} = 1$, let $t_0 \in \mathbf{R}$ be the unique real number such that $\chi = |\cdot|_F^{-it_0}$.*

Let τ be a choice of continuous homomorphic section to the idelic norm.

We define the closed subspace:

$$\mathcal{S}_{\infty,0}(\mathbf{A}_F) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \mathcal{S}(\mathbf{A}_F^f) \subset \mathcal{S}(\mathbf{A}_F) \tag{7.4.3.1}$$

where $\mathcal{S}_{\infty,0}(\mathbf{A}_{F,S_\infty})$ is the closed subspace of $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ defined as the intersection of $\mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ and $\ker(T) \cap \ker(\widehat{T})$, where $T = \delta_0$ and $\widehat{T} = \widehat{\delta_0}$, the Fourier transform of $\delta \in \mathcal{S}(\mathbf{A}_{F,S_\infty})'_\beta$, if $\chi|_{C_F^1}$, and $T = \widehat{T} = 0$ otherwise.

Then:

1. *The linear image $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F)) := \Phi_{F,\chi|_{C_F^1}}(\mathcal{S}_{00}(\mathbf{A}_F))$ is a nonzero closed subspace of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.*
2. *Let $\mathcal{U}_{S_\infty}$ be the subset of $\Omega(C_F)$ defined as in (5.3.4.13):*

$$\mathcal{U}_{S_\infty} := \bigcup_{v \in S_\infty} \psi_v^{-1}(\mathcal{U}_v) \tag{7.4.3.2}$$

for $\psi_v : \Omega(C_F) \to \Omega(F_v^\times)$ the surjective local analytic isomorphism induced by restricting a continuous quasi-character in $\Omega(C_F)$ to $F_v^\times$, and for $\mathcal{U}_v \subset \Omega(F_v^\times)$ the subset of poles of the Euler factor at v of the Hecke L-function $L(c)$. We let N_{S_∞} be the multiplicity function defined as in (5.3.4.14):

$$N_{S_\infty}(c) := \sum_{v \in S_\infty} \mathrm{ord}_{c'=\psi_v(c)} L_v(\psi_v(c')). \tag{7.4.3.3}$$

We write $\chi|\cdot|_F^{s_c}$ for the parametrization of each $c \in \mathcal{U}_{S_\infty}$.

Then the orthogonal $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$ with respect to the duality

$$\langle \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)), \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))'_\beta \rangle$$

equipped with the subspace topology inherited from the strong topology, is linearly and topologically generated by the continuous functionals:

$$D_c^m(f) := \partial_s^m Z(f, \chi|\cdot|_F^s)|_{s=s_c} = \int_{\mathbf{R}_{>0}} \left(\int_{-i\infty}^{+i\infty} t^{-s} Z(\phi, \chi|\cdot|_F^s)\, \mathrm{d}s \right) t^{s_c} (\log(t))^m \frac{\mathrm{d}t}{t} \tag{7.4.3.4}$$

for $0 \leq m \leq N'_{S_\infty}(c) - 1$, and as c varies over the discrete set $\mathcal{U}_{S_\infty} \cap \Omega(C_F)(\chi|_{C_F^1})$ and N'_{S_∞} agrees with the multiplicity function N_{S_∞} over $\mathcal{U}_{S_\infty} - \{|\cdot|_F^{it_0}\}$, and, at $c = |\cdot|_F^{it_0}$, it is redefined to be $N'_{S_\infty}(|\cdot|_F^{it_0}) := N_{S_\infty}(|\cdot|_F^{it_0}) - 1$.

3. *Any enumeration of the countable set $\{D_c^m\}_{(c,m)}$ of continuous functionals on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ from (7.4.3.4), as c varies over the discrete set $\mathcal{U}_{S_\infty} \cap \Omega(C_F)(\chi|_{C_F^1})$, and as m varies over the integers $0 \leq m \leq N'_{S_\infty}(c) - 1$, is a Schauder basis for the orthogonal $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))^\perp$ equipped with the dual strong topology of the quotient topology.*
4. *The linear image of the closed subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$ of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ under the global Mellin transform L from (5.1.1.1), is the closed subspace of $A\mathcal{S}^p(\mathbf{C})$:*

$$\Lambda_f(s,\chi)\, \mathscr{O}(\mathbf{C}) \cap A\mathcal{S}^p(\mathbf{C}) \tag{7.4.3.5}$$

where $\Lambda_f(s,\chi) := (s - 1 - it_0)^{\delta_\chi} L_f(s,\chi)$, with $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and $\delta_\chi = 0$ if $\chi|_{C_F^1} \neq 1$.
5. *The strong dual of the quotient:*

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$$

carries the finest locally convex topology.

PROOF. (1) : We fix an arbitrary continuous character $\chi \in \widehat{C_F}$.

We let S be a finite set of places of F containing all the Archimedean places, and such that χ is unramified outside S. We choose S large enough so as to have $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Assume $\varphi_v \in \mathcal{D}(F_v)$ for each Archimedean place v, and define $\varphi := \bigotimes_{v \in S_\infty} \varphi_v$, a smooth compactly supported function on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

We consider, for each $v \in S_f$, the Schwartz-Bruhat function $\rho_v := 1_{\mathcal{O}_{F_v}}$ if χ is unramified at v, and $\rho_v := \chi_v^{-1} \cdot 1_{\mathcal{O}_{F_v}^\times}$ if χ is ramified at v.

Then $f_0 := \varphi \otimes \bigotimes_{v \in S_f} \rho_v$ is a Schwartz-Bruhat function on $\mathbf{A}_{F,S}$, since the extension by zero of φ is smooth and compactly supported over $\mathbf{A}_{F,S_\infty}$, hence Schwartz on the Euclidean space $\mathbf{A}_{F,S_\infty}$ (we further refer the reader to Lemma 5.3.3.3(3)).

We call χ_S the unique continuous character of $J_{F,S}$ obtained by descending to $J_{F,S}$ the restriction of χ to $C_{F,S}$ by $\widehat{\mathcal{O}}_{S_\infty - S}$-invariance.

Then $f := \pi_S(f_0)$ is such that $\Phi_{F,\chi|_{C_F^1}}(f)$ is an element of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F))$. Assume that $\Phi_{F,\chi|_{C_F^1}}(f)$. Then so is $Z(f,c)$ over the connected component $\Omega(C_F)(\chi|_{C_F^1})$.

We have, parametrizing by $\chi|\cdot|_F^s$ the connected component $\Omega(C_F)(\chi|_{C_F^1})$:

$$Z(f, \chi|\cdot|_F^s) = \prod_{v \in S_\infty} L_v(\varphi_v, \chi_v|\cdot|_v^s) \prod_{v \in \Sigma_{F,f}} L_v(\chi_v|\cdot|_v^s). \tag{7.4.3.6}$$

The vanishing of $Z(f, \chi|\cdot|_F^s)$ forces the vanishing of at least one of the $L_v(\varphi_v, \chi_v|\cdot|_v^s)$ for some $v \in S_\infty$. We now let ψ_v be smooth, real-valued, nonnegative, compactly supported over $F_v^\times$, for each Archimedean place v, and define $\varphi_v := \chi_v^{-1} \cdot \psi_v$. The functions φ_v are also, respectively, smooth, compactly supported over $F_v^\times$, and so φ qualifies to define f as in the foregoing. The vanishing of $L_v(\varphi_v, \chi_v|\cdot|_v^s)$ would, however, force the vanishing of φ_v by Fourier inversion.

Upon assuming that each of the φ_v, equivalently each of the ψ_v, is nonzero, we therefore obtain examples of Schwartz-Bruhat functions $f \in \mathcal{S}_{00}(\mathbf{A}_F)$ such that $\Phi_{F,\chi|_{C_F^1}}(f)$ is nonzero, hence a nonzero element of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F))$.

(2) : We choose S a finite set of places of F containing all the Archimedean places, such that χ is unramified outside S, and such that $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. We denote by χ_S the unique continuous character of $J_{F,S}$ obtained by descending the restriction of χ to $C_{F,S}$ by $\widehat{\mathcal{O}}^\times_{\Sigma_F - S}$-invariance. We denote by $\chi_{S,\infty}$ the restriction of χ to $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$

By Theorem 6.2.2.1, every element ϕ of the linear image of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F))$ under the global Mellin transform L is of the form:

$$\phi(s) = u(s) L_f(s, \chi) \tag{7.4.3.7}$$

where u is an entire function on the complex s-plane of the following form: there exists a Schwartz-Bruhat function $f_0 \in \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty})$ such that, in the special case $\chi|_{C_F^1} = 1$, f_0 additionally satisfies the linear condition $\widehat{f_0}(0) = 0$, and such that we have:

$$u(s) = \int_{\mathbf{R}_{>0}} t^s \, \Psi_{F,S,\chi_{S,\infty}}(f_0)(\tau(t)) \, \frac{\mathrm{d}t}{t}$$

as entire functions on the complex s-plane (see Theorem 6.2.2.1 for a study of the operator $\Psi_{F,S,\chi_{S,\infty}}$).

The Γ-factor

$$\Gamma_F(\chi, s) := \prod_{v \in S_\infty} L_v(\psi_v(\chi|\cdot|_F^s))$$

of the Hecke L-function $L(c)$ on $\Omega(C_F)(\chi|_{C_F^1})$ is such that its reciprocal is entire as a function of s, and has zeros at the trivial zeros of the finite Hecke L-function $L_f(s,\chi)$, with matching multiplicities.

We have, for every $s \in \mathbf{C}$ and upon defining

$$\phi(s) = \frac{u(s)}{\Gamma_F(\chi,s)((s+it_0)(s-1-it_0))^{\delta_\chi}}((s+it_0)(s-1-it_0))^{\delta_\chi}L(s,\chi)$$

where $\delta_\chi = 0$ if $\chi|_{C_F^1}$ is non-trivial, and $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and the ratio $\frac{u(s)}{\Gamma_F(\chi,s)((s+it_0)(1-s-it_0))^{\delta_\chi}}$ has only removable singularities.

We denote by u_0 the unique analytic continuation of such ratio to an entire function on the complex s-plane. It has zeros at each of the trivial zeros of the finite Hecke L-function $L_f(s,\chi)$ with at least the same multiplicity.

We deduce that the image of $\mathcal{S}_{\infty,0}(\mathbf{A}_F)$ under the global Mellin transform L to $A\mathcal{S}^p(\mathbf{C})$ is contained within the closed subspace:

$$\frac{\Lambda(s,\chi)}{\Gamma_F(\chi,s)(s+it_0)^{\delta_\chi}}\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}^p(\mathbf{C}) = (s-1-it_0)^{\delta_\chi}L_f(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}^p(\mathbf{C}). \tag{7.4.3.8}$$

We will be granting for the moment that it in fact agrees with this closed subspace.

We define:

$$\Lambda_f(s,\chi) := (s-1-it_0)^{\delta_\chi}L_f(s,\chi)$$

an entire function of s with at most polynomial growth as $|s| \to +\infty$ uniformly over each vertical strip of bounded width of the complex s-plane. By Dirichlet's Unit Theorem and the Class Number Formula, the order of zero N at $s = -it_0$ of $\Lambda_f(s,\chi)$ is exactly $N_{S_\infty}(|\cdot|_F^{it_0}) - 1$, where $N_{S_\infty}(|\cdot|_F^{it_0})$ is the order of pole of $\Gamma_F(\chi,s)$ at the same point.

The quotient $\mathbf{C}$-linear space:

$$\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}) \tag{7.4.3.9}$$

is Hausdorff, since $\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C})$ is closed in $\mathscr{O}(\mathbf{C})$, and it is complete, since $\mathscr{O}(\mathbf{C})$ is a Fréchet space, and Hausdorff quotients of Fréchet spaces are complete by [**Scha**, Ch.I,§6,6.3,p.31].

We define:

$$K_z := \mathbf{C}[s]/(1-s/z)^{N(z)} \tag{7.4.3.10}$$

equipped with the Euclidean topology, for z in the discrete set of zeros of $\Lambda_f(s,\chi)$, with multiplicity $N(z)$.

K_z is linearly and topologically isomorphic to $\mathscr{O}(\mathbf{C})/(1-s/z)^{N(z)}\,\mathscr{O}(\mathbf{C})$, which is seen by the same argument as for Theorem 7.1.4.2(1), using faithful flatness of $\mathbf{C}[s] \to \mathscr{O}(\mathbf{C})$ and denseness of $\mathbf{C}[s] \subset \mathscr{O}(\mathbf{C})$, by [**Cnw-b**, Ch.VIII,Cor.1.15,p.200].

As a $\mathbf{C}$-vector space, the quotient $\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C})$ is linearly isomorphic to the direct product of the K_z, i.e. to

$$\prod_{z\in\mathbf{C}:\Lambda_f(\rho,\chi)=0} K_z.$$

This is seen as in the argument for Theorem 7.1.4.2(1), since $\mathbf{C}$ is Stein and the set of zeros of $\Lambda_f(s,\chi)$ is countable and discrete: upon denoting by $\mathscr{J} \subset \mathscr{O}$ the ideal sheaf globally generated by $\Lambda_f(s,\chi)$, we have

$$\check{\mathrm{H}}(\mathbf{C},\mathscr{O}/\mathscr{J}) = \mathrm{H}^1(\mathbf{C},\mathscr{O}/\mathscr{J}) = 0$$

and hence upon choosing a locally finite countable open cover $\{B_m\}_{m\geq 0}$ of the complex s-plane by open balls each of which contains at most one of the zeros z of $\Lambda_f(s,\chi)$, then, upon denoting by $B_{\underline{k}}$ the intersection $\bigcap_{m\in\underline{k}} B_m$ for $\underline{k}$ varying over all non-empty finite subsets of the nonnegative integer, we deduce that $\mathrm{H}^0(B_{\underline{k}},\mathscr{J})$ is trivial if $\underline{k}$ contains at least two elements, and it is either trivial or K_z if $\underline{k}$ contains one element exactly.

This computes

$$\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}) = \check{\mathrm{H}}^0(\mathbf{C}, \mathscr{O}/\mathscr{J})$$

to be $\prod_{z:\Lambda_f(z,\chi)=0} K_z$.

The quotient topology $\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C})$ is the product topology, since any compact subset $K \subset \mathbf{C}$ is contained in the union of at most finitely many of the B_m, and hence the linear bijection $\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}) \to \prod_{z:\Lambda_f(z,\chi)=0} K_z$ is continuous. Being source and target Fréchet spaces, it is a linear topological isomorphism by [**NB**, Thm.14.4.5(b)].

The orthogonal of $\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C})$ in $\mathscr{O}(\mathbf{C})'_\beta$, equipped with the subspace topology of the strong topology, is linearly isomorphic to the direct sum $\bigoplus_{z:\Lambda_f(z,\chi)=0} K'_z$. The subspace topology on the orthogonal $(\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp$ inherited from the strong topology on $\mathscr{O}(\mathbf{C})'_\beta$ is the finest locally convex topology, and hence the continuous linear bijection $(\mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}))'_\beta \to (\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp$ is a linear topological isomorphism.

The continuous linear operator

$$(\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}))^\perp \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))^\perp \tag{7.4.3.11}$$

adjoint to the continuous linear map:

$$A\mathcal{S}^p(\mathbf{C})/L\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)) \to \mathscr{O}(\mathbf{C})/\Lambda_f(s,\chi)\,\mathscr{O}(\mathbf{C}).$$

which is, in turn, injective with dense linear image, by denseness of $A\mathcal{S}^p(\mathbf{C})$ in $\mathscr{O}(\mathbf{C})$ and openness of the quotient linear operators, is injective with weakly dense range by [**NB**, Thm.8.13.1].

Since $A\mathcal{S}^p(\mathbf{C})$ is a Fréchet space by Lemma D.3.1.2, nuclear by Proposition D.3.2.5, its quotient by the closed subspace $L\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$ is also nuclear Fréchet, hence Montel, and by [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.230] the range of the continuous linear operator from (7.4.3.11) is strongly dense. This implies that the continuous functionals from (7.4.3.4) are indeed linear topological generators for $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))^\perp$ in $A\mathcal{S}^p(\mathbf{C})'_\beta$.

It remains to show that the image of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$ under the Mellin transform L into $A\mathcal{S}^p(\mathbf{C})$ agrees with the closed subspace from (7.4.3.5).

(3), (4), (5) : By the global functional equation from (3.2.2.2) of Theorem 3.2.2.1, $Z(f,c')$ has a zero at $c'=c$ if and only if $Z(\widehat{f},c')$ has a zero at $c'=\widehat{c}$, with matching multiplicities.

We now choose a large enough finite set S of places of F containing all the Archimedean places, such that χ is unramified outside S and $\mathrm{Pic}(\mathcal{O}_{F,S})=1$. By Theorem 6.2.1.1(2), the operator:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

is a linear topological isomorphism, compatible with passage to the closed codimension-2 subspaces $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))$ on the source, with $\mathcal{S}_0(\mathbf{A}_{F,S}) = \ker(\delta_0) \cap \ker(\widehat{\delta_0})$, on the source, and $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$ on the target.

We can write, for each $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$, $\phi = \overline{\pi}_S \circ \mathrm{PV}^{-1}_{S,S_f}(\phi_0)$ for a unique $\phi_0 \in \mathcal{S}_{00}(\mathbf{A}_{F,S})$, and hence the linear topological embedding with closed range (by Lemma 5.3.1.5) $\overline{\pi}_S \circ \mathrm{PV}^{-1}_{S,S_f}$ induces a linear topological isomorphism $\overline{\pi}_S \circ \mathrm{PV}^{-1}_{S,S_f}$:

(7.4.3.12)

$$\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))/\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}) \cap \ker(\widehat{\delta_0})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)).$$

In particular, the orthogonal of the subquotient on the rightmost side of (7.4.3.12) carries the finest locally convex topology, and it is linearly and topologically isomorphic to the strong dual of the quotient:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$$

by Theorem 7.3.4.1(3).

We now use calculations similar to those made in (8.1.0.10) from the proof of Theorem 7.5.0.1 and the formula for the coefficients of the principal parts of the S-idelic Zeta functions provided by Theorem 7.2.1.4.

We write, for $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, $Z(\phi, \chi|\cdot|_F^s) = u(s)L_f(s,\chi)$ where $u(s) = Z_S(\phi_0, \chi_S|\cdot|_{F,S}^s)$ for a unique $\phi_0 \in \mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J_{F,S}^1}, \mathcal{S}_0(\mathbf{A}_{F,S}))$.

We therefore have, upon denoting by $\widehat{u}(s) := Z_S(\widehat{\phi}_0, \overline{\chi}_S|\cdot|_{F,S}^{1-s})$:

$$Z(\widehat{\phi}, \overline{\chi}|\cdot|_F^{1-s}) = \widehat{u}(s)\widehat{L}_f(\chi, s)$$

and by the same calculation in the proof of Theorem 7.5.0.1 (see (8.1.0.10)) using that $\widehat{L}_f(\chi,s)$ does not vanish for $s \in i\mathbf{R}$, equivalently $L_f(\overline{\chi}, s)$ does not vanish for $s \in 1 + i\mathbf{R}$, we deduce that the continuous functionals in (7.4.3.4) are indeed topologically free by Theorem 7.3.4.1(4), and hence any enumeration is a Schauder basis for $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))^\perp$ equipped with the subspace topology inherited from the strong topology, by [**Jar**, Part II,Ch.14,§14.2,Prop.3,p.292] (since, in this special case, such topology is the finest locally convex topology, and the convergence axiom for Schauder bases is satisfies since the possibly infinite sums involved in it are all convergent, being finite).

This also shows that the linear image of the continuous linear operator in (7.4.3.11) is strongly dense, because all the functionals D_c^m from (7.4.3.4) are in the image. By [**NB**, Thm.8.10.6(e)], the linear image of $\mathcal{S}_{\infty,0}(\mathbf{A}_F)$ under the global Mellin transform L to $A\mathcal{S}^p(\mathbf{C})$, equivalently of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F))$, is weakly dense in the closed subspace of $A\mathcal{S}^p(\mathbf{C})$ from (7.4.3.5) (because its continuous adjoint maps a sequence of topologically free topological generators into a topologically free set and it is therefore injective, and hence the Theorem applies).

By [**Jar**, Part II,Ch.11,§11.5,Prop.1,p.230], it is strongly dense, and hence it agrees with the closed subspace from (7.4.3.5), since it is itself closed (since L has closed range by Theorem 7.1.4.3 and $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ is closed).

This finishes the proof of (2) and hence of (3), (4) and (5). □

We recall that we defined the closed subspace:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

in (7.4.3.1), and Theorem 7.4.3.1 exhibits a Schauder basis for the strong dual of the subquotient:

$$\mathscr{E}_{S_\infty} := \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)) \tag{7.4.3.13}$$

We recall from Theorem 7.4.3.1 that $\mathscr{E}_{S_\infty}$ is linearly and topologically isomorphic to the Hausdorff locally convex topological $\mathbf{C}$-vector space of formal power series in finitely many indeterminates, topologized with the topology of Euclidean-norm convergence of the coefficients, and its strong dual is linearly and topologically isomorphic to the Hausdorff locally convex topological $\mathbf{C}$-vector space of polynomials in the same number of indeterminates, carrying the finest locally convex topology.

The following is an Archimedean analog of Theorem 7.4.2.1.

THEOREM 7.4.3.2 (Strong Topologies and Nuclearity - III). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let τ be a choice of continuous homomorphic section to the idelic norm.*

(1) *The linear subspace*

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{\infty,0}(\mathbf{A}_F)) \subset \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

defined in (7.4.3.1)*, is a closed $\mathcal{D}(C_F)$-submodule.*

(2) *Let $\varphi \in \mathcal{D}(C_F)$ be a test function. Assume φ is supported over $C_F^{>1}$.*

Upon denoting by $A_{\mathrm{geom},\infty,\chi}(\varphi)$ the operator induced on $\mathscr{E}_{S_\infty}$ by $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, then $A_{\mathrm{geom},\infty,\chi}(\varphi)$ is a continuous operator whose matrix trace with respect to the Schauder basis provided by

Theorem 7.4.3.1(3), is finite, with value:

$$\text{Trace}(A_{\text{geom},\infty,\chi}(\varphi)) = \sum_z N(\chi|\cdot|_F^z) \int_{C_F} \varphi(g)\chi(g)|g|_F^z \, \mathrm{d}\overline{\mu}(g) - \delta_\chi \cdot \int_{C_F} \varphi(g)|\cdot|_F^{-ix_0} \, \mathrm{d}\overline{\mu}(g) \tag{7.4.3.14}$$

where $x_0 \in \mathbf{R}$ is the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions on $\mathbf{R}_{>0}$, and where the sum is over the complex numbers z such that $\chi|\cdot|_F^z$ is a zero of the entire function $\prod_{v\in S_\infty} L_v(\chi|\cdot|_F^s)^{-1}$ of multiplicity $N(\chi|\cdot|_F^z)$, contained in the connected component $\Omega(C_F)(\chi|_{C_F^1})$, with $L_v(\chi|\cdot|_F^s)$ the local Euler factor at v of the Hecke L-function $L(s,\chi)$, and $\delta_\chi = 0$ if $\chi|_{C_F^1} \neq 1$, and $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$.

Remark 7.4.3.3. We note that since the strong dual $(\mathscr{E}_{S_\infty})'_\beta$ carries the finest locally convex topology, then $A_{\text{geom},\infty,\chi}(\varphi)$ cannot be nuclear, since the only nuclear operators on $(\mathscr{E}_{S_\infty})'_\beta$ are the finite rank operators. Nonetheless, $A_{\text{geom},\infty,\chi}(\varphi)$ has a matrix trace, computed with respect to the chosen Schauder basis provided by Theorem 7.4.3.1(3).

Proof. We are to show that the sum in the right side of (7.4.3.14) is absolutely convergent: it formally agrees with the matrix trace of $A_{\text{geom},\infty,\chi}(\varphi)$ by following the same calculation in (10.3.1.8) from the proof of (10.3.1.3) of Theorem 10.3.1.1.

This is a consequence of the assumption made on the support of φ.

Indeed, upon fixing real numbers $a, b > 0$, then by Proposition B.2.1.9(4), upon writing:

$$\Phi(s) := \int_{C_F} \varphi(g)\chi(g)|g|_F^s \, \mathrm{d}\overline{\mu}(g)$$

the entire function Φ is an element of $H^p(\mathbf{C}_{<b})$ for every $p \geq 1$ (see §B.2.1), and hence the Asymptotic Residue Theorem (see Theorem B.3.2.1) implies that we have, upon defining $\Gamma_F(\chi,s) := \prod_{v\in S_\infty} L_v(\chi_v|\cdot|_v^s)$:

$$-\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} \Phi(s)\frac{\Gamma_F(\chi,s)'}{\Gamma_F(\chi,s)}\,\mathrm{d}s + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Phi(s)\frac{\Gamma_F(\chi,s)'}{\Gamma_F(\chi,s)}\,\mathrm{d}s = \sum_{z:\mathrm{Re}(z)\in(-a,b)} N(\chi|\cdot|_F^z)\Phi(z) \tag{7.4.3.15}$$

and since Φ is an element of the Hardy space $H^2(\mathbf{C}_{<b})$, then the line integral over the $-a$-line tends to zero as $a \to +\infty$, the sum in the right side of (7.4.3.15) being finite and indexed by the singularities of $\Gamma_F(\chi,s)'/\Gamma_F(\chi,s)$ contained within the vertical strip $\mathbf{C}_{(-a,b)}$.

Upon choosing $b > 1$, the sum in the right side of (7.4.3.15) tends to the sum in (7.4.3.14) with the $\Phi(-ix_0)$ summand removed, as $a \to +\infty$, showing that the desired convergence assertion holds.

To see that the mode of convergence is absolute, we proceed as in [**Den**, §2,(2.16),(2.17),pp.9-10]. We have:

$$\frac{\Gamma(s)'}{\Gamma(s)} = -\gamma - \sum_{m\leq -1}\left(\frac{1}{m} + \frac{1}{s-m}\right) - \frac{1}{s} \tag{7.4.3.16}$$

which is valid over $\mathbf{C} - \mathbf{N}_{<0}$ and is a consequence of Euler's Duplication Formula, the convergence of the sum in the right side of (7.4.3.16) being uniform over the compact subsets of $\mathbf{C} - \mathbf{N}_{<0}$.

By (B.4.1.22) from Lemma B.4.1.9, the sum also satisfies the estimate:

$$\sum_{m\leq -1}\left|\frac{1}{m} + \frac{1}{s-m}\right| \ll |s|.$$

We further refer the reader to [**Den**, §2,Lemma 2.11,p.8] for a stronger estimate.

Since $\Phi(s)$ is Schwartz uniformly on the vertical strips of bounded width of the complex s-plane, the integral $\int_{b-i\infty}^{b+i\infty} |\Phi(s)|\,|s|\,|\mathrm{d}s|$ converges, and bounds from above the sum:

$$\sum_{m\leq -1}|\Phi(m)| = \frac{1}{2\pi}\sum_{m\leq -1}\left|\int_{b-i\infty}^{b+i\infty}\Phi(s)\left\{\frac{1}{m}+\frac{1}{s-m}\right\}\mathrm{d}s\right| \leq \frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}|\Phi(s)|\,|s|\,|\mathrm{d}s| \tag{7.4.3.17}$$

where we used Theorem B.2.2.4 (see also [Dur, Ch.11,Thm.11.8]). The estimate in (7.4.3.17) implies that the sum in the right side of (7.4.3.15) tends to its limit absolutely, and hence the sum in the right side of (7.4.3.14) is absolutely convergent. □

7.5. Strong Co-Primality

The Theorems contained within this subsection and the next are not logically necessary for the rest of the arguments leading to the main trace calculation, which is the subject of this work.

However, we will be using them to explain the presence and role of certain structures underlying the nuclear Fréchet spaces

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})), \quad \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$$

for $\chi \in \widehat{C_F}$ a continuous character, and S a finite set of places of the number field F containing all the Archimedean places, and such that χ is unramified outside S.

Specifically, we will be dicussing how predictions of Deninger on the existence of a "space" exhibiting periodic orbits and fixed points related to the primes, both infinite and finite, of F, are indeed correct, and the role that such structures play behind the scenes (without directly entering the arguments).

The next Theorem is an important ingredient leading to the construction of such periodic orbits and fixed points, and the space hosting them: a topological stack built as a quotient of a sub-locus within the S-adeles $\mathbf{A}_{F,S}$ for all large S, serving as the equivariant support for the eigendistributions and their derivatives, of which the unconditional Schauder bases we have identified throughout the previous sections on large subquotients of

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

are comprised.

In other words, the periodic orbits and fixed points whose existence and role Deninger predicted, serve, in a precise sense to be explained in the next subsection, as the supports of the Orbital Schauder bases, therefore justifying the terminology.

The Mellin transform, both S-local and global, sets up a correspondence between these, and certain divisors on the Riemann surfaces $\Omega(J_{F,S})$ and $\Omega(C_F)$.

The following Theorem, uses the existence of classical zero-free regions (through the non-vanishing and polynomial growth of the reciprocal of the finite Hecke L-function on the 1-line; see Theorem B.6.1.6), together with the existence and properties of the S-local Orbital Schauder Bases provided by Theorem 7.3.1.6, as an input.

THEOREM 7.5.0.1 (Strong Co-Primality). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.*
Let S_0 be a finite set of finite places of F.
The canonical continuous linear map:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})/P_{S_0}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F}) \tag{7.5.0.1}$$

induced by the linear topological embedding:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F}) \tag{7.5.0.2}$$

is a linear topological isomorphism.

PROOF. We choose $N \geq 1$ distinct finite places $v_1, \ldots, v_N$ of F, and choose local uniformizers π_{v_k} for $1 \leq k \leq N$.

We recall that for G an $\mathbf{R}_{>0}$-modulated group (in particular, G is Hausdorff, locally compact, abelian), and $|\cdot|_G : G \to \mathbf{R}_{>0}$ the proper surjective module homomorphism with compact kernel G^1 (see Definition D.2.1.1), and for $\gamma \in G$ such that $|\gamma|_G > 1$, we defined in (4.5.1.3) from Proposition 4.5.1.1 the continuous

linear operator with closed range:

$$P_\gamma : \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1}) \to \mathcal{S}_{\mathrm{E}}(G)(\chi|_{G^1})$$

for each $\chi \in \widehat{G}$, assigned by sending $\phi \in \mathcal{S}_{\mathrm{E}}(G)$ to $\phi(x) - \phi(\gamma x)$.

For $\gamma = \pi_{v_k}^{-1}$ we therefore denote by P_k the operator $P_{\pi_k^{-1}}$, with $G = C_F$, $|\cdot|_G = |\cdot|_F$.

We retain the notation from Situation 4.6.1.1, where $P_{(\pi_1^{-1},\pi_2^{-1})}$ denotes the composition P_1P_2 on $\mathcal{S}_{\mathrm{E}}(C_F)$ and its C_F^1-isotypes.

We define:

$$\mathcal{E}_{\{v_1,\dots,v_N\}}(\chi|_{C_F^1}) := \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{\{v_1,\dots,v_N\}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \tag{7.5.0.3}$$

and:

$$\mathcal{E}_{N-1,N}(\chi|_{C_F^1}) := \mathrm{PV}_{\{v_1,\dots,v_{N-1}\}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{\{v_1,\dots,v_N\}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \tag{7.5.0.4}$$

and analyze the following diagram of continuous linear operators, all induced by the linear topological embedding from (7.5.0.2):

(7.5.0.5)

$$\begin{array}{ccccc}
\mathscr{E}_{1,2}(\chi|_{C_F^1}) & \longrightarrow & \mathscr{E}_{\{v_1,v_2\}}(\chi|_{C_F^1}) & \longrightarrow & \mathscr{E}_{\{v_1\}}(\chi|_{C_F^1}) \\
\downarrow & & \downarrow & & \downarrow \\
P_1\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{(\pi_{v_1}^{-1},\pi_{v_2}^{-1})}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{(\pi_{v_1}^{-1},\pi_{v_2}^{-1})}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_1\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})
\end{array}$$

where we observe that the operator PV_{v_1} on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ and the operator P_1 on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ induce the following commutative diagram of continuous linear operators:

(7.5.0.6)

$$\begin{array}{ccc}
\mathscr{E}_{\{v_2\}}(\chi|_{C_F^1}) & \longrightarrow & \mathscr{E}_{1,2}(\chi|_{C_F^1}) \\
\downarrow & & \downarrow \\
\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_2\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) & \longrightarrow & P_1\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{(\pi_{v_1}^{-1},\pi_{v_2}^{-1})}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})
\end{array}$$

By (4.6.1.7) from Lemma 4.6.1.3, the bottom horizontal arrow in (8.1.0.5) is a linear topological isomorphism, and since reciprocal Euler factors are bounded uniformly on vertical strips of bounded width, and Euler factors are periodic meromorphic functions whose poles are located on $i\mathbf{R}$ only and with, therefore, slow growth near their poles in the sense of Definition B.3.1.1, an application of the Asymptotic Residue Theorem (see Theorem B.3.2.1) shows that the top horizontal arrow is surjective.

If the leftmost vertical arrow in (8.1.0.5) is also injective, then it is a linear topological isomorphism by [**NB**, Thm.14.4.5(a)], being source and target Fréchet spaces, and so are the remaining three arrows in (8.1.0.5), since they would then all be continuous bijections between Fréchet spaces.

This observation, combined with commutativity of the diagram in (8.1.0.4), whose rows are linearly and topologically exact by another application of the Asymptotic Residue Theorem (see the argument provided for Lemma 4.6.1.3, which applies to the bottom row in (8.1.0.4) directly, and adapts to the top row), shows that for the continuous linear operator in (7.5.0.1) to be a linear topological isomorphism, it is enough to show that it is so if $S = \{v\}$, a single finite place of F.

From Theorem 7.1.4.3 combined with Lemma 4.4.2.6, we deduce that the rightmost vertical arrow in (8.1.0.4) is injective, by an application of [**NB**, Thm.8.10.6(e)] (which applies by reflexivity of source and target, which in turn holds since both source, by Corollary 6.2.1.2, and target, by Theorem D.3.3.7(1), are nuclear, by [**Scha**, Ch.III,§7,Thm.7.4,p.103], Fréchet spaces, hence Montel by [**Tre**, Part III,Ch.50,Cor.3,p.520], hence reflexive), since the dual Schauder basis on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_1\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ provided by Lemma 4.4.2.6 (which discusses the Exponential System on the G^1-isotypes of $\mathcal{S}(G/\gamma^{\mathbf{Z}})$ for G an $\mathbf{R}_{>0}$-modulated group,

$\gamma \in G$ an element with $|\gamma|_G > 1$), maps under the continuous adjoint of the continuous linear operator which is the rightmost vertical arrow in (8.1.0.4), onto a topologically free set of continuous functionals in the strong dual of $\mathscr{E}_{\{v_1\}}(\chi|_{C_F^1})$ (because any of its enumerations admits a bi-orthogonal sequence by the Theorem, and [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289] applies).

It follows that in order to prove that the rightmost vertical arrow in (8.1.0.4) is surjective, hence a linear topological isomorphism by [**NB**, Thm.14.4.5(a)], being source and target Fréchet spaces, it is enough to show that the middle vertical arrow in (8.1.0.4) is a linear topological isomorphism.

It is therefore enough to show that the continuous linear map:

$$\mathscr{E}_{\{v_1,\ldots,v_N\}}(\chi|_{C_F^1}) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{S_p}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \tag{7.5.0.7}$$

where p is the residue characteristic of $v_1 := v$, and $S_p = \{v_1, \ldots, v_N\}$ is the finite set of all those finite places of F whose residue characteristic is p.

By Theorem 7.1.4.3, the continuous linear operator in (8.1.0.7) is injective, since the continuous functionals given by evaluation at the zeros z of $\prod_{v\in S_p}(1-\chi(\mathfrak{p}_v)q_v^{-s})$ and their derivatives of order $0 \leq m \leq N(z)-1$, for N the multiplicity function of the zeros of $\prod_{v\in S_p}(1-\chi(\mathfrak{p}_v)q_v^{-s})$, pre-composed by the global Mellin transform L from (5.1.1.1) uniquely determined by the assignment $f \mapsto Z(f, \chi|\cdot|_F^s)$ for $f \in \mathcal{S}_0(\mathbf{A}_F)$, map under the continuous adjoint to the continuous linear operator (8.1.0.7) to a subset of a topologically free set of continuous functionals in the strong dual of $\mathscr{E}_{\{v_1,\ldots,v_N\}}(\chi|_{C_F^1})$, hence itself topologically free (because any of its enumerations admits a bi-orthogonal sequence by the Theorem, and hence [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.289] applies).

We may therefore replace S_0 with S_p in the statement of Theorem 7.5.0.1. We observe that by Theorem 7.1.4.3 and [**NB**, Thm.8.10.6(e)], which applies, as in the foregoing, by reflexivity of source and target, that the continuous linear operator in (7.5.0.1) is injective.

Upon showing it is surjective, it is therefore a linear topological isomorphism by [**NB**, Thm.14.4.5(a)].

We will now use the S-local Orbital Schauder Basis provided by Theorem 7.3.1.6, with $S := S_\infty \cup S_p \cup S_1$, to show that the continuous injective linear operator from (7.5.0.1) is surjective, with S_1 a finite set of finite places of F, disjoint from S_p, chosen so as to arrange S to be large enough so that χ is unramified outside S, and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Under these conditions on S, the transition operator:

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

from Theorem 5.2.1.1(3), is a linear topological isomorphism. We denote by χ_S the unique continuous character of $J_{F,S}$ obtained from the restriction of χ to $C_{F,S} \subset C_F$ by descent by $\widehat{\mathcal{O}}^\times_{F,\Sigma_F - S}$-invariance.

The idea is to show that the image of the absolute Orbital Schauder Basis on the p-primary Orbital Quotients $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ from (7.3.1.5) (see Definition 7.3.1.3) not only remains a Schauder basis in the global subquotient:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

which it does since $\overline{\pi}_S$ is a linear topological isomorphism compatible with Weil's Principal Value operators PV_{S,S_p} on the source, and PV_{S_p} on the target, but also in $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{S_p}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$.

This is where the estimates in (7.3.1.17) from Theorem 7.3.1.6(4) are used, together with the rapid decay uniformly on vertical strips of bounded width of the complex s-plane, of the Mellin transform of the elements of $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$.

We will show below that, in order to prove the desired basis property, such estimates need to be combined with the polynomial growth of the derivatives of the Hecke L-function on vertical lines, and with, most importantly, the polynomial growth of the reciprocal of the finite Hecke L-function on the 1-line (for which we refer the reader to Remark B.1.1.4 and Theorem B.6.1.6).

This will indeed show surjectivity of (7.5.0.1) (we will be providing the argument below), therefore finishing the proof of the Theorem.

By Lemma 5.3.1.5, we have:

$$\overline{\pi}_S \circ \mathrm{PV}_{S,S_p} = \mathrm{PV}_{S_p} \circ \overline{\pi}_S$$

on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

By Theorem 6.2.1.1(2):

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

is a linear topological isomorphism.

We call, for c the arbitrary zero of $L_v(c)^{-1}$ contained in $\Omega(C_F)(\chi|_{C^1_F})$, parametrized by $\chi|\cdot|_F^{s_c}$ (equivalently in $\Omega_S(J_{F,S})(\chi|_{J^1_{F,S}})$, since, in light of the assumption $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, all the zeros of $L_v(c)^{-1}$ in $\Omega(C_F)(\chi|_{C^1_F})$ for each $v \in S$ are contained in $\Omega_S(C_F)(\chi|_{C^1_F})$, which admits a surjective local analytic isomorphism onto $\Omega(J_{F,S})(\chi|_{J^1_{F,S}})$ whose fibres are single-point discrete sets):

$$\gamma_c^{(k)}(\phi) := \partial_s^k Z(\mathcal{F}(\phi), \overline{\chi}|\cdot|_F^s)|_{s=1-s_c} \tag{7.5.0.8}$$

for $\mathcal{F} : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(\overline{\chi}|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$.

We have $\gamma_c^{(k)} \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))'_\beta$ for each zero c of $L_v(c)^{-1}$ contained in $\Omega(C_F)(\chi|_{C^1_F})$ for each $v \in S_0$, and $0 \leq k \leq N_S(c) - 1$.

Assume $\phi = \overline{\pi}_S(\phi_0)$ for a unique $\phi_0 \in \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$.

We have:

$$\gamma_c^{(k)}(\phi) = \partial_s^k (Z_S(\mathcal{F}_S(\phi_0), \overline{\chi}|\cdot|_{F,S}^s) L_S(\overline{\chi}, s))|_{s=1-s_c}$$

where $L_S(\overline{\chi}, s) = \prod_{v \in S} L_v(\overline{\chi}_v|\cdot|_v^s)^{-1} \cdot L(\overline{\chi}, s)$ for $s \in \mathbf{C}_{>0}$, extended meromorphically to the complex s-plane.

$$\gamma_c^{(k)}(\phi) = \left\{ \sum_{0 \leq m \leq k} \binom{k}{m} \partial_s^m (Z_S(\mathcal{F}_S(\phi_0), \overline{\chi}|\cdot|_{F,S}^s) \cdot \partial_s^{k-m} L_S(\overline{\chi}, s) \right\} |_{s=1-s_c} \tag{7.5.0.9}$$

$$= \sum_{0 \leq m \leq k} \binom{k}{m} \partial_s^m (Z_S(\mathcal{F}_S(\phi_0), \overline{\chi}|\cdot|_{F,S}^s)|_{s=1-s_c} \cdot \partial_s^{k-m} L_S(\overline{\chi}, s)|_{s=1-s_c} \tag{7.5.0.10}$$

$$= \sum_{0 \leq m \leq k} \binom{k}{m} \delta_{S,c}^m(\phi_0) \cdot \partial_s^{k-m} L_S(\overline{\chi}, s)|_{s=1-s_c} \tag{7.5.0.11}$$

where we have denoted by $\delta_{S,c}^m$ the continuous functionals:

$$\widehat{\delta}_c^{(m)}$$

from (7.3.1.13) of Theorem 7.3.1.6, which, by the Theorem, constitute, upon choosing an enumeration of their set, an unconditional Schauder basis for $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})'_\beta$.

We rewrite (8.1.0.8) in matrix form

$$\overline{\pi}'_S(\gamma_c^{(k)}) = (\delta_{S,c}^m)_{1 \leq m \leq k} \cdot A_{c,k,\chi} \tag{7.5.0.12}$$

with $A_{c,k,\chi}$ the $k \times k$ triangular matrix with complex coefficients:

(7.5.0.13)

$$\begin{bmatrix} L_S(\overline{\chi},1-s_c) & L_S^{(1)}(\overline{\chi},1-s_c) & L_S^{(2)}(\overline{\chi},1-s_c) & L_S^{(3)}(\overline{\chi},1-s_c) & L_S^{(4)}(\overline{\chi},1-s_c) & \cdots & L_S^{(k)}(\overline{\chi},1-s_c) \\ 0 & L_S(\overline{\chi},1-s_c) & 2L_S^{(1)}(\overline{\chi},1-s_c) & 3L_S^{(2)}(\overline{\chi},1-s_c) & 4L_S^{(3)}(\overline{\chi},1-s_c) & \cdots & kL_S^{(k-1)}(\overline{\chi},1-s_c) \\ 0 & 0 & L_S(\overline{\chi},1-s_c) & 3L_S^{(1)}(\overline{\chi},1-s_c) & 6L_S^{(2)}(\overline{\chi},1-s_c) & \dots & \binom{k}{2}L_S^{(k-2)}(\overline{\chi},1-s_c) \\ 0 & 0 & 0 & L_S(\overline{\chi},1-s_c) & 4L_S^{(1)}(\overline{\chi},1-s_c) & \cdots & \binom{k}{3}L_S^{(k-3)}(\overline{\chi},1-s_c) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & L_S(\overline{\chi},1-s_c) \end{bmatrix}$$

Its determinant is $L_S(\overline{\chi},1-s_c)^k \neq 0$, where its non-vanishing is a consequence of the non-vanishing of the Hecke L-function $L(\overline{\chi},s)$ on the 1-line, which holds by [**CF**, Ch.VIII,§2,Thm.2,p.212].

We now observe that we have $\gamma_c^{(m)} = U(\delta_c^{(m)})$ (the notation being as in Theorem 7.1.4.3(2)), by the global functional equation from (3.2.2.2) of Theorem 3.2.2.1, where $\delta_c^{(m)}$ and $U(\delta_c^{(m)})$ are the continuous functionals in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))'_\beta$ from (7.1.4.27) of Theorem 7.1.4.3.

By Theorem 4.6.1.5 applied to $G = C_F$, $|\cdot|_G = |\cdot|_F$, to the choice of continuous character χ, with the set S of elements of G chosen to be $(\pi_v^{-1})_{v\in S_p}$, upon choosing an ordering for the finite set S_p, and adopting the notation from Situation 4.6.1.1(3), we deduce that the set $\{\gamma_c^{(k)}\}_{c\in\mathcal{U}_{S_p},0\leq k\leq N_p(c)-1}$ is the image under the continuous adjoint of (7.5.0.1) of the unconditional Schauder basis given by the functionals defined in (4.6.1.14), for the strong dual of $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{S_p}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$.

We now rewrite:

$$\gamma_c^{(k)} = ((\overline{\pi}_S^{-1})'(\delta_{S,c}^m))_{1\leq m\leq k} \cdot A_{c,k,\chi} \tag{7.5.0.14}$$

The bi-orthogonal set to $\{\gamma_c^{(k)}\}_{c\in\mathcal{U}_{S_p},0\leq k\leq N_p(c)-1}$ in $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{S_p}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ is therefore obtained from the bi-orthogonal set to the unconditional Schauder basis $\{(\overline{\pi}_S^{-1})'(\delta_{S,c}^m)\}_{c\in\mathcal{U}_{S_p},0\leq m\leq N_p(c)-1}$ by right multiplication of each finite subset of $N_p(c)$ indexed by the same fixed $c \in \mathcal{U}_{S_p}$, by $A^{-1}_{c,N_p(c),\chi}$.

The operator norm of each matrix $A^{-1}_{c,N_p(c),\chi}$ (whose rank is, recall, bounded above by $[F:\mathbf{Q}]$, the degree of the number field, since $N_p(c) \leq [F:\mathbf{Q}]$) grows polynomially with respect to the ordinate of c with respect to the holomorphic parametrization $c = \chi|\cdot|_F^s$, by Theorem B.6.1.6 and Remark B.1.1.4.

By the estimates from (4.6.1.17) of Theorem 4.6.1.5 and from (7.3.1.17) of Theorem 7.3.1.6 respectively, and the exponential decay of every element ϕ of $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, which implies that the sequence of complex numbers $(\overline{\pi}_S^{-1})'(\delta_{S,c}^m)(\phi)$ for each of the finitely many fixed $0 \leq m \leq N_p(c)-1 \leq [F:\mathbf{Q}]$ decays rapidly (with respect to the integer m_v parametrizing the point $c = \chi|\cdot|_F^{\frac{2\pi i m_v}{\log(q_v)}-it_v}$ as per Lemma 7.1.2.2, for a, equivalently any, choice of $v \in S_p$ such that $c \in \psi_v^{-1}(\mathcal{U}_v)$, as $|m_v| \to +\infty$) the image in $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})/P_{S_p}\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ of the absolute Schauder basis $\{(\overline{\pi}_S^{-1})'(\delta_{S,c}^m)\}_{c\in\mathcal{U}_{S_p},0\leq m\leq N_p(c)-1}$ is an absolute Schauder basis.

This implies surjectivity of (7.5.0.1) in the special case $S_0 = S_p$, which finishes the proof of the Theorem. □

Remark 7.5.0.2. Theorem 7.5.0.1 implies that for each prime p, the subquotients:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \tag{7.5.0.15}$$

all nuclear Fréchet spaces, carry Hausdorff locally convex topologies that can be understood concretely.

Indeed, by Theorem 6.2.1.1, they are linearly and topologically isomorphic to the nuclear Fréchet spaces:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S}))/\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_F)) \tag{7.5.0.16}$$

for S a finite set of finite places of F containing all the Archimedean places, and such that χ is unramified outside S, and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ (upon denoting by χ_S the unique continuous character of $J_{F,S}$ obtained by descending by $\widehat{\mathcal{O}}^\times_{\Sigma_F-S}$-invariance the restriction of χ to $C_{F,S}$), and the subquotients in (7.5.0.16) are in turn

linearly and topologically isomorphic to closed subspaces of quotients by a closed subspace of a finite product of copies of $\mathcal{S}(\mathbf{R}_{>0}/q^{\mathbf{Z}})$ for q ranging over finitely many powers of p.

Under these assumptions, $\overline{\pi}_S$ is a linear topological isomorphism (see Theorem 6.2.1.1) compatible with the operators PV_{S,S_p} and PV_{S_p} on source, respectively target (see Lemma 5.3.1.5).

In particular, they are Köthe Sequence Spaces.

The strong dual of the subquotient in (7.5.0.15) is in turn therefore linearly and topologically isomorphic to a quotient by a closed subspace of a closed subspace of a finite locally convex direct sum of copies of

$$\mathcal{S}\left(\frac{2\pi i}{\log(q)}\mathbf{Z}\right) \simeq \mathcal{S}(\mathbf{Z}).$$

We will be collecting the following Corollary, which is a strong form of the non-vanishing of the Hecke L-function $L(s,\chi)$ on the 1-line. By solving the a Second Cousin Problem for the pair of entire functions $(\Lambda(s,\chi),(1-\chi(\mathfrak{p}_v)q_v^{-s}))$, we know from the non-vanishing of $L(s,\chi)$ (see [**CF**, Ch.VIII,Thm.2,p.212]) that there exist entire functions $u,v\in\mathscr{O}(\mathbf{C})$ such that:

$$\Lambda(s,\chi)\cdot u(s)+(1-\chi(\mathfrak{p}_v)q_v^{-s})\cdot v(s)=1$$

as entire functions over the complex s-plane.

We have no way to tell anything about the growth behavior of u,v. Using the Schauder Bases identified throughout this Chapter, and as a consequence of Theorem 7.5.0.1, we have the following:

Corollary 7.5.0.3 (Strong Co-Primality). *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let S be a finite set of finite places of F at which χ is unramified. We write, for each $v\in S$, $q_v:=N(\mathfrak{p}_v)$, with $\mathfrak{p}_v\subset\mathcal{O}_F$ the unique prime ideal corresponding to the finite place v.*

Let τ be a choice of continuous homomorphic section to the idelic norm.

For any entire function $\phi\in A\mathcal{S}(\mathbf{C})$, there exist entire functions $u\in\mathscr{O}(\mathbf{C})$ such that $\Lambda(s,\chi)\cdot u\in A\mathcal{S}(\mathbf{C})$, and $v\in A\mathcal{S}(\mathbf{C})$, such that:

$$\Lambda(s,\chi)\cdot u(s)+\prod_{v\in S}(1-\chi(\mathfrak{p}_v)q_v^{-s})\cdot v(s)=\phi(s) \tag{7.5.0.17}$$

with $\Lambda(s,\chi):=((s+ix_0)(s+ix_0-1))^{\delta_\chi}L(s,\chi)$, for $\delta_\chi=1$ if $\chi|_{C_F^1}=1$, and $\delta_\chi=0$ otherwise, for $x_0\in\mathbf{R}$ the unique real number such that $\chi(\tau(t))=t^{ix_0}$ as smooth functions of $t\in\mathbf{R}_{>0}$.

Proof. The Corollary follows upon combining Theorem 7.5.0.1 with Theorem 6.3.2.2. □

CHAPTER 8

Periodic Orbits and Fixed Points

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Let F be a number field.

In this section we collect the main results of this Chapter and explain them through the lens of Theorem 7.1.2.3, which was the starting point of the Chapter itself. We also compute the support of the distributions that comprise the dual Schauder bases of the absolute bases that we found on the subquotients $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$.

The supports will be loci within the S-adeles, for any finite set of places of F containing all the Archimedean places, and they will be stable under the continuous left-translation action by $\mathcal{O}^{\times}_{F,S}$ and $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

Since such distributional basis elements are obtained from an eigendistribution through derivation of uniformly bounded above order, their supports only enter through their topological quotient stacks, which we compute explicitly: they are either fat points (equipped with a canonical concept of multiplicity, much like the closed point of a formal unit disc over a discrete field) or periodic orbits: quotient stacks whose coarse space is a loop with a natural length inherited form the Haar measure on the S-ideles, of $\log N(\mathfrak{p}_v)|$ for each finite place $v \in S$.

This shows how exactly predictions of Deninger turn out to feature and play a role within this work.

8.1. On Orbital Supports - I

The following Theorem uses Theorem 7.5.0.1 as an input, and it should be thought of as a refinement of Theorem 7.3.1.6: it informs us on the existence of absolute Schauder bases on the p-primary Orbital Subquotients

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}), \quad \mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}})$$

defined in (7.3.1.5) and (7.3.4.6) that lift to elements of $\mathcal{S}(\mathbf{A}_{F,S})$ supported over certain specific loci within the S-adeles.

This enables us to compute the support of the tempered distributions on $\mathbf{A}_{F,S}$ that are the images of the dual basis elements in $\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})'_\beta$.

The computation of their supports will lead to the finding of the periodic orbits and fixed points whose existence and role was alluded to in the previous subsection, and will enable us to "organize" the nuclear Fréchet spaces by rethinking them as Fréchet bundles on a certain topological quotient stack to be defined below, each of whose subquotients are supported each on a stack-theoretic periodic orbit, respectively on a stack-theoretic fixed point.

We will conclude this Chapter by explaining the geometric meaning of the Paley-Wiener Levels, and of Theorem 7.3.3.4, which asserts that in the (sole) case of the subquotients:

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}})$$

i.e. only for p a finite prime, and not for the Archimedean places, the absolute Orbital Schauder Bases and their dual bases are inherited by the corresponding subquotients and their strong duals of the Paley-Wiener Levels:

$$\mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi|_{J^1_{F,S'}}),\ \widetilde{\mathscr{E}}_{S',S_p,Y}(\chi|_{J^1_{F,S'}})$$

from (7.3.3.5) and (7.3.3.6).

THEOREM 8.1.0.1 (On Orbital Supports - I). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.*

Let S be a finite set of places of F containing all the Archimedean places. Assume χ is unramified outside S, and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

(1) *Let p be a prime number such that $S_p \subset S$. The canonical continuous linear map:*

(8.1.0.1)

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_p(\mathbf{A}_{F,S}))/\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_p(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

where $\mathcal{S}_p(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ is the closed subspace:

$$\mathcal{S}_p(\mathbf{A}_{F,S}) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \bigotimes_{v \in S_f - S_p} \mathcal{S}(F_v^\times) \otimes_\pi \bigotimes_{v \in S_p} \mathcal{S}(F_v) \tag{8.1.0.2}$$

is a linear topological isomorphism.

(2) *Let $v \in S$ be a finite place. The canonical continuous linear map:*

(8.1.0.3)

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_v(\mathbf{A}_{F,S}))/\mathrm{PV}_v\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_p(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))/\mathrm{PV}_v\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

where $\mathcal{S}_v(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ is the closed subspace:

$$\mathcal{S}_v(\mathbf{A}_{F,S}) := \mathcal{S}_{00}(\mathbf{A}_{F,S_\infty}) \otimes_\pi \mathcal{S}(F_v) \otimes_\pi \bigotimes_{w \in S_f - \{v\}} \mathcal{S}(F_w^\times) \tag{8.1.0.4}$$

is a linear topological isomorphism.

PROOF. We choose a continuous homomorphic section τ to the idelic norm, with image contained within $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \{1\} \subset \mathrm{GL}_1(\mathbf{A}_F)$.

(1) : Under the stated assumptions on $\overline{\pi}_S$, by Theorem 6.2.1.1(2):

$$\overline{\pi}_S : \mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$$

is a linear topological isomorphism. By Theorem 5.4.1.1, we deduce that L_S from (5.1.1.2) induces a linear topological embedding:

$$U_S : \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_0(\mathbf{A}_{F,S})) \to A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{>0}) \tag{8.1.0.5}$$

where, upon defining:

$$\Lambda(s,\chi) := ((s+ix_0)(s+ix_0-1))^{\delta_\chi} L(s,\chi)$$

for $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$, with $\delta_\chi = 0$ if $\chi|_{C^1_F} \neq 1$, and $\delta_\chi = 1$ otherwise, the range of U_S is identified with the closed subspace of pairs $(\phi_0, \psi_0) \in A\mathcal{S}(\mathbf{C}_{>0}) \times A\mathcal{S}(\mathbf{C}_{>0})$ such that (ϕ, ψ) satisfies $\phi(s) = \rho_S(\chi_S|\cdot|^s_{F,S})\psi(1-s)$.

The unique analytic continuation of ϕ_0 to a meromorphic function $\Phi(s)$ on the complex s-plane with poles located at the poles of ρ_S, i.e. over the subset $\mathcal{U}_S$, with multiplicity bounded above by the multiplicity

function m_S of $\mathcal{U}_S$ (see (7.2.2.2) and (3.3.1.3)), is contained in the linear subspace of $\mathscr{M}(\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}}))$:

$$\frac{1}{L_{\Sigma_F-S}(\chi,s)}\cdot(\Lambda(s,\chi)\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C}))\subset\mathscr{M}(\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})) \tag{8.1.0.6}$$

where:

$$L_{\Sigma_F-S}(\chi,s):=\prod_{v\in S_f}L_v(\chi_v|\cdot|_v^s)^{-1}\cdot L_f(s,\chi) \tag{8.1.0.7}$$

where $L_f(s,\chi)$ is the finite Hecke L-function (see (3.2.1.7) from Definition 3.2.1.4), and where for each finite place $v\in S_f\subset S$, $L_v(\chi_v|\cdot|_v^s)$ is the Euler factor at v of the Hecke L-function $L(s,\chi)$.

Upon writing:

$$L_S(\chi,s):=\prod_{v\in S}L_v(\chi_v|\cdot|_v^s)$$

we observe that we have:

$$\frac{\Lambda(s,\chi)}{L_{\Sigma_F-S}(\chi,s)}=L_S(\chi,s)$$

as meromorphic functions on the complex s-plane.

Let $\phi\in\mathscr{O}(\mathbf{C})$ be such that $\Lambda(s,\chi)\phi(s)\in A\mathcal{S}(\mathbf{C})$, and such that

$$\frac{\Lambda(s,\chi)}{L_{\Sigma_F-S}(\chi,s)}\phi(s)=\Phi(s). \tag{8.1.0.8}$$

Then, since Φ is Schwartz uniformly on every open vertical strip of bounded width of $\mathbf{C}-i\mathbf{R}$, since $\rho_S(\chi|\cdot|_{F,S}^s)$ and all of its meromorphic derivatives grow at most polynomially uniformly on such vertical strips by Proposition 3.3.1.5, and since $\frac{\Lambda(s,\chi)}{L_{\Sigma_F-S}(\chi,s)}=L_S(\chi,s)$ is bounded and holomorphic, as are all of its derivatives, on all such vertical strips, we deduce that we have, as linear subspaces of $\mathscr{M}(\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}}))$:

$$\frac{1}{L_{\Sigma_F-S}(\chi,s)}\cdot(\Lambda(s,\chi)\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C}))=L_S(\chi,s)A\mathcal{S}(\mathbf{C}). \tag{8.1.0.9}$$

By Theorem 7.5.0.1, we have:

$$\Lambda(s,\chi)\mathscr{O}(\mathbf{C})\cap A\mathcal{S}(\mathbf{C})+\prod_{v\in S_p}L_v(\chi_v|\cdot|_v^s)^{-1}A\mathcal{S}(\mathbf{C})=A\mathcal{S}(\mathbf{C}). \tag{8.1.0.10}$$

Let $\phi\in A\mathcal{S}(\mathbf{C})$ be the element satisfying (8.1.0.8).

Then

$$\Phi(s)=L_S(\chi,s)\phi(s)=L_S(\chi,s)\Lambda(s,\chi)u(s)+\prod_{v\in S_p}L_v(\chi_v|\cdot|_v^s)^{-1}L_S(\chi,s)v(s)$$

for $u,v\in\mathscr{O}(\mathbf{C})$ with $\Lambda(s,\chi)u(s)\in A\mathcal{S}(\mathbf{C})$, and $v\in A\mathcal{S}(\mathbf{C})$.

The meromorphic functions $L_S(\chi,s)\Lambda(s,\chi)u(s)$ and $L_S(\chi,s)v(s)$ are in the range of the linear operator L_S from (5.1.1.4), and since L_S is injective, we deduce that we have that the generic element of $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$ can be written as the sum of an element of $\mathrm{PV}_{S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$ and an element of the linear subspace of $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$ linearly generated by the images under $\Phi_{F,S,\chi_S|_{J^1_{F,S}}}$ of those Schwartz-Bruhat functions $f\in\mathcal{S}_0(\mathbf{A}_{F,S})$ such that

$$Z_S(f,\chi|\cdot|_{F,S}^s)=L_S(\chi,s)\Lambda(s,\chi)u(s)$$

for some $u\in\mathscr{O}(\mathbf{C})$ such that $\Lambda(s,\chi)u(s)\in A\mathcal{S}(\mathbf{C})$.

We observe that $\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(f)$ is contained within the closed subspace:

$$\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(\mathcal{S}_\infty(\mathbf{A}_{F,S_\infty})\otimes_\pi\bigotimes_{v\in S_f}\mathcal{S}(F_v))\subset\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S})).$$

Indeed, we have:

$$L(\pi_S(f),\chi|\cdot|_F^s) = Z_S(f,\chi_S|\cdot|_{F,S}^s)L_{\Sigma_F-S}(\chi,s) = L_S(\chi,s)\Lambda(s,\chi)L_{\Sigma_F-S}(\chi,s)\phi(s) = \Lambda(s,\chi)^2u(s)$$

which is indeed an element of

$$\overline{\pi}_S(\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(\mathcal{S}_\infty(\mathbf{A}_{F,S_\infty})\otimes_\pi \bigotimes_{v\in S_f}\mathcal{S}(F_v)))$$

by Theorem 7.3.4.1.

This shows that the continuous linear operator in (8.1.0.1) is surjective, hence open by [**NB**, Thm.14.4.5(a)], being source and target Fréchet spaces. It is also injective, since, as an application of the Asymptotic Residue Theorem, which applies by Lemma 7.1.1.1, we have:

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_p(\mathbf{A}_{F,S}))\cap \mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S})) = \mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_p(\mathbf{A}_{F,S})).$$

Part (2) follows from the same method discussed for (1) upon using Theorem 7.5.0.1 for $S_0=\{v\}$. □

We will be concluding the Chapter by describing the supports of the dual basis elements of the absolute Schauder bases within the orbital quotients

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}),\quad \mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}})$$

defined in (7.3.1.5) from Definition 7.3.1.3 and (7.3.4.3) from Notation 7.3.4.1, respectively.

Describing such supports is the scope of Theorem 8.3.0.1. While it will not be used throughout the rest of this work, we will be using it below throughout the remainder of this section, to explain how expectations formulated by Deninger fit within this treatise.

8.2. Stack-theoretic Periodic Orbits and Fixed Points

We begin by introducing some terminology and notation:

Definition 8.2.0.1. Let F be a number field, S a finite set of places of F containing the Archimedean places. We denote by S_f the subset of S of the finite places of F contained in S.

We define:

(1) The subset of the S-adeles:

$$Z_{F,S} := \{e_1 := 1\}\cup\{e_v\}_{v\in S}\subset \mathbf{A}_{F,S} \tag{8.2.0.1}$$

and the subset of the global adeles:

$$Z_F := \{e_1 := 1\}\cup\{e_v\}_{v\in \Sigma_F}\subset \mathbf{A}_F \tag{8.2.0.2}$$

where $e_v\in\mathbf{A}_{F,S}$ in (8.2.0.1) is the S-adele whose v-indexed coordinate is 0, and whose v'-indexed coordinate is 1 for all $v'\in S-\{v\}$, and $e_v\in\mathbf{A}_F$ from (8.2.0.2) is the global adele whose v-indexed coordinate is 0, and whose v'-indexed coordinate is 1 for all $v'\in\Sigma_F-\{v\}$. We endow both Z_S and Z_F with the respective subspace topologies.

(2) The subsets of the S-adeles:

$$D_{S,1} := \mathbf{A}_{F,S}^\times,\quad D_{S,v} := \mathrm{GL}_1(\mathbf{A}_{F,S})\cdot e_v \tag{8.2.0.3}$$

endowed with the subspace topology, and the subsets of the global adeles:

$$D_1 := \mathbf{A}_F^\times,\quad D_v := \mathrm{GL}_1(\mathbf{A}_F)\cdot e_v \tag{8.2.0.4}$$

endowed with the subspace topology inherited from $\mathbf{A}_F$.

(3) The subsets:

$$D_{F,S} := D_{S,1} \cup \bigcup_{v \in S} D_{S,v} \subset \mathbf{A}_{F,S}, \quad D_F := D_1 \cup \bigcup_{v \in \Sigma_F} D_v \subset \mathbf{A}_F \tag{8.2.0.5}$$

equipped with the respective subspace topologies.

The continuous actions of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathbf{A}_{F,S}$ and $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathbf{A}_F$ by left-translation, stabilize the subsets introduced in (8.2.0.6), (8.2.0.7) and (8.2.0.8) of Definition 8.2.0.1, inducing continuous actions by left-translation on each of them, respectively.

We discuss some preparations around $D_{F,S}$ and D_F. We first need the following subspaces of $D_{F,S}$ and D_F, respectively:

Definition 8.2.0.2. Let F be a number field, S a finite set of places of F containing the Archimedean places. We denote by S_f the subset of S of the finite places of F contained in S.

We define:

(1) the subsets of the S-adeles:

$$D^1_{S,1} := \{g \in \mathbf{A}^\times_{F,S} : |g|_{F,S} = 1\}, \quad D^1_{S,v} := \mathrm{GL}_1(\mathbf{A}_{F,S})^1 \cdot e_v \tag{8.2.0.6}$$

endowed with the subspace topology, and the subsets of the global adeles:

$$D^1_1 := \{g \in \mathbf{A}^\times_F : |g|_F = 1\}, \quad D^1_v := \mathrm{GL}_1(\mathbf{A}_F)^1 \cdot e_v \tag{8.2.0.7}$$

endowed with the subspace topology inherited from $\mathbf{A}_F$.

(2) The subsets:

$$D^1_{F,S} := D^1_{S,1} \cup \bigcup_{v \in S} D^1_{S,v} \subset \mathbf{A}_{F,S}, \quad D^1_F := D^1_1 \cup \bigcup_{v \in \Sigma_F} D^1_v \subset \mathbf{A}_F \tag{8.2.0.8}$$

equipped with the respective subspace topologies.

We begin by describing the topology on $D^1_{F,S}$ and D^1_F.

Lemma 8.2.0.3. *Let F be a number field, S a finite set of places of F containing the Archimedean places. Then:*

(1) *The subspace topology on $D^1_{F,S}$ inherited from $\mathbf{A}_{F,S}$ agrees with the quotient topology inherited from the product topology on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times Z_{F,S}$ under the continuous surjective map $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times Z_{F,S} \to D^1_{F,S}$.*

(2) *The subspace topology on D^1_F inherited from $\mathbf{A}_F$ agrees with the quotient topology inherited from the product topology on $\mathrm{GL}_1(\mathbf{A}_F)^1 \times Z_F$ under the continuous surjective map $\mathrm{GL}_1(\mathbf{A}_F)^1 \times Z_F \to D^1_F$.*

Both $D^1_{F,S}$ and D^1_F are Hausdorff, metrizable, second-countable, perfectly normal, separable, locally compact, σ-compact and paracompact.

Proof. (1) : The map $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times Z_{F,S} \to D^1_{F,S}$ is continuous and surjective. The pre-images in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times Z_{F,S}$ of $D^1_{S,1}$ and $D^1_{S,v}$ are, for each $v \in S$, $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times \{1\}$ and $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times \{e_v\}$, respectively.

Each $D^1_{S,1}$ and $D^1_{S,v}$, for every $v \in S$, is closed in $D^1_{F,S}$. To see this, we proceed as follows. For $D^1_{S,1}$, we observe that $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ is closed in $\mathbf{A}_{F,S}$, and hence in $D^1_{F,S}$.

For $v \in S$, we observe that the stabilizer of $e_v \in \mathbf{A}_{F,S}$ in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ is $\mathcal{O}^\times_{F_v}$, embedded through the v-indexed "coordinate" insertion homomorphism, an injective homomorphism with closed image. Since $|\mathcal{O}^\times_{F_v}|_{F,S} = 1$, then $D^1_{S,v}$ is the subset of $\mathbf{A}_{F,S}$ given by those S-adeles $x \in \mathbf{A}_{F,S}$, $x = (x_w)_{w \in S}$, such that $x_v = 0$, and $\prod_{w \in S - \{v\}} |x_w|_w = 1$, both of which are closed conditions.

By [**Bou-GT**, Ch.I,§5.1,Prop.2,p.51], it is therefore enough to show that each $D^1_{S,1}$ and $D^1_{S,v}$ is equipped with the quotient topology. This is indeed the case, because the idelic topology on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and the subspace topology inherited from $\mathbf{A}_{F,S}$ agree.

(2) : The map $\mathrm{GL}_1(\mathbf{A}_F)^1 \times Z_F \to D_F^1$ is continuous and surjective. The pre-images in $\mathrm{GL}_1(\mathbf{A}_F)^1 \times Z_F$ of D_1^1 and D_v^1 are, for each place v of F, $\mathrm{GL}_1(\mathbf{A}_F)^1 \times \{1\}$ and $\mathrm{GL}_1(\mathbf{A}_F)^1 \times \{e_v\}$, respectively.

Each D_1^1 and D_v^1 is equipped with the quotient topology, because the idelic topology on $\mathrm{GL}_1(\mathbf{A}_F)^1$ and the subspace topology inherited from $\mathbf{A}_F$ agree by [**CF**, Ch.II,§16,Lemma,p.69].

D_1^1 is closed in D_F^1, since it is closed in $\mathbf{A}_F$. For each place v of F, the subset D_v^1 of $\mathbf{A}_F$ is the intersection of the closed subset $H_v := \{x \in \mathbf{A}_F : x_v = 0\}$, with the subset $\Gamma_v \subset \mathbf{A}_F$ such that its projection $\overline{\Gamma}_v$ onto $\widehat{\prod}_{w\in\Sigma_F-\{v\}}(F_w, \mathcal{O}_{F_w})$ is the subset of those elements $(x_w)_{w\neq v}$ such that all but finitely many of the x_w is in $\mathcal{O}_{F_w}^\times$, and $\prod_{w\neq v} |x_w|_w = 1$. $\overline{\Gamma}_v$ is therefore closed, and the continuous surjective map $\Gamma_v \to \overline{\Gamma}_v$ is bijective, hence a homeomorphism. We deduce that Γ_v is closed in $\mathbf{A}_F = \widehat{\prod}_{w\in\Sigma_F-\{v\}}(F_w, \mathcal{O}_{F_w}) \times F_v$, and hence so is $\Gamma_v \cap H_v = D_v^1$.

Since $\{D_v^1\}_{v\in\Sigma_F} \cup \{D_1^1\}$ is a locally finite cover of D_F^1 by closed subsets, then by [**Bou-GT**, Ch.I,§5.1,Prop.2,p.51] we deduce that D_F^1 is equipped with the quotient topology induced by the continuous surjective map $\mathrm{GL}_1(\mathbf{A}_F)^1 \times Z_F \to D_F^1$, since D_1^1 and D_v^1 is.

$D_{F,S}^1$ and D_F^1 are Hausdorff, metrizable and second-countable, since they are subspaces of $\mathbf{A}_{F,S}$, respectively $\mathbf{A}_F$, both Hausdorff, metrizable and second-countable. They are therefore separable, perfectly normal and paracompact. They are countable unions of Hausdorff quotients of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, respectively $\mathrm{GL}_1(\mathbf{A}_F)^1$, and they are therefore σ-compact, since $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and $\mathrm{GL}_1(\mathbf{A}_F)^1$ both are. Since each point x of $D_{F,S}^1$, respectively D_F^1, is contained in some $D_{S,1}^1$, $D_{S,v}^1$, D_1^1, or D_v^1, respectively, and each of these is a locally compact space, then x has a compact neighborhood, and since $D_{F,S}^1$ and D_F^1 are Hausdorff, they are locally compact and σ-compact. □

PROPOSITION 8.2.0.4. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. We let $\tau_S : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be a choice of continuous homomorphic section to the S-idelic norm, and $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ a choice of continuous homomorphic section to the idelic norm.*

We define, for every finite place $v \in S$ and $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ associated to v:

$$\tau_S([1, N(\mathfrak{p}_v))) \cdot e_v \subset \mathbf{A}_{F,S} \tag{8.2.0.9}$$

and for every finite place v of F and $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ associated to v:

$$\tau([1, N(\mathfrak{p}_v))) \cdot e_v \subset \mathbf{A}_F \tag{8.2.0.10}$$

each equipped with the subspace topology. Then:

1. *For every $x \in D_{S,v}$ there exists $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$, unique up to multiplication by $\mathcal{O}_{F_v}^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$, such that $g^{-1}x \in \tau_S([1, N(\mathfrak{p}_v))) \cdot e_v$.*
2. *For every $x \in D_v$ there exists $g \in \mathrm{GL}_1(\mathbf{A}_F)$, unique up to multiplication by $\mathcal{O}_{F_v}^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$, such that $g^{-1}x \in \tau([1, N(\mathfrak{p}_v))) \cdot e_v$.*

PROOF. (1) : $D_{S,v}$ is homeomorphic to the quotient of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ by the stabilizer of e_v, which is $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$ embedded through the v-indexed "coordinate". $D_{S,v}$ is therefore continuously bijective to $\tau_S([1, N(\mathfrak{p}_v))) \cdot \mathrm{GL}_1(\mathbf{A}_{F,S})^1 \cdot e_v$, and since the stabilizer of e_v in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ is $\mathcal{O}_{F_v}^\times$, (1) follows.

(2) : D_v is homeomorphic to the quotient of $\mathrm{GL}_1(\mathbf{A}_F)$ by the stabilizer of e_v, which is $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$ embedded through the v-indexed "coordinate". (2) therefore follows upon observing that the stabilizer of e_v in $\mathrm{GL}_1(\mathbf{A}_F)^1$ is $\mathcal{O}_{F_v}^\times$. □

PROPOSITION 8.2.0.5. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. For every Archimedean place $v \in S$, the action of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ on $D_{S,v}$ is transitive. For every Archimedean place v of F, the action of $\mathrm{GL}_1(\mathbf{A}_F)^1$ on D_v is transitive.*

1. *If v is real, then every point of $D_{S,v}$, respectively of D_v, is stabilized by $\{\pm 1\}$ as a subgroup of $\mathrm{GL}_1(\mathbf{A}_{F,S})$, respectively $\mathrm{GL}_1(\mathbf{A}_F)$, embedded injectively as the subgroup of the units of $F_v^\times$.*

(2) *If v is complex, then every point of $D_{S,v}$, respectively D_v, is stabilized by* $\mathrm{U}(1)$ *as a subgroup of* $\mathrm{GL}_1(\mathbf{A}_{F,S})$*, respectively* $\mathrm{GL}_1(\mathbf{A}_F)$*, embedded injectively as the subgroup of the units of $F_v^\times$.*

PROOF. Since v is Archimedean, we may choose a continuous homomorphic section τ_v to the S-idelic, respectively idelic norm, and hence $\tau_v(\mathbf{R}_{>0})\cdot \mathrm{GL}_1(\mathbf{A}_{F,S})^1$ acts on $D_{S,v}$ only through $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$. Simiarly for $\tau_v(\mathbf{R}_{>0})\cdot \mathrm{GL}_1(\mathbf{A}_F)^1$ acting on D_v only through $\mathrm{GL}_1(\mathbf{A}_F)^1$. Since $D_{S,v}$ is the orbit of e_v under the continuous action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathbf{A}_{F,S}$, as is D_v of e_v under the continuous action of $\mathrm{GL}_1(\mathbf{A}_F)^1$ on $\mathbf{A}_F$, we deduce that $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, respectively $\mathrm{GL}_1(\mathbf{A}_F)^1$, acts transitively on $D_{S,v}$, respectively D_v.

Such actions factor through the quotients of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, respectively $\mathrm{GL}_1(\mathbf{A}_F)^1$, by the stabilizers of e_v, which are $\{\pm 1\}\subset F_v^\times$ if v is real, and $\mathrm{U}(1)$ if v is complex. □

PROPOSITION 8.2.0.6. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let v_0 be an Archimedean place contained in S. We let $\tau_{S,v_0}:\mathbf{R}_{>0}\to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be the choice of continuous homomorphic section to the S-idelic norm based at the v_0-indexed factor of* $\mathrm{GL}_1(\mathbf{A}_{F,S})$*, and $\tau_{v_0}:\mathbf{R}_{>0}\to \mathrm{GL}_1(\mathbf{A}_F)$ the choice of continuous homomorphic section to the idelic norm based at the v_0-indexed factor of* $\mathrm{GL}_1(\mathbf{A}_F)$*, as per Definition A.2.2.5.*

(1) *The closure within $D_{F,S}$ of $\tau_{S,v_0}(\mathbf{R}_{>0})\subset D_{S,1}$ is $\tau_{S,v_0}(\mathbf{R}_{>0})\cup\{e_{v_0}\}$.*
(2) *The closure within D_F of $\tau_{v_0}(\mathbf{R}_{>0})\subset D_1$ is $\tau_{v_0}(\mathbf{R}_{>0})\cup\{e_{v_0}\}$.*

PROOF. In either case, $\tau_{v_0}(\mathbf{R}_{>0})\cup\{e_{v_0}\}$ is homeomorphic to $[0,+\infty)$, a complete metric subspace of the metrizable spaces $D_{F,S}$, respectively D_F, and, as such, sequentially closed, hence closed, being the ambient second-countable, hence sequential, by Lemma 8.2.0.3. □

8.3. On Orbital Supports - II

Let F be a number field, S a finite set of places of F containing all the Archimedean places. Assume $\mathrm{Pic}(\mathcal{O}_{F,S})=1$.

Let $\chi\in\widehat{C_F}$ be a continuous character, and define χ_S the unique continuous character of $J_{F,S}$ obtained from χ by descending the restriction of χ to $C_{F,S}$ to $J_{F,S}$ by $\widehat{\mathcal{O}}^\times_{\Sigma_F-S}$-invariance.

The following Theorem describes the loci of $\mathbf{A}_{F,S}$, modulo the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, over which the Orbital subquotients of $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$ from (7.3.1.5) and (7.3.4.3) are supported.

THEOREM 8.3.0.1 (On Orbital Supports - II). *Let F be a number field, S a finite set of finite places of F containing all the Archimedean places. Assume* $\mathrm{Pic}(\mathcal{O}_{F,S})=1$*. Let $\chi\in\widehat{C_F}$ be a continuous character. Assume χ is unramified outside S.*

Assume S is saturated.

We denote by χ_S the unique continuous character of $J^1_{F,S}$ obtained from χ by descending the restriction of χ to $C_{F,S}$ by $\widehat{\mathcal{O}}^\times_{\Sigma_F-S}$-invariance.

Let $\mathscr{E}$ denote either of the Orbital subquotients of $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))$*:*

$$\mathscr{E}_{S,S_p}(\chi|_{J^1_{F,S}}),\quad \mathscr{E}_{S,S_\infty}(\chi|_{J^1_{F,S}})$$

from (7.3.1.5)*, respectively* (7.3.4.3)*, for p a prime number such that $S_p\subset S$.*

For every tempered distribution T on $\mathbf{A}_{F,S}$ in the linear image of the continuous adjoint of the quotient map

$$\mathcal{S}(\mathbf{A}_{F,S})\to \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}(\mathbf{A}_{F,S}))\to\mathscr{E}$$

and every $f\in\mathcal{S}(\mathbf{A}_{F,S})$, there exists $f_0\in\mathcal{S}(\mathbf{A}_{F,S})$ such that

$$\langle T,f\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta,\mathcal{S}(\mathbf{A}_{F,S})}=\langle T,f_0\rangle_{\mathcal{S}(\mathbf{A}_{F,S})'_\beta,\mathcal{S}(\mathbf{A}_{F,S})}\tag{8.3.0.1}$$

and the support of f_0 is entirely contained within:

(1) *The union of the closed subsets:*

$$H_{S,S_p} := \bigcup_{v\in S_p} \{x \in \mathbf{A}_{F,S} : x_v = 0, x_w \neq 0 \text{ for all } w \in S - \{v\}\} \subset \mathbf{A}_{F,S} \tag{8.3.0.2}$$

if $\mathscr{E} = \mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$. The topological quotient stack:

$$[H_{S,S_p}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] \tag{8.3.0.3}$$

is 2-isomorphic to the quotient stack:

$$\coprod_{v\in S_p} [\mathbf{R}_{>0}/F_v^\times] \tag{8.3.0.4}$$

where $F_v^\times$ acts on $\mathbf{R}_{>0}$ by left translation through the S-idelic norm, with $\mathcal{O}_{F_v}^\times$ inertia. Its coarse space is the topological disjoint union of compact Hausdorff loops:

$$\coprod_{v\in S_p} \mathbf{R}_{>0}/N(\mathfrak{p}_v)^{\mathbf{Z}} \tag{8.3.0.5}$$

where $\mathfrak{p}_v \subset \mathcal{O}_F$ is the unique prime ideal corresponding to the finite place $v \in S_p$.

(2) *The union of the closed subsets:*

$$H_{S,S_\infty} := \bigcup_{v\in S_\infty} \{x \in \mathbf{A}_{F,S} : x_v = 0, x_w \neq 0 \text{ for all } w \in S - \{v\}\} \subset \mathbf{A}_{F,S} \tag{8.3.0.6}$$

if $\mathscr{E} = \mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$. The topological quotient stack:

$$[H_{S,S_\infty}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] \tag{8.3.0.7}$$

is 2-isomorphic to the quotient stack:

$$\coprod_{v\in S_\infty} BU_v \tag{8.3.0.8}$$

where $U_v \subset F_v^\times$ is the compact subgroup $\ker(|\cdot|_v)$, and BU_v is its topological classifying stack. Its coarse space is the disjoint union of $r_1 + r_2$ points, one per each Archimedean place of F.

Proof. (1) : By Theorem 8.1.0.1(1), the continuous functionals in $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})'_\beta$ map under the continuous adjoint to the quotient map $\mathcal{S}(\mathbf{A}_{F,S}) \to \mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ to the image of a unique continuous functional on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_p(\mathbf{A}_{F,S}))$, where $\mathcal{S}_p(\mathbf{A}_{F,S}) \subset \mathcal{S}(\mathbf{A}_{F,S})$ is the closed subspace defined in (8.1.0.4).

This means that, for each such T and for every $f \in \mathcal{S}(\mathbf{A}_{F,S})$, there exists $f_1 \in \mathrm{PV}_{S,S_p}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ such that $f - f_1 \in \mathcal{S}_p(\mathbf{A}_{F,S})$, and $T(f - f_1) = T(f)$.

The elements of $\mathcal{S}_p(\mathbf{A}_{F,S})$ are supported on H_{S,S_p}. The remaining assertions in (1) follow from Proposition 8.2.0.4 and the observation that we have, as topological subspaces of $\mathbf{A}_{F,S}$:

$$H_{S,S_p} = \bigcup_{v\in S_p} D_{S,v}$$

where $D_{S,v}$ was defined in (8.2.0.6) from Definition 8.2.0.1(2).

(2) : The continuous dual of $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$ is comprised of continuous functionals on $\mathrm{PV}_{S,S_f}\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ that vanish on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$.

The elements of $\mathrm{PV}_{S,S_f}\mathcal{S}(\mathbf{A}_{F,S})$ are supported over H_{S,S_∞}. The remaining assertions of (2) follow from Proposition 8.2.0.5, upon observing that we have, as topological subspaces of $\mathbf{A}_{F,S}$:

$$H_{S,S_\infty} = \bigcup_{v\in S_\infty} D_{S,v}$$

where $D_{S,v}$ is defined in (8.2.0.6) from Definition 8.2.0.1(2). □

8.4. Orbital Traces

Let F be a number field, S a finite set of places of F containing all the Archimedean places, and such that $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. Assume S is saturated.

We let $\chi \in \widehat{C_F}$ be a continuous character, unramified outside S.

We let χ_S be the unique continuous character of $J_{F,S}$ obtained by descending the restriction of χ to $C_{F,S}$ by $\widehat{\mathcal{O}}^{\times}_{\Sigma_F - S}$-invariance.

We return to Theorem 7.1.2.3, which was the starting point of this Chapter.

The strong recurrence on vertical strips of bounded width, of the finite partial reciprocal Euler products of the finite Hecke L-function (see Lemma 7.1.1.1 and Proposition B.5.3.3, ultimately an application of the Kronecker-Weyl Theorem; see Theorem B.5.2.1), led to the asymptotic expansion: We reorganize the statement as follows: let $f \in \mathcal{S}(\mathbf{A}_{F,S})$ be any Schwartz-Bruhat function. Then, for all $g \in J_{F,S}$ as $|g|_{F,S} \to 0^+$, we have, for any real number $A > 0$:

(8.4.0.1)

$$\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(f)(g) = \sum_{v \in S_\infty} \sum_{m \in \mathbf{N}_{>0}, m<A} \sum_{1 \le k \le m_S(f,c_{v,m,\chi})} \frac{(-1)^{k-1}\, a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} (\log|g|_{F,S})^{k-1} \chi |g|_{F,S}^{m+it_v}$$

$$+ \sum_{p : S_p \subset S_f - S_\chi} \sum_{v \in S_p} \sum_{m \in \mathbf{Z}} \sum_{1 \le k \le m_S(f,c_{v,m,\chi})} \frac{(-1)^{k-1}\, a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} (\log|g|_{F,S})^{k-1}\, \chi| \cdot |_{F,S}^{-\frac{2\pi i m}{\log(q_v)} + it_v} \tag{8.4.0.2}$$

$$+ O_A(|g|^A_{F,S}) \tag{8.4.0.3}$$

which we have reorganized in light of Theorem 7.3.1.6, to highlight the absolute convergence of the sums indexed by each prime number p such that $S_p \subset S$, a consequence of the Theorem.

As a consequence of Theorem 8.3.0.1 and Theorem 7.3.4.1(4) and (5), is that for $v \in S$ a finite place, we can find Schwartz-Bruhat functions f supported entirely over

$$\{x \in \mathbf{A}_{F,S} : x_v = 0, x_w \neq 0 \text{ for all } w \in S - \{v\}\} \subset \mathbf{A}_{F,S}$$

and such that their own asymptotic expansion as $|g|_{F,S} \to 0^+$ takes the form:

$$\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(f)(g) = \frac{(-1)^{k-1}\, a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} (\log|g|_{F,S})^{k-1} \chi |g|_{F,S}^{m+it_v} + O_A(|g|^A_{F,S}) \tag{8.4.0.4}$$

with $v \in S_\infty$ fixed and $m < A$, and:

$$\Phi_{F,S,\chi_S|_{J^1_{F,S}}}(f)(g) = \frac{(-1)^{k-1}\, a_{k-1}(f, c_{v,m,\chi})}{(k-1)!} (\log|g|_{F,S})^{k-1}\, \chi| \cdot |_{F,S}^{-\frac{2\pi i m}{\log(q_v)} + it_v} + O_A(|g|^A_{F,S}) \tag{8.4.0.5}$$

for $v \in S_f$ at which χ is unramified, with $q_v := N(\mathfrak{p}_v)$ where $\mathfrak{p}_v \subset \mathcal{O}_F$ is the unique prime ideal corresponding to the finite place v of F.

A consequence of Theorem 8.3.0.1 and hence of Theorem 8.1.0.1, is that we may organize the nuclear Fréchet space

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$$

and its suquotients $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$ and $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ according to the following symbolism:

(8.4.0.6)

$$\begin{array}{ccccc}
\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S})) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \longrightarrow & \mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}}) \times \prod'_{p:S_p\subset S}\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}}) \\
\downarrow & & \downarrow & & \downarrow \\
[D_{S,1}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] & \xrightarrow[j]{} & [D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] & \xleftarrow[i]{} & \coprod'_{v\in S}[D_{S,v}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1] \\
\simeq\downarrow & & & & \downarrow\simeq \\
\underline{J_{F,S}/J^1_{F,S}} & & & & \left(\coprod_{v\in S_\infty} BU_v\right) \coprod \left(\coprod_{p:S_p\subset S}\coprod'_{v\in S_p}[\mathbf{R}_{>0}/F_v^\times]\right)
\end{array}$$

The top row in (8.4.0.6) is a short exact sequence (on the left and in the middle), both linearly and topologically. This is the content of Proposition 5.3.4.4 and Proposition 5.3.4.5.

The row below it, indicates which loci of the quotient stack $[D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$ each nuclear Fréchet space is supported on.

The Orbital quotients $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$, respectivey $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ (where we omit mentioning the continuous character χ_S, identifying the χ_S-isotype of each such subquotient), are thought of as being supported over the fat points BU_v for each Archimedean place $v \in S_\infty$, respectively over the periodic orbits $[\mathbf{R}_{>0}/F_v^\times]$ (whose coarse spaces are, respectively, a loop $\mathbf{R}_{>0}/N(\mathfrak{p}_v)^{\mathbf{Z}}$, for each finite place $v \in S$.

We now look into the effect of the S-idelic Mellin transform L_S (see (5.1.1.2)) on the diagram in (8.4.0.6).

It turns it into the following divisor exact sequence:

(8.4.0.7)

$$\begin{array}{ccccc}
L_S\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S})) & \longrightarrow & L_S\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S})) & \longrightarrow & L_S\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}}) \times \prod'_{p:S_p\subset S}L_S\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}}) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}}) & \xrightarrow[j]{} & X_S(\chi_S|_{J^1_{F,S}}) & \xleftarrow[i]{} & \mathcal{U}_{S_\infty} \coprod \left(\coprod'_{p:S_p\subset S}\mathcal{U}_{S_p}\right)
\end{array}$$

where $\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})$ is the connected component parametrized by $\chi_S|_{J^1_{F,S}}$ of the Riemann surface $\Omega(J_{F,S})$ of S-idelic quasi-characters, $X_S(\chi_S|_{J^1_{F,S}})$ is a topological completion whose boundary is the divisor $\mathcal{U}_{S_\infty} \coprod \left(\coprod_{p:S_p\subset S}\mathcal{U}_{S_p}\right)$, where each divisor $\mathcal{U}_{S_\infty}$, respectively $\mathcal{U}_{S_p}$, is defined in (5.3.4.13) from Proposition 5.3.4.5, respectively (7.2.2.1).

The Second Structure Theorem (see Theorem 5.4.1.1), provides an explicit description of the nuclear Fréchet spaces in the top row of (8.4.0.7) in terms of spaces of meromorphic functions on $\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})$, Schwartz uniformly on the vertical strips of bounded width of the complex s-plane not containint $i\mathbf{R}$ in their interior, and satisfying the S-local functional equation from (3.3.1.2) of Theorem 3.3.1.1.

The product of the principal part continuous functionals factors through the top horizontal arrow in (8.4.0.7).

The S-local function spaces and their supporting geometric objects, whether it be the quotient stacks $[D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$ on the backward side of the Mellin transform, or the extended topological space $X_S(\chi_S|_{J^1_{F,S}})$, its open sub-domain $\Omega(J_{F,S})(\chi_S|_{J^1_{F,S}})$ and complementary boundary divisor, on the forward side of the Mellin transform, coalesce together into the global ones, to give the following global analogs of the diagrams in (8.4.0.6) and (8.4.0.7):

$$\begin{array}{ccccc}
\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F)) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F)) & \longrightarrow & \mathscr{E}_{S_\infty}(\chi|_{C_F^1}) \times \prod'_p \mathscr{E}_{S_p}(\chi|_{C_F^1}) \\
\downarrow & & \downarrow & & \downarrow \\
[D_{F,1}/\mathrm{GL}_1(\mathbf{A}_F)^1] & \xrightarrow[j]{} & [D_F/\mathrm{GL}_1(\mathbf{A}_F)^1] & \xleftarrow[i]{} & \coprod'_{v\in\Sigma_F} [D_{F,v}/\mathrm{GL}_1(\mathbf{A}_F)^1] \\
\simeq \downarrow & & & & \downarrow \simeq \\
\underline{C_F/C_F^1} & & & & \left(\coprod_{v\in S_\infty} BU_v\right) \coprod \left(\coprod_p \coprod'_{v\in S_p} [\mathbf{R}_{>0}/F_v^\times]\right)
\end{array} \tag{8.4.0.8}$$

where now the coproduct $\coprod_p \coprod_{v\in S_p} [\mathbf{R}_{>0}/F_v^\times]$ is indexed by all prime numbers, and its coarse space is a countable disjoint union of loops $\coprod_p \coprod_{v\in S_p} \mathbf{R}_{>0}/N(\mathfrak{p}_v)^{\mathbf{Z}}$ organized according to splitting patterns.

The global idelic Mellin transform turns the diagram in (8.4.0.8) into the following:

$$\begin{array}{ccccc}
L\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_{00}(\mathbf{A}_F)) & \longrightarrow & L\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F)) & \longrightarrow & L\mathscr{E}_{S_\infty}(\chi|_{C_F^1}) \times \prod'_p L\mathscr{E}_{S_p}(\chi|_{C_F^1}) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega(C_F)(\chi|_{C_F^1}) & \xrightarrow[j]{} & X(\chi|_{C_F^1}) & \xleftarrow[i]{} & \mathcal{U}_{S_\infty} \coprod \left(\coprod'_p \mathcal{U}_{S_p}\right)
\end{array} \tag{8.4.0.9}$$

where $X(\chi|_{C_F^1})$ is a completion of the connected component $\Omega(C_F)(\chi|_{C_F^1})$ with boundary divisor $\mathcal{U}_{S_\infty} \coprod \left(\coprod_p \mathcal{U}_{S_p}\right)$.

In all four diagrams (8.4.0.6), (8.4.0.7), (8.4.0.8) and (8.4.0.9), the notation $\coprod'$ denotes the coproduct over only those places at which χ_S, respectively χ, is unramified.

Remark 8.4.0.1. The topological spaces $X_S(\chi_S|_{J^1_{F,S}})$ and $X(\chi|_{C_F^1})$ can be assembled together into families parametrized by the discrete character groups $\widehat{J^1_{F,S}}$, respectively $\widehat{C_F^1}$:

$$p_S : \mathcal{X}_S \to \widehat{J^1_{F,S}}, \quad p : \mathcal{X} \to \widehat{C_F^1}.$$

The boundaries of $X_S(\chi_S|_{J^1_{F,S}})$ and $X(\chi|_{C_F^1})$ also relativize over $\widehat{J^1_{F,S}}$, respectively $\widehat{C_F^1}$.

The vertical arrows in (8.4.0.6), (8.4.0.7), (8.4.0.8), respectively (8.4.0.9), are not morphisms, but rather merely refer to the intuition, guided by Theorem 8.1.0.1 and Theorem 7.3.4.1(4), that each nuclear Fréchet space in the top row of each diagram respectively, should be thought of as a Fréchet bundle over the corresponding quotient stack in the bottom row, respectively topological space/divisor.

The idelic Mellin transform, both S-local and global, sets up a correspondence between closed substacks of the quotient stacks $[D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$, respectively boundary divisors of $X_S(\chi_S|_{J^1_{F,S}})$, and $[D_F/\mathrm{GL}_1(\mathbf{A}_F)^1]$, respectively boundary divisors of $X(\chi|_{C_F^1})$.

We now refocus on the S-local Fréchet spaces in the diagram from (8.4.0.6).

They all carry compatible continuous left-translation actions by $J_{F,S}$, and hence by convolution under the convolution algebra $\mathcal{D}(J_{F,S})$.

We denote by E any of the three Fréchet spaces in the top row of (8.4.0.6).

For each $\varphi \in \mathcal{D}(J_{F,S})$:

$$A(\varphi) : E \to E$$

defined as $u \mapsto A(\varphi)(u) := \int_{J_{F,S}} \varphi(g) u(g^{-1}x)\, \mathrm{d}\overline{\mu}_S(g)$ is a continuous linear endomorphism, which, on $E = \mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, admits an integral kernel, namely

$$\varphi(xy^{-1}) : J_{F,S} \times J_{F,S} \to \mathbf{C}$$

by identifying $\Phi_{F,S}(\mathcal{S}_{00}(\mathbf{A}_{F,S}))$, equipped with the quotient topology inherited from $\mathcal{S}_{00}(\mathbf{A}_{F,S})$, with a copy of the regular representation of $J_{F,S}$ on $\mathcal{S}_{\mathrm{E}}(J_{F,S})$.

We now assume φ is supported over the closed semigroup $J_{F,S}^{\geq 1}$ of S-idele classes of S-idelic norm at least 1.

Then $\varphi(1) = 0$ by smoothness of φ, and hence the kernel of $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$ vanishes along the diagonal.

We will now proceed with a discussion aimed at fostering intuition for certain constructions and arguments to be carried out in Chapter 6 and Chapter 7.

While $A(\varphi)$ is not a nuclear operator on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, nor on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, it is so on most of the Orbital subquotients, namely, it is so on $\mathscr{E}_{S,S_p}(\chi_S|_{J^1_{F,S}})$ for every prime p such that $S_p \subset S$, by Theorem 7.4.2.1.

It also has a convergent matrix trace on $\mathscr{E}_{S,S_\infty}(\chi_S|_{J^1_{F,S}})$ (this is where we use the assumption on the support on φ, through Theorem 7.4.3.2).

While not nuclear, it has a trace even on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, which is zero by the above observation on the integral kernel of $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$.

The effect the continuous left-translation action of $J_{F,S}^{\geq 1}$ is, in a way, "small" and controlled along the diagonal over $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, and mostly concentrated over the orbital subquotients, on each of which its trace is computable explicitly, for each test function $\varphi \in \mathcal{D}(J_{F,S})$ supported over $J_{F,S}^{\geq 1}$.

We note that the quotient stacks $BU_v \simeq [D_{S,v}/F_v^\times]$, for each Archimedean place $v \in S$, are now promoted to "basins of attraction" by the condition on the support that we have imposed on test functions. Namely, on the level of coarse spaces, the continuous left-translation action by $J_{F,S}$ on the (non-Hausdorff) coarse space of the quotient stack $[D_{F,S}/\mathrm{GL}_1(\mathbf{A}_{F,S})^1]$, promotes the finitely many points underlying the stacks BU_v, to attractive limit points under the action of $\mathbf{R}_{>0}^{\geq 1} \simeq J_{F,S}^{\geq 1}/J_{F,S}^1$.

The quotient stacks $[D_{S,v}/F_v^\times] \simeq [\mathbf{R}_{>0}/F_v^\times]$ for each non-Archimedean place $v \in S$, are instead promoted to periodic orbits: namely, on the level of coarse spaces, the continuous left-translation action by $J_{F,S}^{\geq 1}$ preserves them.

The trace of $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ will be defined and computed rigorously only in Chapter 7.

We can proceed heuristically and combine the calculations performed in Theorem 7.4.2.1 and Theorem 7.4.3.2, to infer that its value will be:

$$\mathrm{Trace}(A(\varphi)) = \sum_{p:S_p\subset S}\sum_{\varepsilon_p} N(\chi_S|\cdot|_{F,S}^{\varepsilon_p}) \int_{J_{F,S}} \varphi(g)\chi_S(g)|g|_F^{\varepsilon_p}\,\mathrm{d}\overline{\mu}_S(g) \tag{8.4.0.10}$$

$$+ \sum_{z} N(\chi_S|\cdot|_{F,S}^{z}) \int_{J_{F,S}} \varphi(g)\chi_S(g)|g|_F^{z}\,\mathrm{d}\overline{\mu}_S(g) \tag{8.4.0.11}$$

where ε_p varies over the poles of $\prod_{v\in S_p} L_v(\chi_v|\cdot|_v^s)$, for each prime p such that $S_p \subset S$, and z varies over the poles of the Γ-factor of the Hecke L-function $L(s,\chi)$, $N(\chi_S|\cdot|_{F,S}^{\varepsilon_p})$ and $N(\chi_S|\cdot|_{F,S}^{z})$ being the respective multiplicity functions.

The reader will now recognize, through the lens of the Poisson Summation Formula (in its form from Proposition B.4.1.2) and the Asymptotic Residue Theorem (see Theorem B.3.2.1 and Theorem B.4.2.2), the geometric side of Weil's Explicit Formulas, namely:

$$\sum_{v\in S_f - S_\chi}\sum_{m>0} \frac{\log N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)^{m/2}} \left\{\chi(\mathfrak{p}_v^m)\,\varphi(N(\mathfrak{p}_v)^m) + \overline{\chi}(\mathfrak{p}_v^m)\,\varphi(N(\mathfrak{p}_v)^{-m})\right\} \tag{8.4.0.12}$$

$$- \sum_{v\in S_\infty} \int_0^{+\infty} \left\{ \frac{N_v\phi(1)}{x} - (\phi(e^x)e^{s_v x} + \phi(e^{-x})e^{-s_v x}) \frac{e^{\left(\frac{2}{N_v}-\frac{1}{2}\right)x}}{1-e^{-\frac{2}{N_v}x}} \right\} \mathrm{d}x \tag{8.4.0.13}$$

which agrees with the right side of (8.4.0.10) applied to $\varphi|\cdot|_{F,S}^{-1/2}$.

We further refer the reader to [Den, Thm.1.7(4)], where Deninger formulated expectations that a trace-theoretic interpretation to Weil's Explicit Formulas for Hecke L-functions could be taking the form presented, so far only heuristically, in (8.4.0.10).

We have so far learned that the continuous left-translation action of $J_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ is, when restricted to an action of the closed semigroup $J^{\geq 1}_{F,S}$, traceable, in a sense to be made rigorous within the next two Chapters, and the Orbital Schauder bases identified throughout this Chapter enable us to compute the trace explicitly.

We now explain how the Paley-Wiener levels fit into this scenario.

It is a consequence of the classical Paley-Wiener Theorem, that the intersection of

$$\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

with $S' = S \cup S_0$ for some saturated finite set of finite places S_0 of F, with $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}_{00}(\mathbf{A}_{F,S}))$, is trivial for all large enough $X > 0$. This is so, because the only Schwartz-Bruhat function $f \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ whose S_∞-adelic Fourier transform is compactly supported, and that vanishes at any of the hyperplanes $\{x_v = 0\}$, $v \in S_\infty$, with infinite order, is constant with value zero.

We may therefore complete the diagram in (8.4.0.6) into the following:

(8.4.0.14)

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \mathscr{E}_{S',S_\infty}(\chi_{S'}|_{J^1_{F,S'}}) \times \prod'_{p:S_p\subset S'} \mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi_{S'}|_{J^1_{F,S'}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_{00}(\mathbf{A}_{F,S'})) & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'})) & \longrightarrow & \mathscr{E}_{S',S_\infty}(\chi_{S'}|_{J^1_{F,S'}}) \times \prod'_{p:S_p\subset S'} \mathscr{E}_{S',S_p}(\chi_{S'}|_{J^1_{F,S'}}) \\
\downarrow & & \downarrow & & \downarrow \\
[D_{S',1}/\mathrm{GL}_1(\mathbf{A}_{F,S'})^1] & \xrightarrow[j]{} & [D_{F,S'}/\mathrm{GL}_1(\mathbf{A}_{F,S'})^1] & \xleftarrow[i]{} & \coprod'_{v\in S'} [D_{S',v}/\mathrm{GL}_1(\mathbf{A}_{F,S'})^1] \\
\downarrow \simeq & & & & \downarrow \simeq \\
\underline{J_{F,S'}/J^1_{F,S'}} & & & & \left(\coprod_{v\in S_\infty} BU_v\right) \amalg \left(\coprod_{p:S_p\subset S'} \coprod'_{v\in S_p} [\mathbf{R}_{>0}/F_v^\times]\right)
\end{array}$$

for all large enough $X > 0$.

The content of Theorem 7.3.3.4 is that the Orbital subquotients

$$\mathscr{E}^{\mathrm{ABS}}_{S',S_p,X}(\chi_{S'}|_{J^1_{F,S'}})$$

inherit the Orbital Schauder bases from $\mathscr{E}_{S',S_p}(\chi_{S'}|_{J^1_{F,S'}})$ for all $X > 0$ large enough, and the directed union of the $\mathrm{PW}_{S,X}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$ is sequentially dense within $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

This means that the Paley-Wiener levels retain knowledge of the trace of $A(\varphi)$ asymptotically as $X \to +\infty$, missing exactly the closed subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_{00}(\mathbf{A}_{F,S'}))$ which is the main cause of the non-nuclearity of $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$.

We are going to use the Paley-Wiener levels, in their invariant form (see Definition 7.3.2.3), in the next two Chapters, to define and compute the trace of $A(\varphi)$ rigorously, doing so with respect to coarser normed topologies on the ambient $\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}_{00}(\mathbf{A}_{F,S'}))$, than its intrinsic nuclear Fréchet space topology.

A choice of test function $\varphi \in \mathcal{D}(C_F)$, which we again assume supported over the closed semigroup $C^{\geq 1}_F$, induces a test function $\varphi_S \in \mathcal{D}(J_{F,S})$ for all large enough S. Indeed, for S any finite set of places of F containing all the Archimedean places, such that χ is unramified outside S, and with $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, we have that φ is k-invariant for $k \subset C_F$ a compact subgroup with open compactly generated orthogonal $k^\perp \subset \widehat{C_F}$.

In particular, φ is invariant under the compact subgroup given by the image in C_F of $\widehat{\mathcal{O}}^{\times}_{\Sigma_F - S'} \subset \mathrm{GL}_1(\mathbf{A}_F)$ for all large enough S'. Upon replacing S' with a larger set S' if necessary, so as to have that χ is unramified outside S' and $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$, we may descend φ to a unique $\varphi_{S'} \in \mathcal{D}(J_{F,S'})$ and $A(\varphi_{S'})$ acts on

$$\mathrm{H}_{0,\mathrm{cont}}(\chi_{S'}|_{J^1_{F,S'}}, \mathcal{S}(\mathbf{A}_{F,S'}))$$

compatibly with the action of $A(\varphi)$ on the global $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$ under the transition operator $\overline{\pi}_{S'}$, which, under the above conditions on S', is a linear topological isomorphism.

We conclude this Chapter by observing that as S grows to encompass all places of F, the trace of $A(\varphi_S)$ stabilizes in finite time, i.e. for all large enough finite sets of finite places S' of F containing all the Archimedean places, the trace of $A(\varphi_{S'})$ remains constant with respect to S'. The largeness of S' depends on the character χ and on the support of the test function $\varphi \in \mathcal{D}(C_F)$.

We are led to eyeball the following situation: the continuous operator $A(\varphi)$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}(\mathbf{A}_F))$ is traceable. Its trace will be defined rigorously in the next two Chapters, for then to be computed in Chapter 7.

Its trace is concentrated at the S-local level, on the orbital subquotients of $\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$, for S large enough.

The Paley-Wiener levels approximate the orbital subquotients from within, filling them in asymptotically, and they inhert the full Orbital Schauder bases on the primary Orbital quotients.

At the global level, the topologically short exact sequence:

$$0 \to \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F}) \to \mathscr{H}_F(\chi|_{C^1_F}) \to 0$$

leads to computing the trace of $A(\varphi)$ by computing the trace of operators induced by the continuous left-translation action of C_F on the spectral subquotient $\mathscr{H}_F(\chi|_{C^1_F})$ (see Theorem 7.4.1.1).

This is a first look into the elements that enter the trace calculation in Chapter 7. The idea is to perform it not within the nuclear Fréchet spaces from above, but rather within normed completions, whose topology is much coarser.

The mere fact that the trace calculation can be performed with respect to such coarser topologies, is what puts constraints on the zeros that are allowed to contribute to the spectral side of the trace formula: only some of the point-evaluation functionals from (7.1.4.6) of Theorem 7.1.4.2, will remain continuous with respect to coarser normed topologies, and only those, will be contributing to the spectral side.

It turns out that the coarsest possible normed completion of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$ which enables the trace calculation, is a Sobolev completion of degree $2\,[F : \mathbf{Q}]$. We will be studying these Sobolev completions in the next Chapter.

The Orbital Schauder bases on the Paley-Wiener levels will remain in place after such normed completions, turning into Markushevich bases. They will enable us to reduce the trace calculation into a calculation to be done S-locally only, as opposed to globally, and this is where the choice of Sobolev degree $2\,[F : \mathbf{Q}]$ is forced on us: it ensures the orbital Schauder bases on the absolute and invariant Paley-Wiener levels, are inherited by the normed completions, at least in the weaker sense of Markushevich bases, insofar as all the continuous dual basis elements remain continuous on the Sobolev completions.

8.5. The big picture

CHAPTER 9

Hardy-Sobolev topologies

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Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let χ be a continuous character of C_F, respectively of $J_{F,S}$.

In this Chapter, we coarsen the nuclear Fréchet topologies on $\mathcal{S}_{\mathrm{E}}(C_F)$, $\mathcal{S}_I(J_{F,S})$ and their C_F^1-, respectively $J_{F,S}^1$-isotypes, for I an appropriate choice of non-empty connected interval of the real line, as well as the nuclear Fréchet topologies on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J_{F,S}^1}, \mathcal{S}(\mathbf{A}_{F,S}))$, to normed topologies arising by integration along a chosen vertical line within the Riemann surfaces $\Omega(C_F)$, respectively $\Omega(J_{F,S})$, with continuity requirements on finitely many degrees of holomorphic differentiation.

In other words, we form completions of the above Hausdorff locally convex topological $\mathbf{C}$-vector spaces, to Sobolev spaces.

These will serve as the ambients for the trace calculation which is the main subject of study of this work.

9.1. Sobolev spaces $W^{s,2}(G)$

We discuss here general constructions of normed topologies on the Hausdorff locally convex topological complex vector spaces $\mathcal{S}_{\mathrm{E}}(C_F)$ and $\mathcal{S}_I(J_{F,S})$ of Schwartz-Bruhat functions on the idele class group C_F, respectively on the S-idele class group $J_{F,S}$, with exponential, respectively I-exponential decay, for some appropriate chocie of non-empty connected interval $I \subset \mathbf{R}$.

Equivalently, this amounts to constructing continuous norms on $\mathcal{S}_{\mathrm{E}}(C_F)$, respectively $\mathcal{S}_I(J_{F,S})$, and study the Banach spaces obtained by completion.

We will further study the closed subspaces defined as the closures of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J_{F,S}^1}, \mathcal{S}(\mathbf{A}_{F,S}))$, for χ a continuous character of C_F, respectively of $J_{F,S}$.

We will be discussing the resulting Banach spaces in the generality of $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian groups G.

9.1.1. Sobolev spaces of Hausdorff locally compact abelian groups. We begin with a definition, following [**GR**, §1,2], with an eye on [**Co**, Appendix I,p.59].

Definition 9.1.1.1. Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G, $\mu_{\widehat{G}}$ the corresponding Plancherel measure. Let $s \in \mathbf{R}$ a real number. We define a *Sobolev weight* on G to be a continuous function:

$$w : \widehat{G} \to [0, +\infty) \tag{9.1.1.1}$$

such that there exists a real number $C_w > 0$ such that:

$$w(\chi\chi') \leq C_w(w(\chi) + w(\chi')) \tag{9.1.1.2}$$

for all $\chi, \chi' \in \widehat{G}$. We define $H_w^s(G, \mu_G)$ to be the completion of $C_c(\widehat{G})$ with respect to the linear topology induced by the norm:

$$\|f\|_{H_w^s(G,\mu_G)} := \left(\int_{\widehat{G}} |\widehat{f}(\chi)|^2 \, (1 + w(\chi)^2)^{s/2} \, \mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2}. \tag{9.1.1.3}$$

Lemma 9.1.1.2. *Let G be a Hausdorff locally compact abelian group, $w : \widehat{G} \to [0, +\infty)$ a Sobolev weight, $s \in \mathbf{R}$ a real number. Then $H_w^s(G, \mu_G)$ is a Hilbert space. If G is second-countable, then $H_w^s(G, \mu_G)$ is a separable Hilbert space.*

Proof. Since w is continuous, so is $(1 + w^2)^{s/2}$: this is so if $s \geq 0$, and since $1 + w^2$ is real valued, nonnegative and nowhere vanishing over $\widehat{G}$, $(1 + w^2)^{s/2}$ is continuous for negative s too. As such, $\mu := (1 + w^2)^{s/2}\mu_{\widehat{G}}$ is a Radon measure on $\widehat{G}$.

By [**Fol**, Prop.7.9], we have $H_w^s(G, \mu_G) = L^2(\widehat{G}, (1 + w^2)^{s/2}\mu_{\widehat{G}})$. We deduce that $H_w^s(G, \mu_G)$ is a Hilbert space for any $s \in \mathbf{R}$. Being $\widehat{G}$ Hausdorff locally compact, if G is second-countable then so is $\widehat{G}$. We deduce, by [**Fol**, Thm.7.8], that μ is a regular σ-finite Radon measure on $\widehat{G}$, and by [**Coh**, Prop.3.4.5] we deduce that $H_w^s(G, \mu_G)$ is a separable metric space, hence separable as a Hilbert space. □

We refer the reader to [**GR**, §1,2] for more on the Sobolev spaces $H_w^s(G, \mu_G)$ attached to a Hausdorff locally compact abelian group, and to [**FG**, §2]. We observe that the Sobolev weight w from Definition 9.1.1.1 is denoted γ in [**GR**, §1(3),Def.1], and that the real number s in [**GR**, §1], assumed to be only nonnegative in loc.cit., and the results contained therein, is related to our s by doubling, i.e. the space $H_\gamma^s(G)$ from [**GR**, §1,Def.1] is our $H_w^{s/2}(G, \mu_G)$ for $w = \gamma$.

We have the following basic properties of $H_w^s(G, \mu_G)$:

Proposition 9.1.1.3. *Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G, $\mu_{\widehat{G}}$ its Plancherel measure on $\widehat{G}$. Let $s \in \mathbf{R}$ be a real number, and $w : \widehat{G} \to [0, +\infty)$ a Sobolev weight.*

Assume s is nonnegative.

(1) *The inclusion:*

$$H_w^s(G, \mu_G) \to L^2(G, \mu_G) \tag{9.1.1.4}$$

is a continuous linear operator, satisfying, for every $f \in H_w^s(G, \mu_G)$:

$$\|f\|_{L^2(G,\mu_G)} \leq \|f\|_{H_w^s(G,\mu_G)}. \tag{9.1.1.5}$$

(2) *If $s' > s$, then for any $f \in H^s_w(G,\mu_G)$ we have:*

$$\|f\|_{H^s_w(G,\mu_G)} \leq \|f\|_{H^{s'}_w(G,\mu_G)} \tag{9.1.1.6}$$

and the inclusion $H^{s'}_w(G,\mu_G) \subset H^s_w(G,\mu_G)$ is continuous.

(3) *(Sobolev Embedding) If $(1+w^2)^{-s/2} \in L^1(\widehat{G},\mu_{\widehat{G}})$, then every element of $H^s_w(G,\mu_G)$ is represented by a unique continuous function on G vanishing at infinity, resulting in a canonical linear inclusion:*

$$H^s_w(G,\mu_G) \to C_0(G). \tag{9.1.1.7}$$

There exists a real number $C(w,s) > 0$ with the property that, for every $f \in H^s_w(G,\mu_G)$, we have:

$$\|f\|_u \leq C(w,s)\,\|f\|_{H^s_w(G,\mu_G)}. \tag{9.1.1.8}$$

PROOF. (1) : We have, by [**DE**, Thm.3.4.8], for every $f \in H^s_w(G,\mu_G)$:

$$\begin{aligned}
\|f\|_{L^2(G,\mu_G)} &= \|\widehat{f}\|_{L^2(\widehat{G},\mu_{\widehat{G}})} \\
&= \left(\int_{\widehat{G}} |\widehat{f}(\chi)|^2 \,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \\
&\leq \left(\int_{\widehat{G}} |\widehat{f}(\chi)|^2 (1+w(\chi)^2)^{s/2} \,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \\
&= \|f\|_{H^s_w(G,\mu_G)}
\end{aligned}$$

where we have used the assumption $s \geq 0$ to deduce the inequality $1 \leq (1+w(\chi)^2)^{s/2}$ as functions on $\widehat{G}$.

(2) : Since $s' > s$, we have $(1+w(\chi)^2)^{s/2} \leq (1+w(\chi)^2)^{s'/2}$ since $s \geq 0$. The inequality in (9.1.1.6) therefore follows from the definition of $\|\cdot\|_{H^s_w(G,\mu_G)}$, which proves that $H^{s'}_w(G,\mu_G)$ is contained in $H^s_w(G,\mu_G)$ as a linear subspace of $L^2(G,\mu_G)$, and the inclusion $H^{s'}_w(G,\mu_G) \subset H^s_w(G,\mu_G)$ is continuous.

(3) : Let $f \in H^s_w(G,\mu_G)$ be an arbitrary element, and call $u := \widehat{f}$. By [**Ru**, Thm.1.2.4(d)] (see also the proof of [**DE**, Thm.3.2.1] on [**DE**, p.65]), we have, for μ_G-almost all $g \in G$:

$$\begin{aligned}
|u(g)| &\leq \|\widehat{u}\|_{L^1(\widehat{G},\mu_{\widehat{G}})} \\
&= \int_{\widehat{G}} |\widehat{u}(\chi)| \frac{(1+w(\chi)^2)^{s/4}}{(1+w(\chi)^2)^{s/4}} \,\mathrm{d}\mu_{\widehat{G}}(\chi) \\
&\leq \left(\int_{\widehat{G}} |\widehat{u}(\chi)|^2 (1+w(\chi)^2)^{s/2} \,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \left(\int_{\widehat{G}} \frac{1}{(1+w(\chi)^2)^{s/2}} \,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \\
&= \|f\|_{H^s_w(G,\mu_G)}\, \|(1+w^2)^{-s/2}\|^{1/2}_{L^1(\widehat{G},\mu_{\widehat{G}})}.
\end{aligned}$$

and we deduce that u is uniformly bounded over G. By (1) and [**DE**, Thm.3.4.8], we have $u \in L^2(G,\mu_G)$. From the above inequalities, we deduce that $\widehat{u}$ is absolutely $\mu_{\widehat{G}}$-integrable over $\widehat{G}$, and so u is an element of $C_0(G)$ by [**DE**, Thm.3.2.1]. Upon calling $C(w,s) := \|(1+w^2)^{-s/2}\|^{1/2}_{L^1(\widehat{G},\mu_{\widehat{G}})}$, the above inequalities imply that (9.1.1.8) holds. Formation of u is linear with respect to $f \in H^s_w(G,\mu_G)$, and (9.1.1.8) implies that the resulting linear operator $H^s_w(G,\mu_G) \to C_0(G)$ is continuous. It is injective by [**DE**, Thm.3.2.1] (see the proof of injectivity on [**DE**, p.65]). □

REMARK 9.1.1.4. We further refer the reader to [**GR**, Prop.1,2] for (1) and (2) in Proposition 9.1.1.3.

LEMMA 9.1.1.5. *Let G be a Hausdorff locally compact abelian group. Let $s \geq 0$ be a real number. The pairing:*

$$C_c(\widehat{G}) \times C_c(\widehat{G}) \to \mathbf{C} \tag{9.1.1.9}$$

defined by the assignment:

$$(\phi,\psi)\mapsto\int_{\widehat{G}}\phi(\chi)\psi(\chi)\,\mathrm{d}\mu_{\widehat{G}}(\chi) \tag{9.1.1.10}$$

extends uniquely to a continuous non-degenerate bi-linear form:

$$\langle\cdot,\cdot\rangle_{s,w}:H^s_w(G,\mu_G)\times H^{-s}_w(G,\mu_G)\to\mathbf{C} \tag{9.1.1.11}$$

inducing a canonical linear isometry

$$H^s_w(G,\mu_G)'_\beta\to H^{-s}_w(G,\mu_G) \tag{9.1.1.12}$$

from the strong dual of $H^s_w(G,\mu_G)$, whose topology is normed by the dual norm, to the Sobolev space $H^{-s}_w(G,\mu_G)$, whose topology is normed by the norm $\|\cdot\|_{H^{-s}_w(G,\mu_G)}$ from (9.1.1.3).

Proof. By [**Fol**, Thm.6.15], the pairing:

$$\langle\cdot,\cdot\rangle:L^2(\widehat{G},\mu)\times L^2(\widehat{G},\mu)\to\mathbf{C}$$

defined by:

$$\langle\phi,\psi\rangle:=\int_{\widehat{G}}\phi(\chi)\psi(\chi)\mu(\chi) \tag{9.1.1.13}$$

upon choosing the measure space $(X,\mathcal{M},\mu)$ to be $X:=\widehat{G}$, $\mathcal{M}$ the Borel σ-algebra, and $\mu_s:=(1+w^2)^{s/2}\mu_{\widehat{G}}$, with $p=q=2$, is separately continuous and induces an isometry between $L^2(\widehat{G},\mu_s)'_\beta=H^s_w(G,\mu_G)'_\beta$ equipped with the dual norm (defined in [**Fol**, Prop.6.13]) and $L^2(\widehat{G},(1+w^2)^{s/2}\mu_{\widehat{G}})$ itself, which is $H^s_w(G,\mu_G)$.

For every $\phi\in L^2(\widehat{G},\mu_s)$ and $\psi\in L^2(\widehat{G},\mu_{-s})$, the integral:

$$\int_{\widehat{G}}|\phi(\chi)\psi(\chi)|\,\mathrm{d}\mu_{\widehat{G}}(\chi) \tag{9.1.1.14}$$

converges, i.e. $\phi\cdot\psi$, as a complex-valued measurable function on $\widehat{G}$, is absolutely integrable.

Indeed, we have:

$$\int_{\widehat{G}}|\phi(\chi)\psi(\chi)|\,\mathrm{d}\mu_{\widehat{G}}(\chi)=\int_{\widehat{G}}|\phi(\chi)\psi(\chi)|\frac{(1+w(\chi)^2)^{s/4}}{(1+w(\chi)^2)^{s/4}}\,\mathrm{d}\mu_{\widehat{G}}(\chi)\le\|\phi\|^{1/2}_{L^2(\widehat{G},\mu_s)}\,\|\psi\|^{1/2}_{L^2(\widehat{G},\mu_{-s})}. \tag{9.1.1.15}$$

We define $\langle\phi,\psi\rangle_{s,w}$ to be the value of the integral $\int_{\widehat{G}}\phi(\chi)\psi(\chi)\,\mathrm{d}\mu_{\widehat{G}}(\chi)$. The notation is justified since the restriction of $\langle\cdot,\cdot\rangle_{s,w}$ to $C_c(\widehat{G})\times C_c(\widehat{G})$ is the pairing defined in (9.1.1.10).

We have, for all such $\phi,\psi\in L^2(\widehat{G},\mu_s)$:

$$\langle\phi(1+w^2)^{-s/2},\psi\rangle=\langle\phi,\psi(1+w^2)^{-s/2}\rangle=\langle\phi,\psi\rangle_{s,w} \tag{9.1.1.16}$$

where the products $\phi(1+w^2)^{-s/2}$, respectively $\psi(1+w^2)^{-s/2}$, are as complex-valued measurable functions on $\widehat{G}$, noting that the pairing $\langle\cdot,\cdot\rangle$ in (9.1.1.16) is defined on $(\phi,\psi(1+w^2)^{-s/2})$, respectively $(\phi(1+w^2)^{-s/2},\psi)$.

By Proposition 9.1.1.3(1), the pairing $\langle\cdot,\cdot\rangle_{s,w}$ is sequentially continuous with respect to each argument, hence indeed separately sequentially continuous with respect to the norm topology defined by $\|\cdot\|_{H^s_w(G,\mu_G)}$ in each argument, since $\langle\cdot,\cdot\rangle$ is.

Sequential continuity of linear functionals implies continuity, the topology on $H^s_w(G,\mu_G)$ being metric.

Since $H^s_w(G,\mu_G)$ is a Banach space, being a Hilbert space by Lemma 9.1.1.2, hence so is its strong dual, normed with the dual norm, and by the Uniform Boundedness Theorem the pairing $\langle\cdot,\cdot\rangle_{s,w}$ is continuous, being separately continuous (see [**Bou-TVS**, Ch.IV,§4.6,IV.26,Thm.2]). The same conclusion also follows from Proposition C.2.3.11.

The linear endomorphism $C_c(\widehat{G})\to C_c(\widehat{G})$ sending ψ to $(1+w^2)^{-s/2}\psi$ therefore extends to a continuous linear operator $H^s_w(G,\mu_G)'_\beta\to H^{-s}_w(G,\mu_G)$.

Indeed, for continuous functional $T \in H^s_w(G,\mu_G)'_\beta$, by [Fol, Prop.6.13] there exists a unique $\psi \in H^s_w(G,\mu_G)$ such that $T = T_\psi$, with T_ψ the functional defined as:

$$T_\psi : \phi \mapsto T_\psi(\phi) := \langle \phi, \psi \rangle.$$

The functional on $H^s_w(G,\mu_G)$:

$$(1+w^2)^{-s/2}T$$

assigned by:

$$(1+w^2)^{-s/2}T(\phi) := \langle (1+w^2)^{-s/2}\phi, \psi\rangle$$

is continuous on $H^{-s}(G,\mu_G)$, and hence there exists a unique $\psi_0 \in H^{-s}(G,\mu_G)$ representing it. The continuous linear operator $H^s_w(G,\mu_G)'_\beta \to H^{-s}_w(G,\mu_G)$ induced by $\langle\cdot,\cdot\rangle_{s,w}$ sends $T = T_\psi$ to ψ_0. We have $\psi_0 = (1+w^2)^{s/2}\psi$, and we compute:

$$\|\psi_0\|_{H^{-s}_{w^-}(G,\mu_G)} = \|\psi\|_{H^s_w(G,\mu_G)} = \|T\|_{H^s_w(G,\mu_G)'_\beta} \tag{9.1.1.17}$$

the last equality being provided by [Fol, Prop.6.13].

The identities in (9.1.1.17) imply that the continuous linear operator from (9.1.1.12) is an isometry, hence injective.

It is bijective on $C_c(\widehat{G})$ being $(1+w^2)^{-s/2}$ continuous and nowhere vanishing, with continuous and nowhere vanishing reciprocal. It follows that it has dense range, and it is therefore surjective, being its range closed (as is the range of any linear topological embedding of a complete Hausdorff locally convex topological vector space into another). It is therefore an isometric isomorphism. □

9.1.2. Sobolev spaces $W^{s,2}$ of $\mathbf{R}_{>0}$-modulated groups. We will be specializing the Sobolev spaces defined in Definition 9.1.1.1 and discussed in Proposition 9.1.1.3, to the case $G = \widehat{C_F}$, for F a number field, for $\mu_G = \widehat{\overline{\mu}}$ the Plancherel measure associated to the Haar measure $\overline{\mu}$ on C_F, whose Plancherel measure is $\overline{\mu}$ by Fourier inversion, and for $w : \widehat{G} = C_F \to [0,+\infty)$ of the following form:

$$w(g) := |\log|g|_F| \tag{9.1.2.1}$$

which indeed satisfies (9.1.1.2) by multiplicativity of $|\cdot|_F$, additivity of $\log : \mathbf{R}_{>0} \to \mathbf{R}$, and the triangle inequality for the Euclidean absolute value $|\cdot|$ on $\mathbf{R}$, which gives $C_w = 1$.

Since we are choosing $G = \widehat{C_F}$, and not $G = C_F$, we introduce notation for $H^2_w(G,\mu_G)$, which we will be retaining throughout the remainder of this work.

We do so by discussing the intermediate case of $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian groups. We recall from Definition D.2.1.1 (see also §D.3.3).

We recall that a Hausdorff locally compact abelian group G is $\mathbf{R}_{>0}$-modulated if it is equipped with a continuous surjective proper homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$. The fibre G^1 over the neutral element $1 \in \mathbf{R}_{>0}$ is therefore compact Hausdorff.

We refer the reader to Theorem D.3.3.7 for more on function spaces of Schwartz-Bruhat functions (with I-exponential decay) on $\mathbf{R}_{>0}$-modulated groups.

Definition 9.1.2.1 (Sobolev Spaces of $\mathbf{R}_{>0}$-modulated groups). Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. We let μ_G be a choice of Haar measure on G, and $\mu_{\widehat{G}}$ its Plancherel measure.

Let $s \geq 0$ be a real number.

We define:

$$W^{s,2}(G) := H^s_w(\widehat{G}, \mu_{\widehat{G}}) \tag{9.1.2.2}$$

for w the choice of Sobolev weight from (9.1.2.1). We denote by $\|\cdot\|_{s,2}$ the norm:

$$\|f\|_{s,2} := \left(\int_G |f(g)|^2 (1+|\log|g|_G|^2)^{s/2}\, \mathrm{d}\mu_G(g)\right)^{1/2}. \tag{9.1.2.3}$$

We refer to s as the *Sobolev degree* of $W^{s,2}(G)$.

Before stating the next Lemma, we observe the following. For G an $\mathbf{R}_{>0}$-modulated group, and for any choice τ of continuous homomorphic section to the module $|\cdot|_G$ (noting that such sections exist by Remark D.2.1.2), we have an isomorphism, induced by the choice of τ, of Hausdorff locally compact abelian groups $G \simeq \mathbf{R}_{>0} \times G^1$, with $G^1 := \ker(|\cdot|_G)$ a compact Hausdorff abelian group.

The choice of Haar measure μ_G on G is pushed forward to $\frac{\mathrm{d}t}{t} \otimes \mu_1$ for a unique Haar measure μ_1 on G^1.

This isomorphism is compatible with formation of dual groups (see [**DE**, Cor.3.6.2] combined with functoriality of formation of the dual group with respect to continuous group homomorphisms), giving an isomorphism of Hausdorff locally compact abelian groups $\widehat{G} \simeq \widehat{\mathbf{R}_{>0}} \times \widehat{G^1}$ where $\widehat{\mathbf{R}_{>0}} \simeq \mathbf{R}$ upon choosing the basic character $t \mapsto t^{2\pi i}$ to choose the isomorphism $\widehat{\mathbf{R}_{>0}} \simeq \mathbf{R}$.

Since $\widehat{G^1}$ is a discrete abelian group, every continuous 1-parameter group $\mathbf{R}_{>0} \to \widehat{G^1}$ is constant (since its image is connected and contains the neutral element), and every continuous 1-parameter group $\mathbf{R}_{>0} \to \widehat{G}$ has image contained within $\widehat{\mathbf{R}_{>0}} \times 1$, which is the connected component of the neutral element of $\widehat{G}$.

The $\mathbf{C}$-algebra of differential operators on $\widehat{G}$ is therefore generated by a single degree-1 differential operator, which we call D. Under the isomorphism $\widehat{\mathbf{R}_{>0}} \simeq \mathbf{R}$, D matches a generator for the $\mathbf{C}$-algebra of differential operators on $\mathbf{R}$, and we can renormalize D so as to arrange such generator to be ∂_x.

We further refer the reader to Definition D.1.6.5 and Definition D.1.6.8 of §D.1.6 from Appendix D, for more on differential operators on Hausdorff locally compact abelian groups.

LEMMA 9.1.2.2. *Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. We let μ_G be a choice of Haar measure on G, and $\mu_{\widehat{G}}$ its Plancherel measure.*

Let $s = 2N \geq 0$ an even integer. The normed topology on the Hilbert space $W^{s,2}(G)$ is equivalently induced by the finite family of seminorms:

$$\{\|\cdot\|_{s,k,2}\}_{0\leq k\leq N} \tag{9.1.2.4}$$

assigned by, for each $f \in W^{s,2}(C_F)$:

$$\|f\|_{s,k,2} := \left(\int_{\widehat{G}} |D^k \widehat{f}(\chi)|^2 \,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \tag{9.1.2.5}$$

where D is any generator of the $\mathbf{C}$-algebra of differential operators on $\widehat{G}$.

PROOF. An equivalent topology on $W^{s,2}(G)$ is induced by the finite family of seminorms:

$$\{\|\cdot\|'_{s,k,2}\}_{0\leq k\leq s/2} \tag{9.1.2.6}$$

with, for any $f \in W^{s,2}(G)$:

$$\|f\|'_{s,k,2} := \left(\int_G |f(g)|^2 \,|\log|g|_G|^{2k} \,\mathrm{d}\mu_G(g)\right)^{1/2}. \tag{9.1.2.7}$$

Indeed, by [**NB**, Thm.5.2.2(e)], each $\|\cdot\|'_{s,k,2}$ is continuous on $W^{s,2}(G)$, and therefore induces a locally convex topology that is at most as fine as the intrinsic normed topology on $W^{s,2}(G)$. On the other hand, we have:

$$(1 + |\log|g|_G|^2)^{s/2} = \sum_{0\leq k\leq s/2} \binom{s/2}{k} |\log|g|_G|^{2k} \tag{9.1.2.8}$$

and hence the following inequality holds for any $f \in W^{s,2}(G)$:

$$\|f\|_{s,2}^2 \leq \left(\sum_{0\leq k\leq s/2} \binom{s/2}{k} (\|f\|'_{s,k,2})^2\right)^{1/2} \leq C_s \left(\sum_{0\leq k\leq s/2} \|f\|'_{s,k,2}\right) \tag{9.1.2.9}$$

where $C_s > 0$ is an absolute constant, which implies by [**Bou-TVS**, Ch.II,§1.4,II.7,Cor.1] that the topology induced on $W^{s,2}(G)$ by the finite family of seminorms $\{\|\cdot\|'_{s,k,2}\}_{0\leq k\leq s/2}$ is at least as fine as the intrinsic norm topology on $W^{s,2}(G)$, and hence the two topologies are equivalent.

The constant $C_s > 0$ is obtained as follows: upon calling $a_k^2 := \binom{s/2}{k}$, we let $A > 0$ be any real number such that, on the Euclidean space $\mathbf{R}^{1+(s/2)}$, the two norms $\|\cdot\|_2$ and $\|\cdot\|_1$, are related by the inequality $\|\cdot\|_2 \leq A\|\cdot\|_1$.

Then we have

$$\|(a_k\|f\|'_{s,k,2})_{0\leq k\leq s/2}\|_2 \leq A\|(a_k\|f\|'_{s,k,2})_{0\leq k\leq s/2}\|_1 = A\sum_{0\leq k\leq s/2} a_k\|f\|'_{s,k,2} \leq AA_0\sum_{0\leq k\leq s/2}\|f\|'_{s,k,2}$$

where $A_0 := \sup_{0\leq k\leq s/2} a_k$. We then define $C_s := AA_0$.

By [**DE**, Thm.3.4.8], we have:

$$\|f\|'_{s,k,2} = \left(\int_{\widehat{G}} |\mathcal{F}(f(g)(\log|g|_G)^k)(\chi)|^2\,\mathrm{d}\mu_{\widehat{G}}(\chi)\right)^{1/2} \tag{9.1.2.10}$$

for each integer $0 \leq k \leq s/2$, where $\mathcal{F} : L^2(G,\mu_G) \to L^2(\widehat{G},\mu_{\widehat{G}})$ is the Fourier transform, normalized according to the assignment $\phi \mapsto \mathcal{F}(\phi)(\chi) := \int_G \phi(g)\overline{\chi}(g)\,\mathrm{d}\mu_{\widehat{G}}(\chi)$.

Upon choosing a continuous homomorphic section τ to the idelic norm, we normalize the choice of generator D of the $\mathbf{C}$-algebra of differential operators on $\widehat{G}$, so as to match under the isomorphism of Hausdorff locally compact abelian groups $\widehat{G} \simeq \mathbf{R}\times\widehat{G^1}$ induced by τ and by the choice of basic character $t \mapsto t^{2\pi i}$ on $\mathbf{R}_{>0}$, the differential operators $-(2\pi i)^{-1}\partial_x \otimes 1$.

We have, as functions on $\widehat{G}$:

$$\mathcal{F}(f(g)(\log|g|_G)^k)(\chi) = \int_{\mathbf{R}_{>0}} t^{-2\pi ix}\left(\int_{G^1}\overline{\chi}(g_0)f(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\right)(\log(t))^k\,\frac{\mathrm{d}t}{t} \tag{9.1.2.11}$$

where $x \in \mathbf{R}$ is a real number uniquely determined by χ such that $\chi(\tau(t)) = t^{-2\pi ix}$ as smooth functions of $t \in \mathbf{R}_{>0}$, and $g_0 \in C_F^1$ is the unique element such that $g = \tau(t)g_0$.

We now assume $f \in C_c^\infty(C_F)$. The rightmost integral from (9.1.2.11) then in turn equals, by [**Fol**, Thm.8.22(d)]:

$$(-2\pi i)^{-k}\partial_x^k\int_{\mathbf{R}_{>0}} t^{-2\pi ix}\left(\int_{G^1}\overline{\chi}(g_0)f(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\right)\frac{\mathrm{d}t}{t} = (D^k\widehat{f})(\chi). \tag{9.1.2.12}$$

We further refer the reader to Lemma D.1.6.19.

Upon combining this with (9.1.2.10) we obtain $\|f\|'_{s,k,2} = \|f\|_{s,k,2}$ on all $f \in C_c^\infty(G)$ for each integer $0 \leq k \leq s/2$, up to a positive constant multiplier independent of f (and depending only on the normalization of D). By continuity of $\|\cdot\|_{s,k,2}$ and $\|\cdot\|'_{s,k,2}$ for each $0 \leq k \leq s/2$, the identity $\|f\|'_{s,k,2} = \|f\|_{s,k,2}$ remains valid, up to a positive constant multiplier depending only on the normalization of D, on all $f \in W^{s,2}(G)$.

This shows that the seminorms $\{\|\cdot\|_{s,k,2}\}_{0\leq k\leq s/2}$ generate the same topology defined on $W^{s,2}(G)$ by the norm $\|\cdot\|_{s,2}$. □

Remark 9.1.2.3. Lemma 9.1.2.2 holds more generally for all Hausdorff locally compact abelian groups, upon generalizing the definition of $\|\cdot\|_{s,k,2}$. Namely, upon using all monomial elements of the $\mathbf{C}$-algebra of differential operators of G, in place of the sole D^k from (9.1.2.5). This recovers the definition in (9.1.2.5) for $\mathbf{R}_{>0}$-modulated G, the kernel G^1 of the module $|\cdot|_G$ is compact Hausdorff, with, therefore, discrete dual. The only non-trivial differential operators on $\widehat{G}$ are therefore contributed by $G/G^1 \simeq \mathbf{R}_{>0}$, and generated by D, up to a choice of normalization for D.

For $G = \mathbf{R}$, the seminorms $\|\cdot\|_{s,k,2}$ recovers the classical Sobolev seminorms on $\mathbf{R}$, i.e. $\|f\|_{s,k,2} = \|\partial_x^k f\|_{L^2(\mathbf{R},\mathrm{d}x)}$ for all $0 \leq k \leq s/2$.

LEMMA 9.1.2.4. *Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. Let μ_G be a choice of Haar measure on G.*

Let $s \geq 0$ be a real number.

The function of $g \in G$:

$$(1 + |\log|g|_G|^2)^{-s/2} \tag{9.1.2.13}$$

is absolutely integrable if and only if $s > 1$.

PROOF. We call $\phi(g) := (1 + |\log|g|_G|^2)^{-s/2}$. The function ϕ is continuous over G, and invariant under the left-translation action of the compact Hausdorff closed subgroup G^1, since $|\cdot|_G$ is.

To check integrability of ϕ it therefore suffices to choose a continuous homomorphic section τ to the module, and analyze the integrability of the continuous function $\psi(t) := \phi(\tau(t)) = (1 + |\log(t)|^2)^{-s/2}$ on $\mathbf{R}_{>0}$ with respect to the normalized Haar measure $\mathrm{d}t/t$.

Since $\psi(t) = \psi(1/t)$, it is enough to analyze the integrability of ψ over $(1, +\infty) \subset \mathbf{R}_{>0}$ only. We have:

$$\int_0^1 \frac{1}{(1 + |\log(t)|^2)^{s/2}} \frac{\mathrm{d}t}{t} = \int_0^{+\infty} \frac{1}{(1 + |x|^2)^{s/2}} \mathrm{d}x$$

and the rightmost integral converges if and only if $s > 1$. □

PROPOSITION 9.1.2.5. *Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. Let μ_G be a choice of Haar measure on G.*

The linear topological isomorphism:

$$T_s : W^{s,2}(G)'_\beta \to W^{-s,2}(G) \tag{9.1.2.14}$$

is an isometry with respect to the dual norm on the source, and the norm $\|\cdot\|_{-s,2}$ from (9.1.2.3) on the target, equivariant with respect to the adjoint action of the continuous left-translation action ρ_G of G on $W^{s,2}(G)$, and the right-translation action of G on $W^{-s,2}(G)$.

Let $s > 1$ be a real number. For $t \in \mathbf{R}$ and $m \geq 0$, the continuous function:

$$\eta_{t,m}(g) := |g|^{it}(\log|g|_G)^m \tag{9.1.2.15}$$

of $g \in G$, is an element of $W^{-s,2}(G)$ if and only if the following condition holds:

$$m < \frac{s-1}{2}. \tag{9.1.2.16}$$

PROOF. By Lemma 9.1.1.5, indeed the linear operator in (9.1.2.14) exists and it is defined in (9.1.1.12) applied to $\widehat{G}$ and to the Plancherel measure of the Haar measure μ_G, and $w = |\log| \cdot |_G|$. By the Lemma, it is an isometry.

To identify the unique continuous action on $W^{-s,2}(G)$ promoting (9.1.2.14) into an equivariant isometry, we observe that the continuous pairing $\langle \cdot, \cdot \rangle_{s,w}$ from (9.1.1.11) which uniquely determines (9.1.2.14) is invariant upon letting G act through ρ_G on the leftmost argument, and through the right-translation action of G on $W^{-s,2}(G)$, by invariance of the Haar measure μ_G. Since the pairing is nondegenerate, this uniquely identifies the desired action on $W^{-s,2}(G)$ with the continuous right-translation action of G on $W^{-s,2}(G)$.

The function $\eta_{t,m}$ from (9.1.2.15) is continuous on G, and it is an element of $W^{-s,2}(G)$ if and only if the integral:

$$\int_G |\log|g|_G|^{2m}(1 + |\log|g|_G|^2)^{-s/2} \mathrm{d}\mu_G(g)$$

is finite. This is the case if and only if $s - 2m > 1$, which reads $m < \frac{s-1}{2}$. □

PROPOSITION 9.1.2.6. *Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. Let μ_G be a choice of Haar measure on G.*

Let $s > 1$ be a real number.

Then every $f \in W^{s,2}(G)$ is absolutely μ_G-integrable. There exists a real number $C_s > 0$ such that:

$$\|f\|_{L^1(G,\mu_G)} \leq C_s \|f\|_{s,2}. \tag{9.1.2.17}$$

In particular, the linear inclusion $W^{s,2}(G) \to L^1(G,\mu_G)$ is continuous.

PROOF. We have:

$$\begin{aligned}\int_G |f(g)|\,\mathrm{d}\mu_G(g) &= \int_G |f(g)| \frac{(1+|\log|g|_G|^2)^{s/4}}{(1+|\log|g|_G|^2)^{s/4}}\,\mathrm{d}\mu_G(g)\\ &\leq \left(\int_G |f(g)|^2(1+|\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g)\right)^{1/2}\left(\int_G \frac{1}{(1+|\log|g|_G|^2)^{s/2}}\,\mathrm{d}\mu_G(g)\right)^{1/2}\\ &= \|(1+|\log|g|_G|^2)^{-s/2}\|^{1/2}_{L^1(G,\mu_G)}\left(\int_G |f(g)|^2(1+|\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g)\right)^{1/2}\end{aligned}$$

and since $s > 1$, then $(1+|\log|g|_G|^2)^{-s/2}$ is absolutely μ_G-integrable by Lemma 9.1.2.4. Upon calling $C_s := \|(1+|\log|g|_G|^2)^{-s/2}\|^{1/2}_{L^1(G,\mu_G)}$, we see that (9.1.2.17) holds, and hence the resulting linear inclusion $W^{s,2}(G) \to L^1(G,\mu_G)$ is continuous. □

9.2. Hardy-Sobolev spaces $W_I^s(G)$ and Paley-Wiener conditions

We recall from Appendix D.3.2, that for G a Hausdorff locally compact abelian group, modulated by a proper surjective module homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$ whose kernel we denote by G^1 (i.e. $\mathbf{R}_{>0}$-modulated; see §D.3.3) and for $I \subset \mathbf{R}$ a non-empty connected open interval equal to either $(0,+\infty)$ or $I = \mathbf{R}$, the Fourier-Mellin transform:

$$M_I : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I)\,\overline{\otimes}\,\mathcal{S}(\widehat{G^1}) \tag{9.2.0.1}$$

from (D.3.2.2) is a linear topological isomorphism (recall that the Hausdorff locally convex topology on the Schwartz-Bergman space $A\mathcal{S}(\mathbf{C}_I)$ was defined as the topology of L^p-convergence on vertical lines within $\mathbf{C}_I$ for functions ϕ and $s^N\phi$ for all integers $N \geq 0$, for a, equivalently any, $p > 1$. In particular, the topology is independent of $p \geq 1$, similarly to the independence of $p \geq 1$ of the topology of L^p-convergence of functions $f : \mathbf{R} \to \mathbf{C}$ and $q_1(x)q_2(\partial_x)f$ for any polynomials $q_1, q_2 \in \mathbf{C}[X]$, which is equivalent to the topology of the Schwartz space; see §D.1.10.1, and [**KG**, Part I,Thm.29,p.72]. We refer the reader to Definition D.3.1.3 and Lemma D.3.1.2).

We recall that M_I was defined, upon describing each continuous quasi-character $c \in \Omega(G)$ as the corresponding unique pair (s,ω), once a choice of continuous section τ to the module $|\cdot|_G$ is fixed, for $s \in \mathbf{C}$ and $\omega \in \widehat{G^1}$:

$$M_I(\phi)(s,\omega) := \int_{\mathbf{R}_{>0}} t^s \int_{G^1} \phi(\tau(t)g)\,\mathrm{d}\overline{\mu}_1(g)\,\frac{\mathrm{d}t}{t}$$

where $\overline{\mu}_1$ and $\mathrm{d}t/t$ uniquely determine and are determined, as per Theorem D.1.2.1, by the canonical Haar measure μ_G on G. $M_I(\phi)(s,\omega)$ is holomorphic over $\mathbf{C}_I$ and Schwartz uniformly on every vertical strip of bounded width contained within $\mathbf{C}_I$ (see Proposition D.3.2.5).

It is identified with $\mathcal{F}_{G^1}\,\overline{\otimes}\,M_I$ where $\mathcal{F}_{G^1}$ is the Fourier transform $\mathcal{S}(G^1) \to \mathcal{S}(\widehat{G^1})$ and $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I)$ is the Fourier-Mellin transform in the special case $G = \mathbf{R}_{>0}$ (see Theorem D.3.3.13).

REMARK 9.2.0.1. Recall that our choice of Haar measure $\overline{\mu}$ on C_F is not the canonical measure (see Remark D.2.1.4 and [**Weil-e**, §3,p.6]), which requires the normalization giving C_F^1 volume 1. The canonical measure μ_{C_F} is $\overline{\mu}_1(C_F^1)^{-1}\,\overline{\mu}$.

We made this choice to remain consistent with [**CF**, Ch.XV,§2.3]. In particular, the operator L from (5.1.1.1) and $M_{\mathbf{R}}$ on C_F restricted to $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ are equal up to the constant multiplier $\overline{\mu}_1(C_F^1)$.

Throughout this Chapter we will be mentioning the choice of normalization for the Haar measure on the $\mathbf{R}_{>0}$-modulated groups under consideration, explicitly.

We refer the reader to Proposition D.3.2.5.

9.2.1. The special case $I = \{b\}$: b-line Hardy-Sobolev spaces and Paley-Wiener Theorem. As a warmup, we will now be entering a brief interlude, discussing completions of both source and target of M_I from (9.2.0.1) with respect to the Sobolev topologies discussed in Definition 9.1.1.1. We will be terming the corresponding completions of $A\mathcal{S}(\mathbf{C}_I)\,\overline{\otimes}\,\mathcal{S}(\widehat{G^1})$ and of its isotypes, *Hardy-Sobolev spaces* of G.

Definition 9.2.1.1 (b-line Hardy-Sobolev spaces). Let $b \in \mathbf{R}$ be a real number, $N \geq 1$ an integer.
We define:

(1) The completion $W_b^{2N,2}(\mathbf{R}_{>0})$ of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ with respect to the finite family of continuous norms:

$$\|\cdot\|_{b,k} : \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathbf{R}_{\geq 0} \tag{9.2.1.1}$$

for each $0 \leq k \leq N-1$, defined by:

$$\|\phi\|_{b,k} := \left(\int_{\mathbf{R}_{>0}} |\phi(t)\,(\log(t))^k|^2 t^{2b}\,\frac{\mathrm{d}t}{t}\right)^{1/2}. \tag{9.2.1.2}$$

We refer to the Hilbert space $W_b^{2N,2}(\mathbf{R}_{>0})$ as the *b-line Sobolev space* of $\mathbf{R}_{>0}$.

(2) The completion $\mathrm{HS}_b^{2N,2}(\mathbf{C})$ of the Schwartz-Bergman space $A\mathcal{S}(\mathbf{C})$ with respect to the metrizable topology induced by the finite family of continuous norms $\{\|\cdot\|_{b,k}^{\mathrm{HS}}, 0 \leq k \leq N\}$:

$$\|\phi\|_{b,k}^{\mathrm{HS}} := \frac{1}{2\pi}\left(\int_{b-i\infty}^{b+i\infty} |\partial_s^k M(\phi)(s)|^2\,|\mathrm{d}s|\right)^{1/2}. \tag{9.2.1.3}$$

We refer to the Hilbert space $\mathrm{HS}_b^{2N,2}(\mathbf{C})$ as the *b-line Hardy-Sobolev space*.

In the next subsection, §9.2.2, we will rethink $\mathrm{HS}_b^{2N,2}(\mathbf{C})$ as the special case of the Hardy-Sobolev space $\mathrm{HS}_I^{2N,2}(\mathbf{C})$, to be defined therein, with base interval $I = \{b\}$, to be identified, under the Mellin transform, with the Hardy-Sobolev space $W_I^{2N,2}(\mathbf{C})$, for $I = \{b\}$.

Theorem 9.2.1.2 (Paley-Wiener Theorem for Hardy-Sobolev Spaces - I). *Let $a \in \mathbf{R}$ be a real number, $a > 1$. We denote by M the Mellin transform $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I)$ for $I := (-\infty, +\infty)$.*
We denote by:

$$\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}, (1/a, +\infty)) \subset \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \tag{9.2.1.4}$$

the linear subspace of those Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ of exponential decay, whose support is contained within the interval $(1/a, +\infty)$.
Let, for $b \in \mathbf{R}$ a real number, and $N \geq 1$ an integer.
Then:

(1) *The closure of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}, (1/a, +\infty))$ in the Sobolev space $W_b^{2N,2}(\mathbf{R}_{>0})$ is the closed subspace obtained as the closed range of the idempotent multiplication operator $M_{\rho_{a,+}} : W_b^{2N,2}(\mathbf{R}_{>0}) \to W_b^{2N,2}(\mathbf{R}_{>0})$ obtained by multiplying by the indicator function $\rho_{a,+}(t) := 1_{[1/a,+\infty)}(t)$.*
(2) *The Mellin transform M extends by continuity to a linear isometric isomorphism:*

$$M : W_b^{2N,2}(\mathbf{R}_{>0}) \to \mathrm{HS}_b^{2N,2}(\mathbf{C}) \tag{9.2.1.5}$$

where the target is the b-line Hardy-Sobolev space.

(3) *The closure* $\mathrm{HS}_b^{2N,2}(\mathbf{C},(1/a,+\infty))$ *of* $M(\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0},(1/a,+\infty)))$ *in* $\mathrm{HS}_b^2(\mathbf{C})$ *satisfies the following property: there exists a commutative diagram of continuous linear operators with linearly short exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{[1/a,1]}W_b^{2N,2}(\mathbf{R}_{>0}) & \longrightarrow & M_{\rho_{a,+}}W_b^{2N,2}(\mathbf{R}_{>0}) & \longrightarrow & M_{\rho_{1,+}}W_b^{2N,2}(\mathbf{R}_{>0}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \\ 0 & \longrightarrow & \mathrm{PW}_b^{2N,2}(\mathbf{C},a) & \longrightarrow & \mathrm{HS}_b^{2N,2}(\mathbf{C},(1/a,+\infty)) & \longrightarrow & H^{2N,2}(\mathbf{C}_{<b}) & \longrightarrow & 0 \end{array} \tag{9.2.1.6}$$

where $M_{[1/a,1]}$ *is the multiplication by* $1_{[1/a,1]}$ *operator, the leftmost operators in* (9.2.1.6) *are continuous, with closed range, the rightmost are surjective and open, the vertical arrows are isometric isomorphisms induced by the Mellin transform, and where:*

(1) $H^{2N,2}(\mathbf{C}_{<b})$ *is the Hardy-Sobolev space of the left half-plane.*

(2) $\mathrm{PW}_b^{2N,2}(\mathbf{C},a)$ *is the closed linear subspace of the Hardy-Sobolev space* $H^{2N,2}(\mathbf{C}_{>b})$ *of those holomorphic functions* $\phi(s)$ *over* $\mathbf{C}_{>b}$ *that extend to entire functions and satisfy:*

$$\phi(s) = O\left(\frac{(1-a^{-2\,\mathrm{Re}(s)})^{1/2}}{(\mathrm{Re}(s))^{1/2}}\right). \tag{9.2.1.7}$$

We refer to $\mathrm{PW}_b^{2N,2}(\mathbf{C},a)$ *as the* b-line Paley-Wiener space of weight a.

Proof. (1) : The assertion follows from the following: $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0},(1/a,+\infty))$ is sequentially dense within the closed subspace $M_{\rho_{a,+}}W_b^{2N,2}(\mathbf{R}_{>0}) \subset W_b^{2N,2}(\mathbf{R}_{>0})$. The Hausdorff locally convex topology induced on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by the finitely many norms $\{\|\cdot\|_{b,k}\}_{0\leq k\leq N}$ is equivalent to the normed topology induced by the norm:

$$\|\cdot\|_{2N,2} : \phi \mapsto \|\phi\|_{2N,2} := \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 t^{2b} |1+|\log(t)|^2|^N \frac{\mathrm{d}t}{t}\right)^{1/2}$$

the proof of the equivalence of the two topologies being seen by an argument analogous to that provided for Lemma 9.1.2.2.

We define

$$\nu_{a,b,N} := \rho_{a,+}\cdot|\cdot|_{\mathbf{R}}^{2b}|1+|\log|\cdot|_{\mathbf{R}}^2|^N \frac{\mathrm{d}t}{t}$$

a Radon measure on $\mathbf{R}_{>0}$. The closed subspace $M_{\rho_{a,+}}W_b^{2N,2}(\mathbf{R}_{>0})$ is isometric to the completion of the linear space of smooth compactly supported functions on $(1/a,+\infty)$ with respect to the $L^2(\mathbf{R}_{>0},\nu_{a,b,N})$-norm, which shows that $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0},(1/a,+\infty))$ is sequentially dense within $M_{\rho_{a,+}}W_b^{2N,2}(\mathbf{R}_{>0})$.

(2) : The assertion is a consequence of the formula:

$$\int_{\mathbf{R}_{>0}} \phi(t)(\log(t))^k t^{b+ix}\frac{\mathrm{d}t}{t} = \partial_s^k M(\phi)(s)|_{b+ix} \tag{9.2.1.8}$$

(see [**Fol**, Thm.8.22(d)]) and Plancherel's Theorem (see [**DE**, Thm.3.4.8] and [**Fol**, Thm.8.29]), noting that the Plancherel measure associated to $\mathrm{d}t/t$ is $\frac{1}{2\pi}\mathrm{d}x$ (which is seen upon changing the variable x in the definition of the Fourier transform in [**Fol**, Ch.8,p.249] into $x/2\pi$ and using [**Fol**, Thm.8.29]).

(3) : Since $\rho_{a,+} = \rho_{1,+} + 1_{(1/a,1]}$, a sum of nonnegative real valued piece-wise continuous functions on $\mathbf{R}_{>0}$, each multiplication operator M_ρ for $\rho \in \{\rho_{a,+},\rho_{1,+},1_{(1/a,1]}\}$ is a positive idempotent operator, hence, by [**Cnw-a**, Ch.II,§3,Prop.3.3], an orthogonal projector. The topmost extension of $\mathbf{C}$-linear spaces is therefore exact and the leftmost horizontal arrow is an isometric isomorphism onto its image, with closed linearly and topologically complemented range. To prove (3) it is therefore sufficient to show that the Mellin transform induces linear isometric isomorphisms on all three vertical arrows. This is indeed the case for the rightmost vertical arrow, a consequence of Plancherel's Theorem (see [**Fol**, Thm.8.29]) and [**Fol**, Thm.8.22(d)], in light of the description of the normed topology on $H^{2N,2}(\mathbf{C}_{<b})$ as equivalent to the topology generated by the family of seminorms from (9.2.2.2).

We now show that the Mellin transform

$$M : M_{\rho_{a,+}} W_b^{2N,2}(\mathbf{R}_{>0}) \to \mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$$

maps the closed subspace $M_{[1/a,1]} W_b^{2N,2}(\mathbf{R}_{>0})$ onto $\mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$.

For any $f \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, the function $u_a := 1_{[1/a,1]} \cdot f$ is compactly supported and piece-wise smooth. Its support is contained within the interval $[1/a, 1] \subset \mathbf{R}_{>0}$ and its set of singular points is contained in $\{1/a, 1\} \subset \mathbf{R}_{>0}$.

Since the support of u_a is contained within $(0, 1]$, then the Fourier transform of $|\cdot|_{\mathbf{R}}^b u_a$ extends uniquely to an entire function $M(u_a)(b' + s)$ of $s \in \mathbf{C}$ that is an element of $H^{2N,2}(\mathbf{C}_{>0})$, equivalently $M(u_a)$ is an element of $H^{2N,2}(\mathbf{C}_{>b'})$, for any real number b'. In particular, for $b' = b$, we deduce that $M(u_a)$ is an element of $H^{2N,2}(\mathbf{C}_{>b})$. This choice of b' ensures that the leftmost horizontal arrow in the bottom row of the diagram in (9.2.1.6) is defined and injective.

We have:

$$|M(u_a)(s)| = \left| \int_{-\log(a)}^{0} u_a(e^{\xi}) e^{s\xi} \, \mathrm{d}\xi \right| \tag{9.2.1.9}$$

$$\leq \left(\int_{-\log(a)}^{0} |u_a(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \left(\int_{-\log(a)}^{0} |e^{s\xi}|^2 \, \mathrm{d}\xi \right)^{1/2} \tag{9.2.1.10}$$

$$= \|u_a\|_{L^2(\mathbf{R}_{>0}, \mathrm{d}t/t)} \frac{(1 - a^{-2\,\mathrm{Re}(s)})^{1/2}}{(2\,\mathrm{Re}(s))^{1/2}}. \tag{9.2.1.11}$$

This shows that M, restricted to $M_{[1/a,1]} W_b^{2N,2}(\mathbf{R}_{>0})$, is valued in $\mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$.

Let $\phi \in \mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$ be an arbitrary element. By [**Dur**, Ch.11,§11.2,Thm.11.4,p.192] applied to ϕ and its holomorphic derivatives of order $1 \leq k \leq N$, the boundary value functions ϕ_b and $(\partial_s^k \phi)_b$ obtained by asymptotic restriction of $\phi(s)$ to the b-line (i.e. by formation of the limit $\lim_{\sigma \to 0^-} (\partial_s^k \phi)(\sigma + b + ix)$, which converges for $\mathrm{d}x$-almost all x and each $0 \leq k \leq N$), is an element of the Sobolev space $W^{2N,2}(\mathbf{R})$ (using Fourier Inversion to show that indeed $(\partial_s^k \phi)_b$ represents the weak kth derivative of ϕ_b, since it is the Fourier transform of $|t|_{\mathbf{R}}^a u(t) \, (\log(t))^k$ for $|\cdot|_{\mathbf{R}}^b u$ the Fourier anti-transform of ϕ_b). It follows that the Mellin anti-transform u of ϕ is an element of $W_b^{2N,2}(\mathbf{R}_{>0})$.

By the proof of [**Dur**, Ch.11,§11.5,Corollary,p.197], $|\cdot|_{\mathbf{R}}^b u$ is supported over $(0, 1]$, and hence so is u. The entire function $a^s \phi(s)$ is now an element of $H^{2N,2}(\mathbf{C}_{<b})$, and hence the Mellin anti-transform of $a^{-s}\phi(-s)$, i.e. $u(a/t)$ is such that $|\cdot|_{\mathbf{R}}^b u(a/t)$ is supported over $(0, 1]$, by another application of the proof of [**Dur**, Ch.11,§11.5,Corollary,p.197], which implies that u is supported over $[1/a, +\infty)$. This shows that $u \in M_{[1/a,1]} W_b^{2N,2}(\mathbf{R}_{>0})$, where u is uniquely determined almost everywhere, and hence M maps $M_{[1/a,1]} W_b^{2N,2}(\mathbf{R}_{>0})$ onto $\mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$. We further refer the reader to the argument in [**Yos**, Ch.VI,§4,Thm.2,p.164].

We have therefore proved that M is surjective onto $\mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$, and since M is isometric, we deduce that $\mathrm{PW}_b^{2N,2}(\mathbf{C}, a)$ is closed in $\mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$ (which is also a consequence of the Asymptotic Residue Theorem; see Theorem B.3.2.1) and that the bottom leftmost horizontal arrow in (9.2.1.6) is itself an isometric isomorphism onto its closed range. □

COROLLARY 9.2.1.3. *Let $a \in \mathbf{R}$ be a real number, $a > 1$. Let $b \in \mathbf{R}$ be a real number $N \geq 1$ an integer. We denote by M the Mellin transform $M : W_b^{2N,2}(\mathbf{R}_{>0}) \to \mathrm{HS}_b^{2N,2}(\mathbf{C})$ provided by Theorem 9.2.1.2(2). Let $\phi \in W_b^{2N,2}(\mathbf{R}_{>0})$ be the arbitrary element. Then $M(\phi)(s)$ is a holomorphic function on $\mathbf{C}_{<b}$, square-integrable on all vertical lines of $\mathbf{C}_{<b}$ with all of its holomorphic derivatives of order $0 \leq k \leq N$.*

There exist a holomorphic function $u_1 \in H^{2N,2}(\mathbf{C}_{<b})$ and an entire function $u_2 \in H^{2N,2}(\mathbf{C}_{>b})$, canonically associated to $M(\phi)$, such that:

$$u_2(s) = O\left(\frac{(1-a^{-2\,\mathrm{Re}(s)})^{1/2}}{(\mathrm{Re}(s))^{1/2}}\right) \tag{9.2.1.12}$$

and:

$$M(\phi) = u_1 + u_2 \tag{9.2.1.13}$$

as holomorphic functions over $\mathbf{C}_{<b}$.

PROOF. Every element $\phi \in M_{\rho_{a,+}} W_b^{2N,2}(\mathbf{R}_{>0})$ is, in particular, square-integrable with respect to the Radon measure on $\mathbf{R}_{>0}$ defined as $|\cdot|_{\mathbf{R}}^{2b'}\,\frac{\mathrm{d}t}{t}$ for every real number $b' < b$. By Plancherel's Theorem (see [**DE**, Thm.3.4.8] and [**Fol**, Thm.8.29]), this integrability condition reads:

$$\int_{b'-i\infty}^{b'+i\infty} |M(\phi)(s)|^2 |\mathrm{d}s| < +\infty.$$

Following the proof of Theorem 9.2.1.2(2), we have, for each integer $0 \leq k \leq N-1$:

$$\int_{\mathbf{R}_{>0}} \phi(t)(\log(t))^k t^{b'+ix}\,\frac{\mathrm{d}t}{t} = \partial_s^k M(\phi)(s)|_{b'+ix} \tag{9.2.1.14}$$

by see [**Fol**, Thm.8.22(d)] and another application of Plancherel's Theorem for all real numbers $b' < b$, and hence we have:

$$\int_{\mathbf{R}_{>0}} |\phi(t)(\log(t))^k| t^{2b'}\,\frac{\mathrm{d}t}{t} = \int_{b'-i\infty}^{b'+i\infty} |\partial_s^k M(\phi)(s)|^2\,|\mathrm{d}s| \tag{9.2.1.15}$$

which implies that $\partial_s^k \phi$ is also square-integrable on all vertical lines of $\mathbf{C}_{<b}$.

By Theorem 9.2.1.2(3), the functions $u_1 := M(1_{[1,+\infty)} \cdot \phi)$ and $u_2 := M(1_{[1/a,1]} \cdot \phi)$ satisfy the desired properties. □

COROLLARY 9.2.1.4. *Let $a, b \in \mathbf{R}$ be real numbers, $a > 1$. We denote by M the Mellin transform $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I)$ for $I := (-\infty, +\infty)$.*

For every point $z \in \mathbf{C}_{<b}$, the point-evaluation functional δ_z and its derivatives $\delta_z^{(m)}$ for $1 \leq m \leq N-1$, are continuous on $\mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$.

For every $\phi \in M_{\rho_{a,+}} W_b^{2N,2}(\mathbf{R}_{>0})$, we have:

$$\delta_z^{(m)}(M(\phi)(s)) = \int_{\mathbf{R}_{>0}} \phi(t)(\log(t))^m t^z\,\frac{\mathrm{d}t}{t} \tag{9.2.1.16}$$

for each integer $0 \leq m \leq N-1$.

PROOF. Every point-evaluation functional on the elements of $\mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$ supported at points $z \in \mathbf{C}_{<b}$ is bounded. Since the linear topology on $\mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$ promotes the holomorphic derivative operators ∂_s^k for $0 \leq k \leq N-1$, to continuous linear endomorphisms, the functionals $\delta_z^{(m)}$ are all continuous on $\mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$ for $0 \leq m \leq N-1$. The formula in (9.2.1.16) is a consequence of Mellin Inversion. □

We can now incorporate the module-1 subgroup of a Hausdorff locally compact abelian $\mathbf{R}_{>0}$-modulated group into the assertions of Theorem 9.2.1.2 and Corollary 9.2.1.4.

THEOREM 9.2.1.5 (Paley-Wiener Theorem for Hardy-Sobolev Spaces - II). *Let $a \in \mathbf{R}$ be a real number, $a > 1$. Let G be an $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group.*

We denote by M the Mellin transform $M : \mathcal{S}_{\mathrm{E}}(G) \to A\mathcal{S}(\mathbf{C}) \,\overline{\otimes}\, \mathcal{S}(\widehat{G^1})$.

We denote by:

$$\mathcal{S}_{\mathrm{E}}(G,(1/a,+\infty))\subset\mathcal{S}_{\mathrm{E}}(G) \tag{9.2.1.17}$$

the linear subspace of those Schwartz-Bruhat functions on G of exponential decay, whose support is sent by the module homomorphism $|\cdot|_G$ into the interval $(1/a,+\infty)\subset\mathbf{R}_{>0}$.

Let, for $b\in\mathbf{R}$ a real number, and $N\geq 0$ an integer:

$$\|\cdot\|_{b,k}:\mathcal{S}_{\mathrm{E}}(G)\to\mathbf{R}_{\geq 0} \tag{9.2.1.18}$$

be the continuous norm, for each $0\leq k\leq N$, defined by, upon choosing a continuous homomorphic section τ to the module $|\cdot|_G$:

$$\|\phi\|_{b,k}:=\left(\int_{\mathbf{R}_{>0}}|\phi(\tau(t)g)\,(\log(t))^k|^2t^{2b}\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t}\right)^{1/2} \tag{9.2.1.19}$$

where μ_1 is the unique volume-1 Haar measure on G^1.

Then:

1. *The closure of $\mathcal{S}_{\mathrm{E}}(G,(1/a,+\infty))$ in the completion $W_b^{2N,2}(G)$ of $\mathcal{S}_{\mathrm{E}}(G)$ with respect to the metrizable topology induced by the norms $\{\|\cdot\|_{b,k},0\leq k\leq N\}$ is the closed subspace obtained as the closed range of the idempotent multiplication operator $M_{\rho_{a,+}}:W_b^{2N,2}(G)\to W_b^{2N,2}(G)$ obtained by multiplying by the indicator function $\rho_{a,+}(g):=1_{[1/a,+\infty)}(g)$, $g\in G$.*
2. *The Mellin transform M extends by continuity to the linear isometric isomorphism:*

$$M:W_b^{2N,2}(G)\to L^2(G^1,\mu_1)\,\widehat{\otimes}_\sigma\mathrm{HS}_b^{2N,2}(\mathbf{C}) \tag{9.2.1.20}$$

   where the target is the Hilbert-space tensor product obtained as the normed completion of the algebraic tensor product with respect to the Hilbert-space structure assigned by the inner product:

$$\langle u_1\otimes\phi_1,u_2\otimes\phi_2\rangle_\sigma:=\langle u_1,u_2\rangle_{L^2(G^1,\mu_1)}\,\langle\phi_1,\phi_2\rangle_{\mathrm{HS}_b^{2N,2}(\mathbf{C})} \tag{9.2.1.21}$$

   for $u_1,u_2\in L^2(G^1,\mu_1)$ and $\phi_1,\phi_2\in\mathrm{HS}_b^{2N,2}(\mathbf{C})$.
3. *The closure of $M(\mathcal{S}_{\mathrm{E}}(G,(1/a,+\infty)))$ in $L^2(G^1,\mu_1)\,\widehat{\otimes}_\sigma\mathrm{HS}_b^{2N,2}(\mathbf{C})$ satisfies the following property. There exists a commutative diagram of continuous linear operators with linearly short exact rows:*

(9.2.1.22)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{[1/a,1]}W_b^{2N,2}(G) & \longrightarrow & M_{\rho_{a,+}}W_b^{2N,2}(G) & \longrightarrow & M_{\rho_{1,+}}W_b^{2N,2}(G) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \\
0 & \longrightarrow & L^2(G^1,\mu_1)\,\widehat{\otimes}_\sigma\mathrm{PW}_b^{N,2}(\mathbf{C},a) & \longrightarrow & L^2(G^1,\mu_1)\,\widehat{\otimes}_\pi\mathrm{HS}_b^{2N,2}(\mathbf{C},(1/a,+\infty)) & \longrightarrow & L^2(G^1,\mu_1)\,\widehat{\otimes}_\sigma H^{N,2}(\mathbf{C}_{<b}) & \longrightarrow & 0
\end{array}$$

   where $M_{[1/a,1]}$ is the multiplication by $1_{[1/a,1]}$ operator, the leftmost operators in (9.2.1.22) are continuous, with closed range, the rightmost are surjective and open, the vertical arrows are isometric isomorphisms induced by the Mellin transform.

Proof. We observe that for $\phi=u\otimes\phi_0$, with $u\in W_b^{2N,2}(\mathbf{R}_{>0})$ and $\phi_0\in L^2(G^1,\mu_1)$, then $|\phi|^2$ satisfies the assumptions of Theorem D.1.2.1, exhibiting $W_b^{2N,2}(G)$ as the completion of $\mathcal{S}_{\mathrm{E}}(G,(1/a,+\infty))\simeq\mathcal{S}(G^1)\,\widehat{\otimes}_\pi\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0},(1/a,+\infty))$ with respect to the tensor topology induced by the norm uniquely determined by the inner product in (9.2.2.46), since such norm restricts on elementary tensors, by Theorem D.1.2.1, to the norm $\|\cdot\|_{b,k}$.

The Theorem is then a consequence of Theorem 9.2.1.2. □

The following Corollary follows from Theorem 9.2.1.5 and Corollary 9.2.1.3.

Corollary 9.2.1.6. *Let $a\in\mathbf{R}$ be a real number, $a>1$. Let $b\in\mathbf{R}$ be a real number $N\geq 1$ an integer.*

We denote by M the Mellin transform $M : W_b^{2N,2}(G) \to L^2(G^1, \mu_1) \mathbin{\widehat{\otimes}}_\sigma \mathrm{HS}_b^{2N,2}(\mathbf{C})$ provided by Theorem 9.2.1.5(2).

Let $\phi \in W_b^{2N,2}(G)$ be the arbitrary element. Then for each $\omega \in \widehat{G^1}$, the function $M(\phi)(s,\omega)$ is a holomorphic function on $\mathbf{C}_{<b}$, square-integrable on all vertical lines of $\mathbf{C}_{<b}$ with all of its holomorphic derivatives of order $0 \leq k \leq N-1$.

There exist a holomorphic function $u_1 \in H^{2N,2}(\mathbf{C}_{<b})$ and an entire function $u_2 \in H^{2N,2}(\mathbf{C}_{>b})$, canonically associated to $M(\phi,\omega)$, such that:

$$u_2(s) = O\left(\frac{(1-a^{-2\,\mathrm{Re}(s)})^{1/2}}{(\mathrm{Re}(s))^{1/2}}\right) \tag{9.2.1.23}$$

and:

$$M(\phi) = u_1 + u_2 \tag{9.2.1.24}$$

as holomorphic functions over $\mathbf{C}_{<b}$.

The following is an analog of Corollary 9.2.1.4.

Corollary 9.2.1.7. *Let G be a Hausdorff locally compact $\mathbf{R}_{>0}$-modulated group.*

Let $a, b \in \mathbf{R}$ be real numbers, $a > 1$. We denote by M the Mellin transform $M : \mathcal{S}_{\mathrm{E}}(G) \to A\mathcal{S}(\mathbf{C}) \mathbin{\overline{\otimes}} \mathcal{S}(\widehat{G^1})$.

For every point $z \in \mathbf{C}_{<b}$ and every continuous character $\omega \in \widehat{G^1}$, equivalently every point $c \in \Omega(G)_{<b}$ of the Riemann surface of continuous quasi-characters of G, the point-evaluation functional $\delta_{(z,\omega)}$ and its derivatives $\delta_{(z,\omega)}^{(m)}$ for $1 \leq m \leq N-1$, induce continuous functionals on $L^2(G^1,\mu_1) \mathbin{\widehat{\otimes}}_\sigma \mathrm{HS}_b^{2N,2}(\mathbf{C}, (1/a, +\infty))$ uniquely determined by the following formula: for every $\phi \in M_{\rho_{a,+}} W_b^{2N,2}(G)$, we have, for a, equivalently any, choice of continuous homomorphic section τ to the module $|\cdot|_G$:

$$\delta_{(z,\omega)}^{(m)}(M(\phi)(s,\omega)) = \int_{\mathbf{R}_{>0}} \int_{G^1} \omega(g)\phi(\tau(t)g)(\log(t))^m t^z \,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t} \tag{9.2.1.25}$$

for each integer $0 \leq m \leq N-1$.

Proof. This is a consequence of Theorem 9.2.1.5 and Corollary 9.2.1.4. □

9.2.1.1. *Duality.* We now provide a description of the continuous dual of $W_I^s(G)$ for G a second-countable $\mathbf{R}_{>0}$-modulated group, $I \subset \mathbf{R}$ a closed non-empty interval, $s \geq 0$ a real number. We discuss here below the special case $G = \mathbf{R}_{>0}$.

Proposition 9.2.1.8 (Hardy-Sobolev duality - I). *Let $I \subset \mathbf{R}$ be a closed non-empty interval, $s \geq 0$ a real number. Assume I is compact, of the form $I = [a,b]$, for $a < b$ real numbers.*

For every open interval $J \subset \mathbf{R}$ containing I, the pairing:

$$\langle \cdot, \cdot \rangle_{I,s} : \mathcal{S}_J(\mathbf{R}_{>0}) \times \mathcal{S}_J(\mathbf{R}_{>0}) \to \mathbf{C} \tag{9.2.1.26}$$

defined by the assignment:

$$\langle \phi, \psi \rangle_{I,s} := \int_{\mathbf{R}_{>0}} \phi(t)\psi(t)\,\frac{\mathrm{d}t}{t} \tag{9.2.1.27}$$

is continuous and it extends to a continuous duality pairing:

$$\langle \cdot, \cdot \rangle_{I,s} : W_I^s(\mathbf{R}_{>0}) \times W_I^{-s}(\mathbf{R}_{>0}) \to \mathbf{C} \tag{9.2.1.28}$$

where $W_I^{-s}(\mathbf{R}_{>0})$ is the completion of $\mathcal{S}_J(\mathbf{R}_{>0})$ with respect to the norm:

$$\phi \mapsto \|\phi\|_{I,-s} := \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2\, w_{I,s}(t)^{-1}\,\frac{\mathrm{d}t}{t}\right)^{1/2} \tag{9.2.1.29}$$

with $w_{I,s}(t)$ the function:

$$w_{I,s}(t) := \max\{t^{2a}, t^{2b}\}\,(1+|\log(t)|^2)^{s/2}. \tag{9.2.1.30}$$

In particular, the induced linear map:

$$T_{I,s} : W_I^s(\mathbf{R}_{>0})'_\beta \to W_I^{-s}(\mathbf{R}_{>0}) \tag{9.2.1.31}$$

determined by (9.2.1.28)*, given explicitly by $T \mapsto w_{I,s}^{-1} \cdot T$.*

Proof. By [**Fol**, Thm.6.15], the pairing:

$$\langle \cdot, \cdot \rangle : L^2(\mathbf{R}_{>0}, \mu_{I,s}) \times L^2(\mathbf{R}_{>0}, \mu_{I,s}) \to \mathbf{C}$$

defined by:

$$\langle \phi, \psi \rangle := \int_{\mathbf{R}_{>0}} \phi(t)\psi(t)\,\frac{\mathrm{d}t}{t} \tag{9.2.1.32}$$

upon choosing the measure space $(X, \mathcal{M}, \mu)$ to be $X := \mathbf{R}_{>0}$, $\mathcal{M}$ the Borel σ-algebra, and $\mu_s := w_{I,s}\frac{\mathrm{d}t}{t}$, with $p = q = 2$, is separately continuous and induces an isometry between $L^2(\mathbf{R}_{>0}, w_{I,s}\mathrm{d}t/t)'_\beta$ equipped with the dual norm (defined in [**Fol**, Prop.6.13]) and $L^2(\mathbf{R}_{>0}, w_{I,s}\mathrm{d}t/t)$ itself, which is $W_I^s(\mathbf{R}_{>0})$.

We write $\mu_{-s} := w_{I,s}^{-1}\frac{\mathrm{d}t}{t}$

For every $\phi \in L^2(\mathbf{R}_{>0}, \mu_s)$ and $\psi \in L^2(\mathbf{R}_{>0}, \mu_{-s})$, the integral:

$$\int_{\mathbf{R}_{>0}} |\phi(t)\psi(t)|\,\frac{\mathrm{d}t}{t} \tag{9.2.1.33}$$

converges, i.e. $\phi \cdot \psi$, as a complex-valued measurable function on $\mathbf{R}_{>0}$, is absolutely integrable.

Indeed, we have:

$$\int_{\mathbf{R}_{>0}} |\phi(t)\psi(t)|\,\frac{\mathrm{d}t}{t} = \int_{\mathbf{R}_{>0}} |\phi(t)\psi(t)|\frac{w_{I,s}(t)^{1/2}}{w_{I,s}(t)^{1/2}}\,\frac{\mathrm{d}t}{t} \leq \|\phi\|^{1/2}_{L^2(\mathbf{R}_{>0},\mu_s)}\,\|\psi\|^{1/2}_{L^2(\mathbf{R}_{>0},\mu_{-s})}. \tag{9.2.1.34}$$

We define $\langle \phi, \psi \rangle_{I,s}$ to be the value of the integral $\int_{\mathbf{R}_{>0}} \phi(t)\psi(t)\frac{\mathrm{d}t}{t}$. The notation is justified since the restriction of $\langle \cdot, \cdot \rangle_{I,s}$ to $\mathcal{S}_J(\mathbf{R}_{>0}) \times \mathcal{S}_J(\mathbf{R}_{>0})$ is the pairing defined in (9.2.1.27).

We have, for all such $\phi, \psi \in L^2(\mathbf{R}_{>0}, \mu_s)$:

$$\langle \phi w_{I,s}^{-1}, \psi \rangle = \langle \phi, \psi w_{I,s}^{-1} \rangle = \langle \phi, \psi \rangle_{I,s} \tag{9.2.1.35}$$

where the products $\phi w_{I,s}^{-1}$, respectively $\psi w_{I,s}^{-1}$, are as complex-valued measurable functions on $\mathbf{R}_{>0}$.

The pairing $\langle \cdot, \cdot \rangle_{I,s}$ is sequentially continuous with respect to each argument, hence indeed separately sequentially continuous with respect to the normed topology defined by $\| \cdot \|_{I,s}$ in each argument.

Sequential continuity of linear functionals implies continuity, the topology on $W_I^s(\mathbf{R}_{>0})$ being metric.

Since $W_I^s(\mathbf{R}_{>0})$ is a Banach space, being a Hilbert space by Lemma 9.2.2.2, hence so is its strong dual, normed with the dual norm, and by the Uniform Boundedness Theorem the pairing $\langle \cdot, \cdot \rangle_{I,s}$ is continuous, being separately continuous (see [**Bou-TVS**, Ch.IV,§4.6,IV.26,Thm.2]). The same conclusion also follows from Proposition C.2.3.11.

The linear endomorphism $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) \to \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ (i.e. we choose $J = \mathbf{R}$ for the rest of the argument) sending ψ to $w_{I,s}^{-1}\psi$ therefore extends to a continuous linear operator $W_I^s(\mathbf{R}_{>0})'_\beta \to W_I^{-s}(\mathbf{R}_{>0})$.

Indeed, for continuous functional $T \in W_I^s(\mathbf{R}_{>0})'_\beta$, by [**Fol**, Prop.6.13] there exists a unique $\psi \in W_I^s(\mathbf{R}_{>0})$ such that $T = T_\psi$, with T_ψ the functional defined as:

$$T_\psi : \phi \mapsto T_\psi(\phi) := \langle \phi, \psi \rangle.$$

The functional on $W_I^s(\mathbf{R}_{>0})$:

$$w_{I,s}^{-1}\,T$$

assigned by:

$$w_{I,s}^{-1}T(\phi) := \langle w_{I,s}^{-1}\phi, \psi\rangle$$

is continuous on $W_I^{-s}(\mathbf{R}_{>0})$, and hence there exists a unique $\psi_0 \in W_I^{-s}(\mathbf{R}_{>0})$ representing it. The continuous linear operator $W_I^s(\mathbf{R}_{>0})'_\beta \to W_I^{-s}(\mathbf{R}_{>0})$ induced by $\langle\cdot,\cdot\rangle_{I,s}$ sends $T = T_\psi$ to ψ_0. We have $\psi_0 = w_{I,s}\,\psi$, and we compute:

$$\|\psi_0\|_{W_I^{-s}(\mathbf{R}_{>0})} = \|\psi\|_{I,s} = \|T\|_{W_I^s(\mathbf{R}_{>0})'_\beta} \tag{9.2.1.36}$$

the last equality being provided by [Fol, Prop.6.13].

The identities in (9.2.1.36) imply that the continuous linear operator from (9.2.1.31) is an isometry, hence injective.

It is bijective on $C_c(\mathbf{R}_{>0})$ being $w_{I,s}^{-1}$ continuous and nowhere vanishing, with continuous and nowhere vanishing reciprocal. It follows that it has dense range, and it is therefore surjective, being its range closed (as is the range of any linear topological embedding of a complete Hausdorff locally convex topological vector space into another). It is therefore an isometric isomorphism. □

9.2.2. General base intervals: the Hardy-Sobolev spaces $W_I^s(G)$. We briefly recall here the definition of the Hardy-Sobolev space $H^{2N,2}(\mathbf{C}_{<b})$, for $b \in \mathbf{R}$ a real number, and $N \geq 1$ an integer.

This is the $\mathbf{C}$-linear space of holomorphic functions on the half-plane $\mathbf{C}_{<b}$ that are square-integrable with their holomorphic derivatives of order $0 \leq k \leq N$ on each vertical line of $\mathbf{C}_{<b}$ and such that the resulting L^2-norms are bounded uniformly with respect to the abscissa $\sigma < b$, topologized with the metrizable topology generated by the finitely many norms:

$$\|\cdot\|_{\mathrm{HS},2N,k} : \phi \mapsto \sup_{\sigma<b}\left(\int_{\sigma-i\infty}^{\sigma+i\infty} |\partial_s^k\phi(s)|^2\,|\mathrm{d}s|\right)^{1/2} \tag{9.2.2.1}$$

for $0 \leq k \leq N$. We refer to $H^{2N,2}(\mathbf{C}_{<b})$ topologized with this seminormed topology, as the Hardy-Sobolev space of degree $2N$. It is a complex Hilbert space (since the finitely many norms can be reassembled into a single norm inducing a Hilbert space structure on $H^{2N,2}(\mathbf{C}_{<b})$).

We note that the norms, for each $0 \leq k \leq N$ and as σ varies over the real numbers $\sigma < b$:

$$\|\cdot\|_{\mathrm{HS},2N,k,\sigma} : \phi \mapsto \left(\int_{\sigma-i\infty}^{\sigma+i\infty} |\partial_s^k\phi(s)|^2\,|\mathrm{d}s|\right)^{1/2} \tag{9.2.2.2}$$

are all continuous on $H^{2N,2}(\mathbf{C}_{<b})$, and hence their collection as σ varies over the real numbers $\sigma < b$ induces a Hausdorff locally convex topology on $H^{2N,2}(\mathbf{C}_{<b})$ that is coarser than the topology induced by the finitely many norms from (9.2.2.1).

By [Dur, Ch.11,Cor.2,p.191] and [Dur, Ch.11,§11.4,Thm.11.8,p.195], the Asymptotic Cauchy Integral Formula (see Theorem B.2.2.4) applies, and it implies that $H^{2N,2}(\mathbf{C}_{<b})$ topologized with this coarser topology, is complete, and that a countable subfamily of the seminorms from (9.2.2.2) indexed by a countable subset of $\sigma < b$ such that b is an accumulation point, generates an equivalent Hausdorff locally convex topology, which therefore exhibits $H^{2N,2}(\mathbf{C}_{<b})$, re-topologized in this manner, as a Fréchet space (this can also be seen using [Dur, Ch.11,§11.2,Thm.11.5,p.192]).

The identity map on $H^{2N,2}(\mathbf{C}_{<b})$ with the Hardy-Sobolev topology on the source, and the coarser Fréchet topology defined above on the target, is therefore open by [NB, Thm.14.4.5(b)], and hence a linear topological isomorphism.

We will therefore describe the topology on the Hardy-Sobolev space $H^{2N,2}(\mathbf{C}_{<b})$ by either the finite collection of norms from (9.2.2.1), or by the topology of L^2-convergence of holomorphic functions and their kth holomorphic derivatives for all $0 \leq k \leq N$, on all vertical lines strictly contained in $\mathbf{C}_{<b}$ (equivalently on a countable collection of vertical lines strictly contained in $\mathbf{C}_{<b}$ whose abscissas are a countable subset of

$(-\infty, b)$ with b as an accumulation point). We further refer the reader to Lemma D.3.1.2 and its proof, for a similar equivalence of Hausdorff locally convex topologies, resting upon the Asymptotic Cauchy Integral Formula from Theorem B.2.2.4).

Similar assertions hold on the topology of $H^{2N,2}(\mathbf{C}_{>b})$ and are obtained from the foregoing upon acting on the complex s-plane using the bi-holomorphic involution $s \mapsto -s$ (and applying the result to $-b$).

9.2.2.1. *The Hardy-Sobolev spaces $W_I^s(\mathbf{R}_{>0})$.* We will now be revisiting the Hardy-Sobolev spaces $H^{2N,2}(\mathbf{C}_{>b})$ and $H^{2N,2}(\mathbf{C}_{<b})$, allowing general real numbers $s \geq 0$ in place of the even integer degree $2N$, and allowing arbitrary vertical strips of base interval I, as opposed to the half-planes $\mathbf{C}_{>b}$ and $\mathbf{C}_{<b}$.

DEFINITION 9.2.2.1 (Sobolev W_I^s-spaces of $\mathbf{R}_{>0}$). Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number.

We define on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ the following norm:

$$\| \cdot \|_{I,s} : \phi \mapsto \sup_{a \leq b : [a,b] \subset I} \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2a}, t^{2b}\} (1 + |\log(t)|^2)^{s/2} \, \frac{\mathrm{d}t}{t} \right)^{1/2} \tag{9.2.2.3}$$

and denote by $W_I^s(\mathbf{R}_{>0})$ the completion of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ with respect to the norm $\| \cdot \|_{I,s}$. We refer to $W_I^s(\mathbf{R}_{>0})$ as the *Sobolev W_I^s-space* of $\mathbf{R}_{>0}$, and to the norm $\| \cdot \|_{I,s}$ as the Sobolev W_I^s-norm.

We check the following basic properties of the Sobolev W_I^s-norms:

LEMMA 9.2.2.2 (Sobolev W_I^s-norms - I). *Let $I \subset \mathbf{R}$ be a closed interval, $s \geq 0$ a real number. Then:*

(1) *The Sobolev W_I^s-norm on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is continuous on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ and it extends by continuity to $\mathcal{S}_J(\mathbf{R}_{>0})$ for any non-empty open interval $J \subset \mathbf{R}_{>0}$ containing I. The completion $W_I^s(\mathbf{R}_{>0})$ of $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ equals the completion of $\mathcal{S}_J(\mathbf{R}_{>0})$.*

(2) *If I is compact of the form $I = [a, b]$, for $a \leq b$ real numbers, then for any $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ we have:*

$$\|\phi\|_{I,s} = \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2a}, t^{2b}\} (1 + |\log(t)|^2)^{s/2} \, \frac{\mathrm{d}t}{t} \right)^{1/2}. \tag{9.2.2.4}$$

If s is an even integer, then the topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ generated by $\|\phi\|_{I,s}$ is equivalent to the metrizable topology generated by the finitely many continuous seminorms:

$$\|\phi\|_{M,\sigma,k} := \left(\int_{\sigma - i\infty}^{\sigma + i\infty} |\partial_z^k M(\phi)(z)|^2 \, |\mathrm{d}z| \right)^{1/2}, \ \sigma \in \{a, b\}, \ 0 \leq k \leq \left[\frac{s}{2} \right]. \tag{9.2.2.5}$$

(3) *Let $a, b \in \mathbf{R}$ be real numbers. If I is of the form:*

(a) *$I = [a, +\infty)$, then we have, for any $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$:*

$$\|\phi\|_{I,s} = \sup_{\sigma \geq a} \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2a}, t^{2\sigma}\} (1 + |\log(t)|^2)^{s/2} \, \frac{\mathrm{d}t}{t} \right)^{1/2}. \tag{9.2.2.6}$$

If s is an even integer, then for any sequence $\{c_k\}_{k \geq 0} \subset [a, +\infty)$ such that $\limsup_{k \geq 0} c_k = +\infty$ and $c_0 = a$, the topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ generated by $\|\phi\|_{I,s}$ is equivalent to the metrizable topology generated by the countable family of continuous seminorms:

$$\|\phi\|_{M,\sigma,k} := \left(\int_{\sigma - i\infty}^{\sigma + i\infty} |\partial_z^k M(\phi)(z)|^2 \, |\mathrm{d}z| \right)^{1/2}, \ \sigma \in \{c_k, k \geq 0\}, \ 0 \leq k \leq \left[\frac{s}{2} \right] \tag{9.2.2.7}$$

(b) *$I = (-\infty, b]$, then we have, for any $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$:*

$$\|\phi\|_{I,s} = \sup_{\sigma \leq b} \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2\sigma}, t^{2b}\} (1 + |\log(t)|^2)^{s/2} \, \frac{\mathrm{d}t}{t} \right)^{1/2} \tag{9.2.2.8}$$

If s is an even integer, then for any sequence $\{c_k\}_{k\geq 0} \subset (-\infty, b]$ such that $\liminf_{k\geq 0} c_k = -\infty$ and $c_0 = b$, the topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ generated by $\|\phi\|_{I,s}$ is equivalent to the metrizable topology generated by the countable family of continuous seminorms:

$$\|\phi\|_{M,\sigma,k} := \left(\int_{\sigma-i\infty}^{\sigma+i\infty} |\partial_z^k M(\phi)(z)|^2 \,|\mathrm{d}z|\right)^{1/2},\ \sigma \in \{c_k, k \geq 0\},\ 0 \leq k \leq \left[\frac{s}{2}\right]. \tag{9.2.2.9}$$

Proof. We have, for any $a, b \in \mathbf{R}$ such that $a < b$:

$$\left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2a}, t^{2b}\}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t}\right)^{1/2} \leq \left(\int_{[1,+\infty)} |\phi(t)|^2\, t^{\max\{2a,2b\}}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t}\right)^{1/2} \tag{9.2.2.10}$$

$$+ \left(\int_{(0,1)} |\phi(t)|^2\, t^{\min\{2a,2b\}}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t}\right)^{1/2} \tag{9.2.2.11}$$

for all $\phi \in \mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$.

(1) : The inequality in (9.2.2.10) implies continuity of $\|\cdot\|_{I,s}$ on $\mathcal{S}_J(\mathbf{R}_{>0})$ by Monotone Convergence, for any nonempty open interval J containing I. Since $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ is dense in $\mathcal{S}_J(\mathbf{R}_{>0})$ for any such J, the respective completions with respect to the normed topology induced by the norm $\|\cdot\|_{I,s}$ on each are isometric.

(2) : We note that we have, for any $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$:

$$\int_{\mathbf{R}_{>0}} |\phi(t)|^2 \max\{t^{2a}, t^{2b}\}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t} = \int_{[1,+\infty)} |\phi(t)|\, t^{\max\{2a,2b\}}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t} \tag{9.2.2.12}$$

$$+ \int_{(0,1)} |\phi(t)|\, t^{\min\{2a,2b\}}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t}. \tag{9.2.2.13}$$

For any two real numbers $a', b' \in \mathbf{R}$ satisfying $a < a' < b' < b$, we note that we have the inequalities:

$$1_{[1,+\infty)}(t)\, \max\{t^{2a'}, t^{2b'}\} \leq 1_{[1,+\infty)}(t)\, \max\{t^{2a}, t^{2b}\} \tag{9.2.2.14}$$

and:

$$1_{(0,1)}(t)\, \max\{t^{2a'}, t^{2b'}\} = 1_{(0,1)}(t)\, t^{\min\{2a',2b'\}} = 1_{(0,1)}(t)\, t^{2a'} \leq 1_{(0,1)}(t)\, t^{2a} = 1_{(0,1)}(t)\, \max\{t^{2a}, t^{2b}\}. \tag{9.2.2.15}$$

The inequalities in (9.2.2.14) and (9.2.2.15) hold for all $t \in \mathbf{R}_{>0}$ and give the inequality:

$$\max\{t^{2a'}, t^{2b'}\} \leq \max\{t^{2a}, t^{2b}\} \tag{9.2.2.16}$$

which in turn implies that (9.2.2.4) holds.

By Plancherel's Theorem, combined with [**Fol**, Thm.8.22(d)], we have:

$$\|\phi\|_{M,\sigma,k} = \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2\, t^{2\sigma} |\log(t)|^{2k}\,\frac{\mathrm{d}t}{t}\right)^{1/2}. \tag{9.2.2.17}$$

By Lemma 9.1.2.2, the metrizable topology generated on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by the finite collection of continuous seminorms $\|\cdot\|_{M,\sigma,k}$, for $\sigma \in \{a, b\}$ and $0 \leq k \leq [s/2]$, is equivalent to the metrizable topology generated by the two continuous seminorms:

$$\phi \mapsto \left(\int_{\mathbf{R}_{>0}} |\phi(t)|^2\, t^{2\sigma}(1+|\log(t)|^2)^{s/2}\,\frac{\mathrm{d}t}{t}\right)^{1/2},\ \sigma \in \{a, b\}. \tag{9.2.2.18}$$

It follows that such topology is at least as coarse as the topology generated on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by $\|\cdot\|_{I,s}$. By the estimates in (9.2.2.16), the two topologies are equivalent for I compact.

Both parts of (3) follow from an argument similar to the argument provided for (2). □

In analogy with Definition 9.2.2.1 and building on Lemma 9.2.2.2, we introduce the following:

DEFINITION 9.2.2.3 (Absolute Hardy-Sobolev space of degree s and base I). Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number. We define on $A\mathcal{S}(\mathbf{C})$ the following continuous norms:

$$u \mapsto \frac{1}{2\pi}\left(\int_{\sigma-i\infty}^{\sigma+i\infty} |\partial_z^k u(z)|^2 \, |\mathrm{d}z|\right)^{1/2} \tag{9.2.2.19}$$

as σ varies over a sequence $\{c_k\}_{k\geq 0} \subset \mathbf{R}$ such that the boundary of I is a set of accumulation points for $\{c_k\}_{k\geq 0}$. We denote by $\mathrm{HS}^s(\mathbf{C}_I)$ the completion of $A\mathcal{S}(\mathbf{C})$ with respect to the metrizable topology generated by the countable family of continuous norms from (9.2.2.19) as σ varies over $\{c_k\}_{k\geq 0}$. We refer to $\mathrm{HS}^s(\mathbf{C}_I)$ as the *absolute Hardy-Sobolev space of degree s and base I*.

PROPOSITION 9.2.2.4 (Fourier-Mellin transform on $W_I^s(\mathbf{R}_{>0})$). *Let $s \geq 0$ be a real number. Then:*

(1) *The topology on $A\mathcal{S}(\mathbf{C})$ generated by the countable family of continuous norms from* (9.2.2.19) *as σ varies over $\{c_k\}_{k\geq 0}$ does not depend on the choice of $\{c_k\}_{k\geq 0}$ and it is Hilbertizable.*

(2) *$\mathrm{HS}^s(\mathbf{C}_I) \subset \mathscr{O}(\mathbf{C}_{\mathrm{Int}(I)})$, and for every $u \in \mathrm{HS}^s(\mathbf{C}_I)$ and every $b \in \partial I$ finite, the limit:*

$$u_b(\tau) := \lim_{\sigma \to b, \sigma \in \mathrm{Int}(I)} u(\sigma + i\tau) \tag{9.2.2.20}$$

exists for almost all $\tau \in \mathbf{R}$ and it is an element of $W^{s,2}(\mathbf{R})$. If s is an even integer, then u_b is an element of $\mathrm{HS}_b^{s,2}(\mathbf{C})$, the b-line Hardy-Sobolev space.

(3) *Assume s is an even integer. Then, for any non-empty open interval J containing I, the Mellin transform:*

$$M : \mathcal{S}_J(\mathbf{R}_{>0}) \to A(\mathbf{C}_J) \tag{9.2.2.21}$$

from (D.3.2.2) *extends by continuity to a linear topological isomorphism:*

$$M : W_I^s(\mathbf{R}_{>0}) \to \mathrm{HS}^s(\mathbf{C}_I). \tag{9.2.2.22}$$

PROOF. Proposition 9.2.2.4 follows from Lemma 9.2.2.2 in the case I is a compact connected interval, and the general case reduces to this special case using Lemma 9.2.2.2 again and the Asymptotic Cauchy Integral Formula (see Theorem B.2.2.4).[1] □

COROLLARY 9.2.2.5 (Continuous point-evaluation functionals on Hardy-Sobolev spaces). *Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number. Let $z_0 \in \mathbf{C}_I$. Then:*

(1) *if $z_0 \in \mathbf{C}_{\mathrm{Int}(I)}$, then for any integer $k \geq 0$, the functional:*

$$u \mapsto (\partial_z^k u)(z_0) \tag{9.2.2.23}$$

is continuous.

(2) *If $\mathrm{Re}(z_0) \in \partial I$, then the functional:*

$$u \mapsto (\partial_z^k u)(z_0) \tag{9.2.2.24}$$

is continuous if and only if the integer $k \geq 0$ satisfies: for every integer

$$0 \leq k \leq \left[\frac{s-1}{2}\right].$$

PROOF. The proof of Corollary 9.2.2.5 mirrors the argument for Corollary 9.2.1.4, using Proposition 9.2.2.4.[2] □

[1] Return here.
[2] Return here.

THEOREM 9.2.2.6 (Paley-Wiener Theorem for Hardy-Sobolev Spaces - III). *Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ an even integer. Let $J \subset \mathbf{R}$ be any non-empty open interval containing I. We denote by M the Mellin transform $M : \mathcal{S}_J(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_J)$.*

We denote by:

$$\mathcal{S}_J(\mathbf{R}_{>0}, (1/a, +\infty)) \subset \mathcal{S}_J(\mathbf{R}_{>0}) \tag{9.2.2.25}$$

the linear subspace of those Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ of J-exponential decay, whose support is contained within the interval $(1/a, +\infty)$.

Then:

(1) *The closure of $\mathcal{S}_J(\mathbf{R}_{>0}, (1/a, +\infty))$ in the Sobolev space $W_I^s(\mathbf{R}_{>0})$ is the closed subspace:*

$$W_I^s(\mathbf{R}_{>0}, (1/a, +\infty)) \subset W_I^s(\mathbf{R}_{>0}) \tag{9.2.2.26}$$

obtained as the closed range of the idempotent multiplication operator $M_{\rho_{a,+}} : W_I^s(\mathbf{R}_{>0}) \to W_I^s(\mathbf{R}_{>0})$ obtained by multiplying by the indicator function $\rho_{a,+}(t) := 1_{[1/a,+\infty)}(t)$.

(2) *The closure $\mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty))$ of $M(\mathcal{S}_J(\mathbf{R}_{>0}, (1/a, +\infty)))$ in the Hardy-Sobolev space $\mathrm{HS}^s(\mathbf{C}_I)$ satisfies the following property: there exists a commutative diagram of continuous linear operators with linearly short exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{[1/a,1]}W_I^s(\mathbf{R}_{>0}) & \longrightarrow & M_{\rho_{a,+}}W_I^s(\mathbf{R}_{>0}) & \longrightarrow & M_{\rho_{1,+}}W_I^s(\mathbf{R}_{>0}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \\ 0 & \longrightarrow & \mathrm{PW}^s(\mathbf{C}_I, a) & \longrightarrow & \mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty)) & \longrightarrow & \mathrm{HS}^s(\mathbf{C}_{I'}) & \longrightarrow & 0 \end{array} \tag{9.2.2.27}$$

where $M_{[1/a,1]}$ is the multiplication by $1_{[1/a,1]}$ operator, the leftmost operators in (9.2.2.27) are continuous, with closed range, the rightmost are surjective and open, the vertical arrows are linear topological isomorphisms induced by the Mellin transform, and where:

(1) *$I' \subset \mathbf{R}$ is the non-empty closed interval $I' := (-\infty, \sup(I)]$.*

(2) *$\mathrm{PW}^s(\mathbf{C}_I, a)$ is the closed linear subspace of the Hardy-Sobolev space $\mathrm{HS}^s(\mathbf{C}_I)$ of those holomorphic functions $\phi(s)$ over $\mathbf{C}_{\mathrm{Int}(I)}$ that extend to entire functions and satisfy:*

$$\phi(s) = O\left(\frac{(1 - a^{-2\,\mathrm{Re}(s)})^{1/2}}{(\mathrm{Re}(s))^{1/2}}\right). \tag{9.2.2.28}$$

DEFINITION 9.2.2.7 (Paley-Wiener space of weight a, degree s, base I). Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ an even integer. Let $a > 1$ be a real number. We denote by $\mathrm{PW}^s(\mathbf{C}_I, a)$ the closed subspace of $\mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty))$ defined in Theorem 9.2.2.6(2). We refer to $\mathrm{PW}^s(\mathbf{C}_I, a)$ as the *Paley-Wiener space of weight a, degree s and base I.*

PROOF OF THEOREM 9.2.2.6. The proof of Theorem 9.2.2.6 proceeds similarly as the proof of Theorem 9.2.1.2.[3] □

COROLLARY 9.2.2.8. *Let $I \in \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ an even integer. Let $a > 1$ be a real number.*

Let $z_0 \in \mathbf{C}_{\leq \sup(I)}$. Then:

(1) *if $z_0 \in \mathbf{C}_{<\sup(I)}$, then for any integer $k \geq 0$, the functional:*

$$u \mapsto (\partial_z^k u)(z_0) \tag{9.2.2.29}$$

is continuous.

[3]Return here.

(2) *If* $\sup(I)$ *is finite and* $\mathrm{Re}(z_0) = \sup(I)$*, then the functional:*

$$u \mapsto (\partial_z^k u)(z_0) \tag{9.2.2.30}$$

is continuous if and only if the integer $k \geq 0$ *satisfies: for every integer*

$$0 \leq k \leq \left[\frac{s-1}{2}\right].$$

PROOF. This is a consequence of Corollary 9.2.2.5.[4] □

9.2.2.2. *The Hardy-Sobolev spaces* $W_I^s(G)$. We now return to the generality of G an $\mathbf{R}_{>0}$-modulated group with module $|\cdot|_G$ and compact subgroup of module-1 elements G^1 (i.e. $G^1 := \ker(|\cdot|_G)$). Recall that the Mellin transform is a linear topological isomorphism:

$$M : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I) \,\overline{\otimes}\, \mathcal{S}(G^1)$$

for any non-empty open interval $I \subset \mathbf{R}$, obtained as follows. We choose a unitary base-point χ for each connected component of the Riemann surface $\Omega(G)$, and for any $\phi \in \mathcal{S}_I(G)$, $M(\phi)$ sends ϕ to the element:

$$M(\phi)((s,\omega)) = \int_{\mathbf{R}_{>0}} t^{ix_0(\chi)+s} \int_{G^1} \phi(\tau(t)g)\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t}$$

with $\omega := \chi|_{G^1}$, with μ_1 is the volume-1 Haar measure on G^1, τ a fixed choice of continuous homomorphic section to the module $|\cdot|_G$, and $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0(\chi)}$ as smooth functions of $t \in \mathbf{R}_{>0}$.

DEFINITION 9.2.2.9 (Sobolev W_I^s-spaces of $\mathbf{R}_{>0}$-modulated groups). Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number. Let μ_G be the canonical Haar measure on G (i.e. the unique Haar measure normalized so as to have, for every real number $\lambda \geq 1$, $\mu_G(\{g \in G : |g|_G \in [1/\lambda, \lambda]\}) = 2\log(\lambda) + O(1)$ as $\lambda \to +\infty$).

We define on $\mathcal{S}_{\mathrm{E}}(G)$ the following norm:

$$\|\cdot\|_{I,s} : \phi \mapsto \sup_{a \leq b : [a,b] \subset I} \left(\int_G |\phi(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\}(1 + |\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g) \right)^{1/2} \tag{9.2.2.31}$$

and denote by $W_I^s(G)$ the completion of $\mathcal{S}_{\mathrm{E}}(G)$ with respect to the norm $\|\cdot\|_{I,s}$. We refer to $W_I^s(G)$ as the *Sobolev* W_I^s*-space* of G, and to the norm $\|\cdot\|_{I,s}$ as the Sobolev W_I^s-norm.

We check the following basic properties of the Sobolev W_I^s-norms in the general case of $\mathbf{R}_{>0}$-modulated groups G, as opposed to the special case $G = \mathbf{R}_{>0}$, which is discussed in Lemma 9.2.2.2:

LEMMA 9.2.2.10 (Sobolev W_I^s-norms - II). *Let* G *be a second-countable* $\mathbf{R}_{>0}$*-modulated group. Let* $I \subset \mathbf{R}$ *be a closed interval,* $s \geq 0$ *a real number. Then:*

(1) *The Sobolev* W_I^s*-norm on* $\mathcal{S}_{\mathrm{E}}(G)$ *is continuous on* $\mathcal{S}_{\mathrm{E}}(G)$ *and it extends by continuity to* $\mathcal{S}_J(G)$ *for any non-empty open interval* $J \subset \mathbf{R}$ *containing* I*. The completion* $W_I^s(G)$ *of* $\mathcal{S}_{\mathrm{E}}(G)$ *equals the completion of* $\mathcal{S}_J(G)$*. Moreover, if* s *is an even integer, then for any choice of continuous homomorphic section* τ *to the module* $|\cdot|_G$*, the pull-back map:*

$$W_I^s(G) \to W_I^s(\mathbf{R}_{>0}) \,\widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) \tag{9.2.2.32}$$

where μ_1 *is the volume-*1 *Haar measure on* G^1 *and* $\widehat{\mu}_1$ *is its Plancherel measure. The Mellin transform on* $\mathcal{S}_{\mathrm{E}}(G)$ *induces a linear topological isomorphism:*

$$W_I^s(G) \to \mathrm{HS}^s(\mathbf{C}_I) \,\widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) \tag{9.2.2.33}$$

[4]Return here.

where $\widehat{\mu}_1$ is the Plancherel measure of μ_1, and the topology on the σ-tensor product is equivalent to the metrizable topology generated by the finitely many continuous seminorms:

$$\|\cdot\|'_{I,s} \otimes_\sigma \|\cdot\|_{L^2(\widehat{G^1},\widehat{\mu}_1)}$$

where, for each $u \in \mathrm{HS}^s(\mathbf{C}_I)$, $\|u\|'_{I,\sigma,k} := \left(\int_{\sigma-i\infty}^{\sigma+i\infty} |\partial_z^k u(z)|^2 \,|\mathrm{d}z|\right)^{1/2}$ as σ varies over the countably many elements of a sequence $\{c_k\}_{k\geq 0} \subset \mathbf{R}$ such that the finite boundary points of ∂I are limit points of $\{c_k\}_{k\geq 0}$, and such that the integer $k \geq 0$ satisfies $0 \leq k \leq \left[\frac{s}{2}\right]$.

(2) *If I is compact of the form $I = [a,b]$, for $a \leq b$ real numbers, then for any $\phi \in \mathcal{S}_{\mathrm{E}}(G)$ we have:*

$$\|\phi\|_{I,s} = \left(\int_G |\phi(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\}(1+|\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g)\right)^{1/2}. \tag{9.2.2.34}$$

(3) *Let $a,b \in \mathbf{R}$ be real numbers. If I is of the form:*

(a) *$I = [a,+\infty)$, then we have, for any $\phi \in \mathcal{S}_{\mathrm{E}}(G)$:*

$$\|\phi\|_{I,s} = \sup_{\sigma\geq a}\left(\int_G |\phi(g)|^2 \max\{|g|_G^{2a}, t^{2\sigma}\}(1+|\log|g|_G|^2)^{s/2}\,\mu_G(g)\right)^{1/2}. \tag{9.2.2.35}$$

(b) *$I = (-\infty,b]$, then we have, for any $\phi \in \mathcal{S}_{\mathrm{E}}(G)$:*

$$\|\phi\|_{I,s} = \sup_{\sigma\leq b}\left(\int_G |\phi(g)|^2 \max\{|g|_G^{2\sigma}, |g|_G^{2b}\}(1+|\log|g|_G|^2)^{s/2}\,\mu_G(g)\right)^{1/2} \tag{9.2.2.36}$$

Proof. Upon proving (1), the rest of the Lemma follows from Lemma 9.2.2.2. Part (1) follows from the definition of the norm $\|\cdot\|_{I,s}$ combined with Theorem D.1.2.1.[5] □

Proposition 9.2.2.11 (Hardy-Sobolev duality - II). *Let G be an $\mathbf{R}_{>0}$-modulated group. Assume G is second-countable. Let μ_G be a choice of Haar measure on G. Let $I \subset \mathbf{R}$ be a closed non-empty interval, $s \geq 0$ a real number. Assume I is compact, of the form $I = [a,b]$, for $a < b$ real numbers.*

For every open interval $J \subset \mathbf{R}$ containing I, the pairing:

$$\langle\cdot,\cdot\rangle_{I,s} : \mathcal{S}_J(G) \times \mathcal{S}_J(G) \to \mathbf{C} \tag{9.2.2.37}$$

defined by the assignment:

$$\langle\phi,\psi\rangle_{I,s} := \int_G \phi(g)\psi(g)\,\mu_G(g) \tag{9.2.2.38}$$

is continuous and it extends to a continuous duality pairing:

$$\langle\cdot,\cdot\rangle_{I,s} : W_I^s(G) \times W_I^{-s}(G) \to \mathbf{C} \tag{9.2.2.39}$$

where $W_I^{-s}(G)$ is the completion of $\mathcal{S}_J(G)$ with respect to the norm:

$$\phi \mapsto \|\phi\|_{I,-s} := \left(\int_G |\phi(g)|^2\, w_{I,s}(g)^{-1}\,\mu_G(g)\right)^{1/2} \tag{9.2.2.40}$$

with $w_{I,s}(g)$ the function:

$$w_{I,s}(g) := \max\{|g|_G^{2a}, |g|_G^{2b}\}\,(1+|\log|g|_G|^2)^{s/2}. \tag{9.2.2.41}$$

In particular, the induced linear map:

$$T_{I,s} : W_I^s(G)'_\beta \to W_I^{-s}(G) \tag{9.2.2.42}$$

determined by (9.2.2.39), given explicitly by $T \mapsto w_{I,s}^{-1} \cdot T$.

[5] Return here.

The Proposition is proved in a manner analogous to the proof of Proposition 9.2.1.8, or by using Lemma 9.2.2.10(1).

COROLLARY 9.2.2.12. *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $a, s \in \mathbf{R}$ be real numbers, $a > 1, s \geq 0$. Let $I \subset \mathbf{R}$ be a non-empty closed interval. We denote by M the Mellin transform $M : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I) \overline{\otimes} \mathcal{S}(\widehat{G^1})$.*

(1) *For every point $z \in \mathbf{C}_{\mathrm{Int}(I)}$, the point-evaluation functional $\delta_z \otimes \delta_\omega$ and its derivatives $\delta_z^{(k)} \otimes \delta_\omega$, with $\omega \in \widehat{G^1}$ and δ_ω the point-mass at ω on the space of square-summable sequences on the countable discrete abelian group $\widehat{G^1}$, are continuous on $\mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty)) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1)$.*

For every $\phi \in M_{\rho_{a,+}} W_I^s(G)$, we have:

$$\delta_z^{(k)}(M(\phi)(s)) = \int_G \phi(g)(\log|g|_G)^m |g|_G^z \, \mu_G(g) \tag{9.2.2.43}$$

for each integer $k \geq 0$.

(2) *For any $z \in \mathbf{C}_I$ with $\mathrm{Re}(z) \in \partial I$, the point-evaluation functional $\delta_z \otimes \delta_\omega$ and its derivatives $\delta_z^{(k)} \otimes \delta_\omega$ of order $0 \leq k \leq \left[\frac{s-1}{2}\right]$, are continuous on $\mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty)) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1)$.*

PROOF. The Corollary follows from Lemma 9.2.2.10 combined with Corollary 9.2.2.5. □

THEOREM 9.2.2.13 (Paley-Wiener Theorem for Hardy-Sobolev Spaces - IV). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated abelian group. Let $a \in \mathbf{R}$ be a real number, $a > 1$. Let $I \subset \mathbf{R}$ be a non-empty closed interval. We denote by M the Mellin transform $M : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I) \overline{\otimes} \mathcal{S}(\widehat{G^1})$.*

We denote by:

$$\mathcal{S}_I(G, (1/a, +\infty)) \subset \mathcal{S}_I(G) \tag{9.2.2.44}$$

the linear subspace of those Schwartz-Bruhat functions on G of I-exponential decay, whose support is sent by the module homomorphism $|\cdot|_G$ into the interval $(1/a, +\infty) \subset \mathbf{R}_{>0}$.

Then:

(1) *The closure of $\mathcal{S}_{\mathrm{E}}(G, (1/a, +\infty))$ in $W_I^s(G)$ is the closed subspace obtained as the closed range of the idempotent multiplication operator $M_{\rho_{a,+}} : W_I^s(G) \to W_I^s(G)$ obtained by multiplying by the indicator function $\rho_{a,+}(g) := 1_{[1/a,+\infty)}(g)$, $g \in G$.*

(2) *The Mellin transform M extends by continuity to the linear topological isomorphism:*

$$M : W_I^s(G) \to \mathrm{HS}^s(\mathbf{C}_I) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) \tag{9.2.2.45}$$

where the completion of the algebraic tensor product with respect to the Hausdorff locally topology generated by the family of seminorms associated to the inner products:

$$\langle \phi_1 \otimes u_1, \phi_2 \otimes u_2 \rangle_\sigma := \langle \phi_1, \phi_2 \rangle_{\mathrm{HS}_b^s(\mathbf{C})} \langle u_1, u_2 \rangle_{L^2(\widehat{G^1}, \widehat{\mu}_1)}, \ b \in I \tag{9.2.2.46}$$

for $\phi_1, \phi_2 \in \mathrm{HS}^s(\mathbf{C}_I)$ and $u_1, u_2 \in L^2(\widehat{G^1}, \widehat{\mu}_1)$, with ϕ_1, ϕ_2 elements of $\mathrm{HS}_b^s(\mathbf{C}_I)$ by restriction to the vertical lines $b + i\mathbf{R}$ for each $b \in I$.

(3) *The closure of $M(\mathcal{S}_I(G, (1/a, +\infty)))$ in $\mathrm{HS}^s(\mathbf{C}_I) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1)$ satisfies the following property. There exists a commutative diagram of continuous linear operators with linearly short exact rows:*

(9.2.2.47)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{[1/a,1]} W_I^s(G) & \longrightarrow & M_{\rho_{a,+}} W_I^s(G) & \longrightarrow & M_{\rho_{1,+}} W_I^s(G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} & & \\
0 & \longrightarrow & \mathrm{PW}^s(\mathbf{C}_I, a) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) & \longrightarrow & \mathrm{HS}^s(\mathbf{C}_I, (1/a, +\infty)) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) & \longrightarrow & \mathrm{HS}^s(\mathbf{C}_I) \widehat{\otimes}_\sigma L^2(\widehat{G^1}, \widehat{\mu}_1) & \longrightarrow & 0
\end{array}$$

where $M_{[1/a,1]}$ is the multiplication by $1_{[1/a,1]}$ operator, the leftmost operators in (9.2.2.47) *are continuous, with closed range, the rightmost are surjective and open, the vertical arrows are linear topological isomorphisms induced by the Mellin transform.*

PROOF. Part (2) of the Theorem is Lemma 9.2.2.10(1). The remaining assertions follow from Theorem 9.2.2.6. □

9.3. Module structures over $\mathcal{D}(G)$ and approximate units

In this section we discuss, for G a second-countable $\mathbf{R}_{>0}$-modulated group, the continuous $\mathcal{D}(G)$-module structure on $W_I^s(G)$, with $\mathcal{D}(G)$ acting by convolution. We further construct, according to whether I is a single point or has non-empty interior, approximate units for $W_I^s(G)$, i.e. a net of continuous linear endomorphisms converging to the identity with respect to the topology of simple convergence.

9.3.1. Module structure over $\mathcal{D}(G)$. Let G be a Hausdorff locally compact abelian group. We recall that we denote by $\mathcal{D}(G)$ the convolution $\mathbf{C}$-algebra of smooth compactly supported test functions, equipped with its locally convex inductive limit topology as in [**Bru**, §4]. Similarly, $\mathcal{S}(G)$ is the convolution $\mathbf{C}$-algebra of Schwartz-Bruhat functions on G (see [**Bru**, §8] and Appendix D of this work).

For any $\varphi \in \mathcal{D}(G)$, we may regularize the continuous left-translation action ρ_G of G on $\mathcal{S}(G)$ and its closed G-invariant subspaces, to obtain an operator $A(\varphi)$. Such operator may extend by continuity to completions of $\mathcal{S}(G)$ with respect to coarser topologies.

For G a second-countable $\mathbf{R}_{>0}$-modulated group, G acts continuously by left-translation on $\mathcal{S}_I(G)$, for $I \subset \mathbf{R}$ any non-empty open interval. We denote by $\rho_G : G \times \mathcal{S}_I(G) \to \mathcal{S}_I(G)$ such action, i.e. $\rho_G(g,\phi)(x) := \phi(g^{-1}x)$ as functions of $x \in G$. The same construction as in the foregoing defines a module structure of $\mathcal{D}(G)$ on $\mathcal{S}_I(G)$ which is continuous, giving the convolution operator $A(\varphi)$ associated to each $\varphi \in \mathcal{D}(G)$.

The operator $A(\varphi)$ may extend by continuity to completions of $\mathcal{S}_I(G)$ with respect to coarser topologies. This is the subject of study of this subsection: we discuss herein the convolution action of G on $W_I^s(G)$ for any real number $s \geq 0$.

DEFINITION 9.3.1.1. Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let μ_G be a choice of Haar measure on G. Let I be a non-empty closed interval, $s \geq 0$ a real number

We define the $\mathcal{D}(G)$-*module structure* on $E := \mathcal{S}_I(G)$, respectively $E := W_I^s(G)$, to be the linear map:

$$A : \mathcal{D}(G) \to \mathscr{L}_\beta(E) \tag{9.3.1.1}$$

assigned by, for each $\varphi \in \mathcal{D}(G)$:

$$A(\varphi) := \int_G \varphi(g)\rho_G(g,\cdot)\,\mathrm{d}\mu_G(g) \tag{9.3.1.2}$$

where the integral is in the sense of Pettis. We further refer to $A(\varphi)$ as the φ-*regularized action* of G by continuous left-translation on E.

REMARK 9.3.1.2. Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, $E := \mathcal{S}_I(G), W_I^s(G)$ respectively, for I a non-empty closed interval. Since φ is compactly supported, we deduce that indeed A is valued in $\mathscr{L}(E)$, the $\mathbf{C}$-algebra of continuous endomorphisms of E, by Theorem C.4.3.5, since E is complete (see Theorem D.3.3.7(2) for the case $E = \mathcal{S}_I(G)$), hence quasi-complete.

LEMMA 9.3.1.3. *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I \subset \mathbf{R}$ be a closed non-empty interval, $s \geq 0$ be a real number. Then the linear operators A from* (9.3.1.1) *from $\mathcal{D}(G)$ to $\mathscr{L}(\mathcal{S}_I(G))$, and to $\mathscr{L}(W_I^s(G))$, respectively, is continuous with respect to the locally convex inductive limit topology on the source, and the topology of uniform convergence on bounded subsets on the target.*

PROOF. By design of the locally convex inductive limit topology on $\mathcal{D}(G)$, it is enough to pre-compose A with the linear topological embedding $\mathcal{D}_K(G,k) \to \mathcal{D}(G)$ for $K \subset G$ a compact subset with non-empty interior, $k \subset G^1$ a compact subgroup with compactly generated orthogonal $k^\perp \subset \widehat{G}$ such that $k \cdot K \subset K$, and with $\mathcal{D}_K(G,k)$ the linear subspace of $\mathcal{D}(G)$ of those k-invariant smooth compactly supported functions whose support is contained within K.

The linear space $\mathcal{D}_K(G,k)$ is topologized by transport of structure under the linear isomorphism $\mathcal{D}(G,k) \simeq \mathcal{D}(G/k)$ given by composition with the continuous quotient homomorphism $G \to G/k$.

Let $\{\varphi_i\}_{i\in I}$ be a net of test functions in $\mathcal{D}_K(G,k)$ converging to $\varphi \in \mathcal{D}_K(G,k)$.

We now consider the net of vector valued continuous functions $\{\alpha_i : K \to E\}_{i\in I}$, for $E := \mathscr{L}_\beta(\mathcal{S}_I(G))$:

$$\alpha_i(g) := \varphi_i(g)\rho_G(g,\cdot),\ g \in G.$$

Since $\varphi_i \to \varphi$ uniformly as continuous functions on K, and since K is compact, then the convergence of φ_i to φ is also in the L^1-topology on K with respect to μ_G.

To check that $\alpha_i \to \alpha$ uniformly, we compose with the operator $M : \mathcal{S}_I(G) \to A\mathcal{S}(\mathbf{C}_I) \overline{\otimes} \mathcal{S}(\widehat{G^1})$, the Mellin transform from Theorem D.3.3.13, which gives, upon calling $\alpha := \varphi\rho_G$, $M(\alpha_i(f))(c) = M(\varphi_i)(c)M(f)(c)$ and $M(\alpha(f))(c) = M(\varphi)(c)M(f)(c)$ as holomorphic functions on $\Omega(G)$, for all $f \in \mathcal{S}_I(G)$.

If $B \subset \mathcal{S}_I(G)$ is a bounded subset then $M(\alpha_i(f))(c) \to M(\alpha(f))(c)$ uniformly as $f \in B$, because $M(\alpha(f))(c) = M(\varphi)(c)M(f)(c)$, and $M(\varphi_i)(c) \to M(\varphi)(c)$ by $L^1(G, |\cdot|_G^{\sigma(c)}\mu_G)$-convergence of φ_i to φ.

By Lemma C.4.3.12, we have $A(\varphi_i) \to A(\varphi)$ in E.

Upon choosing $E = W_I^s(G)$, the same argument as in the foregoing applies. □

REMARK 9.3.1.4. Let F be a number field, $G = C_F$ modulated by the idelic norm $|\cdot|_F$. Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number. We denote by $W_I^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ the closure of the closed subspace of global adelic θ-functions $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ into $\mathcal{S}_{\mathrm{E}}(C_F)$. This is a closed invariant subspace of $W_I^s(C_F)$, and so the convolution action A of $\mathcal{D}(C_F)$ on $\mathcal{S}_{\mathrm{E}}(C_F)$ and $\mathcal{S}_I(C_F)$ extends uniquely by continuity to a continuous linear map:

$$A : \mathcal{D}(C_F) \to \mathscr{L}_\beta(W_I^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)))$$

upon observing that it induces by composition a continuous operator:

$$A : \mathcal{D}(C_F) \to \mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)), W_I^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)))$$

and, using the Fourier transform over C_F, we deduce that A is valued in the image of the continuous linear operator from $\mathscr{L}_\beta(W_I^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)))$.

Let S be a finite set of places of F containing all the Archimedean places. Then the space of S-local adelic θ-functions $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ itself carries a continuous $\mathcal{D}(J_{F,S})$-module structure. It is given by the linear map:

$$A_S : \mathcal{D}(J_{F,S}) \to \mathscr{L}_\beta(\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))) \tag{9.3.1.3}$$

assigned by, for each $\varphi \in \mathcal{D}(J_{F,S})$:

$$A_S(\varphi) := \int_{J_{F,S}} \varphi(g)\rho_{F,S}(g,\cdot)\,\mathrm{d}\overline{\mu}_S(g) \tag{9.3.1.4}$$

where the integral is in the sense of Pettis, and the same argument from Lemma 9.3.1.3 implies its continuity by the Second Structure Theorem (see Theorem 5.4.1.1), which reduces the continuity assertion to the case of the convolution action of $\mathcal{D}(J_{F,S})$ on $\mathcal{S}_{(0,+\infty)}(J_{F,S})$.

Let G be a second-countable $\mathbf{R}_{>0}$-modulated group.

The continuous left-translation action ρ_G of G on $W_I^s(G)$ is not unitary: it grows at most exponentially (where polynomial and ultra-polynomial growth, in the multiplicative sense, i.e. in the sense of functions

over the Hausdorff locally compact abelian group G, is defined in terms of the function $g \mapsto |\log|g|_G|$, which measures rapid decay over G).

LEMMA 9.3.1.5. *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I = [a,b] \subset \mathbf{R}$ be a compact non-empty interval, $s \geq 0$ a real number. The function:*

$$f : G \to \mathscr{L}_\beta(W_I^s(G)) \tag{9.3.1.5}$$

defined as $f(g) := \rho_G(g,\cdot)$, satisfies, for any continuous seminorm p on $\mathscr{L}_\beta(W_I^s(G))$:

$$p(f(g)) \leq C_p \max\{|g|_G^a, |g|_G^b\} |\log|g|_G|^{s/2} \tag{9.3.1.6}$$

for a positive real number C_p depending on p and on G, I and s. Moreover, $p(f)$ is μ_G-measurable as a real-valued function on G.

In particular, for $I = \{b\}$, a single point, then the operator norm of $\rho_G(g,\cdot)$ as g varies over the second-countable $\mathbf{R}_{>0}$-modulated group G, grows at most polynomially over G.

PROOF. We first treat the case $I = \{b\}$, i.e. $a = b$. We choose p the operator norm.

We have:

$$\begin{aligned}
\|f(g)\| &= \sup_{\|u\|_{s,2}=1} \left(\int_G |u(g^{-1}h)|^2 (1 + |\log|h|_G|^2)^{s/2} \, \mathrm{d}\mu_G(h) \right)^{1/2} \\
&= \sup_{\|u\|_{s,2}=1} \left(\int_G |u(h)|^2 (1 + |\log|g|_G + \log|h|_G|^2)^{s/2} \, \mathrm{d}\mu_G(h) \right)^{1/2} \\
&\leq \sup_{\|u\|_{s,2}=1} \left(\int_G |u(h)|^2 (1 + |\log|g|_G|^2 + 2|\log|g|_G||\log|h|_G| + |\log|h|_G|^2)^{s/2} \, \mathrm{d}\mu_G(h) \right)^{1/2} \\
&\leq C_0 \sup_{\|u\|_{s,2}=1} \left(\|u\|_{0,2}^2 + |\log|g|_G|^s \|u\|_{0,2}^2 + \int_G |u(h)|^2 |1 + \log|h|_G|^2|^{s/2} \, \mathrm{d}\mu_G(h) + C_1(|\log|g|_G|^{s'}) \right)^{1/2}
\end{aligned}$$

for $s' < s$ and the constant C_1 being absolute, as a consequence of (9.1.1.4) from Proposition 9.1.1.3, and for an absolute constant $C_0 > 0$ depending only on s.

By (9.1.1.4) from Proposition 9.1.1.3, the last term in the above chain of inequalities is bounded above by:

(9.3.1.7)

$$C_0 \sup_{\|u\|_{s,2}=1} \left(C_2 + |\log|g|_G|^s C_2 + 1 + C_1(|\log|g|_G|^{s'}) \right)^{1/2} = C_0 \left(C_2 + |\log|g|_G|^s C_2 + 1 + C_1(|\log|g|_G|^{s'}) \right)^{1/2}$$

where C_2 is provided by (9.1.1.4) from Proposition 9.1.1.3 and depends on G.

We may write G as the countable increasing union of the compact subsets $\tau([1/N, N]) \cdot G^1$, where τ is a choice of continuous homomorphic section to the module $|\cdot|_G$, and G^1 is the compact subgroup of module-1 elements.

We may then write, for each $g \in G$ and $u \in W_I^s(G)$ with $\|u\|_{s,2} = 1$:

$$\|\rho_G(g,u)\|_{s,2} = \lim_{N \to +\infty} \left(\int_{\tau([1/N,N]) \cdot G^1} |u(g^{-1}h)|^2 (1 + |\log|h|_G|^2)^{s/2} \, \mathrm{d}\mu_G(h) \right)^{1/2}$$

and since each of the functions of g:

$$g \mapsto \left(\int_{\tau([1/N,N]) \cdot G^1} |u(g^{-1}h)|^2 (1 + |\log|h|_G|^2)^{s/2} \, \mathrm{d}\mu_G(h) \right)^{1/2}$$

is continuous over G, we deduce that $\|\rho_G(g,u)\|_{s,2}$ is a μ_G-measurable function of $g \in G$ for each $u \in W_I^s(G)$ of norm one (see [**Ru-2**, p.15,Corollaries,(a)]).

Since $\|\rho_G(g,\cdot)\|$ is realized as the supremum over a countable subset $\{u_m\}_{m\geq 0}$ of the unit ball in $W_I^s(G)$ of the values $\|\rho_G(g,u_m)\|_{I,s}$, where $\{u_m\}_{m\geq 0}$ can be chosen to be an orthonormal basis for the Hilbert space $W_I^s(G)$, independent of $g\in G$, we see that $\|\rho_G(g,\cdot)\|$, as a function of $g\in G$, is the supremum of a sequence of μ_G-measurable functions, hence itself μ_G-measurable (see [**Ru-2**, p.14,Thm.1.14]).

The general case follows from a similar argument, choosing the continuous seminorm p to be the operator norm:

$$p(T) := \sup_{\phi\in\mathscr{L}_\beta(W_I^s(G)):\|\phi\|_{I,s}\leq 1} \|T(\phi)\|_{I,s}$$

and using Lemma 9.2.2.10(2). □

9.3.2. Approximate units. Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I\subset\mathbf{R}$ be a closed non-empty interval, $s\geq 0$ a real number.

Within $W_I^s(G)$, we have the ability to approximate the identity with respect to the strong operator topology, by means of special continuous operators. We record this approximation result for later use.

9.3.2.1. *The case $I=\{b\}$.* The following Lemma is a consequence of [**Co**, Lemma 5].

LEMMA 9.3.2.1 (Bounded Approximation of the identity). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, μ_G a choice of Haar measure on G. Let $b,s\in\mathbf{R}$ be real numbers, $s\geq 0$. The continuous linear operator A from* (9.3.1.1) *with target $\mathscr{L}_\beta(W^s_{\{b\}}(G))$, extends uniquely to a continuous linear operator*

$$A : \mathcal{S}(G)\to\mathscr{B}_{\mathrm{NT}}(W^s_{\{b\}}(G)). \tag{9.3.2.1}$$

For each continuous character $\omega\in\widehat{G^1}$, we choose a continuous character $\chi\in\widehat{G}$ such that $\chi|_{G^1}=\omega$.

There exists a countable subset $B=\{\varphi_{m,\omega}\}_{(m,\omega)\in\mathbf{N}\times\widehat{G^1}}\subset C^\infty(G)$ of smooth functions, such that

$$|\cdot|_G^b\cdot B\subset\mathcal{S}(G)$$

(i.e. $B\subset\mathcal{S}_I(G)$ for $I=\{b\}$) satisfying the following properties:

1. *$A(B)$ is norm-bounded by 2 in $W^s_{\{b\}}(G)$.*
2. *For each $(m,\omega)\in\mathbf{N}\times\widehat{G^1}$, $\varphi_{m,\omega}$ is ω-homogeneous.*
3. *For each integer $m\geq 0$, the Fourier transform of $|\cdot|_G^b\,\varphi_{m,\omega}\in B$ is supported over the compact subset of $\widehat{G}$ of characters of the form $\chi|\cdot|_G^{ix}$ for $x\in[-m,m]$.*
4. *The closure of $A(B)$ with respect to the strong operator topology on $\mathscr{L}(W^s_{\{b\}}(G))=\mathscr{B}(W^s_{\{b\}}(G))$, contains the identity.*

We note that $\mathscr{B}_{\mathrm{NT}}(W^s_{\{b\}}(G))$, the $\mathbf{C}$-algebra of continuous $\mathbf{C}$-linear endomorphisms of the Hilbert space $W^s_{\{b\}}(G)$ topologized with the operator-norm topology (see Remark C.2.1.8), is $\mathscr{L}_\beta(W^s_{\{b\}}(G))$ as per (9.3.1.1).

REMARK 9.3.2.2. We note that the nomenclature "Bounded Approximation" does not refer to the BAP, the Bounded Approximation Property, for Hausdorff locally convex topological $\mathbf{C}$-vector spaces. All normed spaces in this Chapter will be separable Hilbert spaces, and they have the Metric Approximation Property (see [**Scha**, Ch.III,§9]), since every countable orthonormal topological basis is a Schauder basis with basis constant (the supremum of the operator-norm of the projectors onto the line spanned by each basis element) equal to one.

PROOF. The linear operator $\phi\mapsto|\cdot|_G^b\phi$ induces a linear isometry $W^s_{\{b\}}(G)\to W^{s,2}(G)$. Similarly, $\phi\mapsto|\cdot|_G^b\phi$ induces a linear topological isomorphism $\mathcal{S}(G)\to\mathcal{S}_{\{b\}}(G)$. It is therefore enough to treat the case $b=0$. Let $\varphi\in\mathcal{S}(G)$ be a Schwartz-Bruhat function. We define the assignment:

$$A(\varphi) : u\mapsto A(\varphi)(u) := \int_G \varphi(g)\rho_G(g,u)\,\mathrm{d}\mu_G(g) \tag{9.3.2.2}$$

on each element $u \in W^s_{\{b\}}(G)$. Upon defining the Radon measure $\mu := |\varphi|\,\mu_G$ on G, a Hausdorff locally compact topological space, μ is finite Radon on G, and the function $g \mapsto \rho_G(g,u)$ is weakly measurable on G for every $u \in W^{s,2}(G)$ (see Definition C.4.2.1), since it is continuous, being the left-translation action of G on $W^{s,2}(G)$ continuous with respect to the strong operator topology.

The function $g \mapsto \|\rho_G(g,\cdot)\|$, the norm being the operator norm on $\mathscr{B}(W^{s,2}(G))$, is μ- measurable and μ-integrable, since, by Lemma 9.3.1.5, it is bounded above by the function $C\,|\log|g|_G|^s$ for a positive absolute constant C, and φ is Schwartz-Bruhat over G, and hence decays faster than any positive power of $|\log|g|_G|^{-1}$ as $|\log|g|_G| \to +\infty$.

From Lemma C.4.3.11 applied to $E := W^{s,2}(G)$, to our choice of μ dependent on φ, and to the function $f : g \mapsto \rho_G(g,\cdot)$, valued in $\mathscr{B}(E)$, we deduce that $A(\varphi)$ from (9.3.2.2) is a bounded $\mathbf{C}$-linear endomorphism on $W^{s,2}(G)$.

The assignment $\varphi \mapsto A(\varphi)$ defined in (9.3.2.2) therefore provides the desired extension of A from (9.3.1.1) to the Schwartz-Bruhat functions $\mathcal{S}(G)$ on G, from $\mathcal{D}(G)$ (i.e. it extends A from (9.3.1.1) from $\mathcal{D}(G)$ to the larger domain $\mathcal{S}(G)$).

We now show that $A : \mathcal{S}(G) \to \mathscr{B}_{\mathrm{NT}}(W^{s,2}(G))$ is continuous (with respect to the operator-norm topology on the target). This follows from Lemma C.4.3.9: indeed, the continuous norm on $\mathscr{B}_{\mathrm{NT}}(W^{s,2}(G))$ given by the operator-norm $\|\cdot\|$ satisfies:

$$\int_G \|\rho_G(g,\cdot)\||\varphi(g)|\,\mu_G(g) \leq C\int_G |\varphi(g)||\log|g|_G|^s\,\mathrm{d}\mu_G(g)$$

for a positive absolute constant C. Since the function $f(g) := \int_G |\varphi(g)||\log|g|_G|^s\,\mathrm{d}\mu_G(g)$ is itself a continuous seminorm on $\mathcal{S}(G)$, we deduce from Lemma C.4.3.9 that A is continuous with respect to the Schwartz-Bruhat topology on the source, and the operator-norm topology on the target.

Let $\varepsilon > 0$ be a fixed real number such that $-1+\varepsilon < 1-\varepsilon$, and $\phi_0 \in C_c^\infty(\mathbf{R})$ be a smooth cut-off function supported on $[-1,1]$, such that ϕ_0 is bounded above by 1, and equal to 1 over the interval $[-1+\varepsilon, 1-\varepsilon]$. Such ϕ_0 exists by [**Horm**, Ch.I,Thm.1.4.1,p.25].

Let $\alpha_0 \in \mathcal{S}(\mathbf{R}_{>0})$ be the inverse Fourier transform of ϕ_0, i.e.:

$$\alpha_0(t) := \int_{-1}^{1} \phi_0(x)t^{2\pi i x}\,\mathrm{d}x.$$

We define, for each integer $m \geq 0$:

$$\beta_m(t) := m\alpha_0(t^m). \tag{9.3.2.3}$$

For every integer $m > 0$, β_m is a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ such that the Fourier transform $\widehat{\beta_m}$ is supported on $[-m,m]$. Indeed, we have

$$\int_{\mathbf{R}_{>0}} \beta_m(t)t^{-2\pi i x}\,\frac{\mathrm{d}t}{t} = m\int_{-\infty}^{+\infty} \alpha_0(e^{my})e^{-2\pi i x y}\,\mathrm{d}y = \int_{-\infty}^{+\infty} \alpha_0(e^y)e^{-2\pi i x y/m}\,\mathrm{d}y = \phi_0(x/m)$$

which is supported over $[-m,m]$.

We now define, for each $\omega \in \widehat{G^1}$:

$$\varphi_{m,\omega} := \varphi_m \otimes \psi_\omega \tag{9.3.2.4}$$

where ψ_ω is any choice of smooth compactly supported function on G^1 whose uniform norm is bounded above by $1/\mu_1(G^1)$, for μ_1 the unique choice of Haar measure on G^1 such that $\mathrm{d}t/t$ is the push-forward of the quotient measure induced as per Theorem D.1.2.1 by the choices μ_G on G and μ_1 on G^1, and which is ω-homogeneous.

We have:

$$\|A(\varphi_{m,\omega})\| = \sup_{\|u\|_{s,2}=1} \left(\int_G \left| \int_G \varphi_{m,\omega}(g) u(g^{-1}h)\, \mathrm{d}\mu_G(g) \right|^2 (1+|\log|h|_G|^2)^{s/2}\, \mathrm{d}\mu_G(h) \right)^{1/2}$$

$$= \sup_{\|u\|_{s,2}=1} \left(\int_G \left| \int_G \left\{ \frac{\varphi_{m,\omega}(g)}{(1+|\log|g|_G|^2|^{s/2}} \right\} u(g^{-1}h)\, (1+|\log|g|_G|^2)^{s/2} \mathrm{d}\mu_G(g) \right|^2 (1+|\log|h|_G|^2)^{s/2}\, \mathrm{d}\mu_G(h) \right)^{1/2}.$$

We now call

$$\phi(g) := \left\{ \frac{\varphi_{m,\omega}(g)}{(1+|\log|g|_G|^2|^{s/2}} \right\} \tag{9.3.2.5}$$

a continuous, compactly supported function of $g \in G$, and $\nu := (1+|\log|\cdot|_G|^2)^{s/2}\, \mu_G$, a regular Radon measure on G.

We then have:

(9.3.2.6)

$$\left(\int_G \left| \int_G \left\{ \frac{\varphi_{m,\omega}(g)}{(1+|\log|g|_G|^2|^{s/2}} \right\} u(g^{-1}h)\, (1+|\log|g|_G|^2)^{s/2} \mathrm{d}\mu_G(g) \right|^2 (1+|\log|h|_G|^2)^{s/2}\, \mathrm{d}\mu_G(h) \right)^{1/2} = \|\phi * u\|_{L^2(G,\nu)}.$$

The rightmost side of (9.3.2.6) is, by [**HR-1**, Cor.20.14,p.293], bounded from above as:

$$\|\phi * u\|_{L^2(G,\nu)} \leq \|\phi\|_{L^1(G,\nu)}\, \|u\|_{L^2(G,\nu)} = \|\phi\|_{L^1(G,\nu)}\, \|u\|_{s,2}. \tag{9.3.2.7}$$

Moreover, by design of ϕ from (9.3.2.5), we have:

$$\|\phi\|_{L^1(G,\nu)} = \|\varphi_{m,\omega}\|_{L^1(G,\overline{\mu})} \leq \int_{\mathbf{R}_{>0}} |\beta_m(t)|\, \frac{\mathrm{d}t}{t} \|\psi_\omega\|_{L^1(G^1,\mu_1)} \leq \int_{\mathbf{R}_{>0}} |\beta_m(t)|\, \frac{\mathrm{d}t}{t} = \|\alpha_0\|_{L^1(\mathbf{R}_{>0},\mathrm{d}t/t)}. \tag{9.3.2.8}$$

Since we choose α_0 to be the inverse Fourier transform of ϕ_0, which is in turn supported over $[-1,1]$ and bounded above by 1, we see that we have $\|\alpha_0\|_{L^2(\mathbf{R}_{>0},\mathrm{d}t/t)} \leq 2$.

We deduce that we have, for every $(m,\omega) \in \mathbf{N} \times \widehat{G^1}$:

$$\|A(\varphi_{m,\omega})\| \leq 2. \tag{9.3.2.9}$$

We have therefore shown that the net $B := \{\varphi_{m,\omega}\}_{(m,\omega)\in\mathbf{N}\times\widehat{G^1}}$ fulfills (1),(2) and (3) from the statement of the Lemma.

Since, by Lemma 4.3.2.2, the operators $\{1/\overline{\mu}_1(G^1) P_{F,\omega}\}_{\omega\in\widehat{G^1}}$ are projectors and the closure of their convex hull within $\mathscr{B}_{\mathrm{SOT}}(W^{s,2}(G))$ contains the identity, in order to show that B fulfills (4) it is enough to show that the net $\{\varphi_{m,\omega}\}_{m\geq 0}$, for each fixed ω, of ω-homogeneous Schwartz-Bruhat functions on G is such that the identity is in the closure in $\mathscr{B}_{\mathrm{SOT}}(W^{s,2}(G)(\omega))$ of $A(\{\varphi_{m,\omega}\}_{m\geq 0})$.

In turn, it is enough to show that the identity in $\mathscr{B}_{\mathrm{SOT}}(L^2(\mathbf{R}_{>0}, (1+|\log(t)|^2)^{s/2}\, \frac{\mathrm{d}t}{t}))$ is in the closure of the linear span of the sequence of operators:

$$u \mapsto T(\beta_m)(u) := \int_{\mathbf{R}_{>0}} \beta_m(t) u(t^{-1}x)\, \frac{\mathrm{d}t}{t}, x \in \mathbf{R}_{>0}.$$

Since the identity in $\mathscr{B}_{\mathrm{SOT}}(L^2(\mathbf{R}_{>0}, (1+|\log(t)|^2)^{s/2}\, \frac{\mathrm{d}t}{t}))$ is in the closure of the convex hull of the orthogonal projectors defined by multiplication by $\mathcal{F}^{-1} \circ M_{1_{[-N,N]}} \circ \mathcal{F}$, for $\mathcal{F}$ the Fourier transform on $L^2(\mathbf{R}_{>0}, \mathrm{d}t/t)$, and $N \geq 1$ an integer, then it is enough to show that for any integer $N \geq 1$ and for any $u \in L^2(\mathbf{R}_{>0}, (1+|\log(t)|^2)^{s/2}\, \frac{\mathrm{d}t}{t})$ whose Fourier transform is supported over $[-N,N]$, then $T(\beta_m)(u) \to u$ as $m \to +\infty$ in $\|\cdot\|_{L^2(\mathbf{R}_{>0},(1+|\log(t)|^2)^{s/2}\, \frac{\mathrm{d}t}{t})}$-norm. This holds by Dominated Convergence and uniform boundedness (by 2) of the operator norms $\|T(\beta_m)\|$ for every $m \geq 1$. □

We record the following consequence of Lemma 9.3.2.1:

COROLLARY 9.3.2.3. *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group, μ_G a choice of Haar measure on G. Let $b, s \in \mathbf{R}$ be real numbers, $s \geq 0$.*

(1) *Let $\alpha, \beta \in \mathbf{R}$ be such that $\alpha < \beta$, and $\chi \in \widehat{G}$ a continuous character. Let $\varepsilon > 0$ be a real number such that $\alpha + \varepsilon < \beta - \varepsilon$. There exists $u \in W^{s,2}(G)$ such that the Fourier transform $\widehat{u}$ is smooth, compactly supported over the subset $\chi|\cdot|_G^{i[\alpha,\beta]}$ of $\widehat{G^1}$, and is identically 1 over the subset $\chi|\cdot|_G^{i[\alpha+\varepsilon,\beta-\varepsilon]}$.*
(2) *The set of those elements $u \in W^{s,2}(G)$ such that $\widehat{u}$ is compactly supported, is dense in $W^{s,2}(G)$.*

PROOF. (1) : The indicator function $1_{\{\chi|_{G^1}\}}$ of the subset $\{\chi|_{G^1}\} \subset \widehat{G^1}$ is continuous and compactly supported, being $\widehat{G^1}$ discrete by [**DE**, Thm.3.2.1], as G^1 is compact. As such, its inverse Fourier transform F is continuous, and hence continuous and compactly supported.

Let τ be a continuous homomorphic section to the idelic norm. We define x_0 to be the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ for all $t \in \mathbf{R}_{>0}$. We let ϕ_0 be a smooth cut-off function on $\mathbf{R}$ whose support is contained in $x_0 + [\alpha, \beta] \subset \mathbf{R}$ and such that $\phi_0|_{x_0+[\alpha+\varepsilon,\beta-\varepsilon]} = 1$. Such ϕ_0 exists by [**Horm**, Ch.I,Thm.1.4.1,p.25].

We let φ be the inverse Fourier transform on $\mathbf{R}_{>0}$ of ϕ_0. The function $u := \varphi \otimes F$ is an element of $W^{s,2}(G)$ for any $s \geq 0$, and $\widehat{u}$ is supported only over $\chi|\cdot|_G^{i[\alpha,\beta]}$, and it is identically 1 over the subset $\chi|\cdot|_F^{i[\alpha+\varepsilon,\beta-\varepsilon]}$.

(2) : Let $u \in W^{s,2}(G)$ be an arbitrary element. By Lemma 9.3.2.1, there exists a countable net $B \subset \mathcal{S}(G)$ such that $A(B)$ is norm-bounded in $\mathscr{B}(W^{s,2}(G))$, the Fourier transform of each element of B is compactly supported, and the identity is contained in the closure of $A(B)$ with respect to the strong operator topology. In particular, for any $\varepsilon > 0$ there exists $b \in B$ such that $\|A(b)(u) - u\|_{s,2} < \varepsilon$. Since $\widehat{A(b)(u)} = \widehat{b}\widehat{u}$, and $\widehat{b}$ is compactly supported over $\widehat{G}$, then so is $\widehat{A(b)(u)}$. □

9.3.2.2. *The general case: approximate units in $W_I^s(G)$.* In this subsection we construct approximate units in $\mathscr{L}_\sigma(W_I^s(G))$ using a divisor on the Riemann surface $\Omega(G)$ supported on a vertical strip of bounded width contained in a single connected component.

We begin with the following:

LEMMA 9.3.2.4. *Let G be an $\mathbf{R}_{>0}$-modulated group. Let $\chi \in \widehat{G}$ be a fixed continuous character. Let $I \subset \mathbf{R}$ be a compact interval with non-empty interior.*

Let $\phi \in \mathcal{S}_{\mathrm{E}}(G)$. Let $T \in \mathbf{R}$ be a real number. Then the number of zeros of $M(\phi)(\chi|\cdot|_G^z)$ contained in the box $I \times i[T, T+1]$ is $O(\log|T|)$, where the implied constant depends on ϕ and I.

PROOF. Let $a \in \mathbf{R}$ be a real number such that $M(\phi)(\chi|\cdot|_G^z)$ has no zeros on the boundary of $\mathbf{C}_I$ and $a + i\mathbf{R}$ is not contained in $\mathbf{C}_I$. We define $z_0 := a + iT + i/2$, $r \geq 0$ a real number large enough so as to arrange that the closed ball centered at z_0 with radius r contains the box $I + i[T, T+1]$. We fix a real number $R > r$. By [**Lang-b**, Ch.XVII,§2,Lemma 2.2,p.333], the Maximum Principle and since $R/r > 1$, we have, upon calling $N(T)$ the number of zeros of $M(\phi)(\chi|\cdot|_G^z)$ contained in $B(z_0, r)$:

$$N(T)\log(R/r) \leq \log(B) - \log|M(\phi)(\chi|\cdot|_G^{z_0})|$$

for a large enough real number $B > 0$, and hence $N(T) = O(\log|M(\phi)(\chi|\cdot|_G^{a+iT+i/2})|)$. Since $M(\phi)$ is uniformly Schwartz on every vertical strip of bounded width of each connected component of $\Omega(G)$, we deduce that for every integer $N \geq 0$, there exists a real number $C_N > 0$ such that $M(\phi)(\chi|\cdot|_G^{a+iT+i/2}) = O(|T|^N)$ as $|T| \to +\infty$.

We deduce that $N(T) = O(\log|T|)$ holds, with a possibly different implied constant, depending on ϕ and I. □

LEMMA 9.3.2.5. *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $\chi \in \widehat{G}$ be a fixed continuous character. Let $I \subset \mathbf{R}$ be a compact interval with non-empty interior.*

Let $\phi \in \mathcal{S}_{\mathrm{E}}(G)$. Assume that all the zeros of $M(\phi)(\chi|\cdot|_G^z)$ are contained in the interior of the vertical strip $\mathbf{C}_I$. Then the quotient of $\mathcal{S}_{\mathrm{E}}(G)$ by the convolution ideal $\phi * \mathcal{S}_{\mathrm{E}}(G)$ admits an absolute Schauder basis whose set of coefficient functionals is given by the point-evaluation functionals at the zeros of $M(\phi)(\chi|\cdot|_G^z)$ and their derivatives, i.e. $\delta_\rho^{(k)}$ as ρ varies over the zeros of $M(\phi)(\chi|\cdot|_G^z)$ and $0 \leq k \leq N(\rho)-1$, with $N(\rho)$ the multiplicity of ρ.

PROOF. The exponential decay condition defining $\mathcal{S}_{\mathrm{E}}(G)$ implies the convergence axiom defining Schauder bases, using Lemma 9.3.2.4 and Theorem D.3.3.13, combined with the description of the topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ provided by Lemma D.3.1.2. By Theorem 4.1.2.6(1) and (3), indeed the Schauder basis is absolute.[6] □

THEOREM 9.3.2.6 (Approximate units in $\mathscr{L}_\sigma(W_I^s(G))$). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I \subset \mathbf{R}$ be a closed interval with non-empty interior, $s \geq 0$ a real number.*

For $T > 0$ a real number, we define the closed subspace:

$$W_I^s(G)_{\leq T} := \bigcap_{\rho : M(\phi)(\chi|\cdot|_G^\rho)=0, |\mathrm{Im}(\rho)|\leq T} \bigcap_{0\leq k\leq N(\rho)-1} \ker(\delta_\rho^{(k)}). \tag{9.3.2.10}$$

The orthogonal projectors P_T onto each $W_I^s(G)_{\leq T}$ form a net of orthogonal projectors whose limit as $T \to +\infty$ with respect to the topology of simple convergence is the identity, i.e. $P_T \to \mathrm{Id}$ as $T \to +\infty$ with respect to the strong operator topology on the space of continuous endomorphisms of the Hilbert space $W_I^s(G)$.

PROOF. Using the basis projectors provided by Lemma 9.3.2.5 we see that $\bigcup_{T>0} W_I^s(G)_{\leq T}$ is dense in $W_I^s(G)$, which implies that the net of orthogonal projectors $\{P_T\}_{T>0}$ indeed converges strongly to the identity, since $W_I^s(G)_{\leq T} \subset W_I^s(G)_{\leq T'}$ for every pair of real numbers $0 < T < T'$. □

9.3.3. Trace of the regular representation on $W_I^s(G)$. Let G be an $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group, with module $|\cdot|_G$. Let μ_G be a choice of Haar measure on G. Let $I \subset \mathbf{R}$ be a non-empty closed interval, $s \geq 0$ a real number.

The continuous left-translation action ρ_G of on $\mathcal{S}_{\mathrm{E}}(G)$ extends by continuity to $W_I^s(G)$.

We refer to this continuous representation of G as the *regular representation* of G on $W_I^s(G)$.

9.3.3.1. *Cutoffs.* The purpose of this subsection is to introduce truncation operators, to be reprised later.

We let $\varphi : G \to \mathbf{C}$ be a bounded function, whose support is compact.

The operator

$$A_G(\varphi) : \psi \mapsto \int_G \varphi(g)\psi(g^{-1}x)\,\mathrm{d}\mu_G(g)$$

is well defined and it is continuous as an operator on $W^{s,2}(G)$ by Theorem C.4.3.5, which applies by boundedness of φ and compactness of its support, which implies that $g \mapsto \varphi(g)\rho_G(g,\cdot)$, as a uniformly continuous (see Lemma 4.3.3.1 and the discussion preceding Lemma D.2.2.1) $W^{s,2}(G)$-valued function, is μ_G-integrable in the sense of Pettis.

We will be computing the trace of the nuclear obtained from $A_G(\varphi)$ by composition with a truncation operator.

For any non-empty connected interval $I \subset \mathbf{R}$ and $a \in \mathbf{R}$ such that $-a + I$ contains $0 \in \mathbf{R}$, we will be making the calculation also on the closure of $|\cdot|_G^a \mathcal{S}_I(G)$ in $W^{s,2}(G)$, noting that every element ψ of $|\cdot|_G^a \mathcal{S}_I(G)$ is absolutely square-integrable with respect to the Radon measure $(1+|\log|\cdot|_G|^2)^{s/2}\,\mu_G$ for any $s \geq 0$, since it is a Schwartz-Bruhat function on G, i.e. an element of $\mathcal{S}(G)$, by definition of $\mathcal{S}_I(G)$, the Hausdorff locally convex topological $\mathbf{C}$-vector space of Schwartz-Bruhat functions on G with I-exponential decay (see Definition D.3.3.1 from §D.3).

[6] Return here.

The truncation operators alluded to in the foregoing are obtained by integration on growing relatively compact domains of the $\mathbf{R}_{>0}$-modulated group G of the form $\{g \in G : |g|_G \in (1/\Lambda, \Lambda)\}$ for $\Lambda > 0$.

We refer the reader to Definition D.2.1.1 for more on modulated groups. Recall that an $\mathbf{R}_{>0}$-modulated group is a Hausdorff locally compact abelian group G equipped with a *module* homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$ which is assumed to be surjective and proper. Its fibre over $1 \in \mathbf{R}_{>0}$, i.e. its kernel $G^1 := \ker(|\cdot|_G)$, is a closed compact Hausdorff abelian group.

Let $K \subset \mathbf{R}_{>0}$ be a compact subset. We define G^K to be the compact subset:

$$G^K := \{g \in G : |g|_G \in K\} \subset G.$$

For instance, for any real number $\Lambda > 0$:

$$G^{[1/\Lambda, \Lambda]} := \{g \in G : |g|_G \in [1/\Lambda, \Lambda]\} \subset G$$

a compact subset of G with non-empty interior.

Definition 9.3.3.1 (Smooth cutoffs). Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. We let μ_G be a choice of Haar measure on G. Let I be a closed non-empty interval.

Let $\Lambda, \varepsilon > 0$ and $s \geq 0$ be real numbers. Assume $0 < 1/\Lambda - \varepsilon < \Lambda - \varepsilon$.

We let $\rho : G \to \mathbf{C}$ be a function such that:

(1) ρ is bounded with $\|\rho\|_u = 1$, real-valued and nonnegative.
(2) ρ is constant on the fibres of the module $|\cdot|_G : G \to \mathbf{R}_{>0}$, equivalently ρ is G^1-invariant.
(3) The unique function induced by ρ on $\mathbf{R}_{>0}$ by G^1-invariance and through the module $|\cdot|_G : G/G^1 \simeq \mathbf{R}_{>0}$, is smooth away from finitely many points, at each of which its derivatives of any nonnegative integer order have at worst jump discontinuities, such that both one-ended limits exist and are finite.
(4) ρ is *multiplicatively symmetric*, i.e. $\rho(g) = \rho(g^{-1})$.
(5) ρ is supported over $G^{[1/\Lambda, \Lambda]}$.
(6) ρ is identically 1 over $G^{[1/\Lambda+\varepsilon, \Lambda-\varepsilon]}$.

We refer to ρ as a (Λ, ε)-*cutoff*. If ρ is smooth, then we refer to ρ as a *smooth* (Λ, ε)-cutoff.

For ρ a (Λ, ε)-cutoff, we define the ρ-*truncation* operator to be the multiplication operator:

$$M_\rho : W_I^s(G) \to W_I^s(G) \tag{9.3.3.1}$$

defined as:

$$M_\rho(\phi)(g) := \rho(g)\phi(g) \tag{9.3.3.2}$$

as functions of $g \in G$.

Remark 9.3.3.2. We note that indeed M_ρ is a continuous linear endomorphism of $W_I^s(G)$, since for any $\phi \in W_I^s(G)$, we have:

$$\int_G |(M_\rho\phi)(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\} (1 + |\log|g|_G|^2)^{s/2} \, \mathrm{d}\mu_G(g) \leq \int_G |\phi(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\} (1 + |\log|g|_G|^2)^{s/2} \, \mathrm{d}\mu_G(g) < +\infty$$

and hence M_ρ is valued in $W_I^s(G)$ and bounded, and since ρ is multiplicatively symmetric, it satisfies:

$$M_\rho = J \circ M_\rho \circ J \tag{9.3.3.3}$$

where J is the linear topological isomorphism of $W_I^s(G) \to W_{-I}^s(G)$ given by inversion on G.

If $I = [a, b]$ for real numbers $a < b$, then M_ρ is a self-adjoint operator on the Hilbert space $W_I^s(G)$. Indeed, for any $\phi \in W_I^s(G)$ we have:

$$\langle M_\rho\phi, \phi\rangle_{W_I^s(G)} = \int_G \rho(g)|\phi(g)|^2 \max\{|g|_G^{2a}, |g|_G^{2b}\} (1 + |\log|g|_G|^2)^{s/2} \, \mathrm{d}\mu_G(g) \geq 0$$

which implies that M_ρ is a positive operator. By [**Cnw-a**, Ch.II,§2,Prop.2.12,p.33], it is self-adjoint (more generally, every multiplication operator is normal by [**Cnw-a**, Ch.II,§2,Prop.2.16(b),p.35], and by [**Cnw-a**, Ch.II,§2,Example 2.8,p.32] it is self-adjoint if and only if it is real valued).

If G is second-countable, then G is σ-compact, and μ_G is σ-finite. By [**Cnw-a**, Ch.II,§1,Thm.1.5,p.28], we have $\|M_\rho\| = \|\rho\|_\infty = \|\rho\|_u = 1$.

M_ρ is injective if the zero-set of ρ is μ_G-negligible as a subset of G.

The operator M_ρ is not generally an idempotent operator, unless ρ is constant over G on an open subset of the form $G - \bigcup_{1\leq k\leq N}\{g \in G : |g|_G = t_k\}$ for a finite set $\{t_k\}_{1\leq k\leq N}$. M_ρ has closed range if ρ is strictly bounded away from 0 on a subset of G whose complement is μ_G-negligible.

Definition 9.3.3.3 (Absolute cutoffs). Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. We let μ_G be a choice of Haar measure on G. Let I be a closed non-empty interval.

Let $\Lambda > 0$ and $s \geq 0$ real numbers.

We let $\rho_{\Lambda,+}, \rho_{\Lambda,-} : G \to \mathbf{R}_{\geq 0}$ be the bounded functions:

$$\rho_{\Lambda,+}(x) := 1_{G^{\geq 1/\Lambda}}(x),\ \rho_{\Lambda,-}(x) := 1_{G^{\leq \Lambda}}(x),\ x \in G \tag{9.3.3.4}$$

and $\rho_\Lambda : G \to \mathbf{R}_{\geq 0}$ be the bounded function:

$$\rho_\Lambda(x) := 1_{G^{[1/\Lambda,\Lambda]}}(x) = \rho_{\Lambda,+}(x)\rho_{\Lambda,-}(x),\ x \in G. \tag{9.3.3.5}$$

We refer to $\rho_{\Lambda,+}$, respectively $\rho_{\Lambda,-}$, as a *positive*, respectively *negative* Λ-*cutoff*, and to ρ_Λ as a Λ-*cutoff*.

For $\rho \in \{\rho_{\Lambda,+}, \rho_{\Lambda,-}, \rho_\Lambda\}$, we define the multiplication operator:

$$M_\rho : W^s_I(G) \to W^s_I(G) \tag{9.3.3.6}$$

defined as:

$$M_\rho(\phi)(g) := \rho(g)\phi(g) \tag{9.3.3.7}$$

as functions of $g \in G$.

Remark 9.3.3.4. As in Remark 9.3.3.2, for $\rho \in \{\rho_{\Lambda,+}, \rho_{\Lambda,-}, \rho_\Lambda\}$, we note that indeed M_ρ is a continuous linear endomorphism of $W^{s,2}(G)$, since for any $\phi \in W^{s,2}(G)$, we have:

$$\int_G |(M_\rho\phi)(g)|^2(1 + |\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g) \leq \int_G |\phi(g)|^2(1 + |\log|g|_G|^2)^{s/2}\,\mathrm{d}\mu_G(g) < +\infty$$

and hence M_ρ is valued in $W^{s,2}(G)$ and bounded.

Each M_ρ is a positive operator. By [**Cnw-a**, Ch.II,§2,Prop.2.12,p.33], it is self-adjoint (as noted in Remark 9.3.3.2, more generally, every multiplication operator is normal by [**Cnw-a**, Ch.II,§2,Prop.2.16(b),p.35], and by [**Cnw-a**, Ch.II,§2,Example 2.8,p.32] it is self-adjoint if and only if it is real valued).

It is injective, since each choice of ρ vanishes only on a μ_G-negligible subset of G.

By [**Cnw-a**, Ch.II,§1,Thm.1.5,p.28], we have $\|M_\rho\| = \|\rho\|_\infty = 1$. Moreover, since each of the three given choices of ρ is, for each $\Lambda > 0$, essentially bounded below away from 0, then M_ρ has closed range.

Another way to see this is to observe that $M_\rho^2 = M_\rho$, from which it follows by [**Cnw-a**, Ch.II,§3,Pop.3.2,(b),p.37] that M_ρ has closed range. In fact, by [**Cnw-a**, Ch.II,§3,Pop.3.3,p.37], it is an orthogonal projector.

For G an $\mathbf{R}_{>0}$-modulated group, and μ_G a choice of Haar measure on G, continuous homomorphic sections τ to the module $|\cdot|_G$ exist (see Remark D.2.1.2). We define μ_1 to be the unique Haar measure on G^1 such that, under the isomorphism of Hausdorff locally compact abelian groups $G \simeq \mathbf{R}_{>0} \times G^1$ induced by one such choice of τ, μ_G is pushed-forward to $\frac{\mathrm{d}t}{t} \otimes \mu_1$.

Theorem A.5.4.1, which is Lebesgue's Differentiation Theorem for Hausdorff locally compact abelian groups, and which rests on an analog of the Hardy-Littlewood Maximal Theorem (see Theorem A.5.3.6 and

its global form, Theorem A.5.3.7) leads to the concept of "smoothening" of a locally integrable function on a Hausdorff locally compact abelian group at a point (see Terminology A.5.4.1).

We refer the reader to §A.5.3, §A.5.4, and to [**Bris-a**, §3], where a generalization of the same instruments leads to trace formulas for nuclear integral operators on the L^2-space of a Hausdorff locally compact second-countable topological space equipped with a Radon measure; see also [**Bris-a**, §3.1,Thm.3.1], and the special case of subdomains of Euclidean spaces from [**Bris-b**, Thm.3.1].

9.3.3.2. *Calculation of the trace of $M_\rho \circ A(\varphi)$.* We have the following:

THEOREM 9.3.3.5 (Nuclearity of the smoothly truncated regular representation on $W_I^s(G)$). *Let G be an $\mathbf{R}_{>0}$-modulated group, with module $|\cdot|_G$. Assume G is second-countable. Let I be a non-empty closed interval. We let μ_G be a choice of Haar measure on G.*

Let $\Lambda, \varepsilon > 0$ and $s \geq 0$ real numbers. Assume $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$. Let $\rho : G \to \mathbf{R}_{\geq 0}$ be a (Λ, ε)-cutoff. Then:

(1) *for every bounded compactly supported complex-valued function φ on G, the following endomorphism of $W_I^s(G)$:*

$$M_\rho \circ A_G(\varphi) \tag{9.3.3.8}$$

is an integral Hilbert-Schmidt operator whose integral kernel has an absolutely μ_G-integrable smoothening along the diagonal.

(2) *For every smooth compactly supported $\varphi \in \mathcal{D}(G)$ and every smooth (Λ, ε)-cutoff, then the Hilbert-Schmidt operator in* (9.3.3.8) *is nuclear, and its nuclear trace equals the integral of the integral kernel along the diagonal.*

(3) *As $\rho \to 1$ pointwise over G with $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$, then the integral of the smoothening along the diagonal of the integral kernel of the operator in* (9.3.3.8) *is:*

$$\mathrm{Trace}(M_\rho \circ A_G(\varphi)) = 2\log(\Lambda)\, \mu_1(G^1)\, \varphi(1) + o(1). \tag{9.3.3.9}$$

PROOF. Assume first that $0 \in I$ holds.

We denote by $\rho_{I,s} : G \times W_I^s(G) \to W_I^s(G)$ the continuous left translation action ρ_G of G on $W_I^s(G)$ for any real number $s \geq 0$.

We now note that $M_\rho \circ \int_G \varphi(g)\rho_{\{0\},0}(g,\cdot)\,\mathrm{d}\mu_G(g)$, factors through $W_I^s(G)$ for any $s > 0$ and any non-empty closed interval containing 0.

Explicitly, given the injective continuous (by Proposition 9.1.1.3(1)) linear operator $U_{I,s} : W_I^s(G) \to L^2(G, \mu_G)$, we have

$$M_\rho \circ \int_G \varphi(g)\rho_s(g,\cdot)\,\mu_G(g) = U_{I,s} \circ M_\rho \circ \int_G \varphi(g)\rho_{\{0\},0}(g,\cdot)\,\mu_G(g). \tag{9.3.3.10}$$

By [**Scha**, Ch.III,Cor.3,p.100], if $M_\rho \circ \int_G \varphi(g)\rho_{\{0\},0}(g,\cdot)\,\mathrm{d}\mu_G(g)$ is nuclear, then so is

$$M_\rho \circ \int_G \varphi(g)\rho_{I,s}(g,\cdot)\,\mathrm{d}\mu_G(g). \tag{9.3.3.11}$$

Moreover, their traces are the same by Lemma C.3.2.2.

We therefore may and do assume $s = 0$ and $I = \{0\}$. We abbreviate $L^2(G, \mu_G)$ as $L^2(G)$. We rename $\rho_{\{0\},0}$ as ρ_0.

We analyze the commutator operator:

$$[M_\rho \circ \left(\int_G \varphi(g)\rho_0(g,\cdot)\,\mathrm{d}\mu_G(g)\right) - \left(\int_G \varphi(g)\rho_0(g,\cdot)\,\mathrm{d}\mu_G(g)\right) \circ M_\rho] : L^2(G) \to L^2(G). \tag{9.3.3.12}$$

This operator is Hilbert-Schmidt on $L^2(G)$, with integral kernel given by the element $K_{\varphi,\rho}(x,y) \in L^2((G)_x \times (G)_y)$:

$$K_{\varphi,\rho}(x,y) := \varphi(xy^{-1})(\rho(x) - \rho(y)). \tag{9.3.3.13}$$

We check the square-integrability property of $K_{\varphi,\rho}(x,y)$ over $G \times G$ explicitly.

Let $B \subset G$ be the compact support of φ. The support of $K_{\varphi,\rho}$ is the union within $(G)_x \times (G)_y$

$$[1/\Lambda, \Lambda] \times (B^{-1} \cdot [1/\Lambda, \Lambda]) \cup (B \cdot [1/\Lambda, \Lambda]) \times [1/\Lambda, \Lambda]$$

which is compact within $(G)_x \times (G)_y$. $K_{\varphi,\rho}$ is therefore p-integrable over $(G)_x \times (G)_y$ with respect to the product measure $\mu_G \otimes \mu_G$, for every $p \geq 1$.

We now check the assertion that indeed the operator from (9.3.3.13) is Hilbert-Schmidt with $K_{\varphi,\rho}$ as its integral kernel.

We have, for any $u \in L^2(G)$:

$$\begin{aligned}\int_{y\in G} K_{\varphi,\rho}(x,y)u(y)\,\mathrm{d}\mu_G(y) &= \int_{y\in G} \varphi(xy^{-1})(\rho(x)-\rho(y))u(y)\,\mathrm{d}\mu_G(y)\\ &= \rho(x)\int_G \varphi(xy^{-1})u(y)\,\mathrm{d}\mu_G(y) - \int_G \varphi(xy^{-1})\rho(y)u(y)\,\mathrm{d}\mu_G(y)\\ &= [M_\rho \circ \left(\int_G \varphi(g)\rho_0(g,\cdot)\,\mathrm{d}\mu_G(g)\right) - \left(\int_G \varphi(g)\rho_0(g,\cdot)\,\mathrm{d}\mu_G(g)\right) \circ M_\rho](u)(x)\end{aligned}$$

for μ_G-almost all $x \in G$, which indeed shows, by of Hilbert-Schmidt kernels, that $K_{\varphi,\rho}$ indeed is the integral kernel of the commutator operator from (9.3.3.12) (explicitly, the convolution action of $K_{\varphi,\rho}$ on the Schwartz-Bruhat functions on G, valued in $L^2(G)$, agrees with that of the commutator operator from (9.3.3.12), and since both operators are Hilbert-Schmidt on $L^2(G)$, their integral kernels agree $(\mu_G \otimes \mu_G)$-almost everywhere on $G \times G$. One of them being already an integral operator with integral kernel $K_{\varphi,\rho}$, we obtain the desired identification of the Hilbert-Schmidt kernel.

When ρ and φ are smooth, see also [**Tre**, Part III,Ch.51,Cor.51.7], Corollary to [**Tre**, Part III,Ch.51,Thm.51.6], and Theorem C.2.4.19), noting that $K_{\varphi,\rho}$ maps from an element of $L^2((G)_x \times (G)_y)$ to the Schwartz kernel of the operator from (9.3.3.13) restricted to $\mathcal{S}(G)$ and valued in tempered distributions over G, as an element of $\mathcal{S}(G)'_\beta \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G)'_\beta$.

As such, the given operator is Hilbert-Schmidt, and $K_{\varphi,\rho}$ is its Hilbert-Schmidt kernel.

The function $K_{\varphi,\rho}$ is smooth over $G \times G$ away from a $(\mu_G \otimes \mu_G)$-negligible subset of the form

$$(\bigcup_{1\leq k\leq N} \{g \in G : |g| = t_k\}) \times (\bigcup_{1\leq k\leq N} \{g \in G : |g| = t_k\})$$

for $\{t_1, \ldots, t_N\} \subset \mathbf{R}_{>0}$ a finite subset, and it is $(\mu_G \otimes \mu_G)$-integrable.

In particular, the diagonal in $G_x \times G_y$ is contained in the Lebesgue set of $K_{\varphi,\rho}$, up to a subset of μ_G-measure zero (see Theorem A.5.4.1 and Remark A.5.4.2).

Along the diagonal in $(G)_x \times (G)_y$, the smoothening $\widetilde{K}_{\varphi,\rho}$ (in the sense of Theorem A.5.4.1 and [**Bris-a**, §3]; see also [**Bris-a**, §3.1,Thm.3.1]) vanishes μ_G-almost everywhere. In fact, $K_{\varphi,\rho}$ is itself already defined μ_G-almost everywhere along the diagonal at its points of continuity, a subset of the diagonal whose complement in the diagonal is μ_G-negligible. Away from a μ_G-negligible subset, $K_{\varphi,\rho}$ and $\widetilde{K}_{\varphi,\rho}$ agree.

The operator from (9.3.3.13) is therefore an integral Hilbert-Schmidt operator whose kernel has an absolutely integrable smoothening along the diagonal, whose mass vanishes.

If both ρ and φ are smooth, then it is nuclear on $L^2(G)$ (because the kernel is smooth and compactly supported over $G \times G$, and hence an element of $\mathcal{S}(G) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G)$ (in fact an element of the smaller $\mathcal{S}(G) \mathbin{\overline{\otimes}} \mathcal{S}(G)$, since φ is contained in some $\mathcal{D}_K(G,k)$ for a compact subgroup $k \subset G$ with compactly generated orthogonal $k^\perp$, with $K \subset G$ a compact subset with non-empty interior, and $\mathcal{D}_K(G,k)$ is a Fréchet space), which therefore

maps to $S_1(G) = L^2(G)\,\widehat{\otimes}_\pi L^2(G)$ by [**Jar**,]) and by Lemma C.3.7.1(2) its trace vanishes by Lemma C.3.7.1(1) (see also [**Bris-a**, Cor.3.2]).

To show that the operator in (9.3.3.11) is an integral Hilbert-Schmidt operator whose kernel has an absolutely integrable smoothening along the diagonal, and, if ρ and φ are smooth, is nuclear, it is enough to show that its sum with the operator in (9.3.3.13) is.

Such sum is the operator:

$$\left(\int_G \varphi(g)\rho_0(g,\cdot)\,\mathrm{d}\mu_G(g)\right)\circ M_\rho : L^2(G)\to L^2(G). \tag{9.3.3.14}$$

This is again a Hilbert-Schmidt operator whose integral kernel is:

$$K'_{\varphi,\rho}(x,y) := \varphi(xy^{-1})\rho(y) \tag{9.3.3.15}$$

whose smoothening is again integrable along the diagonal of $(G)_x \times (G)_y$.

If ρ and φ are smooth, the same argument as in the foregoing implies nuclearity, and Lemma C.3.7.1(2) implies that the trace of the nuclear operator in (9.3.3.14) is computed by the integral of the smoothening of $K'_{\varphi,\rho}$ along the diagonal of $(G)_x \times (G)_y$ (see also [**Bris-a**, Cor.3.2], a generalization of Mercer's Theorem).

It remains to perform the calculation. We choose a continuous homomorphic section τ to the module $|\cdot|_G$, and use the resulting isomorphism of Hausdorff locally compact abelian groups $\mathbf{R}_{>0}\times G^1 \simeq G$ through which $(\mathrm{d}t/t)\otimes\mu_1$ matches μ_G.

We rename the smoothening of $K'_{\varpi,\rho}$ as $K'_{\varphi,\rho}$, and we have:

(9.3.3.16)

$$\int_G K'_{\varphi,\rho}(g,g)\,\mathrm{d}\mu_G(g) = \int_{\mathbf{R}_{>0}}\int_{G^1} K'_{\varphi,\rho}(\tau(t)g_0,\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t}$$

$$= \int_{1/\Lambda}^{1/\Lambda+\varepsilon}\int_{G^1} K'_{\varphi,\rho}(\tau(t)g_0,\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} + \int_{\Lambda-\varepsilon}^{\Lambda}\int_{G^1} K'_{\varphi,\rho}(\tau(t)g_0,\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} \tag{9.3.3.17}$$

$$+ \int_{1/\Lambda+\varepsilon}^{\Lambda-\varepsilon}\int_{G^1} K'_{\varphi,\rho}(\tau(t)g_0,\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} \tag{9.3.3.18}$$

$$= \varphi(1)\int_{1/\Lambda}^{1/\Lambda+\varepsilon}\int_{G^1} \rho(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} + \varphi(1)\int_{\Lambda-\varepsilon}^{\Lambda}\int_{G^1} \rho(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} \tag{9.3.3.19}$$

$$+ \varphi(1)\int_{1/\Lambda+\varepsilon}^{\Lambda-\varepsilon}\left(\int_{G^1}\mathrm{d}\mu_1(g)\right)\frac{\mathrm{d}t}{t}. \tag{9.3.3.20}$$

We have:

(9.3.3.21)

$$\left|\varphi(1)\int_{1/\Lambda}^{1/\Lambda+\varepsilon}\int_{G^1} \rho(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t} + \varphi(1)\int_{\Lambda-\varepsilon}^{\Lambda}\int_{G^1} \rho(\tau(t)g_0)\,\mathrm{d}\mu_1(g_0)\,\frac{\mathrm{d}t}{t}\right| \leq |\varphi(1)|\left\{\log\left(\frac{1/\Lambda+\varepsilon}{1/\Lambda}\right)+\log\left(\frac{\Lambda}{\Lambda-\varepsilon}\right)\right\}$$

and:

$$\varphi(1)\int_{1/\Lambda+\varepsilon}^{\Lambda-\varepsilon}\left(\int_{G^1}\mathrm{d}\mu_1(g)\right)\frac{\mathrm{d}t}{t} = \log\left(\frac{\Lambda-\varepsilon}{1/\Lambda+\varepsilon}\right)\mu_1(G^1)\varphi(1). \tag{9.3.3.22}$$

Since, by assumption, $\varepsilon\Lambda = o(1)$ as $\Lambda\to+\infty$, we deduce that we have:

$$\log\left(\frac{1/\Lambda+\varepsilon}{1/\Lambda}\right)+\log\left(\frac{\Lambda}{\Lambda-\varepsilon}\right) = o(1),\ \log\left(\frac{\Lambda-\varepsilon}{1/\Lambda+\varepsilon}\right) = 2\log(\Lambda)+o(1) \tag{9.3.3.23}$$

as $\Lambda\to+\infty$, which shows that (9.3.3.9) holds. It remains to treat the case when I does not contain 0. We choose $b\in\mathbf{R}$ such that $b+I$ contains 0. Then $\phi\mapsto|\cdot|_G^b\phi$ is a linear topological isomorphism $W^s_{b+I}(G)\to W^s_I(G)$

compatible with the continuous left-translation action by G up to twist by the quasi-character $|\cdot|_G^{-b}$, and since conjugation by linear topological isomorphisms preserves traces of nuclear operators, it is enough to compute the desired trace under the additional assumption $0 \in I$, since the value at $g = 1$ of φ and of $|\cdot|_G^{-b}\varphi$ is the same. $\square$

CHAPTER 10

Trace formula: Spectral Side

10.1. Hardy-Sobolev topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ 434
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10.1. Hardy-Sobolev topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$

In this section we study the completions of the Hausdorff locally convex topological complex vector spaces $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, $\mathcal{S}_{\mathrm{E}}(C_F)$, $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, $\mathcal{S}_I(J_{F,S})$ for I a non-empty connected interval of the real line, with respect to the coarser normed topologies originating from the Sobolev norms on $\mathcal{S}_I(G)$ defined in (9.1.2.3) of Definition 9.1.2.1, for G the choice of $\mathbf{R}_{>0}$-modulated group $G = C_F$, respectively $J_{F,S}$.

10.1.1. Hardy-Sobolev completions of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$. Let F be a number field, $a \in \mathbf{R}$ an arbitrary real number.

By Theorem 2.2.3.4 and Lemma 2.2.4.3, we have an injective $\mathbf{C}$-linear map:

$$|\cdot|_F^a \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to C^\infty_{\mathrm{dir}}(C_F) \tag{10.1.1.1}$$

where the leftmost $\mathbf{C}$-linear space is thought of as a function space on the idele class group, i.e. each of its elements $\phi = \Phi_F(f)$ for some $f \in \mathcal{S}(\mathbf{A}_F)$ with $f(0) = \widehat{f}(0) = 0$, is the function $g \mapsto |g|_F^a \Phi_F(f)(g)$ of $g \in C_F$, and the rightmost space is the $\mathbf{C}$-linear space of those continuous functions on C_F whose directional derivatives of any order along any continuous 1-parameter subgroup of C_F exist and are continuous.

Such function is, by Lemma 2.2.4.3, smooth along any continuous 1-parameter subgroup, and decays more rapidly than any positive power of $e^{-|\log|g|_F|}$ as $|\log|g|_F| \to +\infty$.

We note that the Sobolev norms $\|\cdot\|_{s,2}$ and $\|\cdot\|_{I,s}$ are defined on the image of the injective linear operator in (10.1.1.1) for any real number $s \geq 0$ and any closed interval I with non-empty interior, and induce Hausdorff normed topologies on the source of (10.1.1.1), coarser than the Hausdorff locally convex LF topology (see the First Structure Theorem, Theorem 5.2.1.1(1)) obtained by transport of structure from the case $a = 0$.

In particular, every element in the image of (10.1.1.1) is also an element of $W_I^s(C_F)$. We therefore obtain an injective linear map:

$$|\cdot|_F^a \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to W_I^s(C_F) \tag{10.1.1.2}$$

equivariant with respect to the continuous left-translation action by C_F on the source (sending $(g, |\cdot|_F^a\phi)$ to $|g|_F^{-a}\rho_F(g,\phi)$) and on the target.

Definition 10.1.1.1 (Hardy-Sobolev spaces of global adelic θ-functions). Let F be a number field, S a finite set of places of F containing all the Archimedean places. We define:

(1) for $a \in \mathbf{R}$ a real number, $I \subset \mathbf{R}$ a compact non-empty interval, $s \geq 0$ a real number:

$$W_{a,I}^s \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset W_I^s(C_F) \tag{10.1.1.3}$$

to be the closure of $|\cdot|_F^a \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ in $W_I^s(C_F)$. We refer to it as the *Hardy-Sobolev completion of* $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ *along the a-line.*

(2) For $a \in \mathbf{R}_{>0}$ a real number, $I = [\alpha, \beta] \subset \mathbf{R}$ a compact non-empty interval, $s \geq 0$ a real number:

$$W_{a,I}^s \,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \tag{10.1.1.4}$$

to be the normed completion of $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ with respect to the normed topology defined by the norm assigned by, for every $\phi \in \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$:

$$\|\phi\|_{a,I,s} := \left(\int_{J_{F,S}} |\phi(g)|^2 \,\max\{|g|_{F,S}^{2a+2\alpha}, |g|_{F,S}^{2a+2\beta}\}\,(1 + |\log|g|_{F,S}|^2)^{s/2}\,\mathrm{d}\overline{\mu}_S(g)\right)^{1/2}. \tag{10.1.1.5}$$

We refer to it as the *Sobolev completion of* $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ *along the a-line.*

Remark 10.1.1.2. We observe that by Proposition 2.4.3.1, the integral in (10.1.1.5) converges. We further caution the reader that while the continuous left-translation action of C_F on $W_I^s(C_F)$ induces on $W_{a,I}^s \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ the action $(g,\phi) \mapsto |g|_F^{-a}\rho_F(g,\phi)$, with ρ_F the continuous left-translation action

induced by left-translation on the global adelic θ-functions, the continuous left-translation action of $J_{F,S}$ on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ extends by continuity to the $W_{a,I}^s$-completion: it will be related to the action inherited from $\mathcal{S}_{-a+(0,+\infty)}(J_{F,S})$ by multiplication by $|\cdot|_{F,S}^{-a}$.[1]

In this subsection and the next, we collect basic properties of the completions $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, and discuss continuity of the Fourier transform: specifically, we will see that the only real number a for which the Fourier transform $\mathcal{F}$ defined in Theorem 5.2.4.1 on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$ (and hence on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$) and on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, extends to a continuous linear operator on the Sobolev completions from Definition 10.1.1.1, is $a = 1/2$.

LEMMA 10.1.1.3. *Let F be a number field, $s \geq 0$ a real number, $a \in \mathbf{R}$ a real number. Let $I \subset \mathbf{R}$ be a compact non-empty interval.*

The subspace topology on $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ inherited from $W_I^s(C_F)$ is the norm-topology induced by the norm on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, continuous with respect to the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, defined as:

$$\|\Phi_F(f)\|_{a,I,s} := \left(\int_{C_F} |\Phi_F(f)(g)|^2 \max\{|g|_F^{2a+2\alpha}, |g|_F^{2a+2\beta}\}\,(1+|\log|g|_F|^2)^{s/2}\,\mathrm{d}\overline{\mu}_1(g)\right)^{1/2}. \tag{10.1.1.6}$$

We have:

(1) *$W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is a separable Hilbert space, closed within $W_I^s(C_F)$.*
(2) *The continuous left-translation action of C_F on $W_I^s(C_F)$ preserves $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and restricts to the continuous left-translation action of C_F on $|\cdot|_F^a\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ induced by the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}_0(\mathbf{A}_F)$.*

PROOF. Since C_F is second-countable, $W_I^s(C_F)$ is a separable Hilbert space by Lemma 9.1.1.2[2], applied to $G = \widehat{C_F}$, $\mu := \widehat{\overline{\mu}_1}$. Since $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is closed in $W_I^s(C_F)$, it is itself a separable Hilbert space, and the normed topology on it is induced by the norm $\|\cdot\|_{I,s}$ on the ambient, which gives (10.1.1.6). □

Recall that we have denoted the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_{F,S})$ on $\mathbf{A}_{F,S}$ by $\rho_{F,S} : \mathrm{GL}_1(\mathbf{A}_{F,S}) \times \mathbf{A}_{F,S} \to \mathbf{A}_{F,S}$.

We denote, for each real number $s \geq 0$ and real number $a \in \mathbf{R}_{>0}$, for any compact non-empty interval $I = [\alpha, \beta] \subset \mathbf{R}$, by $\mu_{I,s}$ the Borel measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})$:

$$\mu_{I,s} := \max\{|\cdot|_{F,S}^{2\alpha}, |g|_{F,S}^{2\beta}\}\,(1+|\log|\cdot|_{F,S}|^2)^{s/2}\,\mu_S$$

where μ_S is the multiplicative S-local Tamagawa measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ from (A.3.4.2).

We note that in light of Theorem 2.4.2.1 and Proposition 2.3.1.2(2), $|\cdot|_{F,S}^a \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ is a linear subspace of $W_{-a+(0,+\infty)}^s(J_{F,S})$ for any real number $s \geq 0$ and any real number $a > 0$, and hence of $W_I^s(J_{F,S})$ for any I contained in $-a + (0, +\infty)$.

LEMMA 10.1.1.4. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. Let $s \geq 0$ be a real number, $a \in \mathbf{R}_{>0}$ a real number. Then:*

(1) *$\mu_{I,s}$ is a regular Radon measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ invariant under the left-translation action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})$.*
(2) *For any compact interval $I = [\alpha, \beta] \subset (0, +\infty)$, the subspace topology on $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$ inherited from $W_I^s(J_{F,S})$ is the norm-topology induced by the norm on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, continuous*

[1] Return here.
[2] Return here.

with respect to the Hausdorff locally convex topology on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$*, defined as:*

$$\|\Phi_{F,S}(f)\|_{a,I,s}:=\left(\int_{J_{F,S}}|\Phi_{F,S}(f)(g)|^2\max\{|g|_{F,S}^{2a+2\alpha},|g|_{F,S}^{2a+2\beta}\}\,(1+|\log|g|_{F,S}|^2)^{s/2}\,\mathrm{d}\overline{\mu}_{S,1}(g)\right)^{1/2}. \tag{10.1.1.7}$$

(3) $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ *is a separable Hilbert space, closed within* $W_I^s(J_{F,S})$.

(4) *The continuous left-translation action of* $J_{F,S}$ *on* $W_I^s(J_{F,S})$ *preserves* $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ *and restricts to the continuous left-translation action of* $J_{F,S}$ *on* $|\cdot|_{F,S}^a\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ *induced by the continuous left-translation action of* $\mathrm{GL}_1(\mathbf{A}_{F,S})$ *on* $\mathcal{S}(\mathbf{A}_{F,S})$ *multiplied by the effect of left-translation on the quasi-character* $|\cdot|_{F,S}^a$.

Proof. (1) : The function of $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$:

$$\alpha_{I,s}(g):=\max\{|g|_{F,S}^{2\alpha},|g|_{F,S}^{2\beta}\}\,(1+|\log|g|_{F,S}|^2)^{s/2}$$

is continuous over $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and vanishes at infinity. It follows that $\alpha_{I,s}\,\mu_S$ is a Radon measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and being $\mathrm{GL}_1(\mathbf{A}_{F,S})$ second-countable, by [Fol, Thm.7.8] $\mu_{I,s}$ is regular. Invariance under the left-translation action of $\mathcal{O}_{F,S}^{\times}$ follows from invariance of $|\cdot|_{F,S}$.

(2), (3) and (4) are a consequence of the definition of $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ from (10.1.1.5) and separability of $W_I^s(J_{F,S})$, which follows from Lemma 9.1.1.2[3], since $J_{F,S}$ is a second-countable $\mathbf{R}_{>0}$-modulated group. □

We denote by:

$$\langle\cdot,\cdot\rangle_{L^2(\mathrm{GL}_1(\mathbf{A}_{F,S}),\mu_{I,s})}:L^2(\mathrm{GL}_1(\mathbf{A}_{F,S}),\mu_{I,s})\times L^2(\mathrm{GL}_1(\mathbf{A}_{F,S}),\mu_{I,s})\to\mathbf{C}$$

the pairing:

$$\langle\phi,\psi\rangle_{L^2(\mathrm{GL}_1(\mathbf{A}_{F,S}),\mu_{I,s})}=\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})}\phi(g)\overline{\psi(g)}\,\mathrm{d}\mu_{I,s}(g)$$

inducing the L^2-norm topology on $L^2(\mathrm{GL}_1(\mathbf{A}_{F,S}),\mu_{I,s})$.

We now discuss isotypes of $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}_0(\mathbf{A}_F))$ and $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))$ under the continuous left-translation action of C_F^1, respectively $J_{F,S}^1$.

Proposition 10.1.1.5. *Let* F *be a number field,* S *a finite set of places of* F *containing the Archimedean places. Let* $s\geq 0$ *be a real number,* $a,b\in\mathbf{R}$ *real numbers,* $b>0$. *Let* $I\subset\mathbf{R}$ *be a compact non-empty interval. Let* ω *be a continuous character of* C_F^1*, respectively of* $J_{F,S}^1$.

(1) *The integrals:*

$$P_{F,\omega}:=\int_{C_F^1}\overline{\omega(g)}\rho_F(g,\cdot)\,\mathrm{d}\overline{\mu}_1(g) \tag{10.1.1.8}$$

and, upon assuming $I\subset\mathbf{R}_{>0}$*:*

$$P_{F,S,\omega}:=\int_{J_{F,S}^1}\overline{\omega(g)}\rho_{F,S}(g,\cdot)\,\mathrm{d}\overline{\mu}_{S,1}(g) \tag{10.1.1.9}$$

exist in the sense of Pettis, and define surjective open continuous linear operators:

$$P_{F,\omega}:W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))\to W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}(\mathbf{A}_F))(\omega) \tag{10.1.1.10}$$

and, upon assuming $I\subset\mathbf{R}_{>0}$*:*

$$P_{F,S,\omega}:W_{b,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))\to W_{b,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^{\times},\mathcal{S}(\mathbf{A}_{F,S}))(\omega) \tag{10.1.1.11}$$

[3]Return here.

inducing linear topological isomorphisms:

$$P_{F,\omega} : W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))(\omega) \tag{10.1.1.12}$$

and, upon assuming $I \subset \mathbf{R}_{>0}$:

$$P_{F,S,\omega} : W_{b,I}^s \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to W_{b,I}^s \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))(\omega). \tag{10.1.1.13}$$

(2) *The induced continuous linear operators:*

$$P_F : W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F)) \to \prod_{\omega \in \widehat{C_F^1}} W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_F)) \tag{10.1.1.14}$$

and, upon assuming $I \subset \mathbf{R}_{>0}$:

$$P_{F,S} : W_{b,I}^s \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \to \prod_{\omega \in \widehat{J_{F,S}^1}} W_{b,I}^s \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(\mathbf{A}_{F,S})) \tag{10.1.1.15}$$

to the Hausdorff locally convex products, are continuous and injective, and factor through surjective isometries onto the respective Hilbert sums.

Proposition 10.1.1.5 is a direct consequence of the same argument provided for Proposition 5.2.3.2, upon observing that the actions ρ_F and $\rho_{F,S}$ are continuous with respect to the respective Sobolev topologies, and upon inspection of the definition of Hilbert sum (see §4.3.2 and specifically the proof of Lemma 4.3.2.2).[4]

10.1.2. Fourier transform. In this subsection we study the continuity of the global and S-local Fourier transforms with respect to the Sobolev topologies on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}(\mathbf{A}_F))$, respectively $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$.

THEOREM 10.1.2.1 (Fourier transform on Hardy-Sobolev spaces of global adelic θ-functions). *Let F be a number field, $s \geq 0$ a real number, $a \in \mathbf{R}$ a complex number. Let $I \subset \mathbf{R}$ a compact non-empty interval. The continuous linear automorphism $J : W_I^s(C_F) \to W_I^s(C_F)$ induced by inversion on C_F, induces a continuous linear isometry:*

$$J : W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to W_{1-a-I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \tag{10.1.2.1}$$

compatible with passage to C_F^1-isotypes.

Assume $I = [-\alpha, \alpha]$ for some real number $\alpha \geq 0$. The following conditions on a are equivalent:

(1) *Fourier transform $\mathcal{F}$ on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ defined in* (5.2.4.3)*, induces a unique continuous linear automorphism $\mathcal{F}$ of $W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ making the following diagram of continuous linear operators:*

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) & \xrightarrow{E_{a,I}} & W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \\ \Big\downarrow \mathcal{F} & & \Big\downarrow \mathcal{F} \\ \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) & \xrightarrow[E_{a,I}]{} & W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \end{array} \tag{10.1.2.2}$$

commute, where $E_{a,I}$ is the operator sending $\phi \in \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ to the continuous function

$$E_{a,I}(\phi) : g \mapsto |g|_F^a \phi(g) \tag{10.1.2.3}$$

of $g \in C_F$ as an element of $W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

(2) *$a = 1/2$.*

PROOF. We denote by $J : W_I^s(C_F) \to W_I^s(C_F)$ the continuous isometric automorphism uniquely determined by the continuous linear automorphism $J : \mathcal{S}(C_F) \to \mathcal{S}(C_F)$ defined by sending $\phi \in \mathcal{S}(C_F)$ to $(J\phi)(g^{-1})$, a Schwartz-Bruhat function of $g \in C_F$.

[4]Return here.

Since the Sobolev weight $w = \max\{|g|_F^{2\alpha}, |g|_F^{-2\alpha}\}\,(1 + |\log| \cdot |_F|^2)^{s/2}$ is invariant under inversion on C_F, and so is the Haar measure $\overline{\mu}_1$, J is indeed norm-preserving on $W_I^s(C_F)$, and hence induces a continuous isometric automorphism on $W_I^s(C_F)$.

We observe, as a consequence of Theorem 2.2.3.2, that we have the following identity, as elements of $\mathcal{S}(C_F)$:

$$E_{a,I}(\mathcal{F}(\phi)) = JE_{1-a,I}(\phi) \tag{10.1.2.4}$$

from which we deduce that J completes the following diagram of continuous linear operators:

$$\begin{array}{ccc} \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) & \xrightarrow{E_{1-a,I}} & W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \\ \Big\downarrow{\scriptstyle \mathcal{F}} & & \Big\downarrow{\scriptstyle J} \\ \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) & \xrightarrow[E_{a,I}]{} & W_{1-a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \end{array} \tag{10.1.2.5}$$

rendering it commutative.

(1) : Since $|\cdot|_F^z$ is a bounded multiplier on $W_I^s(C_F)$ if and only if $\mathrm{Re}(z) = 0$[5], we deduce from (10.1.2.4) that continuity of $\mathcal{F}$ in the $\|\cdot\|_{a,I}$ is equivalent to $a = 1/2$, in which case we may define $\mathcal{F} := J$. From commutativity of (10.1.2.5) we deduce that (10.1.2.2) commutes with this choice of $\mathcal{F}$ on $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

(2) : If $a = 1/2$, then the definition $\mathcal{F} := J$, indeed provides a continuous linear automorphism on $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ making the diagram in (10.1.2.2) commute, so long as $I = -I$, which holds since $I = [-\alpha, \alpha]$ for a real number $\alpha \geq 0$. □

We refer the reader to Proposition 5.4.2.1 and the discussion preceding the Proposition, for a discussion on the extension by continuity of the S-local Fourier transform on $\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S}))$, to the L^2-completion

$$L^2\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) = W_{\{1/2\}}^0\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})) \tag{10.1.2.6}$$

(the equal sign denoting the surjective isometry sending the element ϕ to $|\cdot|_{F,S}^{1/2}\,\phi$) ultimately a consequence of the fact (see Theorem 3.3.1.5(2)) that $\rho_S(c)$ from (3.3.1.3) is, when restricted to the relative vertical line $\sigma(c) = 1/2$, valued in U(1).

10.2. Spectral Markushevich Bases

We recall that for a complete Hausdorff locally convex topological $\mathbf{C}$-vector space E with a Schauder basis $\{x_m\}_{m\geq 0}$, and $\{f_m\}_{m\geq 0}$ the bi-orthogonal sequence in E', then if E is semi-reflexive, $\{f_m\}_{m\geq 0}$ is also a Schauder basis (see [**Jar**, Part II,Ch.14,§14.5,Thm.1(a),p.302]), and the datum $\{(x_m, f_m)\}_{m\geq 0}$ is a Markushevich basis.

Being a Schauder basis is an unstable property with respect to coarsening of the Hausdorff locally convex topology. The reader may consider the example of the exponential system on $\mathcal{S}(\mathbf{T}) = C^\infty(\mathbf{T})$, with $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ the circle group, which is an absolute Schauder basis with respect to the topology of uniform convergence on functions and their derivatives of arbitrary order, but is not a Schauder basis on $L^1(\mathbf{T}) := L^1(\mathbf{T}, \mu)$, μ being the volume-1 Haar measure on $\mathbf{T}$.

It is, however, a Markushevich basis. The property that is lost under coarsening of the Hausdorff locally convex topology from $\mathcal{S}(\mathbf{T})$ to $L^1(\mathbf{T})$ is that of the simple convergence of the finite rank basis projections.

We let F be a number field, S a finite set of places of F containing all the Archimedean places. Let χ be a continuous character of C_F, respectively of $J_{F,S}$.

In the remaining sections of this Chapter, we analyze to what extent the Schauder bases constructed in the previous Chapter with respect to the strong topology on the Hausdorff locally convex space $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}(\mathbf{A}_F))$

[5]Return here.

and its subquotients, and on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}}, \mathcal{S}(\mathbf{A}_{F,S}))$ and its orbital subquotients, remain bases with respect to the coarser Sobolev topologies introduced in Definition 10.1.1.1.

10.2.1. The a-line Spectral Markushevich Basis. In the special case the compact non-empty interval $I \subset \mathbf{R}$ from the definition of the Hardy-Sobolev spaces $W^s_I(C_F)$ is of the form $I = \{0\}$, we have

$$W^s_I(C_F) = W^{s,2}(C_F)$$

for any real number $s \geq 0$. In this special case, we denote the Hardy-Sobolev completion on the a-line of the space of global adelic θ-functions by:

$$W^{s,2}_a \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) = W^s_{a,I} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)).$$

In this section, we consider the linear topological extension:

$$0 \to W^{s,2}_a \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to W^{s,2}(C_F)(\chi|_{C^1_F}) \to \mathscr{H}^s_{F,a}(\chi|_{C^1_F}) \to 0$$

where $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$ is the normed completion of $\mathscr{H}_{F,a}(\chi|_{C^1_F})$ in Sobolev degree $s \geq 0$, where $\mathscr{H}_{F,a}(\chi|_{C^1_F})$ is defined as the quotient nuclear Fréchet space:

$$\mathscr{H}_{F,a}(\chi|_{C^1_F}) := \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})/|\cdot|^a_F \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$$

where multiplication by the quasi-character $|\cdot|^a_F$, a linear topological automorphism on $\mathcal{S}_{\mathrm{E}}(C_F)$ and $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})$, is isomorphic to a horizontal translation by a to the left on the Mellin transforms of the elements of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$.

We further refer the reader to (7.1.4.7) for the case $a = 0$.

Equivalently, by the Open Mapping Theorem, the quotient of the normed completion $W^{s,2}(C_F)(\chi|_{C^1_F})$ of $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})$ by the closure of $|\cdot|^a_F \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$.

We will be computing the nuclear traces of nuclear operators induced on the rightmost Hilbert space $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$, given by the operators $A(\varphi)$ from (9.3.1.2), for any $\varphi \in \mathcal{D}(C_F)$.

The methods and results discussed in this section build on [**Co**, §III,pp.9-13] and [**Co**, Appendix I,pp.63-68], and, in particular, rework the spectral interpretation to a subset of the zeros of the Hecke L-function $L(s,\chi)$ using the continuous left-translation action of C_F on $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$.

We will be discussing the spectral interpretation to such subsets of the zeros of the Hecke L-function $L(s,\chi)$ already provided in [**Co**, §III,Thm.1,p.13] and [**Co**, §III,Cor.2,p.14], reworking it through the lens of the Schatten-von Neumann convergence of sequences of certain explicit finite rank operators on $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$ provided by Markushevich bases of point-evaluation functionals supported at the ordinates of the zeros on the a-line.

In [**Co**, §III,p.13,14], the case $a = 1/2$ is discussed and the Sobolev space $\mathscr{H}^s_{F,1/2}(\chi|_{C^1_F})$ is denoted by $\mathcal{H}_\chi$, with the infinitesimal generator of the continuous left-translation action by C_F, which factors through $C_F/C^1_F \simeq \mathbf{R}_{>0}$, being denoted by D_χ (see [**Co**, §III,(27),p.14]).

The Sobolev degree s is denoted by δ in [**Co**, §III,Thm.1,p.13].

10.2.1.1. *The a-line basis.* Let $a \in \mathbf{R}$ be a complex number, $s \geq 0$ a real number. We define:

$$\mathscr{H}^s_{F,a} := W^{s,2}(C_F)/W^{s,2}_a \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)). \tag{10.2.1.1}$$

For every $\omega \in \widehat{C^1_F}$, we will be writing $\mathscr{H}^s_{F,a}(\omega)$ for its ω-isotype, which is isometric to the quotient:

$$\mathscr{H}^s_{F,a}(\omega) := W^{s,2}(C_F)(\omega)/W^{s,2}_a \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) \tag{10.2.1.2}$$

by the Open Mapping Theorem (and an application of Lemma 4.3.3.1).

We recall that we denoted the pairing $\langle\cdot,\cdot\rangle_{s,w}$ from (9.1.1.11) by:

$$\langle\cdot,\cdot\rangle_{s,2} : W^{s,2}(C_F) \times W^{-s,2}(C_F) \to \mathbf{C} \tag{10.2.1.3}$$

where $\langle\cdot,\cdot\rangle_{s,2}$ is assigned by:

$$\langle\phi,\psi\rangle_{s,2} := \int_{C_F} \phi(g)\psi(g)\,\mathrm{d}\overline{\mu}(g).$$

The duality pairing $\langle\cdot,\cdot\rangle_{s,2}$ further identifies, by [**NB**, Thm.8.12.4(b)], the continuous dual $(\mathscr{H}^s_{F,a})'$ of the quotient $\mathscr{H}^s_{F,a}$ with the orthogonal in the sense of [**NB**, Def.8.9.1] of $W_a^{s,2}\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}_0(\mathbf{A}_F))$.

We note that the pairing $\langle\cdot,\cdot\rangle_{s,2}$ respects C_F^1-isotypes, i.e. it induces a non-degenerate duality pairing:

$$\langle\cdot,\cdot\rangle_{s,2} : W^{s,2}(C_F)(\omega)\times W^{-s,2}(C_F)(\overline{\omega})\to\mathbf{C}$$

by invariance of the Haar measure $\overline{\mu}$ on C_F.

Notation 10.2.1.1. Throughout, we fix a choice of holomorphic parametrization for $\Omega(C_F)$.

This amounts to choosing, for every $\omega\in\widehat{C_F^1}$, a continuous unitary character $\chi\in\widehat{C_F}$, and parametrize every point of $\Omega(C_F)(\omega)$, the fibre of $\Omega(C_F)$ over $\omega\in\widehat{C_F^1}$, the discrete parameter space of connected components of $\Omega(C_F)$, as:

$$\chi|\cdot|_F^z,\ z\in\mathbf{C}. \tag{10.2.1.4}$$

We will be denoting by $\widetilde{C_F^1}\subset\widehat{C_F}$ the chosen set of base-points for the connected components of $\Omega(C_F)$.

We will be writing $\sum'_{\omega\in\widehat{C_F^1}}$ to denote the sum of $\mathbf{C}$-valued functions depending on χ (such as countable sums of $M(\varphi)(\chi|\cdot|_F^{it})$ for $t\in\mathbf{R}$ varying over a suitable discrete subset, for $\varphi\in\mathcal{D}(C_F)$ fixed, and interpreting the value of the sum as a function of χ), to mean the sum indexed by the choices $\{\chi\in\widehat{C_F}\}$ lifting each point $\omega\in\widehat{C_F^1}$, choices which we fixed at the outset when parametrizing $\Omega(C_F)$ as $\chi|\cdot|_F^z$, $z\in\mathbf{C}$.

Theorem 10.2.1.1 (a-line Spectral Markushevich Basis). *Let F be a number field, $s,a\in\mathbf{R}$ real numbers, $s>1$. The functions:*

$$\eta_{\chi,t,m}(g) := \chi(g)|g|_F^{it}(\log|g|_F)^m \tag{10.2.1.5}$$

such that:

(1) *for $t\in\mathbf{R}$ such that $\rho_{\chi,t}:=\chi|\cdot|_F^{a+it}$ is a zero of $L(s,\chi)$ of multiplicity $N(\rho_{\chi,t})$.*
(2) *$m\geq 0$ is an integer satisfying $m\leq N_s(\rho_{\chi,t})-1$*

with:

$$N_s(\rho_{\chi,t}) = \min\left\{N(\rho_{\chi,t}),\left[\frac{s-1}{2}+1\right]\right\} \tag{10.2.1.6}$$

are elements of $W^{-s,2}(C_F)$, any enumeration admits a bi-orthogonal sequence, and the resulting sequence of pairs is a Markushevich basis for $\mathscr{H}^s_{F,a}$, for the image of $(\mathscr{H}^s_{F,a})'_\beta$ in $W^{-s,2}(C_F)$ under T_s from (9.1.2.14)*, and with respect to the duality pairing $\langle\cdot,\cdot\rangle_{s,2}$ from* (10.2.1.3)*.*

Terminology 10.2.1.1 (a-line Spectral Markushevich Basis). We refer to any sequence of pairs

$$\{(\psi_m,u_m)\}_{m\geq 0}\subset\mathscr{H}^s_{F,a}\times T_s((\mathscr{H}^s_{F,a})'_\beta)$$

obtained from a choice of enumeration of the elements from (10.2.1.5) of $W^{-s,2}(C_F)$ and their bi-orthogonal sequence, as the *a-line Spectral Markushevich Basis*.

Remark 10.2.1.2 (On Markushevich Bases). Since, by [**Jar**, Part II,Ch.14,§14.1,Prop.2,p.290], bi-orthogonal sequences are unique, for a Markushevich basis $\{(x_m,f_m)\}_{m\geq 0}$ for the duality $\langle E,E'\rangle$ with E a Hausdorff locally convex topological $\mathbf{C}$-vector space, we will be omitting either $\{x_m\}_{m\geq 0}$ or $\{f_m\}_{m\geq 0}$ from the notation, referring to the former as a Markushevich basis for E, and to the latter as a Markushevich basis for E'_σ. If E is a Hilbert space, then so is E'_β, and then $\{f_m\}_{m\geq 0}$ linearly generates a subspace of E' which is dense in the strong topology too, by [**Jar**, Part II,Ch.8,§8.2,Prop.1,p.149] and reflexivity. The

sequence $\{x_m\}_{m\geq 0}$ linearly generates a $\sigma(E,E')$-dense subspace, which is, by the same token, dense with respect to the intrinsic topology on E.

Markushevich bases throughout this work will only be with respect to Hilbert normed topologies, and we will therefore be making no distinction between strong and weak denseness of the linear spans of the basis elements.

PROOF OF THEOREM 10.2.1.1. Let $\omega \in \widehat{C_F^1}$ be a unitary character, and $\chi \in \widehat{C_F}$ a unitary character such that $\chi|_{C_F^1} = \omega$. We choose a continuous homomorphic section τ to the idelic norm, and identify C_F/C_F^1 with $\mathbf{R}_{>0}$ as a Hausdorff locally compact abelian group, by means of $|\cdot|_F$.

The following diagram:

$$\begin{array}{ccccc} \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) & \times & \mathcal{S}_{\mathrm{E}}(C_F)(\omega)'_\beta & \longrightarrow & \mathbf{C} \\ \downarrow & & \uparrow & & \| \\ W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) & \times & W^{-s,2}(C_F)(\omega) & \longrightarrow & \mathbf{C} \end{array} \tag{10.2.1.7}$$

is commutative, where the top horizontal arrow is the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathrm{E}}(C_F)(\omega), \mathcal{S}_{\mathrm{E}}(C_F)(\omega)'_\beta}$, noting that by Theorem 6.3.2.1, $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$ is closed as a subspace of $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)$, where $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)$ is the $\mathbf{C}$-linear subspace of the space $\mathcal{S}(\mathbf{R}_{>0})$ of Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ with exponential decay (see Definition D.3.3.1, Lemma D.3.2.7 and Theorem D.3.3.7. Its topology is equivalent to the Hausdorff locally convex topology induced by transport of structure under pull-back along a choice of continuous homomorphic section τ to the idelic norm, by the countable family of seminorms:

$$p_{m,n}(f) := \sup_{t \in \mathbf{R}_{>0}} |e^{-m|\log(t)|}(t\partial_t)^n f(t)|, \quad (m,n) \in \mathbf{N}^2$$

on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$).

The bottom horizontal arrow in (10.2.1.7) is the duality pairing $\langle \cdot, \cdot \rangle_{s,2} : W^{s,2}(C_F) \times W^{-s,2}(C_F) \to \mathbf{C}$ from (10.2.1.3).

The leftmost vertical arrow is the completion map, and the arrow $W^{-s,2}(C_F)(\omega) \to \mathcal{S}_{\mathrm{E}}(C_F)(\omega)'_\beta$ is the continuous linear operator induced as the continuous adjoint of the completion map

$$\mathcal{S}(C_F)(\omega) \to W^{s,2}(C_F)(\omega).$$

As a consequence of commutativity of (10.2.1.7), we obtain a continuous linear map:

$$(\mathscr{H}_{F,a}^s(\omega))'_\beta \to \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))^\perp \tag{10.2.1.8}$$

where the orthogonal $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))^\perp$ within $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)'_\beta$ is being formed with respect to the duality $\langle \mathcal{S}_{\mathrm{E}}(C_F)(\omega), \mathcal{S}_{\mathrm{E}}(C_F)(\omega)'_\beta \rangle$.

Since $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)$ is linearly and topologically isomorphic to $A\mathcal{S}(\mathbf{C})$ under the Mellin transform $\phi \mapsto M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \frac{\mathrm{d}t}{t}$ pre-composed with the linear topological isomorphism induced by pull-back under a fixed choice of continuous homomorphic section τ to the idelic norm, and M restricts to L from (5.1.1.4) on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ up to a vertical translation on the complex s-plane (We further refer the reader to Discussion 4.4.1.1. For $\chi \in \widehat{C_F}$ chosen according to our choice of holomorphic parametrization of $\Omega(C_F)$, and for any $f \in \mathcal{S}_0(\mathbf{A}_F)$, we have, for $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions on $\mathbf{R}_{>0}$:

$$M(\Phi_{F,\chi|_{C_F^1}}(f)(\tau(t)))(s + ix_0) = Z(f, \chi|\cdot|_F^s)$$

as entire functions on the complex s-plane, Schwartz uniformly on the open vertical strips of bounded width) then by Theorem 6.3.2.1 we obtain that $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))^\perp$ is linearly isomorphic to $(((s+a)(s+a-1))^{\delta_\chi} L(a+s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C}))^\perp$ within $A\mathcal{S}(\mathbf{C})'_\beta$, for $\delta_\chi = 0$ for $\chi|_{C_F^1} \neq 1$, and $\delta_\chi = 1$ for $\chi|_{C_F^1} = 1$, which in

turn contains

$$\bigoplus_{\rho: L(a+\rho,\chi)=0} U(J_{a+\rho})$$

as a dense linear subspace, where $J_{a+\rho}$ was defined in (7.1.4.3).

We note that $U(J_{a+\rho})$ is in the image of (10.2.1.8) if and only if $\rho \in i\mathbf{R}$, since otherwise the functionals $U(\delta^{(m)}_{a+\rho})$ are discontinuous with respect to the Sobolev topology on $W^{s,2}(C_F)$, since multiplication by a quasi-character of nonzero exponent is a discontinuous linear operator on $W^{s,2}(C_F)$ (valued in $\overline{\mu}$-measurable functions on C_F. It is only valued in $W^{s,2}(C_F)$ if and only if the exponent of the quasi-character is zero. In which case such multiplication operator is a linear topological automorphism on $W^{s,2}(C_F)$, preserving C^1_F-isotypes).

Explicitly, the element $U(\delta^{(m)}_{a+it})$ is the image of the continuous linear functional on $W^{s,2}(C_F)(\omega)$:

$$D^m_t : \phi \mapsto \int_{C_F} \phi(g)\chi(g)|g|^{it}_F(\log|g|_F)^m \,\mathrm{d}\overline{\mu}(g) = \int_{C_F} \phi(g)\eta_{\chi,t,m}(g)\,\mathrm{d}\overline{\mu}(g) \tag{10.2.1.9}$$

so long as $0 \leq m \leq N_s(\rho_{\chi,t}) - 1$ (by Proposition 9.1.2.5).

Since both $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$ and $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)$ have dense image in $W^{s,2}_a \mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$, respectively $W^{s,2}(C_F)(\omega)$, then the induced continuous linear operator from (10.2.1.8) is injective.

Since the strong topology on $(\mathscr{H}^s_{F,a}(\omega))'_\beta$ is finer than the strong topology on the continuous dual of the nuclear Fréchet space $\mathscr{H}_F(\omega) = \mathcal{S}_{\mathrm{E}}(C_F)(\omega)/\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$ from Theorem 7.1.4.2, then, since the countable set of continuous functionals $U(\delta^{(m)}_\rho)$ on such quotient is topologically free by Theorem 6.3.2.1(3), so is the countable set of continuous functionals $\{D^m_t\}$ from (10.2.1.9) on $\mathscr{H}^s_{F,a}(\omega)$.

It remains to show that the $\{D^m_t\}$ linearly generate a strongly dense subspace of $(\mathscr{H}^s_{F,a}(\omega))'_\beta$, and their bi-orthogonal set, which exists by Theorem 6.3.2.1(3) and is the image of a countable subset of $\mathscr{H}_F(\omega)$, is dense in $\mathscr{H}^s_{F,a}(\omega)$.

By Lemma 9.3.2.1, there exists, for each $\chi \in \widehat{C_F}$ chosen as in the fixed holomorphic parametrization of $\Omega(C_F)$ as in Notation 10.2.1.1, a countable subset $B = \{\varphi_{m,\omega}\}_{(m,\omega)\in\mathbf{N}\times\widehat{C^1_F}} \subset \mathcal{S}(C_F)$ of Schwartz-Bruhat functions satisfying the following properties:

(1) $A(B)$ is norm-bounded by 2 in $W^{s,2}(C_F)$.
(2) For each $(m,\omega) \in \mathbf{N} \times \widehat{C^1_F}$, $\varphi_{m,\omega}$ is ω-homogeneous.
(3) For each integer $m \geq 0$, the Fourier transform of $\varphi_{m,\omega} \in B$ is supported over the compact subset of $\widehat{C_F}$ of characters of the form $\chi|\cdot|^{ix}_F$ for $x \in [-m, m]$.
(4) The closure of $A(B)$ with respect to the strong operator topology on $\mathscr{L}(W^{s,2}(C_F))$ contains the identity.

where A is the continuous linear operator $\mathcal{S}(C_F) \to \mathscr{L}(W^{s,2}(C_F))$ defined by the assignment:

$$A(\varphi) := \int_{C_F} \varphi(g)\psi(g^{-1}x)\,\mathrm{d}\overline{\mu}(g)$$

which was shown to be well-defined and continuous as a linear endomorphism of $W^{s,2}(C_F)$ in Lemma 9.3.2.1

We fix a single $\chi \in \widehat{C_F}$ among the base points of the connected components of $\Omega(C_F)$ according to the chosen holomorphic parametrization of $\Omega(C_F)$ (i.e. we choose a connected component $\Omega(C_F)(\omega)$, and consider the corresponding already chosen lift χ of ω to a continuous character of C_F), and we will be keeping track of this choice through the notation for the rest of this argument.

We define by:

$$P_{\chi,m} : W^{s,2}(C_F)(\chi|_{C^1_F}) \to W^{s,2}(C_F)(\chi|_{C^1_F}) \tag{10.2.1.10}$$

the continuous linear operator $A(\varphi_{m,\chi|_{C^1_F}})$. $P_{\chi,m}$ has range contained within the closed subspace of $W^{s,2}(C_F)(\chi|_{C^1_F})$ of those elements whose Fourier transform (pre-composed with the continuous homomorphic

section τ to the idelic norm) is compactly supported over the interval $[-m,m]$, because it is isomorphic under the Fourier transform on $W^{s,2}(C_F)(\chi|_{C_F^1})$ (which is indeed defined by Proposition 9.1.2.6) to the multiplication operator by the Fourier transform of $\varphi_{m,\chi|_{C_F^1}}$.

We call $Q_{\chi,m}$ the orthogonal projector on $W^{s,2}(C_F)(\chi|_{C_F^1})$ onto such closed subspace, and $\overline{Q}_{\chi,m}$ the orthogonal projector on $\mathscr{H}^s_{F,a}(\chi|_{C_F^1})$ onto the closure of the linear image of such closed subspace under the quotient $W^{s,2}(C_F)(\chi|_{C_F^1}) \to \mathscr{H}^s_{F,a}(\chi|_{C_F^1})$, a closed subspace of $\mathscr{H}^s_{F,a}(\chi|_{C_F^1})$.

Since $P_{\chi,m} \to \mathrm{Id}$ in $\mathscr{B}_{\mathrm{SOT}}(W^{s,2}(C_F)(\chi|_{C_F^1}))$, as $m \to +\infty$, by Lemma 9.3.2.1, the same holds for $Q_{\chi,m}$ as $m \to +\infty$, and therefore $\overline{Q}_{\chi,m} \to \mathrm{Id}$ in $\mathscr{B}_{\mathrm{SOT}}(\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1}))$.

The same therefore holds for the sequence of continuous adjoints $\overline{Q}'_{\chi,m}$ in $(\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1}))'_\beta$, as $m \to +\infty$, in $\mathscr{B}_{\mathrm{SOT}}((\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1}))'_\beta)$.

The range of $\overline{Q}'_{\chi,m}$ in $(\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1}))'_\beta$ is finite dimensional and is linearly generated by the functionals $\{D_t^k\}$ as $t \in [-m,m]$ and $0 \leq k \leq N_s(\rho_{\chi,t}) - 1$. It follows that the range of $\overline{Q}_{\chi,m}$ in $\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1})$ is also finite dimensional and linearly generated by the bi-orthogonal set of the finite set of functionals $\{D_t^k\}$ as $t \in [-m,m]$ and $0 \leq k \leq N_s(\rho_{\chi,t}) - 1$. This proves both denseness conditions for the set of continuous functionals $\{D_t^k\}$ as $0 \leq k \leq N_s(\rho_{\chi,t}) - 1$ and $t \in \mathbf{R}$ varying over the ordinates of the zeros of $L(s,\chi)$ on the a-line, in $(\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1}))'_\beta$, and for its bi-orthogonal set in $\mathscr{H}^s_{F,a}(C_F)(\chi|_{C_F^1})$. □

10.2.1.2. *Basic properties of the a-line basis.* We prove the following properties of the Markushevich basis $\{\eta_{\chi,t,m}\}$ introduced in (10.2.1.5) of Theorem 10.2.1.1, for the orthogonal

$$W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))^\perp$$

in $W^{-s,2}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (10.2.1.3).

We note that the subspace topology on such orthogonal inherited from the normed topology on $W^{-s,2}(C_F)$, is equivalent to the strong topology on the continuous dual of the quotient $\mathscr{H}^s_{F,a}$ equipped with its normed quotient topology: indeed, the latter is, a priori, finer, by [**NB**, Thm.8.11.3(d)]. By [**NB**, Thm.8.12.1(a)] combined with the Open Mapping Theorem, the two topologies are equivalent.

PROPOSITION 10.2.1.3. *Let F be a number field, let a, s be real numbers, $s > 1$.*

There exists a real number $C > 0$ such that the functions:

$$\eta_{\chi,t,m}(g) := \chi(g)|g|_F^{it}(\log|g|_F)^m \tag{10.2.1.11}$$

as:

1. *χ ranges over the base-points for the chosen holomorphic parametrization for $\Omega(C_F)$ from* (6.3.1.1).
2. *t ranges over the imaginary parts of the zeros of the Hecke L-function $L(s,\chi)$ on the vertical line $\mathrm{Re}(s) = a$, with $0 \leq m \leq N(\rho_{\chi,t}) - 1$, where $N_s(\rho_{\chi,t})$ is the function defined in* (10.2.1.6).

all satisfy:

$$\{\chi(g)|g|_F^{it}(\log|g|_F)^m\}_{t,m} \subset B_C := \{\phi \in W^{-s,2}(C_F) : 1/C \leq \|\phi\|_{-s,2} \leq C\}. \tag{10.2.1.12}$$

In particular, the set of functions from (10.2.1.11) *is not contained in the open ball in $W^{-s,2}(C_F)$ of radius $1/C$.*

PROOF. As for Proposition 11.2.0.4, we proceed as follows. For $s > 1$, then:

$$\|\eta_{\chi,t,m}\|_{-s,2} = \left(\int_{C_F} |\log|g|_F|^{2m}(1 + |\log|g|_F|^2)^{-s/2}\,\mathrm{d}\overline{\mu}(g)\right)^{1/2}$$

by Proposition 9.1.2.5, and so $\|\eta_{\chi,t,m}\|_{-s,2}$ can only attain finitely many values, none of which is zero. There therefore exists a real number $C > 0$ such that $1/C$ is a strict lower bound for their minimum, and C is a strict upper bound for their maximum. □

TERMINOLOGY 10.2.1.2 (Normalized Markushevich Bases). As before Corollary 11.2.0.5, we recall the following terminology from [Kal], which applies to Schauder bases.

We recall from [Kal, §3,p.95] that a Schauder basis $\{b_k\}_{k\geq 0}$ in a Hausdorff locally convex topological $\mathbf{C}$-vector space E is *simple* if and only if the finite rank projection operators $P_N := \sum_{0\leq k\leq N}\langle\varphi_k,\cdot\rangle_{E',E}b_k$, for $\{\varphi_k\}_{k\geq 0}$ the dual basic sequence of $\{b_k\}_{k\geq 0}$, are a bounded set in $\mathscr{L}_\beta(E)$, i.e. they are strongly bounded.

We further recall from [Kal, §1,Def.1.1,p.91] that a Schauder basis $\{b_k\}_{k\geq 0}$ of E is *regular* if and only if there exists a 0-neighborhood V such that $b_k \notin V$ for all $k \geq 0$.

Finally, from [Kal, §1,Def.1.2], a Schauder basis is *normalized* if and only if it is bounded and regular.

We extend the concepts of boundedness, regularities and normalization, to Markushevich bases, as follows. Let E be Banach space, and $\{(x_m, f_m)\}_{m\geq 0} \subset E \times E'_\beta$ a Markushevich basis. We say it is:

(1) *Bounded*, if and only if $\{x_m\}_{m\geq 0} \subset E$ is a bounded subset.
(2) *Regular*, if and only if $\{x_m\}_{m\geq 0} \subset E$ is strictly bounded away from $0 \in E$. In particular, $\{f_m\}_{m\geq 0} \subset E'_\beta$ is bounded.
(3) *Normalized*, if and only if it is both bounded and regular.

We have the following:

COROLLARY 10.2.1.4. *Let F be a number field, $s > 1$ a real number. Any enumeration of the set of functions:*

$$\eta_{\chi,t,m}(g) := \chi(g)|g|_F^{it}(\log|g|_F)^m \tag{10.2.1.13}$$

as:

(1) *χ ranges over the base-points for the chosen holomorphic parametrization for $\Omega(C_F)$ from* (6.3.1.1).
(2) *t ranges over the imaginary parts of the zeros of the Hecke L-function $L(s,\chi)$ on the vertical line $\mathrm{Re}(s) = a$, with $0 \leq m \leq N_s(\rho_{\chi,t}) - 1$, where $N_s(\rho_{\chi,t})$ is the function defined in* (10.2.1.6).

is a normalized Markushevich basis for the image of $(\mathscr{H}^s_{F,a})'_\beta$ in $W^{-s,2}(C_F)$ under T_s from (9.1.2.14)*, i.e. for the orthogonal $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))^\perp$ in $W^{-s,2}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{s,2}$ from* (10.2.1.3).

PROOF. As a subset of $W^{-s,2}(C_F)$, there exists a real number $C > 0$ such that the set of functions $\{\eta_{\chi,t,m}\}_{\chi,t,m}$ is contained in the annulus B_C from (10.2.1.12) of Proposition 10.2.1.3.

In particular, as per Proposition 10.2.1.3, any enumeration of $\{\eta_{\chi,t,m}\}_{\chi,t,m}$, a Markushevich basis for $(\mathscr{H}^s_{F,a})'_\beta$ by Theorem 10.2.1.1, is regular and bounded, hence normalized. □

REMARK 10.2.1.5 (Approximations of the identity and basis projectors). We follow the notational choices made in Terminology 10.2.1.2.

In the special case $E = \mathscr{H}^s_{F,a}(\chi|_{C^1_F})$, for $a, s \in \mathbf{R}$, $s > 1$, the a-line spectral Markushevich bases $\{(x_m, f_m)\}_{m\geq 0} \subset E \times E'_\beta$ provided by Theorem 10.2.1.1 are all normalized by Proposition 10.2.1.3; see Corollary 10.2.1.4.

They also satisfy a close analog of the "simplicity" condition: $\{x_m\}_{m\geq 0}$ is not a Schauder basis, and so the basis projections, all finite rank idempotent operators, do not converge strongly to the identity. However, the orthogonal projectors $\overline{Q}_{\chi,N}$ from the proof of Theorem 10.2.1.1, onto the closed subspace of those elements of $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$ which are images of elements in $W^{s,2}(C_F)(\chi|_{C^1_F})$ whose Fourier transform, pre-composed with pull-back along the chosen continuous homomorphic section τ to the idelic norm, is compactly supported over the interval $[-N, N] \subset \mathbf{R}$, serve as a replacement of the basis projectors, and they do converge to the identity on $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$ strongly, as a consequence of Lemma 9.3.2.1.

10.2.2. Spectral Markushevich basis: the general case of a compact interval $I = [\alpha, \beta]$. In this section, we consider the general case of the linear topological extension:

$$0 \to W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \to W^s_I(C_F)(\chi|_{C^1_F}) \to \mathscr{H}^s_{F,a,I}(\chi|_{C^1_F}) \to 0$$

where $\mathscr{H}^s_{F,a,I}(\chi|_{C^1_F})$ is the normed completion of $\mathscr{H}_{F,a,I}(\chi|_{C^1_F})$ in Sobolev degree $s \geq 0$, with I a compact non-empty interval $I = [-\alpha, \alpha]$ for a real number $\alpha \geq 0$, with $\mathscr{H}_{F,a}(\chi|_{C^1_F})$ defined as the quotient nuclear Fréchet space:

$$\mathscr{H}_{F,a}(\chi|_{C^1_F}) := \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})/|\cdot|^a_F \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$$

where multiplication by the quasi-character $|\cdot|^a_F$, a linear topological automorphism on $\mathcal{S}_{\mathrm{E}}(C_F)$ and $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C^1_F})$, is isomorphic to a horizontal translation by a to the left on the Mellin transforms of the elements of $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F))$, and $\mathscr{H}^s_{F,a,I}(\chi|_{C^1_F})$ is the quotient Hilbert space, equivalently (by the Open Mapping Theorem) the completion of $\mathscr{H}^s_{F,a}(\chi|_{C^1_F})$ with respect to the normed topology induced by the quotient $\|\cdot\|_{I,s}$-norm.

We will be computing the nuclear traces of nuclear operators induced on the rightmost Hilbert space $\mathscr{H}^s_{F,a,I}(\chi|_{C^1_F})$, given by the operators $A(\varphi)$ from (9.3.1.2), for any $\varphi \in \mathcal{D}(C_F)$ and $I = [-\alpha, \alpha]$.

10.2.2.1. *The Markushevich basis on the I-based strip.* Let $a, s \in \mathbf{R}$ be real numbers, $s \geq 0$. We define:

$$\mathscr{H}^s_{F,a,I} := W^s_I(C_F)/W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)). \tag{10.2.2.1}$$

For every $\omega \in \widehat{C^1_F}$, we will be writing $\mathscr{H}^s_{F,a,I}(\omega)$ for its ω-isotype, which is isometric to the quotient:

$$\mathscr{H}^s_{F,a,I}(\omega) := W^s_I(C_F)(\omega)/W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) \tag{10.2.2.2}$$

by the Open Mapping Theorem (and an application of Lemma 4.3.3.1).

We recall that we denoted the pairing $\langle\cdot,\cdot\rangle_{I,s}$ from (9.2.1.28) by:

$$\langle\cdot,\cdot\rangle_{I,s} : W^s_I(C_F) \times W^{-s}_I(C_F) \to \mathbf{C} \tag{10.2.2.3}$$

where $\langle\cdot,\cdot\rangle_{I,s}$ is assigned by:

$$\langle\phi,\psi\rangle_{I,s} := \int_{C_F} \phi(g)\psi(g)\,\mathrm{d}\overline{\mu}(g).$$

The duality pairing $\langle\cdot,\cdot\rangle_{I,s}$ further identifies, by [**NB**, Thm.8.12.4(b)], the continuous dual $(\mathscr{H}^s_{F,a,I})'$ of the quotient $\mathscr{H}^s_{F,a,I}$ with the orthogonal in the sense of [**NB**, Def.8.9.1] of $W^s_{a,I}\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

We note that the pairing $\langle\cdot,\cdot\rangle_{I,s}$ respects C^1_F-isotypes, i.e. it induces a non-degenerate duality pairing:

$$\langle\cdot,\cdot\rangle_{I,s} : W^s_I(C_F)(\omega) \times W^{-s}_I(C_F)(\overline{\omega}) \to \mathbf{C}$$

by invariance of the Haar measure $\overline{\mu}$ on C_F.

Notation 10.2.2.1. Throughout, we fix a choice of holomorphic parametrization for $\Omega(C_F)$ as in Notation 10.2.1.1.

This amounts to choosing, for every $\omega \in \widehat{C^1_F}$, a continuous unitary base-point $\chi \in \widehat{C_F}$, and parametrize every point of $\Omega(C_F)(\omega)$ as $\chi|\cdot|^z_F$ for a unique $z \in \mathbf{C}$.

We will be denoting by $\widehat{C^1_F} \subset \widehat{C_F}$ the chosen set of unitary base-points for the connected components of $\Omega(C_F)$.

We will be writing $\sum'_{\omega \in \widehat{C^1_F}}$ to denote the sum of $\mathbf{C}$-valued functions depending on χ (such as countable sums of $M(\varphi)(\chi|\cdot|^{it}_F)$ for $t \in \mathbf{R}$ varying over a suitable discrete subset, for $\varphi \in \mathcal{D}(C_F)$ fixed, and interpreting the value of the sum as a function of χ), to mean the sum indexed by the choices $\{\chi \in \widehat{C_F}\}$ lifting each point $\omega \in \widehat{C^1_F}$, choices which we fixed at the outset when parametrizing $\Omega(C_F)$ as $\chi|\cdot|^z_F$, $z \in \mathbf{C}$.

THEOREM 10.2.2.1 (Spectral Markushevich Basis on the I-based strip). *Let F be a number field, $s, a \in \mathbf{R}$ real numbers, $s \geq 0$. Let $I = [-\alpha, \alpha]$ be a compact interval, $\alpha \geq 0$ a real number. The functions:*

$$\eta_{\chi,t,b,m}(g) := \chi(g)|g|_F^{it+b}(\log|g|_F)^m \tag{10.2.2.4}$$

such that:

(1) *for $t \in \mathbf{R}$ such that $\rho_{\chi,t,b} := \chi|\cdot|_F^{a+it+b}$ is a zero of $L(s,\chi)$ of multiplicity $N(\rho_{\chi,t,b})$.*
(2) *$m \geq 0$ is an integer satisfying $m \leq N_{I,s}(\rho_{\chi,t,b}) - 1$*

with:

$$N_{I,s}(\rho_{\chi,t,b}) = \min\left\{N(\rho_{\chi,t,b}), \left[\frac{s-1}{2}+1\right]\right\},\ b = \pm\alpha \tag{10.2.2.5}$$

and

$$N_{I,s}(\rho_{\chi,t,b}) = N(\rho_{\chi,t,b}),\ b \in (-\alpha, \alpha) \tag{10.2.2.6}$$

are elements of $W_I^{-s}(C_F)$, any enumeration admits a bi-orthogonal sequence, and the resulting sequence of pairs is a Markushevich basis for $\mathscr{H}^s_{F,a,I}$, for the image of $(\mathscr{H}^s_{F,a,I})'_\beta$ in $W_I^{-s}(C_F)$ under $T_{I,s}$ from (9.2.1.31)*, and with respect to the duality pairing $\langle\cdot,\cdot\rangle_{I,s}$ from* (10.2.2.3)*.*

TERMINOLOGY 10.2.2.1 (Spectral Markushevich Basis on the I-bases strip). We refer to any sequence of pairs

$$\{(\psi_m, u_m)\}_{m\geq 0} \subset \mathscr{H}^s_{F,a,I} \times T_{I,s}((\mathscr{H}^s_{F,a,I})'_\beta)$$

obtained from a choice of enumeration of the elements from (10.2.2.4) of $W_I^{-s}(C_F)$ and their bi-orthogonal sequence, as the *Spectral Markushevich Basis on the I-based strip.*

PROOF. The proof of Theorem 10.2.2.1 rests on the same method as in the proof of Theorem 10.2.1.1, with Theorem 9.3.2.6 replacing Lemma 9.3.2.1.[6] □

10.2.2.2. *Basic properties of the Spectral Markushevich Basis on the I-based strip.* We prove the following properties of the Markushevich basis $\{\eta_{\chi,t,b,m}\}$ introduced in (10.2.2.4) of Theorem 10.2.2.1, for the orthogonal

$$W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))^\perp$$

in $W_I^{-s}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{I,s}$ from (10.2.2.3).

We note that the subspace topology on such orthogonal inherited from the normed topology on $W_I^{-s}(C_F)$, is equivalent to the strong topology on the continuous dual of the quotient $\mathscr{H}^s_{F,a,I}$ equipped with its normed quotient topology by the Open Mapping Theorem.

PROPOSITION 10.2.2.2. *Let F be a number field, let a, s be real numbers, $s > 1$. Let $I = [-\alpha, \alpha] \subset \mathbf{R}$ be a compact non-empty interval, $\alpha \geq 0$ a real number.*

There exists a real number $C > 0$ such that the functions:

$$\eta_{\chi,t,b,m}(g) := \chi(g)|g|_F^{it+b}(\log|g|_F)^m \tag{10.2.2.7}$$

as:

(1) *χ ranges over the base-points for the chosen holomorphic parametrization for $\Omega(C_F)$ from* (6.3.1.1)*.*
(2) *t ranges over the imaginary parts of the zeros of the Hecke L-function $L(s,\chi)$ on the vertical line $\mathrm{Re}(s) = a$, with $0 \leq m \leq N_{I,s}(\rho_{\chi,t,b}) - 1$, where $N_{I,s}(\rho_{\chi,t})$ is the function defined in* (10.2.2.5) *and* (10.2.2.5)*.*

all satisfy:

$$\{\chi(g)|g|_F^{it+b}(\log|g|_F)^m\}_{t,b,m} \subset B_C := \{\phi \in W_I^{-s}(C_F) : 1/C \leq \|\phi\|_{I,-s} \leq C\}. \tag{10.2.2.8}$$

[6] Return here.

In particular, the set of functions from (10.2.2.7) *is not contained in the open ball in* $W^{-s,2}(C_F)$ *of radius* $1/C$.

PROOF. For $s > 1$, then:

$$\|\eta_{\chi,t,b,m}\|_{I,-s} = \left(\int_{C_F} \frac{|g|_F^{2b}\,|\log|g|_F|^{2m}}{\max\{|g|_F^{2\alpha}, |g|_F^{-2\alpha}\}\,(1+|\log|g|_F|^2)^{s/2}}\,\mathrm{d}\overline{\mu}(g)\right)^{1/2}$$

by Proposition 9.2.2.11, and so $\|\eta_{\chi,t,m}\|_{I,-s}$ can only attain values within a compact interval which does not contain $0 \in \mathbf{R}$. There therefore exists a real number $C > 0$ such that $1/C$ is a strict lower bound for their minimum, and C is a strict upper bound for their maximum. □

We retain, throughout this subsection, the lexicon introduced in Terminology 10.2.1.2.

We have the following:

COROLLARY 10.2.2.3. *Let* F *be a number field,* $s > 1$ *a real number. Any enumeration of the set of functions:*

$$\eta_{\chi,t,b,m}(g) := \chi(g)|g|_F^{it+b}(\log|g|_F)^m \tag{10.2.2.9}$$

as:

(1) χ *ranges over the base-points for the chosen holomorphic parametrization for* $\Omega(C_F)$ *from* (6.3.1.1).
(2) t *ranges over the imaginary parts of the zeros of the Hecke L-function* $L(s,\chi)$ *on the vertical line* $\mathrm{Re}(s) = a$, *with* $0 \leq m \leq N_{I,s}(\rho_{\chi,t,b}) - 1$, *where* $N_{I,s}(\rho_{\chi,t,b})$ *is the function defined in* (10.2.2.5) *and* (10.2.2.6).

is a normalized Markushevich basis for the image of $(\mathscr{H}^s_{F,a,I})'_\beta$ *in* $W_I^{-s}(C_F)$ *under* $T_{I,s}$ *from* (9.2.1.31), *i.e. for the orthogonal* $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))^\perp$ *in* $W^{-s,2}(C_F)$ *with respect to the pairing* $\langle\cdot,\cdot\rangle_{I,s}$ *from* (9.2.1.28).

PROOF. As a subset of $W_I^{-s}(C_F)$, there exists a real number $C > 0$ such that the set of functions $\{\eta_{\chi,t,b,m}\}_{\chi,t,b,m}$ is contained in the annulus B_C from (10.2.2.8) of Proposition 10.2.2.2.

In particular, as per Proposition 10.2.2.2, any enumeration of $\{\eta_{\chi,t,b,m}\}_{\chi,t,b,m}$, a Markushevich basis for $(\mathscr{H}^s_{F,a,I})'_\beta$ by Theorem 10.2.2.1, is regular and bounded, hence normalized. □

10.3. Spectral Nuclearity

A consequence of Theorem 10.2.1.1 is that it enables us to identify certain nuclear operators on $\mathscr{H}^s_{F,a}$ explicitly, and compute their trace.

The contents of this subsection reprise [**Co**, §III,Cor.2,pp.13,14] and especially [**Co**, Appendix II,pp.62,68], providing only a few complements on the convergence of certain finite rank operators with respect to the appropriate Schatten-von Neumann norms (see Definition C.3.6.2) so as to enable a discussion on explicit integral kernels early in the next Chapter.

We recall from [**Kothe-I**, Part III,Ch.19,§19.9] that for two Banach spaces E_1, E_2, the $\mathbf{C}$-linear space $\mathscr{S}(E_1, E_2)$ denotes the $\mathbf{C}$-linear space of those linear operators $T : E_1 \to E_2$ which can be written as

$$T = \sum_{m\geq 0} \lambda_m \langle a_m, \cdot\rangle_{E_1', E_2} \cdot y_m$$

with $\{a_m\}_{m\geq 0} \subset B_{E_1'}$, $\{y_m\}_{m\geq 0} \subset B_{E_2}$, where B_E denotes the unit ball of the Banach space E, and $\{\lambda_m\}_{m\geq 0} \subset \mathfrak{s}$, where $\mathfrak{s}$ is the Köthe sequence space of sequences of complex numbers $\{\lambda_m\}_{m\in\mathbf{Z}}$ that are rapidly decaying, i.e. such that $\{m^k\lambda_m\}_{m\in\mathbf{Z}} \subset \mathfrak{c}_0$ for every integer $k \geq 0$ (equivalently, $\mathfrak{s} = \mathcal{S}(\mathbf{Z})$, the space of Schwartz-Bruhat functions on the discrete abelian group $\mathbf{Z}$).

The space $\mathfrak{s}$ is a Fréchet space when endowed with the countable family of seminorms:

$$p_k(\{\lambda_m\}_{m\geq 0}) := \sum_{m\in\mathbf{Z}} (1+|m|)^k|\lambda_m|,\ k \geq 0.$$

We further refer the reader to [**Tre**, Part III,Ch.51,Rem.51.1,p.527] for more on $\mathfrak{s} = \mathcal{S}(\mathbf{Z})$.

The linear space $\mathscr{S}(E_1, E_2)$ can be made into a complete metrizable topological vector space (see [**Kothe-I**, Part III,Ch.19,§19.9,Thm.3]).

$\mathscr{S}(E_1, E_2)$ is a linear subspace of the space $S_1(E_1, E_2)$ of nuclear operators, and when $E_1 = E_2$ is a separable Hilbert space, $S_1(E) := S_1(E_1, E_2)$ is the $\mathbf{C}$-linear space of trace class linear endomorphisms on E.

There is a well-defined trace functional

$$\text{Trace} : S_1(E) \to \mathbf{C}$$

which, for E a closed invariant subspace or quotient of $W^{s,2}(C_F)$, we shall denote by Trace to record the dependence on the Sobolev degree s, i.e. the real number $s \geq 0$ built into the Sobolev weight defining $W^{s,2}(C_F)$.

We refer the reader to §C.3 and §C.3.4 for more on the trace class and nuclearity.

We specifically refer the reader to Definition C.3.4.2 for the trace of a nuclear endomorphism of a Hilbert space.

10.3.1. Spectral nuclearity: the case $I = \{0\}$.

THEOREM 10.3.1.1 (Spectral Nuclearity - I). *Let F be a number field. Let $a, s \in \mathbf{R}$ be real numbers, with $s > 1$. We choose a subset $\widetilde{C_F^1} \subset \widehat{C_F}$ of representatives for the countable group of continuous characters $\widehat{C_F^1}$, equivalently, a holomorphic parametrization for the Riemann surface $\Omega(C_F)$ of the form*

$$\chi|\cdot|_F^z, \ (\chi, z) \in \widetilde{C_F^1} \times \mathbf{C}$$

as in (10.2.1.4) *and Notation 10.2.1.1.*

All characters $\chi \in \widehat{C_F}$ occurring below, are assumed to vary over the chosen subset $\widetilde{C_F^1}$, i.e. they are assumed to vary over the chosen base-points of the connected components of $\Omega(C_F)$.

Let $\varphi \in \mathcal{D}(C_F)$ be a test function.

(1) *For any choice of enumeration $\{\eta_{\chi_k,t_{k,j},m_{k,j}}\}$, with $(k,j) \in \mathbf{N} \times \mathbf{N}$ and $0 \leq m_{k,j} \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$, for the set of elements $\{\eta_{\chi,t,m}\} \subset W^{-s,2}(C_F)$ from Theorem 10.2.1.1, the sequence of finite rank operators:*

$$\sum_{0\leq k\leq N_1} \sum_{0\leq j\leq N_2} \sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1} \langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \cdot\rangle_{s,2}\, \psi_{\chi_k,t_{k,j},m_{k,j}} \tag{10.3.1.1}$$

in $\mathscr{L}_\beta(\mathscr{H}^s_{F,a})$ converges in $S_p(\mathscr{H}^s_{F,a})$ for every $p \geq 1$.

(2) *The $\mathcal{D}(C_F)$-module structure:*

$$A : \mathcal{D}(C_F) \to \mathscr{L}_\beta(\mathscr{H}^s_{F,a}) \tag{10.3.1.2}$$

is valued in the Banach space of nuclear operators $S_1(\mathscr{H}^s_{F,a})$ on $\mathscr{H}^s_{F,a}$, and is continuous with respect to the Hausdorff locally convex topology on the source, and the trace-norm topology on the target. In particular, the composition of A and the trace functional on $S_1(\mathscr{H}^s_{F,a})$ is a distribution on C_F, whose value at $\varphi \in \mathcal{D}(C_F)$ is:

$$\text{Trace } A(\varphi) = \sum_{\chi\in\widetilde{C_F^1}} \sum_{t\in\mathbf{R}:L(a+it,\chi)=0} N_s(\rho_{\chi,t}) \int_{C_F} \varphi(g)\chi|g|_F^{it}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.1.3}$$

where N_s is the counting function:

$$N_s(\rho_{\chi,t}) = \min\left\{N(\rho_{\chi,t}), \left[\frac{s-1}{2} + 1\right]\right\} \tag{10.3.1.4}$$

where $[x]$ is the largest integer $N \geq 0$ satisfying $N \leq x$.

(3) *If φ is homogeneous of weight $\overline{\omega} \in \widehat{C_F^1}$ under the continuous left-translation action of C_F on $\mathcal{D}(C_F)$, then $A(\varphi)$ is valued in the linear subspace $\mathscr{S}(\mathscr{H}^s_{F,a}(\omega))$ of strongly nuclear operators on the isotype $\mathscr{H}^s_{F,a}(\omega)$, and its trace is:*

$$\text{Trace } A(\varphi) = \sum_{t \in \mathbf{R}: L(a+it,\chi)=0} N_s(\rho_{\chi,t}) \int_{C_F} \varphi(g)\chi|g|_F^{it}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.1.5}$$

for $\chi \in \widetilde{C_F^1}$ the unique element of $\widetilde{C_F^1}$ such that $\chi|_{C_F^1} = \omega$, i.e. where χ is the base-point of $\Omega(C_F)(\omega)$ for the chosen holomorphic parametrization of $\Omega(C_F)$ from (10.2.1.4).

Proof. We choose an enumeration for the set $\{\eta_{\chi,t,m}\}$ of functions in $W^{-s,2}(C_F)$ from (10.2.1.5), and we denote it:

$$\{\eta_{\chi_k,t_{k,j},m_{k,j}}\}_{(k,j)\in\mathbf{N}\times\mathbf{N}, 0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1}. \tag{10.3.1.6}$$

By Theorem 10.2.1.1, the sequence of vectors in $W^{-s,2}(C_F)$ from (10.3.1.6) is a Markushevich basis for the orthogonal of $\mathscr{H}^s_{F,a}$ with respect to the duality pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (9.1.1.11).

(1) : To see this, we proceed as follows: we first show that the sequence of finite rank operators from (10.3.1.1):

$$\sum_{0\leq k\leq N_1} \sum_{0\leq j\leq N_2} \sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1} \langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \cdot\rangle_{s,2}\, \psi_{\chi_k,t_{k,j},m_{k,j}} \tag{10.3.1.7}$$

in $\mathscr{L}_\beta(\mathscr{H}^s_{F,a})$ converges in $S_p(\mathscr{H}^s_{F,a})$ for every $p \geq 1$.

We fix k, j such that $0 \leq k \leq N_1$, and $0 \leq j \leq N_2$. We have:

(10.3.1.8)

$$A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}(y) = \int_{C_F} \varphi(g)\chi_k(gy)|gh|_F^{it_{k,j}} (\log|gy|_F)^{m_{k,j}}\, \mathrm{d}\overline{\mu}(g)$$

$$= \int_{C_F} \varphi(g)\chi_k(g)\chi_k(y)|g|_F^{it_{k,j}}|y|_F^{it_{k,j}} (\log|g|_F + \log|y|_F)^{m_{k,j}}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.1.9}$$

$$= \int_{C_F} \varphi(g)\chi_k(g)\chi_k(y)|g|_F^{it_{k,j}}|y|_F^{it_{k,j}} \sum_{0\leq i\leq m_{k,j}} \binom{m_{k,j}}{i} (\log|g|_F)^i (\log|y|_F)^{m_{k,j}-i}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.1.10}$$

$$= \int_{C_F} \varphi(g)\chi_k(g)|g|_F^{it_{k,j}} \sum_{0\leq i\leq m_{k,j}} \binom{m_{k,j}}{i} (\log|g|_F)^i \left\{\chi_k(h)|y|_F^{it_{k,j}} (\log|y|_F)^{m_{k,j}-i}\right\} \mathrm{d}\overline{\mu}(g) \tag{10.3.1.11}$$

$$= \sum_{0\leq i\leq m_{k,j}} \binom{m_{k,j}}{i} \eta_{\chi_k,t_{k,j},m_{k,j}-i}(y) \int_{C_F} \varphi(g)\chi_k(g)|g|_F^{it_{k,j}} (\log|g|_F)^i\, \mathrm{d}\overline{\mu}(g). \tag{10.3.1.12}$$

We now compute:

(10.3.1.13)

$$\langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \psi_{\chi_k,t_{k,j},m'_{k,j}}\rangle_{s,2} = \binom{m_{k,j}}{m_{k,j}-m'_{k,j}} \int_{C_F} \varphi(g)\chi_k(g)|g|_F^{it_{k,j}} (\log|g|_F)^{m_{k,j}-m'_{k,j}}\, \mathrm{d}\overline{\mu}(g)$$

for each fixed $0 \leq k \leq N_1$, $0 \leq j \leq N_2$, and for each $0 \leq m'_{k,j} \leq m_{k,j}$, and

$$\langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \psi_{\chi_k,t_{k,j},m'_{k,j}}\rangle_{s,2} = 0 \tag{10.3.1.14}$$

if $m_{k,j} < m'_{k,j} \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$.

From the Markushevich basis $\{\psi_{\chi_k,t_{k,j},m_{k,j}}\}$ for $\mathscr{H}^s_{F,a}$, we define the subset:

$$B_K := \{\psi_{\chi_{k'},t_{k',j'},m_{k',j'}}\}_{0\leq m_{k',j'}\leq N_s(\rho_{\chi_{k'},t_{k',j'}})-1, (k',j')\in K}$$

for each fixed finite subset $K \subset \mathbf{N}\times\mathbf{N}$.

For each such K and each fixed $k \in \mathbf{N}$ indexing the elements of $\widetilde{C_F^1}$, we further define the interval of the real line $[a_{K,k}, b_{K,k}]$ where, upon defining $a'_K := \inf\{t_{k,j} : (k', j) \in K, k' = k\}$, $b'_K := \sup\{t_{k,j} : (k, j) \in K\}$, we define $a_{K,k} := a'_K - \varepsilon$ and $b_{K,k} := b'_K + \varepsilon$, for a choice of $\varepsilon > 0$ such that the vertical segment $a + i[a_{K,k}, b_{K,k}]$ contains no further zeros of $L(s, \chi_k)$ other than $\{a + it_{k,j} : (k, j) \in K\}$.

For each pair of integers (k, j) such that $(k, j) \in K$, we define the finite rank operator:

$$A_{K,(k,j)} := \sum_{0 \leq m_{k,j} \leq N_s(\rho_{\chi_k, t_{k,j}}) - 1} \langle A(\varphi)' \eta_{\chi_k, t_{k,j}, m_{k,j}}, P_{B_K}(\cdot) \rangle_{s,2} \, \psi_{\chi_k, t_{k,j}, m_{k,j}} \tag{10.3.1.15}$$

in $\mathscr{L}(\mathscr{H}^s_{F,a})$, where P_{B_K} is the orthogonal projector onto the closed subspace of $\mathscr{H}^s_{F,a}$ linearly and topologically generated by the linear image of the closed subspace of $W^{s,2}(C_F)$ of those elements whose Fourier transform is supported over the subset:

$$\{\chi_k | \cdot |_F^{i[a_{K,k}, b_{K,k}]}, k \in \mathbf{N}\} \subset \widehat{C_F}.$$

We note that in light of the orthogonality of the $\chi_k|_{C_F^1}$- and $\chi_{k'}|_{C_F^1}$-isotypes, for $k \neq k'$, then for each fixed integer k, the finite rank operator, whose range is indeed contained within $\mathscr{H}^s_{F,a}(\chi_k|_{C_F^1})$, defined in (10.3.1.15), factors through the orthogonal projection $P_{B_K,k}$ onto the closed subspace of $\mathscr{H}^s_{F,a}(\chi_k|_{C_F^1})$ given by the $\chi_k|_{C_F^1}$-isotype of the range of P_{B_K}. Such subspace is finite dimensional, since is dual is linearly and topologically generated by the elements of $W^{-s,2}(C_F)(\overline{\chi}_k|_{C_F^1})$:

$$\eta_{\chi_k, t_{k,j}, m_{k,j}} : t_{k,j} \in [a_{K,k}, b_{K,k}], 0 \leq m_{k,j} \leq N_s(\rho_{\chi_k, t_{k,j}}) - 1. \tag{10.3.1.16}$$

These elements are finitely many, and their number is bounded above by:

$$\left[\frac{s-1}{2} + 1\right] \log(\max\{|a_{K,k}|, |b_{K,k}|\})$$

by [**Lang-b**, Ch.XVII,§2,Prop.2.4,p.334].

The following factorization for $A_{K,(k,j)}$ therefore holds:

$$A_{K,(k,j)} = A^{\text{finite}}_{K,(k,j)} \circ P_{B_K,k}$$

for $A^{\text{finite}}_{K,(k,j)}$ the unique finite rank operator $A^{\text{finite}}_{K,(k,j)} \in \mathscr{L}(\text{Im}(P_{B_K,k}))$ induced by $A_{K,(k,j)}$, because the range of $P_{B_K,k}$ is a finite dimensional indecomposable sub-representation of C_F.

$A^{\text{finite}}_{K,(k,j)} = A^{\text{finite}}_{\{(k,j)\},(k,j)}$ is represented by the $N \times N$ matrix:

(10.3.1.17)

$$\begin{bmatrix}
\widehat{\varphi}(it_{k,j}) & 0 & 0 & 0 & 0 & \cdots & 0 \\
\widehat{\varphi}^{(1)}(it_{k,j}) & \widehat{\varphi}(it_{k,j}) & 0 & 0 & 0 & \cdots & 0 \\
\widehat{\varphi}^{(2)}(it_{k,j}) & \binom{2}{1}\widehat{\varphi}^{(1)}(it_{k,j}) & \widehat{\varphi}(it_{k,j}) & 0 & 0 & \cdots & 0 \\
\widehat{\varphi}^{(3)}(it_{k,j}) & \binom{3}{2}\widehat{\varphi}^{(2)}(it_{k,j}) & \binom{3}{1}\widehat{\varphi}^{(1)}(it_{k,j}) & \widehat{\varphi}(it_k) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\widehat{\varphi}^{(N-1)}(it_{k,j}) & \binom{N-1}{N-2}\widehat{\varphi}^{(N-2)}(it_{k,j}) & \binom{N-1}{N-3}\widehat{\varphi}^{(N-3)}(it_{k,j}) & \binom{N-1}{N-4}\widehat{\varphi}^{(N-4)}(it_{k,j}) & \binom{N-1}{N-5}\widehat{\varphi}^{(N-5)}(it_{k,j}) & \cdots & \widehat{\varphi}(it_{k,j})
\end{bmatrix}$$

where $N := N_s(\rho_{\chi_k, t_{\chi,j}})$, and $\widehat{\varphi}(z)$ is the Mellin transform of φ, i.e. $\widehat{\varphi}(z) := \int_{C_F} \varphi(g)\chi_k(g)|g|_F^z \, \mathrm{d}\overline{\mu}(g)$, an entire function on the complex z-plane, with $\widehat{\varphi}^{(r)}(z) = \frac{\partial^r}{\partial z^r}\widehat{\varphi}(z)$.

We further refer the reader to (D.3.3.11) of Theorem D.3.3.7 from Appendix D.3.2, and to Discussion 4.4.1.1, where, upon choosing a continuous homomorphic section τ to the idelic norm, $\widehat{\varphi}(z) = M(\varphi)(z + ix_k)$ as entire functions of $z \in \mathbf{C}$, for $x_k \in \mathbf{R}$ the unique real number such that $\chi_k(\tau(t)) = t^{ix_k}$ as smooth functions on $\mathbf{R}_{>0}$.

The finite sum:

$$A_K := \sum_{(k,j) \in K} A_{K,(k,j)}$$

is itself a finite rank operator, and by the foregoing it factors as $A_K = A_K^{\mathrm{finite}} \circ P_{B_K}$ where A_K^{finite} is a finite rank operator on the finite dimensional vector space $\mathrm{Im}(B_K)$, itself a finite dimensional direct sum of finitely many indecomposable sub-representations of C_F on $\mathscr{H}^s_{F,a}$.

For a general finite subset $K \subset \mathbf{N}\times\mathbf{N}$ containing (k,j), $A^{\mathrm{finite}}_{K,(k,j)}$ is represented by the $N_K \times N_K$ block matrix, for $N_K := \sum_{(k,j)\in K} N_s(\rho_{\chi_k,t_{k,j}})$, comprised, up to conjugation by a finite product of elementary matrices to interchange the ordering of the finitely many basis elements in $\mathrm{Im}(P_{B_K})$, of an $N_s(\rho_{\chi_k,t_{k,j}}) \times N_s(\rho_{\chi_k,t_{k,j}})$ block equal to the matrix in (10.3.1.17) representing $A^{\mathrm{finite}}_{K,(k,j)}$, and zero elsewhere.

For every finite subset $K \subset \mathbf{N} \times \mathbf{N}$ containing the fixed pair of integers (k,j), the singular values of the linear transformation $A^{\mathrm{finite}}_{K,(k,j)} \in \mathscr{L}(\mathrm{Im}(P_{B_K}))$ satisfy the following estimate:

$$\sigma_m(A^{\mathrm{finite}}_{K,(k,j)}) \leq N_s(\rho_{\chi_k,t_{k,j}}) \cdot C \cdot 2^{N_s(\rho_{\chi_k,t_{k,j}})-1} \sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1} |\widehat{\varphi}^{(i)}(t_{k,j})| \tag{10.3.1.18}$$

where C is such that

$$\|\eta_{\chi_k,t_{k,j},m'_{k,j}}\|_{-s,2} \cdot \|\psi_{\chi_k,t_{k,j},m_{k,j}}\|_{s,2} \leq C$$

for all $0 \leq m'_{k,j} \leq m_{k,j}$, and all $0 \leq m_{k,j} \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$.

The existence of C is ensured by the Markushevich basis $\{\eta_{\chi_k,t_{k,j},m'_{k,j}}\}$, and hence its bi-orthogonal sequence $\{\psi_{\chi_k,t_{k,j},m'_{k,j}}\}$, being normalized (see Terminology 10.2.1.2), which implies that C can be chosen to be independent of $m'_{k,j}$ and of (k,j) (i.e. the countable set $\{\eta_{\chi_k,t_{k,j},m'_{k,j}}\}$ in $W^{-s,2}(C_F)$ and its bi-orthogonal in $W^{s,2}(C_F)$ are both contained in the closed ball of radius $C^{1/2}$ for some large enough $C > 0$, and in fact they are contained in a closed collar bounded away from 0 in each Hilbert space; see Proposition 10.2.1.3).

We deduce from (10.3.1.17) and [**Jar**, Part III,Ch.19,§19.8,Prop.1,438] (see also Lemma C.3.6.3(2)), that the singular values of the finite rank operator in (10.3.1.15) satisfy:

$$\sigma_m(A_{K,(k,j)}) \leq N_s(\rho_{\chi_k,t_{k,j}}) \cdot C \cdot 2^{N_s(\rho_{\chi_k,t_{k,j}})-1} \sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1} |\widehat{\varphi}^{(i)}(t_{k,j})| \, \|P_{B_K}\|_{\mathscr{L}_\beta(\mathscr{H}^s_{F,a})} \tag{10.3.1.19}$$

for $0 \leq m \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$, and $\sigma_m(A_{K,(k,j)}) = 0$ for $m \geq N_s(\rho_{\chi_k,t_{k,j}})$.

Since P_{B_K} is an orthogonal projector, we deduce from (10.3.1.19) that the following estimate holds:

$$\sigma_m(A_{K,(k,j)}) \leq N_s(\rho_{\chi_k,t_{k,j}}) \cdot C \cdot 2^{N_s(\rho_{\chi_k,t_{k,j}})-1} \sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1} |\widehat{\varphi}^{(i)}(t_{k,j})|. \tag{10.3.1.20}$$

Since, for every integer $m \geq 1$, we have $\sigma_m(A_{K,(k,j)}) = \sigma_m(A_{K',(k,j)})$ for every pair of finite subsets $K \subset K' \subset \mathbf{N} \times \mathbf{N}$ with $(k,j) \in K$, then the net of finite rank operators $\{A_{K,(k,j)}\}_K$ converges in $S_p(\mathscr{H}^s_{F,a})$ for every $p \geq 1$ to the finite rank operator:

$$A_{(k,j)} := \sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j},m_{k,j}})-1} \langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \cdot\rangle_{s,2} \, \psi_{\chi_k,t_{k,j},m_{k,j}}. \tag{10.3.1.21}$$

To infer the convergence, we are using (10.3.1.20) upon observing that the rightmost side of (10.3.1.20) does not depend on K.

We deduce from (10.3.1.19) that we have:

$$\sigma_m(A_{(k,j)}) \leq N_s(\rho_{\chi_k,t_{k,j}}) \cdot C \cdot 2^{N_s(\rho_{\chi_k,t_{k,j}})-1} \sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1} |\widehat{\varphi}^{(i)}(t_{k,j})| \tag{10.3.1.22}$$

for $1 \leq m \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$, and $\sigma_m(A_{(k,j)}) = 0$ for $m \geq N_s(\rho_{\chi_k,t_{k,j}})$.

Since $N_s(\rho_{\chi_k,t_{k,j}})$ is bounded independently of (k,j) and only depends on s (see (10.2.1.6)), and since $\widehat{\varphi}$ and all of its derivatives, the first $N_s(\rho_{\chi_k,t_{k,j}}) - 1$ only of which enter the estimates from (10.3.1.19), are

Schwartz functions, being φ smooth and compactly supported, then the sum

$$\sum_{k\geq 0}\sum_{j\geq 0}\left(\sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1}|\widehat{\varphi}^{(i)}(it_{k,j})|\right)^p$$

is convergent for every $p\geq 1$, and hence by (10.3.1.22) we deduce that the sequence of finite rank operators in $\mathscr{L}(\mathscr{H}^s_{F,a})$:

$$\sum_{0\leq k\leq N_1}\sum_{0\leq j\leq N_2}\sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1}\langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}},\cdot\rangle_{s,2}\,\psi_{\chi_k,t_{k,j},m_{k,j}} \tag{10.3.1.23}$$

is Cauchy with respect to the $\|\cdot\|_{S_p(\mathscr{H}^s_{F,a})}$-norm for every $p\geq 1$, and hence it indeed converges in $S_p((\mathscr{H}^s_{F,a})'_\beta)$ for every $p\geq 1$.

Moreover, its limit, pre-composed with P_{B_K} for each finite subset $K\subset\mathbf{N}\times\mathbf{N}$, agrees with the operator $A(\varphi)\circ P_{B_K}$, since the range of P_{B_K} is a finite direct sum of finite dimensional indecomposable sub-representations of C_F on the dual of $\mathscr{H}^s_{F,a}$, isometrically identified with a closed subspace of $W^{-s,2}(C_F)$ under T_s from (10.2.1.3), linearly generated by the finitely many elements from (10.3.1.16) as $k\in\mathbf{N}$ varies over the image of K in $\mathbf{N}$ under the first projection.

The directed union of the ranges of P_{B_K} is dense in $\mathscr{H}^s_{F,a}$ by Corollary 9.3.2.3(2) (i.e. $P_{B_K}\to 1$ with respect to the strong operator topology, as K tends to encompass $\mathbf{N}\times\mathbf{N}$; see Remark C.3.5.2).

It follows that the limit in $S_p(\mathscr{H}^s_{F,a})$, for $p\geq 1$, of the sequence of finite rank operators from (10.3.1.23), agrees with $A(\varphi)$ on a dense linear subspace, and hence the two continuous operators agree on $\mathscr{H}^s_{F,a}$ by continuity.

We further refer the reader to the proof of Theorem 7.4.1.1.

(2) and (3) : Since $W^{s,2}(C_F)$ and $W^{s,2}_a\,\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}_0(\mathbf{A}_F))$ are reflexive Banach spaces, being Hilbert spaces, then by [**Tre**, p.484] $A(\varphi)$ is nuclear if and only if its adjoint $A(\varphi)'$, a continuous linear endomorphism of $(\mathscr{H}^s_{F,a})'_\beta$, is nuclear. Its trace-norm is computed by $\|A(\varphi)\|_{\mathrm{Tr}}=\|A(\varphi)'\|_{\mathrm{Tr}}$.

The countable set $\{\eta_{\chi,t,m}\}_{(\chi,t,m)}$, with the enumeration chosen in (10.3.1.6), is a Markushevich basis within the in $W^{-s,2}(C_F)$ of $(\mathscr{H}^s_{F,a})'_\beta$ under the isometry T_s from (10.2.1.3), by Theorem 10.2.1.1, as χ varies over the chosen (in (10.2.1.4)) base points of the connected components of $\Omega(C_F)$, each over a single point of the discrete parameter space $\widehat{C^1_F}$, $t\in\mathbf{R}$ varies over the imaginary parts of the zeros of $L(s,\chi)$ on the vertical line $\mathrm{Re}(s)=a$, and for each such zero $\rho=\rho_{\chi,t}$, m varying over the integers $0\leq m\leq N_s(\rho_{\chi,t})-1$, where $N_s(\rho_{\chi,t})$ is the counting function defined in (10.2.1.6) from Theorem 10.2.1.1.

From the proof of (1), the singular values of the limit operator $A(\varphi)$ are a sequence of rapid decay, and hence $A(\varphi)$ is strongly nuclear.

We note that, since φ is smooth and compactly supported, the sum in the right side of (10.3.1.3) is absolutely convergent. To see this, we proceed as follows (compare with Lemma B.4.4.8 and its proof, and see the Uniform Riemann-Lebesgue Lemma from Theorem D.3.4.2. Compare also with Proposition B.2.2.3).

By Theorem D.1.2.1, we write it the right side of (10.3.1.3), upon choosing a continuous homomorphic section τ to the idelic norm, and writing for any $g\in C_F$, $g=\tau(t)g_0$ for unique $t\in\mathbf{R}_{>0}$ and $g_0\in C^1_F$:

$$\int_{\omega\in\widehat{C^1_F}}\sum_{\tau\in\mathbf{R}:L(a+i\tau,\chi)=0}N_s(\rho_{\chi,\tau})\int_{\mathbf{R}_{>0}}\chi(\tau(t))t^{i\tau}\left(\int_{C^1_F}\omega(g_0)\varphi(\tau(t)g_0)\,\mathrm{d}\overline{\mu}_1(g_0)\right)\frac{\mathrm{d}t}{t} \tag{10.3.1.24}$$

and we call:

$$\Phi(x,\omega):=\int_{\mathbf{R}_{>0}}\chi(\tau(t))t^{ix}\left(\int_{C^1_F}\omega(g_0)\varphi(\tau(t)g_0)\,\mathrm{d}\overline{\mu}_1(g_0)\right)\frac{\mathrm{d}t}{t} \tag{10.3.1.25}$$

which is the Fourier transform $\widehat{\varphi}$ of φ as a function over $\widehat{C_F}$, composed with the isomorphism of Hausdorff locally compact abelian groups $\mathbf{R}\times\widehat{C_F^1}\simeq\widehat{C_F}$ induced by τ.

Let $\widehat{\overline{\mu}}_1$ be the Plancherel measure of $\overline{\mu}_1$, and $\widehat{\overline{\mu}}$ the Plancherel measure of $\overline{\mu}$ on $\widehat{C_F}$. Then under such isomorphism, the Haar measure $\lambda\otimes\widehat{\overline{\mu}}_1$ matches $\widehat{\overline{\mu}}$, λ being the Lebesgue measure on $\mathbf{R}$, normalized so as to give the unit interval $[0,1]$ volume 1.

Since $\varphi\in\mathcal{D}(C_F)\subset\mathcal{S}(C_F)$, the Fourier transform $\widehat{\varphi}$ is Schwartz-Bruhat over $\widehat{C_F}$, hence in particular absolutely integrable, and so:

$$\sum_{(\omega,t)\in\widehat{C_F^1}\times\{\tau\in\mathbf{R}:L(a+i\tau,\chi)=0\}}|\Phi(\tau,\omega)| \tag{10.3.1.26}$$

is convergent, because it is comparable to the integral $\int_{\mathbf{R}_{>0}}\int_{\widehat{C_F^1}}|\Phi(x,\omega)|\,\mathrm{d}(\lambda\otimes\widehat{\overline{\mu}}_1)(x,\omega)$.

We further observe that the assignment, for each $\omega\in\widehat{C_F^1}$, of the sequence $\{\Phi(\tau,\omega):\tau\in\mathbf{R},L(a+i\tau,\chi)=0\}$, upon choosing any numbering of the imaginary parts of the zeros of $L(s,\chi)$ on the vertical line $\mathrm{Re}(s)=a$, is a sequence of rapid decay. In particular, if φ is $\overline{\omega}$-homogeneous, then $\widehat{\varphi}$ is supported over $\mathbf{R}\times\{\omega\}$.

The formulas for the trace of $A(\varphi)$ provided in (10.3.1.3) and (10.3.1.5) follow from [**Lit-II**, §4,Cor.2,p.61] applied to $A(\varphi)\circ P_{B_K}$ for each finite subset $K\subset\mathbf{N}\times\mathbf{N}$ (the trace of $A(\varphi)\circ P_{B_K}$ is computed by the expression in (10.3.1.3) since $\{\eta_{\chi,t,m}\}$, with the chosen enumeration, is a Schauder basis within the range of P_{B_K}, since this is a finite dimensional $\mathbf{C}$-linear subspace of $\mathscr{H}^s_{F,a}$, and hence the nuclear trace of $A(\varphi)\circ P_{B_K}$ equals its matrix trace) for then using Proposition C.3.6.5 and continuity of the trace functional (see Lemma C.3.4.4) and the convergence $P_{B_K}\to 1$ with respect to the strong operator topology, as K tends to cover $\mathbf{N}\times\mathbf{N}$, upon computing the point-spectrum (i.e. the subset of the complex plane of those complex numbers λ that are eigenvalues of $A(\varphi)$) of $A(\varphi)$ to be exactly given by the countable set

$$\{\Phi(\tau,\omega):(\tau,\omega)\in\mathbf{R}\times\widehat{C_F^1},L(a+i\tau,\chi)=0\}\subset\mathbf{C}$$

each $\Phi(\tau,\omega)$ counted with multiplicity $N_s(\rho_{\chi,\tau})$, which follows from the calculation of the triangular matrix in (10.3.1.17), and hence the spectrum of $A(\varphi)\circ P_{B_K}$ to be given by the subset of $\{\Phi(\tau,\omega):(\tau,\omega)\in\mathbf{R}\times\widehat{C_F^1},L(a+i\tau,\chi)=0\}$ of those values $\Phi(\tau,\omega)$ for $\omega=\chi_k|_{C_F^1}$ with k varying in the image of K under the first projection $\mathbf{N}\times\mathbf{N}\to\mathbf{N}$, and τ varying over the real numbers $t_{k,j}$ for all $(k,j)\in K$. □

10.3.2. Spectral nuclearity: the general case of compact intervals $I=[-\alpha,\alpha]$. The following generalizes Theorem 10.3.1.1 to the case $I=[-\alpha,\alpha]$ with $\alpha\geq 0$ a real number.

THEOREM 10.3.2.1 (Spectral Nuclearity - II). *Let F be a number field. Let $a,s\in\mathbf{R}$ be real numbers, with $s>1$. We choose a subset $\widetilde{C_F^1}\subset\widehat{C_F}$ of representatives for the countable group of continuous characters $\widehat{C_F^1}$, equivalently, a holomorphic parametrization for the Riemann surface $\Omega(C_F)$ of the form*

$$\chi|\cdot|_F^z,\ (\chi,z)\in\widetilde{C_F^1}\times\mathbf{C}$$

as in (10.2.1.4) *and Notation 10.2.1.1.*

All characters $\chi\in\widehat{C_F}$ occurring below, are assumed to vary over the chosen subset $\widetilde{C_F^1}$, i.e. they are assumed to vary over the chosen unitary base-points of the connected components of $\Omega(C_F)$.

Let $\varphi\in\mathcal{D}(C_F)$ be a test function.

(1) *For any choice of enumeration $\{\eta_{\chi_k,t_{k,j},b_{k,j},m_{k,j}}\}$, with $(k,j)\in\mathbf{N}\times\mathbf{N}$ and $0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j},b_{k,j}})-1$, for the set of elements $\{\eta_{\chi,t,b,m}\}\subset W_I^{-s}(C_F)$ from Theorem 10.2.2.1, the sequence of finite rank operators:*

$$\sum_{0\leq k\leq N_1}\sum_{0\leq j\leq N_2}\sum_{0\leq m_{k,j}\leq N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})-1}\langle A(\varphi)'\eta_{\chi_k,t_{k,j},b_{k,j},m_{k,j}},\cdot\rangle_{I,s}\,\psi_{\chi_k,t_{k,j},b_{k,j},m_{k,j}} \tag{10.3.2.1}$$

in $\mathscr{L}_\beta(\mathscr{H}^s_{F,a})$ converges in $S_p(\mathscr{H}^s_{F,a,I})$ for every $p\geq 1$.

(2) *The $\mathcal{D}(C_F)$-module structure:*

$$A : \mathcal{D}(C_F) \to \mathscr{L}_\beta(\mathscr{H}^s_{F,a,I}) \tag{10.3.2.2}$$

is valued in the Banach space of nuclear operators $S_1(\mathscr{H}^s_{F,a,I})$ on $\mathscr{H}^s_{F,a,I}$, and is continuous with respect to the Hausdorff locally convex topology on the source, and the trace-norm topology on the target. In particular, the composition of A and the trace functional on $S_1(\mathscr{H}^s_{F,a,I})$ is a distribution on C_F, whose value at $\varphi \in \mathcal{D}(C_F)$ is:

$$\mathrm{Trace}\, A(\varphi) = \sum_{\chi \in \widetilde{C^1_F}} \sum_{t \in \mathbf{R}, b \in [-\alpha,\alpha] : L(a+b+it,\chi)=0} N_{I,s}(\rho_{\chi,t,b}) \int_{C_F} \varphi(g)\chi|g|_F^{b+it}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.2.3}$$

where $N_{I,s}$ is the counting function:

$$N_{I,s}(\rho_{\chi,t,b}) = \min\left\{ N(\rho_{\chi,t,b}), \left[\frac{s-1}{2}+1\right]\right\},\ b = \pm\alpha,\ N_{I,s}(\rho_{\chi,t,b}) = N(\rho_{\chi,t,b}),\ b \in (-\alpha,\alpha) \tag{10.3.2.4}$$

where $[x]$ is the largest integer $N \geq 0$ satisfying $N \leq x$.

(3) *If φ is homogeneous of weight $\overline{\omega} \in \widehat{C^1_F}$ under the continuous left-translation action of C_F on $\mathcal{D}(C_F)$, then $A(\varphi)$ is valued in the linear subspace $\mathscr{S}(\mathscr{H}^s_{F,a,I}(\omega))$ of strongly nuclear operators on the isotype $\mathscr{H}^s_{F,a,I}(\omega)$, and its trace is:*

$$\mathrm{Trace}\, A(\varphi) = \sum_{t \in \mathbf{R}, b \in [-\alpha,\alpha] : L(a+b+it,\chi)=0} N_{I,s}(\rho_{\chi,t,b}) \int_{C_F} \varphi(g)\chi|g|_F^{b+it}\, \mathrm{d}\overline{\mu}(g) \tag{10.3.2.5}$$

for $\chi \in \widetilde{C^1_F}$ the unique element of $\widetilde{C^1_F}$ such that $\chi|_{C^1_F} = \omega$, i.e. where χ is the base-point of $\Omega(C_F)(\omega)$ for the chosen holomorphic parametrization of $\Omega(C_F)$ from (10.2.1.4).

Proof. The argument for Theorem 10.3.2.1 follows the same method as in the proof of Theorem 10.3.1.1 upon replacing $N_s(\rho)$ defined in (10.3.1.4) with the truncated multiplicity function $N_{I,s}(\rho)$ defined in (10.2.2.5) and (10.2.2.6), noting that $N_{I,s}$, contrary to N_s, is unbounded if $\alpha > 0$. In this case, we need to refine the estimate on:

$$\sum_{k\geq 0}\sum_{j\geq 0}\left(\sum_{0\leq i\leq N_s(\rho_{\chi_k,t_{k,j}})-1} |M(\varphi)^{(i)}(it_{k,j})|\right)^p \tag{10.3.2.6}$$

is convergent for every $p \geq 1$, which is used in the proof of Theorem 10.3.1.1 to infer that the sequence from (10.3.1.1) of finite rank operators converges in the S_1-norm topology.

We aim at deducing that the sequence of finite rank operators in (10.3.2.1) also converges in the S_1-norm, as $N_1, N_2 \to +\infty$.

The sum that replaces (10.3.2.6) is:

$$\sum_{k\geq 0}\sum_{j\geq 0}\left(\sum_{0\leq i\leq N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})-1} |M(\varphi)^{(i)}(b_{k,j}+it_{k,j})|\right)^p. \tag{10.3.2.7}$$

In order to prove that it converges for every $p > 1$, we proceed as follows. We let $N > 1$ be a fixed integer, to be chosen later.

We assume for simplicity and without loss of generality that we have $i \geq N$.

We note that we have, for each $\chi \in \widetilde{C_F}$ and upon choosing a continuous homomorphic section τ to the idelic norm:

(10.3.2.8)

$$|z^N M(\varphi)^{(i)}(z)| \leq \int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} |(t\partial_t)^N (\varphi(\tau(t)g)\log(t)^i) t^{\mathrm{Re}(z)}| \, \mathrm{d}\overline{\mu}(g) \right) \frac{\mathrm{d}t}{t}$$

(10.3.2.9)

$$\leq \sum_{0 \leq n \leq N} \binom{N}{n} \int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} ((t\partial_t)^{N-n} \varphi(\tau(t)g)) \cdot \frac{i!}{(i-n)!} (\log(t))^{i-n} \, t^{\mathrm{Re}(z)}| \, \mathrm{d}\overline{\mu}(g) \right) \frac{\mathrm{d}t}{t}$$

(10.3.2.10)

$$\leq \sum_{0 \leq n \leq N} \binom{N}{n} \prod_{0 \leq n \leq N-1} (i-n) \int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} |(t\partial_t)^{N-n} \varphi(\tau(t)g) \, \max\{t^\alpha, t^{-\alpha}\} \, (1+|\log(t)|^2)^{s/2} \, \mathrm{d}\overline{\mu}(g) \right) \frac{\mathrm{d}t}{t}$$

for any integer $0 \leq i \leq N_{I,s}(z) - 1$. It follows that, upon calling $C_{N,I,s}(\varphi) > 0$ the value of each integral in the last finite sum in (10.3.2.8), we have:

$$|M(\varphi)^{(i)}(z)| \leq \frac{C_{N,I,s}(\varphi) \prod_{0 \leq n \leq N-1} (i-n)}{|z|^N} \tag{10.3.2.11}$$

for all $z \in \mathbf{C}_I$, $\mathrm{Im}(z) \neq 0$. Using [**Lang-e**, Ch.XVII,Prop.2.6], we obtain $N_{I,s}(b + i\tau) = O(\log|\tau|)$, with the implied constant being uniform over all $b \in [-\alpha, \alpha]$, and $\prod_{0 \leq n \leq N-1}(i-n) = C \cdot N \cdot O(\log|\tau)$ with the same implied constant in $O(\log|\tau|)$, and for $C > 0$ a universal constant depending only on $N \geq 0$. We now choose $N > 1$, and obtain the desired convergence of (10.3.2.7) for every $p > 1$. The estimates from the proof of Theorem 10.3.1.1 now carry over, and the rest of the argument for Theorem 10.3.1.1 also establishes Theorem 10.3.2.1. □

10.4. Trace formula: Spectral Side

Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.

We fix a real number $s \geq 0$, and we assume $s > 1$ to ensure that every element of $W^{s,2}(C_F)$ is absolutely $\overline{\mu}$-integrable (see Proposition 9.1.2.6). Let I be a compact interval, $I = [-\alpha, \alpha]$ for a real number $\alpha \geq 0$. Then we still choose $s > 1$ in $W_I^s(C_F)$, to ensure the integrability of the functions $|\cdot|_F^b (\log|g|_F)^m$ for $b = \pm\alpha$ and $m < \left[\frac{s-1}{2} + 1\right]$.

We further fix a real number $a \in \mathbf{R}$. We will be strengthening the assumptions on s and a as the need arises.

In this section we compute the contribution to the trace of the effect of the convolution action of the kernel $\varphi(xy^{-1})$ on $W_I^s(C_F)$, for $\varphi \in \mathcal{D}(C_F)$ a test function, on the dual of the quotient of the (closed) range of the multiplication operator M_ρ by a smooth (Λ, ε)-cutoff ρ, modulo the closure of the range of M_ρ restricted to

$$|\cdot|_F^{1/2} \, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)).$$

We first introduce orthogonal projections P_a^{geom} on $W_I^s(C_F)(\chi|_{C_F^1})$ onto

$$W_{a,I}^s \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

and its orthogonal complement, $P_a^{\mathrm{spec}} := \mathrm{Id} - P_a^{\mathrm{geom}}$, and study the operators $P_a^{\mathrm{geom}} \circ A(\varphi)$ and $P_a^{\mathrm{spec}} \circ A(\varphi)$, for any test function $\varphi \in \mathcal{D}(C_F)$.

We shall prove in Theorem 10.4.2.1 that the latter is a nuclear operator on $W_I^s(C_F)$, and we will be computing its trace in Proposition 10.4.2.4.

We will also be showing in Theorem 10.4.2.2 that its trace can be approximated by the traces of the Spectral Truncations $P^{\mathrm{spec}}_{\rho,a} \circ A(\varphi)$ as $\rho \to 1$, where $P^{\mathrm{spec}}_{\rho,a}$ will be defined as the composition of P^{spec}_{a} followed by multiplication by a (Λ, ε)-cutoff (see Definition 9.3.3.1 and Definition 9.3.3.3).

We will then be revisiting these results in terms of the integral kernels of the operators $P^{\mathrm{spec}}_{\rho,a} \circ A(\varphi)$, $P^{\mathrm{spec}}_{a} \circ A(\varphi)$, and $P^{\mathrm{geom}}_{a} \circ A(\varphi)$.

10.4.1. Truncations. We now begin our study of the $\mathcal{D}(C_F)$-module $W^s_I(C_F)$ and its closed submodules $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, for. Due to non-compactness of the idele class group, and much as in the Selberg Trace Formula, we introduce truncation operators to promote the otherwise non-nuclear operator $A(\varphi)$ for each $\varphi \in \mathcal{D}(C_F)$ (see Definition 9.3.1.1) and its restriction to $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, to nuclear truncated versions, which we will then study as the truncation "level" Λ tends to grow large.

We require the following terminology and notation:

Definition 10.4.1.1. Let F be a number field, let s, a be real numbers, $s \geq 0$, $a > 0$. Let $I = [-\alpha, \alpha]$ be a compact non-empty interval, with $\alpha \geq 0$ a real number. We define:

(1) The *geometric projection* operator to be the orthogonal projector:

$$P^{\mathrm{geom}}_{a} : W^s_I(C_F) \to W^s_I(C_F) \tag{10.4.1.1}$$

assigned by sending $\phi \in W^s_I(C_F)$ to the orthogonal projection onto the closed subspace

$$W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset W^s_I(C_F).$$

(2) The *spectral projection* operator to be the orthogonal projector:

$$P^{\mathrm{spec}}_{a} : W^s_I(C_F) \to W^s_I(C_F) \tag{10.4.1.2}$$

assigned by sending $\phi \in W^s_I(C_F)$ to the orthogonal projection onto the closed subspace of $W^s_I(C_F)$ defined as the orthogonal complement of the closed subspace $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ within $W^s_I(C_F)$.

Lemma 10.4.1.2. *Let F be a number field, let s, a be real numbers, $s \geq 0$, $a > 0$. Let*

$$Q_a : W^s_I(C_F) \to \mathscr{H}^s_{F,a,I} := W^s_I(C_F)/W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \tag{10.4.1.3}$$

be the quotient continuous linear operator by the closed C_F-invariant subspace $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset W^s_I(C_F)$.

Then we have:

$$Q_a = Q_a \circ P^{\mathrm{spec}}_{a},\ Q_a \circ P^{\mathrm{geom}}_{a} = 0 \tag{10.4.1.4}$$

and for every $\psi \in \mathscr{H}^s_{F,a,I}$, the unique $\phi \in W^s_I(C_F)$ orthogonal within $W^s_I(C_F)$ to $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and such that $Q_a(\phi) = \psi$ is, for any choice $\widetilde{\psi} \in W^s_I(C_F)$ such that $Q_a(\widetilde{\psi}) = \psi$, the element $\phi = P^{\mathrm{spec}}_{a}(\widetilde{\psi})$.

The assignment $S_a : \phi \mapsto P^{\mathrm{spec}}_{a}(\widetilde{\psi})$ does not depend on the choice of $\widetilde{\psi}$, and it is an isometric linear topological isomorphism:

$$S_a : \mathscr{H}^s_{F,a,I} \to P^{\mathrm{spec}}_{a} W^s_I(C_F). \tag{10.4.1.5}$$

Proof. Since P^{geom}_{a} is valued in $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, we have $Q_a \circ P^{\mathrm{geom}}_{a} = 0$, and since $\mathrm{Id} = P^{\mathrm{geom}}_{a} + P^{\mathrm{spec}}_{a}$ in $\mathscr{L}_\beta(W^s_I(C_F))$, we have $Q_a = Q_a \circ (P^{\mathrm{geom}}_{a} + P^{\mathrm{spec}}_{a}) = Q_a \circ P^{\mathrm{spec}}_{a}$.

The remaining assertion on ϕ follows from the definition of orthogonal projectors on a Hilbert space: if ϕ is orthogonal to $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and satisfies $Q_a(\phi) = \psi$, then so does $P^{\mathrm{spec}}_{a}(\phi)$ by (10.4.1.4), and so $\phi = P^{\mathrm{spec}}_{a}(\phi)$.

This shows that formation of S_a does not depend on the choices being made in its definition, and it is linear. Since $Q_a \circ S_a$ is the identity on $\mathscr{H}^s_{F,a,I}$, we deduce that $S_a : \mathscr{H}^s_{F,a,I} \to P^{\mathrm{spec}}_{a} W^s_I(C_F)$ is linear and surjective, and Q_a is a continuous left-inverse. Q_a is an isometric linear topological isomorphism from

$P_a^{\mathrm{spec}} W_I^s(C_F)$ to $\mathscr{H}^s_{F,a,I}$, since it is surjective, injective and open, being a quotient map. Its inverse, both left and right, is continuous, and hence S_a is continuous and bijective, hence a linear topological isomorphism by [**NB**, Thm.14.4.5]. Since its inverse is isometric, so is S_a. □

We now introduce truncated versions of P_a^{geom}, respectively P_a^{spec}.

We refer the reader to Definition 9.3.3.1 and Definition 9.3.3.3 for the definition of (Λ, ε)-cutoffs and smooth (Λ, ε)-cutoff, which we will be specializing hereafter to the $\mathbf{R}_{>0}$-modulated group C_F, whose module is the idelic norm $|\cdot|_F$.

Definition 10.4.1.3 (Spectral and geometric truncations - projection after convolution). Let F be a number field. Let $I = [-\alpha, \alpha]$ be a compact non-empty interval, with $\alpha \geq 0$ a real number. Let $a, s, \Lambda, \varepsilon$ be real numbers, with $s \geq 0$ and $\Lambda, \varepsilon > 0$. Assume $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$. Let $\rho : C_F \to \mathbf{C}$ be a (Λ, ε)-cutoff.

Let $\varphi \in \mathcal{D}(C_F)$ be a test function.

(1) We define the *Geometric Truncation Operator* to be the composition:

$$P^{\mathrm{geom}}_{\rho,a} : W_I^s(C_F) \to W_I^s(C_F) \tag{10.4.1.6}$$

of the orthogonal projection P_a^{geom} onto $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, followed by the multiplication operator M_ρ.

We define the *Geometric Truncation of $A(\varphi)$ of level ρ* to be the composition:

$$P^{\mathrm{geom}}_{\rho,a} \circ A(\varphi) : W_I^s(C_F) \to P^{\mathrm{geom}}_{\rho,a} W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to W_I^s(C_F). \tag{10.4.1.7}$$

(2) We define the *Spectral Truncation Operator*:

$$P^{\mathrm{spec}}_{\rho,a} : W_I^s(C_F) \to W_I^s(C_F) \tag{10.4.1.8}$$

to be the composition of P_a^{spec} followed by the multiplication operator M_ρ.

We define the *Spectral Truncation of $A(\varphi)$ of level ρ* to be the composition:

$$P^{\mathrm{spec}}_{\rho,a} \circ A(\varphi) : W_I^s(C_F) \to P^{\mathrm{spec}}_{\rho,a} W_I^s(C_F) \to W_I^s(C_F). \tag{10.4.1.9}$$

Remark 10.4.1.4. We note that since $\mathrm{Id} = P_a^{\mathrm{geom}} + P_a^{\mathrm{spec}}$, and since $P^{\mathrm{geom}}_{\rho,a} = M_\rho \circ P_a^{\mathrm{geom}}$ and $P^{\mathrm{spec}}_{\rho,a} = M_\rho \circ P_a^{\mathrm{spec}}$, we have:

$$M_\rho = P^{\mathrm{geom}}_{\rho,a} + P^{\mathrm{spec}}_{\rho,a}$$

in $\mathscr{L}_\beta(W_I^s(C_F))$.

We further note that since the Sobolev weight function $\max\{|g|_F^\alpha, |g|_F^{-\alpha}\}\,(1 + |\log|\cdot|_F|^2)^{s/2}$ is not C_F-invariant (see Lemma 9.3.1.5), the continuous left-translation action of C_F on $W_I^s(C_F)$ is through linear topological automorphisms, but not through isometries, and in particular it does not preserve orthogonality within the metric space $W_I^s(C_F)$.

It does, however, preserve the pairing $\langle \cdot, \cdot \rangle_{I,s}$ from (9.2.1.28), by invariance of the Haar measure $\overline{\mu}$ on C_F.

To fix ideas, we choose $I = [-\alpha, \alpha] = \{0\}$, i.e. $\alpha = 0$.

If s is an even integer, however, we may change the metric on $W_I^s(C_F) = W^{s,2}(C_F)$ without changing the topology: this is the content of Lemma 9.1.2.2. We further refer the reader to [**Fol**, Ch.9,§9.3,p.301].

The new metric is such that C_F acts by continuous left-translations in a *conformal* manner, i.e. its action preserves angles, although, still, not distances (again, by another application of Lemma 9.1.2.2 combined with Lemma 9.3.1.5. The action of C_F is not unitary, but, rather, it grows slowly with respect to the operator norm according to Lemma 9.3.1.5).

Since P_a^{geom} is valued in $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, and this latter is a closed C_F-invariant subspace of $W_I^s(C_F)$, we have

$$P_a^{\mathrm{geom}} \circ A(\varphi) \circ P_a^{\mathrm{geom}} = A(\varphi) \circ P_a^{\mathrm{geom}} \tag{10.4.1.10}$$

and

$$P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{spec}} = P_a^{\mathrm{spec}} \circ A(\varphi) \tag{10.4.1.11}$$

while we do not have the stronger commutation relations $A(\varphi) \circ P_a^{\mathrm{geom}} = P_a^{\mathrm{geom}} \circ A(\varphi)$ and $A(\varphi) \circ P_a^{\mathrm{spec}} = P_a^{\mathrm{spec}} \circ A(\varphi)$, ultimately because $P_a^{\mathrm{geom}} \circ A(\varphi) \circ P_a^{\mathrm{spec}} \neq 0$ (i.e. the continuous left-translation action of C_F on $W_I^s(C_F)$ does not preserve the orthogonality relations intrinsic within $W_I^s(C_F)$ with respect to the metric defined in (9.1.2.3), with the sole exception of the case $s = \alpha = 0$, in which case such action is through unitary automorphisms).

We do have $P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{geom}} = 0$, which follows from (10.4.1.10).

We define truncations again, but in a different order:

Definition 10.4.1.5 (Spectral and geometric truncations - convolution after projection). Let F be a number field. Let $I = [-\alpha, \alpha]$ be a compact non-empty interval, with $\alpha \geq 0$ a real number. Let $a, s, \Lambda, \varepsilon$ be real numbers, with $s \geq 0$ and $\Lambda, \varepsilon > 0$. Assume $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$. Let $\rho : C_F \to \mathbf{C}$ be a (Λ, ε)-cutoff.

Let $\varphi \in \mathcal{D}(C_F)$ be a test function.

(1) We define the *Geometric Restriction Operator of level* ρ:

$$M_\rho \circ A_a^{\mathrm{geom}}(\varphi) : W_I^s(C_F) \to W_I^s(C_F) \tag{10.4.1.12}$$

to be the composition of $P_a^{\mathrm{geom}} : W_I^s(C_F) \to W_{a,I}^s \, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, followed by the convolution operator $A(\varphi)$, and by the multiplication operator M_ρ.

(2) We define the *Spectral Restriction Operator of level* ρ:

$$M_\rho \circ A_a^{\mathrm{spec}}(\varphi) : W_I^s(C_F) \to W_I^s(C_F) \tag{10.4.1.13}$$

to be the composition of $P_a^{\mathrm{spec}} : W_I^s(C_F) \to W_I^s(C_F)$ followed by the convolution operator $A(\varphi)$ and by the multiplication operator M_ρ.

Geometric and spectral restrictions are those that will be the main focus of this Chapter. It will turn out that the nuclear trace

$$\mathrm{Trace}\, M_\rho \circ A_a^{\mathrm{spec}}(\varphi)$$

satisfies:

$$\lim_{\Lambda \to +\infty, \varepsilon\Lambda = o(1)} \mathrm{Trace}\, M_\rho \circ A_a^{\mathrm{spec}}(\varphi) = \mathrm{Trace}\, A_a^{\mathrm{spec}}(\varphi)$$

as for $P_{a,\rho}^{\mathrm{spec}} \circ A(\varphi)$. This is not formal, but requires explicit estimates using lifts of the Spectral Markushevich bases, combined with Theorem 10.3.2.1.

10.4.2. Spectral Side. In this subsection we study the nuclearity and trace of the spectral truncation of $A(\varphi)$ of level ρ (see (10.4.1.9)) for ρ a (Λ, ε)-cutoff in the sense of Definition 9.3.3.1 and Definition 9.3.3.3, for each test function $\varphi \in \mathcal{D}(C_F)$: we show it is a nuclear operator, and compute the limit of its trace as $\rho \to 1$ in the regime $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$.

Since we computed the trace of $M_\rho \circ A(\varphi)$ in Theorem 9.3.3.5, and since we have

$$\mathrm{Trace}\, M_\rho \circ A(\varphi) = P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi) + P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi) \tag{10.4.2.1}$$

we will obtain the identity:

$$\mathrm{Trace}\,(M_\rho \circ A(\varphi)) = \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)) + \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)) \tag{10.4.2.2}$$

which we will be studying for suitable a, and upon choosing $s \geq 0$ large enough, and as $\rho \to 1$ while having $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$.

To study $P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$, we first prove that it is nuclear for every (Λ, ε)-cutoff ρ, and we then compute its trace as $\rho 1$ while $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$. This is the scope of this subsection and the next.

THEOREM 10.4.2.1. *Let F be a number field, a, s real numbers, $s \geq 0$. Let $I = [-\alpha, \alpha]$ be a compact non-empty interval, with $\alpha \geq 0$ a real number. Let $\Lambda, \varepsilon > 0$ be real numbers such that $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$. Let ρ be a (Λ, ε)-cutoff.*

Then for any $\varphi \in \mathcal{D}(C_F)$, the operator $P_a^{\mathrm{spec}} \circ A(\varphi)$, and the operators $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ from (10.4.1.7) and $P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$ from (10.4.1.9) are nuclear.

Their traces satisfy the identity:

$$\mathrm{Trace}\,(M_\rho \circ A(\varphi)) = \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)) + \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)) \tag{10.4.2.3}$$

PROOF. The projector $P_{\rho,a}^{\mathrm{spec}}$ from (10.4.1.8) satisfies:

$$P_{\rho,a}^{\mathrm{spec}} = M_\rho \circ P_a^{\mathrm{spec}} \tag{10.4.2.4}$$

where P_a^{spec} is the orthogonal projector $W_I^s(C_F)$ onto the orthogonal within $W_I^s(C_F)$ of $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

Recall that the quotient of $W_I^s(C_F)$ by $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ was denoted $\mathscr{H}_{F,a,I}^s$ (see (10.2.1.1) and Theorem 10.3.1.1).

The operator $P_a^{\mathrm{spec}} \circ A(\varphi)$ satisfies:

$$P_a^{\mathrm{spec}} \circ A(\varphi) = S_a \circ A_a^{\mathrm{spec}}(\varphi) \circ Q_a \tag{10.4.2.5}$$

where $S_a : \mathscr{H}_{F,a}^s \to P_a^{\mathrm{spec}} W_I^s(C_F)$ is the surjective isometry induced by the quotient $Q_a : W_I^s(C_F) \to \mathscr{H}_{F,a,I}^s$, and $A_a^{\mathrm{spec}}(\varphi)$ is the operator from (10.3.1.2) of Theorem 10.3.1.1.

Explicitly, for $\psi \in \mathscr{H}_{F,a}^s$, $S_a(\psi)$ is the unique element $\widetilde{\psi} \in P_a^{\mathrm{spec}} W_I^s(C_F)$ such that $Q_a(\widetilde{\psi}) = \psi$. We can compute $S_a(\psi)$ by choosing $\widetilde{\psi} \in W_I^s(C_F)$ such that $Q_a(\widetilde{\psi}) = \psi$, and we have $S_a(\psi) = P_a^{\mathrm{spec}}(\widetilde{\psi})$.

To prove that (10.4.2.5) holds, we proceed as follows.

We let $\phi \in W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ be an arbitrary element. Then, since $W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is a closed C_F-invariant subspace of $W_I^s(C_F)$, $A(\varphi)$ induces a continuous linear endomorphism of $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, and hence $A(\varphi)(\phi) \in W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, which implies that we have:

$$(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi) = 0. \tag{10.4.2.6}$$

Let $\phi \in W_I^s(C_F)$. We have, by (10.4.2.6):

$$(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi + \phi_0) = (P_a^{\mathrm{spec}} \circ A(\varphi))(\phi) \tag{10.4.2.7}$$

for every $\phi_0 \in W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

This implies that we have:

$$(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi) = (S_a^{-1} \circ Q_a \circ A(\varphi))(\phi) = (S_a^{-1} \circ A_a^{\mathrm{spec}}(\varphi) \circ Q_a)(\phi) \tag{10.4.2.8}$$

for all $\phi \in W_I^s(C_F)$, and so (10.4.2.5) holds ($S_a^{-1}(\psi)$ is the unique element $\widetilde{\psi}$ of $P_a^{\mathrm{spec}} W_I^s(C_F)$ such that $Q_a(\widetilde{\psi}) = \psi$. For $\psi = A_a^{\mathrm{spec}}(\varphi)(Q_a(\phi))$, and since $Q_a \circ A(\varphi) = A_a^{\mathrm{spec}}(\varphi) \circ Q_a$, such element is $(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi)$, by (10.4.2.7)).

By Theorem 10.3.1.1, $A_a^{\mathrm{spec}}(\varphi)$ is nuclear, and by [**Cnw-b**, Ch.3,§18,Thm.18.11(a)] (see also Lemma C.3.4.4(1)), so are $P_a^{\mathrm{spec}} \circ A(\varphi)$ and $P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$.

Upon observing that we have $P_{\rho,a}^{\mathrm{geom}} + P_{\rho,a}^{\mathrm{spec}} = M_\rho$ within $\mathscr{L}_\beta(W_I^s(C_F))$, and since $M_\rho \circ A(\varphi)$ is nuclear by Theorem 9.3.3.5, then so is $P_a^{\mathrm{geom}} \circ A(\varphi) = M_\rho \circ A(\varphi) - P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$, and we deduce from Proposition C.3.5.1 that (10.4.2.3) holds. □

We now compute the "Spectral Side" of the Trace Formula, i.e. the limit as $\rho \to 1$ as $\Lambda \to +\infty$ (under the assumption $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$) of $\mathrm{Trace}\,(P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi))$ for any $s \geq 0$ and any $a \in \mathbf{R}$.

We study the $\mathcal{D}(C_F)$-module:

$$\mathscr{H}_{F,a,I}^s := W_I^s(C_F)/W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$$

and its $(\chi|_{C_F^1})$-isotype.

It is linearly and topologically isomorphic to the orthogonal of $W_{a,I}^s \, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ in $W_I^s(C_F)$ with respect to the intrinsic Hilbert pairing $\langle \cdot, \cdot \rangle$ on $W_I^s(C_F)$.

The duality pairing $\langle \cdot, \cdot \rangle_{s,2}$ further identifies, by [**NB**, Thm.8.12.4(b)], the continuous dual $(\mathscr{H}_{F,a}^s)'$ of the quotient $\mathscr{H}_{F,a}^s$ with the orthogonal in the sense of [**NB**, Def.8.9.1] with respect to $\langle \cdot, \cdot \rangle_{I,s} : W_I^s(C_F) \times W^{-s,2}(C_F) \to \mathbf{C}$ from (10.2.1.3).

Such identifications are all compatible with the continuous left-translation action of C_F, whose adjoint on $(\mathscr{H}_{F,a}^s)'$ is the right-translation action of C_F on $(\mathscr{H}_{F,a}^s)'$, and hence with $\mathcal{D}(C_F)$-module structures.

The following is the first of the two main Theorems of this section (the second being Theorem 10.4.2.8:

THEOREM 10.4.2.2 (Spectral side: projection after convolution). *Let F be a number field, a, s real numbers, $s \geq 0$. Let $I = [-\alpha, \alpha] \subset \mathbf{R}$ be a compact interval, $\alpha \geq 0$ a real number. We have, for all real numbers $\Lambda, \varepsilon > 0$ satisfying $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$, and $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$, and for a (Λ, ε)-cutoff ρ for each pair (Λ, ε):*

$$\mathrm{Trace}\,(P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)) = \mathrm{Trace}\,(A_a^{\mathrm{spec}}(\varphi)) + o(1) \tag{10.4.2.9}$$

as $\Lambda \to +\infty$, where $A_a^{\mathrm{spec}}(\varphi)$ is the nuclear operator from (10.3.1.2)*, whose trace is computed by the formula in* (10.3.1.3) *of Theorem 10.3.1.1.*

REMARK 10.4.2.3. Theorem 10.4.2.2 will be proved throughout the remainder of this section. The rest of this Chapter will be devoted to computing the trace of the geometric truncation of $A(\varphi)$ of level ρ, for then using the formula from (10.4.2.3) of Theorem 10.4.2.1 to obtain an asymptotic identity between $\mathrm{Trace}\,(A_a^{\mathrm{spec}}(\varphi))$, which involves the zeros of all Hecke L-functions of F, and the quantity:

$$2\log(\Lambda)\,\overline{\mu}_1(C_F^1)\,\varphi(1) - \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)) \tag{10.4.2.10}$$

for $s \geq 0$ and a to be chosen, and for $\Lambda > 0$ to be sent to $+\infty$ (with $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$, which forces $\rho \to 1$ pointwise over C_F while remaining nonnegative, continuous and uniformly bounded above by the constant function with value 1).

Note that for the moment the unknown value of the expression in (10.4.2.10) can only be deduced to converge to a finite limit as $\Lambda \to +\infty$, as a consequence of Theorem 10.4.2.2 and Theorem 10.3.1.1.

Before we can prove Theorem 10.4.2.2, we need the following:

PROPOSITION 10.4.2.4. *Let F be a number field, a, s real numbers, $s \geq 3$. Let $\varphi \in \mathcal{D}(C_F)$ be a test function.*

The following identity between the traces of the two nuclear operators $P_a^{\mathrm{spec}} \circ A(\varphi)$ on $W_I^s(C_F)$ and $A_a^{\mathrm{spec}}(\varphi)$ on $\mathscr{H}_{F,a,I}^s$ from Theorem 10.3.1.1:

$$\mathrm{Trace}\,(P_a^{\mathrm{spec}} \circ A(\varphi)) = \mathrm{Trace}\,(A_a^{\mathrm{spec}}(\varphi)) \tag{10.4.2.11}$$

holds.

PROOF. By (10.4.1.11) from Remark 10.4.1.4, we have $P_a^{\mathrm{spec}} \circ A(\varphi) = P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{spec}}$ as continuous operators on $W_I^s(C_F)$.

In particular, since both are nuclear, we have:

$$\mathrm{Trace}\,(P_a^{\mathrm{spec}} \circ A(\varphi)) = \mathrm{Trace}(P_a^{\mathrm{spec}} \circ A(\varphi)) = \mathrm{Trace}(P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{spec}}). \tag{10.4.2.12}$$

By Lemma C.3.2.2, since $P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{spec}}$ factors through the subspace $V := P_a^{\mathrm{spec}} W_I^s(C_F)$, its trace equals the trace of the unique nuclear operator on V obtained as the restriction T of $P_a^{\mathrm{spec}} \circ A(\varphi) \circ P_a^{\mathrm{spec}}$ to V.

The continuous linear quotient homomorphism $Q_a : W_I^s(C_F) \to \mathscr{H}_{F,a,I}^s$ from (10.4.1.3) induces a surjective linear isometry $Q_a : V \to \mathscr{H}_{F,a,I}^s$ with inverse S_a from (10.4.1.5) of Lemma 10.4.1.2.

We have:

$$Q_a \circ T \circ S_a = A_a^{\mathrm{spec}}(\varphi) \tag{10.4.2.13}$$

as nuclear operators on $\mathscr{H}^s_{F,a}$. In particular, we have:

$$\mathrm{Trace}(Q_a \circ T \circ S_a) = \mathrm{Trace}\,(A_a^{\mathrm{spec}}(\varphi)) \tag{10.4.2.14}$$

and since $Q_a : V \to \mathscr{H}^s_{F,a}$ is a surjective linear isometry, and S_a is its inverse isometry onto V, we deduce that we have:

$$\mathrm{Trace}(P_a^{\mathrm{spec}} \circ A(\varphi)) = \mathrm{Trace}(Q_a \circ T \circ S_a) \tag{10.4.2.15}$$

and hence (10.4.2.11) holds. □

Proof of Theorem 10.4.2.2. The operator $P_a^{\mathrm{spec}} \circ A(\varphi)$ is nuclear by Theorem 10.4.2.1, and its trace equals $\mathrm{Trace}\,(A_a^{\mathrm{spec}}(\varphi))$, where $A_a^{\mathrm{spec}}(\varphi)$ is the nuclear operator on $\mathscr{H}^s_{F,a,I}$ from Theorem 10.3.2.1.

The multiplication operators M_ρ tend to the identity with respect to the strong operator topology, as $\Lambda \to +\infty$, since the assumption $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$ forces ρ to tend to the constant function with value 1 pointwise over C_F, while remaining nonnegative, multiplicatively symmetric and bounded above by 1.

The multiplication operator M_ρ is self-adjoint, as noted in Remark 9.3.3.2.

By design of $P^{\mathrm{spec}}_{\rho,a}$ from Definition 10.4.1.3, we have $P^{\mathrm{spec}}_{\rho,a} = M_\rho \circ P_a^{\mathrm{spec}}$, and hence, by Corollary C.3.5.5, $P^{\mathrm{spec}}_{\rho,a} \circ A(\varphi) \to P_a^{\mathrm{spec}} \circ A(\varphi)$ in $S_1(W^s_I(C_F))$, i.e. as nuclear operators on $W^s_I(C_F)$. In particular, (10.4.2.9) holds. □

10.4.2.1. *ε-approximate Kernels.* We record here the following Lemma:

Lemma 10.4.2.5. *Let E be a Banach space, $E_0 \subset E$ a closed subspace. Let $\|\cdot\|_E$ be a norm on E generating the Banach space topology on E.*

Assume E/E_0, endowed with the quotient topology, admits a normalized Schauder, respectively Markushevich, basis $\{x_m\}_{m\geq 0}$. Let $\{\varphi_m\}_{m\geq 0}$ be its bi-orthogonal sequence.

Let $r, r' > 0$ be real numbers such that, upon denoting by $\|\cdot\|_{E/E_0}$ the induced norm on the quotient E/E_0, we have:

$$\{x_m\}_{m\geq 0} \subset \{x \in E/E_0 : \|x\|_{E/E_0} \in [r, r']\}. \tag{10.4.2.16}$$

Let $F \subset E$ be a linear subspace such that $F_0 := F \cap E_0$ is dense in E_0 with respect to the normed topology inherited from E. Assume $\{x_m\}_{m\geq 0} \subset F/F_0$.

We denote by $\widetilde{\varphi}_m$ the image of φ_m under $(E/E_0)'_\beta \to E'_\beta$, i.e. the composition of φ_m with the quotient $E \to E/E_0$.

There exists a subset $\{u_m\}_{m\geq 0} \subset F$ satisfying the following properties:

(1) *For every pair of integers $m, n \geq 0$, we have:*

$$\widetilde{\varphi}_m(u_n) = \delta_{m,n}. \tag{10.4.2.17}$$

(2) *There exists a number $\varepsilon > 0$, with $r > \varepsilon$, such that $\{u_m\}_{m\geq 0} \subset E$ satisfies:*

$$\{u_m\}_{m\geq 0} \subset \{u \in E : \|u\|_E \in [r - \varepsilon, r' + \varepsilon]\}. \tag{10.4.2.18}$$

In particular, $\{u_m\}_{m\geq 0}$ is a bounded subset of E contained in F, and a normalized Schauder basis in the closure of its linear span in E.

There exists a sequence $\{\delta_m\}$ of small enough positive real numbers such that for any $\{y_m\}_{m\geq 0} \subset F_0$ satisfying $\|y_m\|_E < \delta_m$ for every integer $m \geq 0$, then $\{u_m + y_m\}_{m\geq 0}$ also satisfies (1) and (2), and the assignment $u_m \mapsto u_m + y_m$ uniquely extends to a linear topological isomorphism from the closure of the linear span of $\{u_m\}_{m\geq 0}$ in E, to the closure of the linear span of $\{u_m + y_m\}_{m\geq 0}$ in E.

TERMINOLOGY 10.4.2.1. Let E be a Banach space, $E_0 \subset E$ a closed subspace, and assume $\{x_m\}_{m\geq 0} \subset E/E_0$ is a normalized Schauder, respectively Markushevich, basis, with bi-orthogonal sequence $\{\varphi_m\}_{m\geq 0}$. Assume $F \subset E$ is a linear subspace such that $F \cap E_0$ is dense in E_0, and the linear image of F in E/E_0 is dense.

We will be referring to the sequence $\{u_m\}_{m\geq 0}$ satisfying (1) and (2) from Lemma 10.4.2.5, whose existence is provided by the Lemma, as an *ε-lift* of $\{x_m\}_{m\geq 0}$.

PROOF OF LEMMA 10.4.2.5. We discuss the case of a Schauder basis.[7]

For each integer $m \geq 0$, we let $u_m \in E$ be a choice of element such that its image under the linear quotient operator $E \to E/E_0$ is x_m.

Since $E \to E/E_0$ is open, the linear image of $\{\varphi_m\}_{m\geq 0}$ in E'_β under the continuous linear operator $(E/E_0)'_\beta \to E'_\beta$ is a Schauder basis in its closure, since $(E/E_0)'_\beta \to E'_\beta$ is a linear topological embedding by [**NB**, Thm.8.13.2] (since by [**NB**, Thm.8.13.2], the range of $(E/E_0)'_\beta \to E'_\beta$ is weakly closed, hence strongly closed).

For any $u \in E_0$, we have $\widetilde{\varphi}_k(u + u_m) = \widetilde{\varphi}_k(u_m)$, and hence $\widetilde{\varphi}_k(u_m) = \varphi_k(x_m) = \delta_{m,n}$.

The quotient topology on E/E_0 is induced, by [**NB**, Thm.5.11.2], by the norm

$$\overline{p}(x) := \inf_{u_0 \in E_0} \|u_0 + u\|_E \tag{10.4.2.19}$$

for u a choice of element of E whose image in E/E_0 is x.

Since $\{x_m\}_{m\geq 0}$ is bounded and regular (i.e. bounded away from $0 \in E/E_0$; see [**Kal**, §1,Def.1.1,p.91]), there exist real numbers $r, r' > 0$ with $r < r'$, such that $r \leq \overline{p}(x_m) \leq r'$ for each $m \geq 0$.

For each integer $m \geq 0$, and for any $u_m \in E$ as in the foregoing, $u + u_m$ also satisfies the conditions in (10.4.2.17) for any $u \in E_0$.

We fix a real number $\varepsilon > 0$.

There therefore exists $u \in E_0$ such that $r \leq \|u + u_m\|_E \leq r' + \varepsilon/2$, and $u + u_m$ satisfies the conditions in (10.4.2.17).

This is indeed the case since the distance function from a fixed point of a metric space, is a metric-continuous function.

By assumption, we have $\{x_m\}_{m\geq 0} \subset F/F_0$, and hence we may and do assume that each u_m is an element of F.

Since $F_0 \subset E_0$ is dense with respect to the normed topology on E_0 inherited from E, there exists $u' \in F_0$ such that $\|u - u'\|_E < \varepsilon/2$.

We therefore have:

$$r - \varepsilon/2 \leq |\|u + u_m\|_E - \|u - u'\|_E| \leq \|u' + u_m\|_E \leq \|u + u_m\|_E + \|u - u'\|_E \leq r' + \varepsilon.$$

Since r does not depend on $m \geq 0$, we may choose $\varepsilon > 0$ small enough so as to have $r - \varepsilon/2 \geq r - \varepsilon > 0$.

We now let $V \subset E$ be the closure of the linear span of $\{u_m\}_{m\geq 0}$. It follows that $\{\widetilde{\varphi}_m\}_{m\geq 0}$ is a sequence in H' such that $\widetilde{\varphi}_m(u_n) = \delta_{m,n}$, which proves that $\{u_m\}_{m\geq 0}$ is a Schauder basis in V by [**Jar**, Part II,Ch.14,§14.2,Prop.3,p.292], since it its topologically free [**Jar**, Part II,Ch.14,§14.1,Prop.1,p.290].

For any sequence $\{y_m\}_{m\geq 0}$, the sequence $\{u_m + y_m\}_{m\geq 0}$ is also contained in F and satisfies (1). For $\{y_m\}_{m\geq 0}$ satisfying $\|y_m\|_E < \delta_m$ for $\delta_m > 0$ small enough, we have $\|u_m + y_m\|_E \in [r - \varepsilon, r' + \varepsilon]$.

Up to choosing a sequence $\{\delta'_m\}_{m\geq 0}$ with $0 < \delta'_m \leq \delta_m$, the $\{u_m\}_{m\geq 0}$ and $\{u_m + y_m\}_{m\geq 0}$ are equivalent basic sequences by [**Meg**, Ch.4,Prop.4.3.4,p.387], and hence by [**Meg**, Ch.4,Prop.4.3.2,p.387] the assignment $u_m \mapsto u_m + y_m$ extends uniquely to a linear topological isomorphism $V \to W$, where W is the closure of the linear span of $\{u_m + y_m\}_{m\geq 0}$ in E. □

[7] Return here.

REMARK 10.4.2.6. We note that the quotient $E \to E/E_0$ in Lemma 10.4.2.5 maps the open unit ball in E surjectively onto the open unit ball in E/E_0. If E is reflexive, then it maps the closed unit ball in E surjectively onto the closed unit ball in E/E_0 (a consequence of [**Jar**, Part II,Ch.11,§11.4,Cor.2,p.280], to deduce that continuous linear functionals attain their maximum on the closed unit ball). The requirement that the lift $\{u_m\}_{m\geq 0}$ of the subset $\{x_m\}_{m\geq 0} \subset E/E_0$ contained in the closed collar $\{x \in E/E_0 : \|x\|_{E/E_0} \in [r, r']\}$, should not only be contained in a similar closed collar in E bounded away from $0 \in E$, but also be contained in F, leads to the necessity of enlarging the closed collar in E from $\{u \in E : \|u\|_E \in [r, r']\}$ to its closed ε-neighborhood for a small enough choice of $\varepsilon > 0$.

REMARK 10.4.2.7. The Lemma says that given a normalized Schauder basis of E/E_0, we can lift it to a Schauder basic sequence of E contained in a given subspace $F \subset E$ such that $F \cap E_0$ is dense, and the image of F in E/E_0 is dense (hence, in particular, F is itself dense in E), and upon perturbing such lift a little, to another that satisfies the same properties, the closures of their linear spans are linearly and topologically isomorphic.

If E is a Hilbert space, then we can find $\{u_m\}_{m\geq 0}$ such that its closed linear span is linearly and topologically isomorphic to the orthogonal complement of E_0, and hence so are the closed linear spans of all small enough perturbations of $\{u_m\}_{m\geq 0}$.

10.4.2.2. *Spectral Side: convolution after projection.* We can now state and prove the last of the main Theorems of this section:

THEOREM 10.4.2.8 (Spectral side: convolution after projection). *Let F be a number field, a, s real numbers, $s \geq 0$. Let $I = [-\alpha, \alpha] \subset \mathbf{R}$ be a compact interval, $\alpha \geq 0$ a real number. We have, for all real numbers $\Lambda, \varepsilon > 0$ satisfying $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$, and $\varepsilon\Lambda = o(1)$ as $\Lambda \to +\infty$, and for a (Λ, ε)-cutoff ρ for each pair (Λ, ε):*

$$\text{Trace}\,(M_\rho \circ A(\varphi) \circ P_a^{\text{spec}}) = \text{Trace}\,(A_a^{\text{spec}}(\varphi)) + o(1) \tag{10.4.2.20}$$

as $\Lambda \to +\infty$, where $A_a^{\text{spec}}(\varphi)$ is the nuclear operator from (10.3.2.2)*, whose trace is computed by the formula in* (10.3.2.3) *of Theorem 10.3.2.1.*

PROOF. We fix a real number $\varepsilon > 0$. Let $\chi \in \widehat{C_F}$ be also fixed. We pick a lift of the Spectral Markushevich basis

$$\psi_{\chi,t,b,m},\ (t,b) \in \mathbf{R}^2 : L(s+b+it,\chi) = 0, 0 \leq m \leq N(\rho_{\chi,t,b,m}) - 1 \tag{10.4.2.21}$$

on the I-based strip (see (10.2.2.4)) provided by Theorem 10.2.2.1, whose continuous coefficient functionals are, as per the theorem, provided by integration with respect to the Haar measure $\overline{\mu}$ on C_F against the functions of $g \in C_F$:

$$\eta_{\chi,t,b,m}(g) = \chi(g)|g|_F^{a+b+it}(\log|g|_F)^m,\ (t,b) \in \mathbf{R}^2 : L(s+b+it,\chi) = 0, 0 \leq m \leq N(\rho_{\chi,t,b,m}) - 1 \tag{10.4.2.22}$$

whose lift we denote by:

$$\widetilde{\psi}_{\chi,t,b,m},\ (t,b) \in \mathbf{R}^2 : L(s+b+it,\chi) = 0, 0 \leq m \leq N(\rho_{\chi,t,b,m}) - 1 \tag{10.4.2.23}$$

such that, upon denoting by $C > 0$ the real number serving as upper bound to the W_I^s-norms of the basis elements such that $1/C$ serves as a lower bound to their norms ($C > 0$ exists by Proposition 10.2.2.2), we have:

$$1/C - \varepsilon < \|\widetilde{\psi}_{\chi,t,b,m}\|_{W_I^s(C_F)} < C + \varepsilon \tag{10.4.2.24}$$

for all $(t,b) \in \mathbf{R}^2$ such that $L(s+b+it,\chi) = 0$ and all integers $0 \leq m \leq N(\rho_{\chi,t,b,m}) - 1$. We choose $\varepsilon > 0$ so as to have $1/C - \varepsilon > 0$, and choose the lifts in (10.4.2.23) accordingly.

The S_1-norm of the finite rank operators:

$$A(\varphi)\widetilde{\psi}_{\chi,t,b,m} \otimes \eta_{\chi,t,b,m} \in W_I^s(C_F) \otimes W_I^{-s}(C_F) \tag{10.4.2.25}$$

equals their S_2-norm (see [**Cnw-b**, Ch.3,§18,Thm.18.13,p.91]), which is in turn computed as the square of the following integral:

$$\int_{C_F \times C_F} \left| \int_{C_F} \varphi(g)\widetilde{\psi}_{\chi,t,b,m}(g^{-1}x)\, \mathrm{d}\overline{\mu}(g)\eta_{\chi,t,b,m}(y) \right|^2 w_{I,s}(x)w_{I,s}(y)^{-1}\, \mathrm{d}(\overline{\mu}\otimes\overline{\mu})(x,y). \tag{10.4.2.26}$$

We note that we can arrange the $\widetilde{\psi}_{\chi,t,b,m}$ to be in $\mathcal{S}_{\mathrm{E}}(C_F)$, by Lemma 10.4.2.5.

We now estimate the integral in (10.4.2.26). We have that the integral in (10.4.2.26) equals:

$$\int_{C_F \times C_F} \int_{C_F \times C_F} \varphi(g_1)\varphi(g_2)\widetilde{\psi}_{\chi,t,b,m}(g_1^{-1}x)\widetilde{\psi}_{\chi,t,b,m}(g_2^{-1}x)\, \mathrm{d}(\overline{\mu}\otimes\overline{\mu})(g_1,g_2)\eta_{\chi,t,b,m}(y)^2\, w_{I,s}(x)w_{I,s}(y)^{-1}\, \mathrm{d}(\overline{\mu}\otimes\overline{\mu})(x,y). \tag{10.4.2.27}$$

By Fubini's Theorem (see [**Fol**, Thm.7.28]) and shearing under the transformation $(x,y) \mapsto (g_1x, g_1y)$, we obtain that the integral in (10.4.2.27) equals:

$$\int_{C_F \times C_F} \widetilde{\psi}_{\chi,t,b,m}(x) \int_{C_F \times C_F} \varphi(g_1)\varphi(g_2)\widetilde{\psi}_{\chi,t,b,m}(g_2^{-1}g_1x)\eta_{\chi,t,b,m}(g_1y)^2\, w_{I,s}(g_1x)w_{I,s}(g_1y)^{-1}\, \mathrm{d}(\overline{\mu}\otimes\overline{\mu})(x,y)\, \mathrm{d}(\overline{\mu}\otimes\overline{\mu})(g_1,g_2). \tag{10.4.2.28}$$

We have:

$$\eta_{\chi,t,b,m}(gy) = \chi(gy)|gy|_F^{b+it}(\log|gy|_F)^m \tag{10.4.2.29}$$

$$= \chi(g)\chi(y)|g|_F^{b+it}|y|_F^{b+it}(\log|g|_F + \log|y|_F)^m \tag{10.4.2.30}$$

$$= \chi(g)\chi(y)|g|_F^{b+it}|y|_F^{b+it} \sum_{0\le i\le m} \binom{m}{i} (\log|g|_F)^i(\log|y|_F)^{m-i} \tag{10.4.2.31}$$

$$= \chi(g)|g|_F^{b+it} \sum_{0\le i\le m} \binom{m}{i} (\log|g|_F)^i \left\{\chi(y)|y|_F^{b+it}(\log|y|_F)^{m-i}\right\} \tag{10.4.2.32}$$

$$= \sum_{0\le i\le m} \binom{m}{i} \chi(g)|g|_F^{b+it}(\log|g|_F)^i \cdot \eta_{\chi,t,b,m-i}(y). \tag{10.4.2.33}$$

Using the estimate in (9.3.1.6) from Lemma 9.3.1.5, combined with (10.4.2.29) and un-doing the shearing, we obtain that up to a positive real constant independent of the choice label (χ,t,b,m), the following sum is an upper bound for the S_2-norm, and hence the S_1-norm, of (10.4.2.25):

$$\sum_{k\ge 0}\sum_{j\ge 0} \left(N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})2^{N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})-1} \sum_{0\le i\le N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})-1} |M(\varphi)^{(i)}(b_{k,j}+it_{k,j})| \right)^2 \tag{10.4.2.34}$$

The argument for the proof of Theorem 10.3.2.1 can now be repeated, giving that the S_1-limit of the sum of the rank-1 operators from (10.4.2.25) exists and agrees with $A(\varphi)\circ P_a^{\mathrm{spec}}$, which is therefore nuclear and has the same trace as $P_a^{\mathrm{spec}} \circ A(\varphi)$. The remaining assertions of the Theorem follow as a consequence.[8] □

10.4.3. Spectral Kernels.

10.4.3.1. *Spectral kernels: the case $I = \{0\}$.* We let F be a number field and $\chi \in \widehat{C_F}$ a continuous character. We choose a continuous homomorphic section τ to the idelic norm.

[8] Return here and complete proof.

We defined:

$$\Lambda(s,\chi) := \left(\frac{1}{2}s(s-1)\right)^{\delta_\chi} L(s,\chi) \tag{10.4.3.1}$$

where $L(s,\chi) = L(\chi|\cdot|_F^s)$ is the Hecke L-function, and $\delta_\chi = 1$ if $\chi|_{C_F^1} = 1$, and $\delta_\chi = 0$ if $\chi|_{C_F^1} \neq 1$.

The completion map $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \to W^{s,2}(C_F)(\chi|_{C_F^1})$ fits into a commutative diagram of continuous linear operators:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_\chi & \longrightarrow & A\mathcal{S}(\mathbf{C}) & \longrightarrow & K_\chi := A\mathcal{S}(\mathbf{C})/W_\chi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle L^{-1}} & & \downarrow{\scriptstyle M^{-1}} & & \downarrow{\scriptstyle M^{-1}} & & \\
0 & \longrightarrow & \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) & \longrightarrow & \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) & \longrightarrow & \check{K}_\chi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) & \longrightarrow & W^{s,2}(C_F)(\chi|_{C_F^1}) & \longrightarrow & \mathscr{H}^s_{F,a}(\chi|_{C_F^1}) & \longrightarrow & 0
\end{array} \tag{10.4.3.2}$$

for:

$$W_\chi := \Lambda(s,\chi)\,\mathscr{O}(\mathbf{C}) \cap A\mathcal{S}(\mathbf{C})$$

topologized, in light of Theorem 6.3.2.1, with the subspace topology inherited from the separable nuclear Fréchet space $A\mathcal{S}(\mathbf{C})$, and where the notation $\mathscr{H}^s_{F,a}(\chi|_{C_F^1})$ was introduced in (10.2.1.1), where K_χ is the subquotient $A\mathcal{S}(\mathbf{C})/W_\chi$, and $\check{K}_\chi$ is its image under the anti-transform M^{-1}, where M is the Mellin transform $\phi \mapsto \int_{C_F} \phi(g)\chi(g)|g|_F^s\,\mathrm{d}\overline{\mu}(g)$, where χ is the fixed character of C_F, noting that M restricts to L from (5.2.3.14) on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

We refer the reader to the discussion on the Mellin transform on $\mathbf{R}_{>0}$-modulated groups from Discussion 4.4.1.1, and we recall that the operator M in the foregoing is related to M_I for $I = \mathbf{R}$ by a vertical translation by ix_0 on the holomorphic coordinate s, which depends on the choice of τ, and on the continuous character $\chi \in \widehat{C_F}$: $x_0 \in \mathbf{R}$ is the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions of $t \in \mathbf{R}_{>0}$.

The bottom vertical arrows are all induced by the norm-completion with respect to the $\|\cdot\|_{s,2}$-norm topology on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, and on $|\cdot|_F^a\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \subset \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$.

They are all injective since the norm topology on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ is Hausdorff, and hence so is on its closed (by Theorem 6.3.2.1) subspace $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, and on the subspace $|\cdot|_F^a \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ (being multiplication by the quasi-character $|\cdot|_F^a$ a continuous continuously (equivalently boundedly, since $\mathcal{S}_{\mathrm{E}}(G)$ is bornological, for G a second-countable $\mathbf{R}_{>0}$-modulated group; see Theorem D.3.3.7(1)) invertible automorphism on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, due to the exponential decay of the elements on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ and being the Hausdorff locally convex topology on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ defined to be, under Mellin transform, that which induces the topology of uniform convergence on the vertical strips of bounded width within $\mathbf{C}$; see Lemma D.3.1.2 and Theorem D.3.3.13).

By Theorem 6.3.2.1), L^{-1}, the top leftmost vertical arrow in (10.4.3.2), is a linear topological isomorphism. So is M^{-1} in the top middle vertical arrow of (10.4.3.2).

It is the content of Theorem 7.1.4.2 that the strong dual $(\check{K}_\chi)'_\beta$ admits a countable set of linear topological generators given by point-evaluation functionals supported at the zeors of $\Lambda(s,\chi)$ (shifted by a to the left), and their derivatives of order strictly bounded above by the multiplicity of the zero.

Let (ρ, m) be a pair comprising a zero of $L(s,\chi)$ of mutiplicity $N(\rho)$, and m an integer satisfying $0 \leq m \leq N(\rho) - 1$.

Such continuous functionals in $(\check{K}_\chi)'_\beta$ are, for each (ρ, m), given explicitly by the assignment:

$$D_{\rho,m} : \phi \mapsto \int_{C_F} \phi(g)\chi(g)|g|_F^\rho (\log|g|_F)^m\,\mathrm{d}\overline{\mu}(g). \tag{10.4.3.3}$$

Multiplication by the quasi-character $|\cdot|_F^a$ on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, which is a linear topological automorphism on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, sends $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ to $|\cdot|_F^a \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ as closed subspaces of $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$, and its adjoint sends each $D_{\rho,m}$ from (10.4.3.3) to the functionals:

$$D_{-a+\rho,m} : \phi \mapsto \int_{C_F} \phi(g)\chi(g)|g|_F^{\rho-a}(\log|g|_F)^m \,\mathrm{d}\overline{\mu}(g) \tag{10.4.3.4}$$

where now $\rho - a$ is a zero of $L(s+a,\chi)$.

In particular, for any real number a, and for each pair (ρ, m) such that $a+\rho$ is a zero of $L(s,\chi)$ of multiplicity $N(a+\rho)$, and m is an integer $0 \leq m \leq N(a+\rho)-1$, there exists a unique bi-orthogonal sequence of elements $\psi_{a+\rho,m} \in \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ satisfying:

$$D_{a+\rho,m}(\psi_{a+\rho',m'}) = \delta_{(a+\rho,m),(a+\rho',m')} \tag{10.4.3.5}$$

with $\delta_{(a+\rho,m),(a+\rho',m')} = 1$ for $(a+\rho,m) = (a+\rho',m')$, and $\delta_{(a+\rho,m),(a+\rho',m')} = 0$ otherwise. This is a consequence of the explicit construction of the functions $\phi_{\rho,m}$ in the proof of Theorem 7.1.4.2.

We denote, for each zero $a+\rho$ of $L(s,\chi)$, by:

$$\check{K}_{\chi,a+\rho} := \sum_{0\leq m\leq N(a+\rho)-1} \mathbf{C}\psi_{a+\rho,m} \subset \mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \tag{10.4.3.6}$$

the finite dimensional $\mathbf{C}$-linear subspace generated by the finite collection $\{\psi_{(a+\rho,m)}, 0 \leq m \leq N(a+\rho)-1\}$.

Since the $\|\cdot\|_{s,2}$-norm topology on $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1})$ is, for every real number $s \geq 0$, separated, the finite dimensional linear subspace $\check{K}_{\chi,a+\rho}$ is mapped injectively into $W^{s,2}(C_F)$ under the completion map $\mathcal{S}_{\mathrm{E}}(C_F)(\chi|_{C_F^1}) \to W^{s,2}(C_F)(\chi|_{C_F^1}) \subset W^{s,2}(C_F)$.

These observations were made throughout the proof of Theorem 10.3.1.1.

In this subsection, we discuss the "spectral side" of the Trace Formula, i.e. we compute the limit as $\Lambda \to +\infty$ of Trace $P_{\Lambda,a}^{\mathrm{spec}} \circ A(\varphi)$, for any $a \in \mathbf{R}$, $s > 1$, and $\varphi \in \mathcal{D}(C_F)$.

We begin with the following consequence of Theorem 10.2.1.1 and Theorem 10.3.1.1.

We recall that for X, Y two Hausdorff locally compact second countable topological spaces, and μ a Radon measure on X and ν a Radon measure on Y, then

$$S_2(L^2(X,\mu), L^2(Y,\nu)) = L^2(X\times Y, \mu\otimes\nu) \tag{10.4.3.7}$$

the source being the Hilbert space of Hilbert-Schmidt operators $L^2(X,\mu) \to L^2(Y,\nu)$, i.e. every Hilbert-Schmidt operator $T : L^2(X,\mu) \to L^2(Y,\nu)$ is obtained by integration against a unique kernel function $k \in L^2(X\times Y, \mu\otimes\nu)$:

$$(T\phi)(y) = \int_X k(x,y)\phi(x)\,\mathrm{d}\mu(x) \tag{10.4.3.8}$$

for almost all $y \in Y$.

We further refer the reader to [**Jar**, Part III,Ch.17,§17.6,p.391], to (C.3.6.5) and to Proposition C.3.6.4.

THEOREM 10.4.3.1 (Spectral and Geometric kernels - I). *Let F be a number field.*
Let a, s be real numbers, with $a > 0$ and $s > 1$. Let $\varphi \in \mathcal{D}(C_F)$ be a test function.
We choose an enumeration for the set of triples $Z := \{\chi, t, m\}$ as follows:

1. *We choose an enumeration for the set of continuous characters χ of the base-points of the connected components of $\Omega(C_F)$ chosen in the holomorphic parametrization from* (10.2.1.4) *of Notation 10.2.1.1, and denote it $\{\chi_k\}_{k\geq 0}$.*
2. *For each integer $k \geq 0$, we choose an enumeration $\{t_{k,j}\}_{j\geq 0}$, for the set of real numbers $t \in \mathbf{R}$ such that $\rho_{\chi_k,t} := \chi_k|\cdot|_F^{a+it}$ is a zero of $L(s,\chi_k)$, of multiplicity denoted by $N(\rho_{\chi_k,t})$, and we choose the enumeration for the $t_{k,j}$ so as to have $|t_{k,j}| \leq |t_{k,j+1}|$ for all integers $j \geq 0$.*

(3) *For each pair* $(\chi_k, t_{k,j})$*, we enumerate the finitely many integers* $m \geq 0$ *satisfying* $m \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1$*, with* $N_s(\rho_{\chi_k,t_{k,j}})$ *defined in* (10.2.1.6) *by:*

$$N_s(\rho_{\chi_k,t_{k,j}}) = \min\left\{N(\rho_{\chi_k,t_{k,j}}), \left[\frac{s-1}{2}+1\right]\right\} \tag{10.4.3.9}$$

according to their natural increasing ordering.

(4) *For each such* χ*, the induced enumeration of the subset* $Z_\chi \subset Z$ *of those triples such that the first entry is* χ*, is lexicographical.*

and denote by $\{\psi_{\chi,t,m}\}_{\chi,t,m} \subset \mathscr{H}^s_{F,a}$ *the dual Markushevich basis in* $\mathscr{H}^s_{F,a}$ *to the Markushevich basis for* $(\mathscr{H}^s_{F,a})'_\beta$ *obtained as the image of the functions:*

$$\eta_{\chi,t,m} := \chi|\cdot|_F^{it}(\log|\cdot|_F)^m \tag{10.4.3.10}$$

under the inverse of the isometry $T_s : W^{s,2}(C_F)'_\beta \to W^{-s,2}(C_F)$ *from* (9.1.2.14)*, provided by Theorem 10.2.1.1 for the strong dual* $(\mathscr{H}^s_{F,a})'_\beta$*, where* $\mathscr{H}^s_{F,a}$ *is the Hilbert space quotient of* $W^{s,2}(C_F)$ *by* $W^{s,2}_a \,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ *defined in* (10.2.1.1)*.*

We denote by:

$$I := \{(k,j,m_{k,j}) \in \mathbf{N}^3 : k \geq 0, j \geq 0, 0 \leq m_{k,j} \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1\} \tag{10.4.3.11}$$

ordered lexicographically from the lexicographic ordering on $\mathbf{N}^3$*.*

Then:

(1) *The effect of the canonical linear isometry* $S_a : \mathscr{H}^s_{F,a} \to P_a^{\mathrm{spec}} W^{s,2}(C_F)$ *from* (10.4.1.5)*, inverse to the quotient* $P_a^{\mathrm{spec}} W^{s,2}(C_F) \to \mathscr{H}^s_{F,a}$*, on the basis elements* $\{\psi_{\chi,t,m}\}_{\chi,t,m} \subset \mathscr{H}^s_{F,a}$*, is described as follows: for each* $(\chi,t,m) \in Z$ *according to the chosen enumeration, we make a choice of an element* $\widetilde{\psi}_{\chi,t,m} \in W^{s,2}(C_F)$ *whose image in* $\mathscr{H}^s_{F,a}$ *is* $\psi_{\chi,t,m}$*. Then:*

$$S_a(\psi_{\chi,t,m}) = P_a^{\mathrm{spec}}(\widetilde{\psi}_{\chi,t,m}). \tag{10.4.3.12}$$

(2) *The sequences of finite rank kernels:*

$$\sum_{i\in I} \langle A(\varphi)'\eta_{(\chi,t,m)_i}, \cdot\rangle_{s,2} \otimes \psi_{(\chi,t,m)_i} = \sum_{0\leq k\leq N_1} \sum_{0\leq j\leq N_2} \sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1} \langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}}, \cdot\rangle_{s,2} \otimes \psi_{\chi_k,t_{k,j},m_{k,j}} \tag{10.4.3.13}$$

in $(\mathscr{H}^s_{F,a})'_\beta \,\widehat{\otimes}_\pi (\mathscr{H}^s_{F,a})$*, and:*

$$\sum_{i\in I} A(\varphi)'\eta_{(\chi,t,m)_i} \otimes S_a(\psi_{(\chi,t,m)_i}) = \sum_{0\leq k\leq N_1} \sum_{0\leq j\leq N_2} \sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1} A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}} \otimes S_a(\psi_{\chi_k,t_{k,j},m_{k,j}}) \tag{10.4.3.14}$$

in $W^{-s,2}(C_F) \,\widehat{\otimes}_\pi W^{s,2}(C_F)$*, converge as* $(N_1,N_2) \to (+\infty,+\infty)$ *in* $\mathbf{N}\times\mathbf{N}$*, equivalently* $i \to +\infty$ *in* I*, with respect to the respective projective tensor product topologies.*

(3) *Let:*

$$\xi_\pi : W^{-s,2}(C_F) \,\widehat{\otimes}_\pi W^{s,2}(C_F) \to \mathscr{L}_\beta(W^{s,2}(C_F)) \tag{10.4.3.15}$$

be the composition of $T_s^{-1} \,\widehat{\otimes}_\pi \mathrm{Id}$ *and the continuous injective linear operator* χ_π *from Proposition C.3.6.4, whose effect on finite rank kernels is that of associating to* $\sum_{1\leq k\leq N} \phi_k \otimes \psi_k$ *the finite rank operator* $\sum_{1\leq k\leq N} \langle \phi_k, \cdot\rangle_{s,2}\, \psi_k$*. Upon denoting by* $K^{\mathrm{spec}}_{\varphi,a,s}$ *the limit in* $W^{-s,2}(C_F) \,\widehat{\otimes}_\pi W^{s,2}(C_F)$ *of the finite rank kernels from* (10.4.3.14)*, then we have, within* $\mathscr{L}_\beta(W^{s,2}(C_F))$*:*

$$\xi_\pi(K^{\mathrm{spec}}_{\varphi,a,s}) = P_a^{\mathrm{spec}} \circ A(\varphi). \tag{10.4.3.16}$$

The sequence of finite rank operators in (10.3.1.1) was shown to converge in $S_1(\mathscr{H}^s_{F,a})$ throughout the proof of Theorem 10.3.1.1.

PROOF. (1) follows from Lemma 10.4.1.2.

Directly, let Q_a be the quotient linear operator from $W^{s,2}(C_F)$ to $\mathscr{H}^s_{F,a}$ from (10.4.1.3).

S_a is the inverse of the restriction of Q_a to $P_a^{\mathrm{spec}}W^{s,2}(C_F)$: such restriction is continuous, open and bijective onto $\mathscr{H}^s_{F,a}$, hence an isometric linear topological isomorphism (see [**Tre**, Part I,Ch.17,Thm.17.1]).

For each basis element $\psi_{\chi,t,m} \in \mathscr{H}^s_{F,a}$, the choice of lift $\widetilde{\psi}_{\chi,t,m}$ satisfies

$$S_a^{-1}(\widetilde{\psi}_{\chi,t,m}) = S_a^{-1}(P_a^{\mathrm{spec}}(\widetilde{\psi}_{\chi,t,m})) = Q_a(P_a^{\mathrm{spec}}(\widetilde{\psi}_{\chi,t,m})) = \psi_{\chi,t,m} \tag{10.4.3.17}$$

where the first identity is a consequence of the containment

$$P_a^{\mathrm{geom}}W^{s,2}(C_F) = W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset \ker(Q_a)$$

and orthogonality in $W^{s,2}(C_F)$ of $P_a^{\mathrm{geom}}W^{s,2}(C_F)$ to the range of S_a.

Upon evaluating S_a on both the sides of the rightmost identity (10.4.3.17), we obtain:

$$P_a^{\mathrm{spec}}(\widetilde{\psi}_{\chi,t,m}) = S_a(\psi_{\chi,t,m}) \tag{10.4.3.18}$$

and so (10.4.3.12) holds.

(2) : Since S_a is a linear isometry, the image of a Markushevich basis for $\mathscr{H}^s_{F,a}$ is a Schauder basis for $P_a^{\mathrm{spec}}W^{s,2}(C_F)$, since linear isometries respect topologically free sequences (the duality $\langle P_a^{\mathrm{spec}}W^{s,2}(C_F), (P_a^{\mathrm{spec}})'W^{-s,2}(C_F)\rangle$ is induced by the duality $\langle\cdot,\cdot\rangle_{s,2}$, with respect to which the continuous adjoint of P_a^{spec} is being defined).

By Theorem 10.2.1.1

$$\{\eta_{\chi_k,t_{k,j},m_{k,j}}\}_{k\geq 0, j\geq 0, 0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1}$$

is the image under the isometric isomorphism $T_s : W^{s,2}(C_F)'_\beta \to W^{-s,2}(C_F)$ of (9.1.2.14), of a Markushevich basis for $(\mathscr{H}^s_{F,a})'_\beta$, and

$$\{\psi_{\chi_k,t_{k,j},m_{k,j}}\}_{k\geq 0, j\geq 0, 0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1}$$

is its dual basis in $\mathscr{H}^s_{F,a}$, itself a Markushevich basis, for $\mathscr{H}^s_{F,a}$, with the dualities on sources and targets defined accordingly.

This means that the functionals:

$$\langle \eta_{\chi_k,t_{k,j},m_{k,j}}, \cdot\rangle_{s,2} : W^{s,2}(C_F) \to \mathbf{C}$$

each of which factors through $\mathscr{H}^s_{F,a}$ defining a unique continuous functional on $\mathscr{H}^s_{F,a}$, i.e. an element of the strong dual $(\mathscr{H}^s_{F,a})'_\beta$, are a Markushevich basis for $(\mathscr{H}^s_{F,a})'_\beta$, as $(k,j,m_{k,j})$ ranges over the countable set

$$I := \{(k,j,m_{k,j}) \in \mathbf{N}^3 : k\geq 0, j\geq 0, 0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1\}$$

as in (10.4.3.11).

We note that the concept of "diverging to infinity" for sequences in I is already intrinsically defined, inherited from $\mathbf{N}\times\mathbf{N}\times\mathbf{N}$. It reads as follows: $i\to+\infty$ if and only if, writing $i=(k(i),j(i),m_{k(i),j(i)})$, we have $(k(i),j(i))\to(+\infty,+\infty)$ in $\mathbf{N}\times\mathbf{N}$. In other words, "infinity" within I is the subset of the closure $\overline{I}$ of I within the one-point compactification of the countable discrete set $\mathbf{N}\times\mathbf{N}\times(\mathbf{N}\cap[0,C_s])$ with $C_s := [(s-1)/2+1]$, obtained as $\overline{I}-I$, which is contained within $(+\infty,+\infty)\times(\mathbf{N}\cap[0,C_s])$.

By Corollary 10.2.1.4, each of the two Markushevich bases is normalized (we refer the reader to Terminology 10.2.1.2 for this lexicon, and to the discussion preceding Corollary 10.2.1.4 and Corollary 11.2.0.5).

The sequence of finite rank kernels from (10.4.3.13):

$$\sum_{0\leq k\leq N_1}\sum_{0\leq j\leq N_2}\sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j}})-1} \langle A(\varphi)'\eta_{\chi_k,t_{k,j},m_{k,j}},\cdot\rangle_{s,2}\otimes\psi_{\chi_k,t_{k,j},m_{k,j}} \tag{10.4.3.19}$$

in $(\mathscr{H}^s_{F,a})'_\beta \otimes \mathscr{H}^s_{F,a}$ converge in $(\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a}$.

Indeed, the sequence of finite rank operators in $\mathscr{L}(\mathscr{H}^s_{F,a})$ from (10.3.1.1) converges in $S_p(\mathscr{H}^s_{F,a})$ for every $p \geq 1$ by Theorem 10.3.1.1(1).

In particular, the sequence of finite rank kernels in (10.4.3.13) converges in $(\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a}$, since $(\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a}$ is isometric to $S_1(\mathscr{H}^s_{F,a})$ under the continuous linear operator χ_π of (C.3.6.7) from Proposition C.3.6.4.

This establishes both convergence assertions in (2).

The Hilbert space $(\mathscr{H}^s_{F,a})'_\beta$ is a closed Hilbert subspace of $W^{s,2}(C_F)'_\beta$, which in turn is mapped isometrically isomorphically onto $W^{-s,2}(C_F)$ by T_s from (9.1.2.14) of Proposition 9.1.2.5.

The effect of T_s on the continuous functional $\langle \eta_{(\chi,t,m)_i}, \cdot \rangle_{s,2}$ on $\mathscr{H}^s_{F,a}$ is that of sending it to the function $\eta_{(\chi,t,m)_i} \in W^{-s,2}(C_F)$, for each index $i \in I$.

T_s is compatible with the adjoint action of C_F on $(\mathscr{H}^s_{F,a})'_\beta$ (induced by the action on $W^{s,2}(C_F)$, on the closed C_F-invariant subspace $W^{s,2}_a \, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, and hence on the quotient $\mathscr{H}^s_{F,a}$), and so we have:

$$T_s(\langle A(\varphi)'\eta_{(\chi,t,m)_i}, \cdot \rangle_{s,2}) = T_s(A(\varphi)'\langle \eta_{(\chi,t,m)_i}, \cdot \rangle_{s,2}) = A(\varphi)'\eta_{(\chi,t,m)_i}$$

for every index $i \in I$.

Since $S_a : \mathscr{H}^s_{F,a} \to P^{\mathrm{spec}}_a W^{s,2}(C_F)$ is a surjective linear isometry, as is T_s onto its own image in $W^{-s,2}(C_F)$, we deduce that the topology induced on the tensor product on the source of

$$T_s \mathbin{\widehat{\otimes}}_\pi S_a : (\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a} \to W^{-s,2}(C_F) \mathbin{\widehat{\otimes}}_\pi W^{s,2}(C_F) \tag{10.4.3.20}$$

by the projective tensor product topology on the target, is the projective tensor product topology.

Indeed, both T_s and S_a are linear topological embeddings whose range is closed and linearly and topologically complemented, being each linear isometries, and being their sources and targets Hilbert spaces. By [**Jar**, Part III,Ch.15,§15.2,Prop.3(b),p.327], combined with [**Jar**, Part III,Ch.15,§15.2,Prop.1,p.326], $T_s \otimes S_a$ is a linear topological isomorphism onto its range, and its range W is a linearly and topologically complemented subspace of $W^{-s,2}(C_F) \otimes_\pi W^{s,2}(C_F)$.

The unique linear topological isomorphism induced by [**Jar**, Part I,Ch.3,§3.4,Thm.2,p.61] between the completions of source and target is $(T_s \otimes S_a)^\wedge : (\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a} \to \widehat{W}$ agrees with $T_s \mathbin{\widehat{\otimes}}_\pi S_a$, because, by [**Jar**, Part I,Ch.3,§3.4,Thm.2,p.61], $\widehat{W} \subset W^{-s,2}(C_F) \mathbin{\widehat{\otimes}}_\pi W^{s,2}(C_F)$ is also closed and linearly and topologically complemented (which is seen by applying [**Jar**, Part I,Ch.3,§3.4,Thm.2,p.61] to the continuous section splitting the linear topological embedding $W \subset W^{-s,2}(C_F) \otimes_\pi W^{s,2}(C_F)$) in particular, it carries the subspace topology. We further refer the reader to [**Tre**, Part III,Ch.43,Rem.43.2,p.442].

The image under $T_s \otimes S_a$ of the finite rank kernel in (10.4.3.19) is, for each index $i_0 \in I$:

$$\sum_{i \in I, i \leq i_0} A(\varphi)'\eta_{(\chi,t,m)_i} \otimes S_a(\psi_{(\chi,t,m)_i}) \in W^{-s,2}(C_F) \otimes_\pi W^{s,2}(C_F) \subset W^{-s,2}(C_F) \mathbin{\widehat{\otimes}}_\pi W^{s,2}(C_F) \tag{10.4.3.21}$$

where we have used that the projective tensor product topology on $W^{-s,2}(C_F) \otimes W^{s,2}(C_F)$ is Hausdorff by [**Jar**, Part III,Ch.15,§15.1,Prop.3,p.325].

Since the sequence of finite rank kernels in (10.4.3.19) converges in $(\mathscr{H}^s_{F,a})'_\beta \mathbin{\widehat{\otimes}}_\pi \mathscr{H}^s_{F,a}$ by Theorem 10.2.1.1, Corollary 10.2.1.4 and Theorem 10.3.1.1, then so does the sequence of finite rank tensors in $W^{-s,2}(C_F) \mathbin{\widehat{\otimes}}_\pi W^{s,2}(C_F)$ in (10.4.3.21), since it is obtained as its image under $T_s \mathbin{\widehat{\otimes}}_\pi S_a$.

(3) : The continuous linear endomorphism $\xi_\pi(K^{\mathrm{spec}}_{\varphi,a,s})$ acts on each element $\phi \in W^{s,2}(C_F)$ as follows:

$$\xi_\pi(K^{\mathrm{spec}}_{\varphi,a,s})(\phi) = \sum_{k \geq 0} \langle A'(\varphi)\eta_{\chi_k,t_k,m_k}, \phi \rangle_{s,2} S_a(\psi_{\chi_k,t_k,m_k}) \tag{10.4.3.22}$$

and since $P^{\mathrm{spec}}_a \circ A(\varphi) = P^{\mathrm{spec}}_a \circ A(\varphi) \circ P^{\mathrm{spec}}_a$, it is enough to check that for each ϕ of the form $S_a(\psi)$ for some $\psi \in \mathscr{H}^s_{F,a}$, the effects of $P^{\mathrm{spec}}_a \circ A(\varphi)$ and of $\xi_\pi(K^{\mathrm{spec}}_{\varphi,a,s})$ agree.

Since $\{\psi_{\chi_k,t_k,m_k}\}_{k\geq 0}$ is a Markushevich basis for $\mathscr{H}^s_{F,a}$, it is enough to show the assertion for $\phi = S_a(\psi_{\chi,t,m})$. The assertion in this case is proved as a consequence of the calculation in (10.3.1.8), and Lemma 10.4.1.2.

The rest of the assertions in (3) are a consequence of Proposition C.3.6.4. □

The continuous linear map:

$$\xi_\pi : W^{-s,2}(C_F)\,\widehat{\otimes}_\pi W^{s,2}(C_F) \to \mathscr{L}_\beta(W^{s,2}(C_F))$$

from (10.4.3.15) factors through the continuous linear operator

$$W^{-s,2}(C_F)\,\widehat{\otimes}_\pi W^{s,2}(C_F) \to W^{-s,2}(C_F)\,\widehat{\otimes}_\sigma W^{s,2}(C_F) \tag{10.4.3.23}$$

the target being the Hilbert-Schmidt tensor product from Proposition C.3.6.4.

There is a continuous linear isometry:

$$W^{-s,2}(C_F)\,\widehat{\otimes}_\sigma W^{s,2}(C_F) \to L^2(C_F\times C_F, (1+|\log|\cdot|_F|^2)^{-s/2}\,\overline{\mu}\otimes(1+|\log|\cdot|_F|^2)^{s/2}\,\overline{\mu}) \tag{10.4.3.24}$$

The image of $K^{\mathrm{spec}}_{\varphi,a,s}$ under the composition of the continuous linear operator from (10.4.3.23) with the continuous linear isometry from (10.4.3.24) is the unique element

$$k \in L^2(C_F\times C_F, (1+|\log|\cdot|_F|^2)^{-s/2}\,\overline{\mu}\otimes(1+|\log|\cdot|_F|^2)^{s/2}\,\overline{\mu})$$

such that, for every $\phi \in W^{s,2}(C_F)$:

$$\int_{C_F} k(x,y)\phi(y)\,(1+|\log|y|_F|^2)^{-s/2}\mathrm{d}\overline{\mu}(y) = (P^{\mathrm{spec}}_a \circ A(\varphi))(\phi)(x) \tag{10.4.3.25}$$

for $\overline{\mu}$-almost all $x \in C_F$.

Definition 10.4.3.2 (Global Spectral and Geometric Kernels - the case $I=\{0\}$). Let F be a number field, a, s real numbers, $s \geq 3$.

We define:

(1) The measurable function on $C_F \times C_F$:

$$k^{\mathrm{spec}}_{\varphi,a,s}(x,y) := k(x,y)(1+|\log|y|_F|^2)^{-s/2}. \tag{10.4.3.26}$$

We refer to $k^{\mathrm{spec}}_{\varphi,a,s}$ as the *Global Spectral Kernel* of the φ-regularized action of C_F on $W^{s,2}(C_F)$.

(2) The measurable function on $C_F \times C_F$:

$$k^{\mathrm{geom}}_{\varphi,a,s}(x,y) := \varphi(xy^{-1}) - k^{\mathrm{spec}}_{\varphi,a,s}(x,y) \tag{10.4.3.27}$$

We refer to $k^{\mathrm{geom}}_{\varphi,a,s}$ as the *Global Geometric Kernel* of the φ-regularized action of C_F on $W^{s,2}(C_F)$.

Remark 10.4.3.3. The function $k^{\mathrm{spec}}_{\varphi,a,s}$ satisfies, for every $\phi \in W^{s,2}(C_F)$:

$$\int_{C_F} k^{\mathrm{spec}}_{\varphi,a,s}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y) = P^{\mathrm{spec}}_a \circ A(\varphi) \tag{10.4.3.28}$$

for $\overline{\mu}$-almost all $x \in C_F$, as a consequence of (10.4.3.25).

Theorem 10.4.3.4. *Let F be a number field, a, s real numbers, $a > 0$, $s > 1$. Let $\Lambda, \varepsilon > 0$ be real numbers, such that $0 < 1/\Lambda + \varepsilon < \Lambda - \varepsilon$. Let $\rho : C_F \to \mathbf{C}$ be a (Λ,ε)-cutoff.*

(1) *For any $\phi \in W^{s,2}_a\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, we have, for $\overline{\mu}$-almost all $x \in C_F$:*

$$A(\varphi)(\phi)(x) = (P^{\mathrm{geom}}_a \circ A(\varphi))(\phi)(x) = \int_{C_F} k^{\mathrm{geom}}_{\varphi,a,s}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y). \tag{10.4.3.29}$$

(2) *The functions:*

$$(1+|\log|y|_F|^2)^{-s/2}\,k^{\mathrm{spec}}_{\varphi,a,s}(x,y), \quad \rho(x)\,(1+|\log|y|_F|^2)^{-s/2}\,k^{\mathrm{spec}}_{\varphi,a,s}(x,y) \tag{10.4.3.30}$$

are representative functions for the Hilbert-Schmidt kernel in $L^2(C_F \times C_F, ((1+|\log|\cdot|_F|^2)^{s/2}\,\overline{\mu})^{\otimes 2})$ *of the operator* $P_a^{\mathrm{spec}} \circ A(\varphi)$ *on* $W^{s,2}(C_F)$*, respectively for the Hilbert-Schmidt kernel in* $L^2(C_F \times C_F, ((1+|\log|\cdot|_F|^2)^{s/2}\,\overline{\mu})^{\otimes 2})$ *of the operator* $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ *on* $W^{s,2}(C_F)$.

(3) *The function:*

$$(1+|\log|y|_F|^2)^{-s/2}\,\rho(x)\,k_{\varphi,a,s}^{\mathrm{geom}}(x,y) \tag{10.4.3.31}$$

is a representative function for the Hilbert-Schmidt kernel of the operator $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ *on* $W^{s,2}(C_F)$.

Proof. (1) : We compute:

$$\int_{C_F} k_{\varphi,a,s}^{\mathrm{geom}}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y) = \int_{C_F} (\varphi(xy^{-1}) - k_{\varphi,a,s}^{\mathrm{spec}}(x,y))\phi(y)\,\mathrm{d}\overline{\mu}(y) \tag{10.4.3.32}$$

$$= \int_{C_F} \varphi(xy^{-1})\phi(y)\,\mathrm{d}\overline{\mu}(y) - \int_{C_F} k_{\varphi,a,s}^{\mathrm{spec}}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y) \tag{10.4.3.33}$$

$$= \int_{C_F} \varphi(y)\phi(y^{-1}x)\,\mathrm{d}\overline{\mu}(y) - (P_a^{\mathrm{spec}} \circ A(\varphi))(\phi)(x) \tag{10.4.3.34}$$

$$= A(\varphi)(\phi)(x) - (P_a^{\mathrm{spec}} \circ A(\varphi))(\phi)(x) \tag{10.4.3.35}$$

where we have used (10.4.3.28) from Remark 10.4.3.3 to identify $\int_{C_F} k_{\varphi,a,s}^{\mathrm{spec}}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y)$ with $(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi)(x)$, for $\overline{\mu}$-almost all $x \in C_F$.

Since $A(\varphi) - P_a^{\mathrm{spec}} \circ A(\varphi) = (\mathrm{Id} - P_a^{\mathrm{spec}}) \circ A(\varphi)$, and $\mathrm{Id} - P_a^{\mathrm{spec}} = P_a^{\mathrm{geom}}$, we deduce that (10.4.3.29) holds.

(2) : Since

$$(P_a^{\mathrm{spec}} \circ A(\varphi))(\phi)(x) = \int_{C_F} (1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)\,(1+|\log|y|_F|^2)^{s/2}\,\mathrm{d}\overline{\mu}(y)$$

we deduce, by uniqueness of the Hilbert-Schmidt kernel associated to the nuclear, hence Hilbert-Schmidt, operator $P_a^{\mathrm{spec}} \circ A(\varphi)$, that $(1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)$ is a representative kernel function for the Hilbert-Schmidt kernel of $P_a^{\mathrm{spec}} \circ A(\varphi)$ (i.e. any other such function agrees with $(1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)$ almost everywhere over $C_F \times C_F$ with respect to the product measure $(1+|\log|\cdot|_F|^2)^{-s/2}\,\overline{\mu})^{\otimes 2}$, equivalently $\overline{\mu} \otimes \overline{\mu}$, since the two measures have the same null sets in $C_F \times C_F$).

A similar argument shows that $\rho(x)\,(1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)$ is a representative function over $C_F \times C_F$ for the Hilbert-Schmidt kernel of $P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$.

(3) : Since $(1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)$ is the Hilbert-Schmidt kernel of the nuclear, hence Hilbert-Schmidt, operator $P_a^{\mathrm{spec}} \circ A(\varphi)$ on $W^{s,2}(C_F)$ by (2), then the function:

$$\rho(x) \cdot ((1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y)) \tag{10.4.3.36}$$

is the Hilbert-Schmidt kernel of $P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$.

Moreover, $\rho(x) \cdot (1+|\log|y|_F|^2)^{-s/2}\,\varphi(xy^{-1})$ is the Hilbert-Schmidt kernel of $M_\rho \circ A(\varphi)$ on $W^{s,2}(C_F)$, and so:

$$\rho(x) \cdot (1+|\log|y|_F|^2)^{-s/2}\,\varphi(xy^{-1}) - \rho(x) \cdot (1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{spec}}(x,y) \tag{10.4.3.37}$$

which is $\rho(x) \cdot (1+|\log|y|_F|^2)^{-s/2}\,k_{\varphi,a,s}^{\mathrm{geom}}(x,y)$, is the Hilbert-Schmidt kernel of $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi) = M_\rho \circ A(\varphi) - P_{\rho,a}^{\mathrm{spec}} \circ A(\varphi)$ on $W^{s,2}(C_F)$. □

10.4.3.2. *Spectral kernels: the general case of compact intervals* $I = [-\alpha, \alpha]$. Using the same estimates on singular values from the proof of Theorem 10.4.2.8, we have the following general version of Theorem 10.4.3.1, which we discuss in the case of the convolution-after-projection spectral restriction operators (see Definition 10.4.1.5):

THEOREM 10.4.3.5 (Spectral and Geometric kernels - II). *Let F be a number field.*

Let a, s be real numbers, with $a > 0$ and $s > 1$. Let $I = [-\alpha, \alpha]$ be a compact interval, $\alpha \geq 0$ a real number. Let $\varphi \in \mathcal{D}(C_F)$ be a test function.

We choose an enumeration for the set of tuples $Z := \{\chi, t, b, m\}$ as follows:

(1) *We choose an enumeration for the set of continuous characters χ of the base-points of the connected components of $\Omega(C_F)$ chosen in the holomorphic parametrization from* (10.2.1.4) *of Notation 10.2.1.1, and denote it $\{\chi_k\}_{k\geq 0}$.*
(2) *For each integer $k \geq 0$, we choose an enumeration $\{t_{k,j}, b_{k,j}\}_{j\geq 0}$, for the set of pairs of real numbers $(t, b) \in \mathbf{R}$ such that $\rho_{\chi_k,t,b} := \chi_k| \cdot |_F^{a+b+it}$ is a zero of $L(s, \chi_k)$, of multiplicity denoted by $N(\rho_{\chi_k,t,b})$, and we choose the enumeration for the $(t_{k,j}, b_{k,j})$ so as to have $|t_{k,j}| \leq |t_{k,j+1}|$ for all integers $j \geq 0$.*
(3) *For each triple $(\chi_k, t_{k,j}, b_{k,j})$, we enumerate the finitely many integers $m \geq 0$ satisfying $m \leq N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}}) - 1$, with $N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})$ defined in* (10.2.2.5) *and* (10.2.2.6) *by:*

(10.4.3.38)
$$N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}}) = \min\left\{N(\rho_{\chi_k,t_{k,j},b_{k,j}}), \left[\frac{s-1}{2}+1\right]\right\},\ b_{k,j} = \pm\alpha, N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}}) = N(\rho_{\chi_k,t_{k,j},b_{k,j}}),\ b_{k,j} \in (-\alpha, \alpha)$$

according to their natural increasing ordering.
(4) *For each such χ, the induced enumeration of the subset $Z_\chi \subset Z$ of those tuples such that the first entry is χ, is lexicographical.*

and denote by $\{\psi_{\chi,t,b,m}\}_{\chi,t,b,m} \subset \mathscr{H}^s_{F,a,I}$ the dual Markushevich basis in $\mathscr{H}^s_{F,a,I}$ to the Spectral Markushevich basis for $(\mathscr{H}^s_{F,a,I})'_\beta$ obtained as the image of the functions:

(10.4.3.39)
$$\eta_{\chi,t,b,m} := \chi| \cdot |_F^{b+it}(\log| \cdot |_F)^m$$

under the inverse of the isometry $T_{I,s} : W_I^s(C_F)'_\beta \to W_I^{-s}(C_F)$ from (9.2.2.42)*, provided by Proposition 9.2.2.11 for the strong dual $(\mathscr{H}^s_{F,a,I})'_\beta$, where $\mathscr{H}^s_{F,a,I}$ is the Hilbert space quotient of $W_I^s(C_F)$ by $W^s_{a,I}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ defined in* (10.2.2.1)*.*

We denote by:

(10.4.3.40)
$$I := \{(k, j, m_{k,j}) \in \mathbf{N}^3 : k \geq 0, j \geq 0, 0 \leq m_{k,j} \leq N_s(\rho_{\chi_k,t_{k,j}}) - 1\}$$

ordered lexicographically from the lexicographic ordering on $\mathbf{N}^3$.

Then:

(1) *The effect of the canonical linear isometry $S_a : \mathscr{H}^s_{F,a,I} \to P_a^{\mathrm{spec}} W_I^s(C_F)$ from* (10.4.1.5)*, inverse to the quotient $P_a^{\mathrm{spec}} W_I^s(C_F) \to \mathscr{H}^s_{F,a,I}$, on the basis elements $\{\psi_{\chi,t,b,m}\}_{\chi,t,b,m} \subset \mathscr{H}^s_{F,a,I}$, is described as follows: for each $(\chi, t, b, m) \in Z$ according to the chosen enumeration, we make a choice of an element $\widetilde{\psi}_{\chi,t,b,m} \in W_I^s(C_F)$ whose image in $\mathscr{H}^s_{F,a,I}$ is $\psi_{\chi,t,b,m}$. Then:*

(10.4.3.41)
$$S_a(\psi_{\chi,t,b,m}) = P_a^{\mathrm{spec}}(\widetilde{\psi}_{\chi,t,b,m}).$$

(2) *The sequences of finite rank kernels:*

(10.4.3.42)
$$\sum_{i\in I}\langle A(\varphi)'\eta_{(\chi,t,b,m)_i}, \cdot\rangle_{I,s}\otimes\psi_{(\chi,t,b,m)_i} = \sum_{0\leq k\leq N_1}\sum_{0\leq j\leq N_2}\sum_{0\leq m_{k,j}\leq N_s(\rho_{\chi_k,t_{k,j},b_{k,j}})-1}\langle A(\varphi)'\eta_{\chi_k,t_{k,j},b_{k,j},m_{k,j}}, \cdot\rangle_{I,s}\otimes\psi_{\chi_k,t_{k,j},b_{k,j},m_{k,j}}$$

in $(\mathscr{H}^s_{F,a,I})'_\beta \mathbin{\widehat{\otimes}}\, \pi(\mathscr{H}^s_{F,a,I})$, and:

(10.4.3.43)
$$\sum_{i\in I} A(\varphi)'\eta_{(\chi,t,b,m)_i}\otimes S_a(\psi_{(\chi,t,b,m)_i}) = \sum_{0\leq k\leq N_1}\sum_{0\leq j\leq N_2}\sum_{0\leq m_{k,j}\leq N_{I,s}(\rho_{\chi_k,t_{k,j},b_{k,j}})-1} A(\varphi)'\eta_{\chi_k,t_{k,j},b_{k,j},m_{k,j}}\otimes S_a(\psi_{\chi_k,t_{k,j},b_{k,j},m_{k,j}})$$

in $W_I^{-s}(C_F)\,\widehat{\otimes}_\pi W_I^s(C_F)$*, converge as* $(N_1,N_2)\to(+\infty,+\infty)$ *in* $\mathbf{N}\times\mathbf{N}$*, equivalently* $i\to+\infty$ *in* I*, with respect to the respective projective tensor product topologies.*

(3) *Let:*

$$\xi_\pi : W_I^{-s}(C_F)\,\widehat{\otimes}_\pi W_I^s(C_F)\to\mathscr{L}_\beta(W_I^s(C_F)) \tag{10.4.3.44}$$

be the composition of $T_{I,s}^{-1}\,\widehat{\otimes}_\pi\mathrm{Id}$ *and the continuous injective linear operator* χ_π *from Proposition C.3.6.4, whose effect on finite rank kernels is that of associating to* $\sum_{1\leq k\leq N}\phi_k\otimes\psi_k$ *the finite rank operator* $\sum_{1\leq k\leq N}\langle\phi_k,\cdot\rangle_{I,s}\,\psi_k$*. Upon denoting by* $K^{\mathrm{spec}}_{\varphi,a,I,s}$ *the limit in* $W_I^{-s}(C_F)\,\widehat{\otimes}_\pi W_I^s(C_F)$ *of the finite rank kernels from* (10.4.3.42)*, then we have, within* $\mathscr{L}_\beta(W_I^s(C_F))$*:*

$$\xi_\pi(K^{\mathrm{spec}}_{\varphi,a,I,s})=A(\varphi)\circ P_a^{\mathrm{spec}}. \tag{10.4.3.45}$$

As a Corollary, we obtain:

THEOREM 10.4.3.6. *Let* F *be a number field,* a,s *real numbers,* $a>0$*,* $s>1$*. Let* $\Lambda,\varepsilon>0$ *be real numbers, such that* $0<1/\Lambda+\varepsilon<\Lambda-\varepsilon$*. Let* $\rho:C_F\to\mathbf{C}$ *be a* (Λ,ε)*-cutoff.*

Let $w_{I,s}$ *is the weight function defined in* (9.2.2.41) *as:*

$$w_{I,s}(g):=\max\{|g|_F^{2a},|g|_F^{2b}\}\,(1+|\log|g|_F|^2)^{s/2},\ g\in C_F.$$

Then:

(1) *For any* $\phi\in W_{a,I}^s\,\mathrm{H}_{0,\mathrm{cont}}(F^\times,\mathcal{S}_0(\mathbf{A}_F))$*, we have, for* $\overline{\mu}$*-almost all* $x\in C_F$*:*

$$A(\varphi)(\phi)(x)=(A(\varphi)\circ P_a^{\mathrm{geom}})(\phi)(x)=\int_{C_F}k^{\mathrm{geom}}_{\varphi,a,I,s}(x,y)\phi(y)\,\mathrm{d}\overline{\mu}(y). \tag{10.4.3.46}$$

(2) *The functions:*

$$w_{I,s}(y)^{-1}\,k^{\mathrm{spec}}_{\varphi,a,I,s}(x,y),\quad \rho(x)\,w_{I,s}(y)^{-1}\,k^{\mathrm{spec}}_{\varphi,a,I,s}(x,y) \tag{10.4.3.47}$$

are representative functions for the Hilbert-Schmidt kernel in $L^2(C_F\times C_F,(w_{I,s}\,\overline{\mu})^{\otimes 2})$ *of the operator* $A(\varphi)\circ P_a^{\mathrm{spec}}$ *on* $W_I^s(C_F)$*, respectively for the Hilbert-Schmidt kernel in* $L^2(C_F\times C_F,(w_{I,s}\,\overline{\mu})^{\otimes 2})$ *of the operator* $M_\rho\circ A(\varphi)\circ P_a^{\mathrm{geom}}$ *on* $W_I^s(C_F)$*.*

(3) *The function:*

$$w_{I,s}(y)^{-1}\,\rho(x)\,k^{\mathrm{geom}}_{\varphi,a,I,s}(x,y) \tag{10.4.3.48}$$

is a representative function for the Hilbert-Schmidt kernel of the operator $M_\rho\circ A(\varphi)\circ P_a^{\mathrm{geom}}$ *on* $W_I^s(C_F)$*.*

The following is the resulting general case of Definition 10.4.3.2, allowing arbitrary compact intervals $I=[-\alpha,\alpha]$, for $\alpha\geq 0$ a real number.

DEFINITION 10.4.3.7 (Global Spectral and Geometric Kernels - the general case). Let F be a number field, a,s real numbers, $s\geq 3$. Let $I=[-\alpha,\alpha]\subset\mathbf{R}$ be a compact interval, $\alpha\geq 0$ a real number.

We define:

(1) The measurable function on $C_F\times C_F$:

$$k^{\mathrm{spec}}_{\varphi,a,I,s}(x,y):=k(x,y)\,w_{I,s}(y)^{-1} \tag{10.4.3.49}$$

where $w_{I,s}$ is the weight function defined in (9.2.2.41) as:

$$w_{I,s}(g):=\max\{|g|_F^{2a},|g|_F^{2b}\}\,(1+|\log|g|_F|^2)^{s/2},\ g\in C_F.$$

We refer to $k^{\mathrm{spec}}_{\varphi,a,I,s}$ as the *Global Spectral Kernel* of the φ-regularized action of C_F on $W_I^s(C_F)$.

(2) The measurable function on $C_F\times C_F$:

$$k^{\mathrm{geom}}_{\varphi,a,I,s}(x,y):=\varphi(xy^{-1})-k^{\mathrm{spec}}_{\varphi,a,I,s}(x,y) \tag{10.4.3.50}$$

We refer to $k^{\mathrm{geom}}_{\varphi,a,I,s}$ as the *Global Geometric Kernel* of the φ-regularized action of C_F on $W^s_I(C_F)$.

CHAPTER 11

The Shell Structure

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11.1. The Shell Structure

The purpose of this subsection is to show that over the unbounded, bounded "below", closed subsets

$$C_F^{\geq 1/\Lambda} := \{g \in C_F : |g|_F \geq 1/\Lambda\}$$

for $\Lambda > 0$ a fixed real number, we may let the non-Archimedean "components" of a Schwartz-Bruhat function on the adele $\mathbf{A}_F$ degenerate from $1_{\mathcal{O}_{F_v}}$ to $1_{\mathcal{O}_{F_v}^\times}$, in a manner that is convergent in $L^p(C_F^{\geq 1/\Lambda}, \overline{\mu})$ for every $p \geq 1$.[1]

11.1.1. General convergence criteria. We will be using this to construct operators from the truncations [2]

$$M_{(1/\Lambda,+\infty)} W_a^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$$

to truncated subspaces of the S-local spaces:

$$W_a^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}(\mathbf{A}_{F,S})).$$

The Geometric Schauder Bases will enable us to control the loss of information of this "degeneration process".

We begin with a few general convergence criteria.

Lemma 11.1.1.1. *We let $\{f_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{A}_F)$ be a sequence of Schwartz-Bruhat functions on $\mathbf{A}_F$. Assume that for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$, $\lim_{k\geq 0} f_k(g)$ exists. Then, as a function of $g \in \mathrm{GL}_1(\mathbf{A}_F)$*

$$f(g) := \lim_{k\to+\infty} f_k(g) \tag{11.1.1.1}$$

is Borel-measurable over $\mathrm{GL}_1(\mathbf{A}_F)$.

[1] Return here.
[2] Return here.

PROOF. Each f_k, being a Schwartz-Bruhat function on $\mathbf{A}_F$, is continuous when pre-composed with the continuous map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{A}_F$, which is in turn the composition of the continuous identity map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{A}_F^\times$, with $\mathbf{A}_F^\times$ endowed with the subspace topology, followed by the topological embedding $\mathbf{A}_F^\times \subset \mathbf{A}_F$. As such, it is Borel measurable over $\mathrm{GL}_1(\mathbf{A}_F)$. By [Ru-2, p.15,Corollaries,(a)] countable limit of complex-valued Borel measurable functions is Borel measurable. □

PROPOSITION 11.1.1.2. *We let $\{f_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{A}_F)$ be a sequence of Schwartz-Bruhat functions on $\mathbf{A}_F$. Let η be a positive Radon measure on $\mathrm{GL}_1(\mathbf{A}_F)$, and $Z \subset \mathrm{GL}_1(\mathbf{A}_F)$ a Borel-measurable subset. Let $r \geq 1$ be a real number. Assume that for every $g \in Z$, $f(g) := \lim_{k\to+\infty} f_k(g)$ exists, and that there exists a Borel-measurable function $\alpha : F^\times Z \to \mathbf{R}_{\geq 0}$ such that:*

(1) There exists a fractional ideal $I_Z \subset F$ such that for every $g \in Z$, $\sum_{\xi\in I_Z\cap F^\times} \alpha(\xi g)$ is absolutely convergent, and its value $J_F(\alpha)(g)$ is, as a function of $g \in Z$, Borel-measurable and an element of $L^r(Z, \eta|_Z)$.

(2) We have, for every $g \in F^\times Z$ and every integer $k \geq 0$

$$\sum_{\xi\in F^\times} |f_k(\xi g)| \leq J_F(\alpha)(g). \tag{11.1.1.2}$$

Then

(1) for all $g \in Z$, the series

$$\sum_{\xi\in F^\times} f(\xi g) \tag{11.1.1.3}$$

is absolutely convergent. The function $\Phi_F(f)$ defined as the sum of the series in (11.1.1.3) over Z, and identically zero over $\mathrm{GL}_1(\mathbf{A}_F) - Z$, is Borel measurable over $\mathrm{GL}_1(\mathbf{A}_F)$.

(2) $\Phi_F(f)$ is an element of $L^r(Z, \eta|_Z)$, and we have

$$\Phi_F(f) = \lim_{k\to+\infty} \Phi_F(f_k) \tag{11.1.1.4}$$

with respect to the norm topology on $L^r(Z, \eta|_Z)$.

REMARK 11.1.1.3. We note that the orbit of the Borel set $Z \subset \mathrm{GL}_1(\mathbf{A}_F)$ under the free and proper left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$ from Proposition 11.1.1.2, is also a Borel set, since $F^\times Z$ is a countable union of Borel sets.

PROOF. (1) : We let $g \in Z$ be fixed, and we denote by $\delta_{\{g\}}$ the finite, positive Radon measure on $\mathrm{GL}_1(\mathbf{A}_F)$ determined under [Fol, Thm.7.2] by the assignment of the functional

$$\delta_{\{g\}} : C_c(\mathrm{GL}_1(\mathbf{A}_F)) \to \mathbf{C}$$

defined by $\delta_{\{g\}}(f) := f(g)$.

Since the continuous action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$ by left-translation is free and discrete, the orbit $F^\times g = \{\xi g\}_{\xi\in F^\times}$ is a discrete subset of $\mathrm{GL}_1(\mathbf{A}_F)$. We endow it with the following Radon measure. We consider the $\mathbf{C}$-linear space $C_c(F^\times g)$ of continuous compactly supported functions, and define:

$$\Delta_{\{g\}} : C_c(F^\times g) \to \mathbf{C}$$

by $\Delta_{\{g\}}(\varphi) := \sum_{\xi\in F^\times} \varphi(\xi g)$. By [Fol, Thm.7.2], $\Delta_{\{g\}}$ determines a unique complex Radon measure on $F^\times g$, which we denote μ_g. The Radon measure μ_g is real, positive, and it agrees with the counting measure on $F^\times g$ by uniqueness (see [Bou-II, INT VII.31,§2,Prop.4(a)], applied to $X = F^\times g$ endowed with the discrete topology, $H = F^\times$ as a discrete topological group acting freely and properly on X by left-translation), since it is $F^\times$-invariant (in the sense of [Bou-II, INT VII.2,§1,Def.1(a)]).

Each function $f_k \in \mathcal{S}(\mathbf{A}_F)$ is continuous over $F^\times g$ with respect to the discrete topology, and it is an element of $L^r(F^\times g, \mu_g)$ for every real number $r \geq 1$ by Lemma 2.2.2.1(2).

By Fatou's Lemma (see [**Fol**, Thm.2.18]), we have

$$\sum_{\xi \in F^\times} |f(\xi g)| = \int_{F^\times g} |f(h)| \, \mathrm{d}\mu_g(h) \leq \liminf_{k \to +\infty} \int_{F^\times g} |f_k(h)| \, \mathrm{d}\mu_g(h). \tag{11.1.1.5}$$

By assumption (2), the rightmost side of (11.1.1.5) is bounded above by $J_F(\alpha)(g)$, which proves that the series in (11.1.1.3) is absolutely convergent pointwise over Z.

We now choose an exhaustion of $\mathrm{GL}_1(\mathbf{A}_F)$ by compact subsets $\{K_m\}_{m \geq 0}$, i.e. a sequence of compact subsets $K_m \subset \mathrm{GL}_1(\mathbf{A}_F)$ such that $K_m \subset \mathrm{Int}(K_{m+1})$ and $\mathrm{GL}_1(\mathbf{A}_F) = \bigcup_{m \geq 0} K_m$. Such exhaustion exists since $\mathrm{GL}_1(\mathbf{A}_F)$ is locally compact, Hausdorff and second-countable. For each integer $m \geq 0$, we choose a continuous compactly supported function $\varphi_m \in C_c(\mathrm{GL}_1(\mathbf{A}_F))$ whose support is contained in K_{m+1}, and such that $\varphi_m(K_m) = 1$. Such sequence $\{\varphi_m\}_{m \geq 0} \subset C_c(\mathrm{GL}_1(\mathbf{A}_F))$ exists by [**Fol**, Thm.4.15].

It satisfies $\lim_{m \to +\infty} \varphi_m(g) = 1$ pointwise over $\mathrm{GL}_1(\mathbf{A}_F)$ as Borel-measurable functions. The series $\sum_{\xi \in F^\times} (\varphi_m f)(\xi g)$ is absolutely convergent for every integer $m \geq 0$ and for every $g \in Z$, since it is bounded above by $\sum_{\xi \in F^\times} |f(\xi g)|$, which converges by the foregoing.

We now observe that the sum

$$\Phi_m(g) := \sum_{\xi \in F^\times} (\varphi_m f)(\xi g) \tag{11.1.1.6}$$

is, for each $g \in Z$, finite, since $\mathrm{supp}(\varphi_m f) \subset K_{m+1}$, and $F^\times g \cap K_{m+1}$ is finite, since $F^\times g$ is discrete and K_{m+1} is compact in $\mathrm{GL}_1(\mathbf{A}_F)$.

We let U_0 be an arbitrary open subset of $\mathrm{GL}_1(\mathbf{A}_F)$ with compact closure. The function Φ_m from (11.1.1.6) is, over $U_0 \cap Z$, of the form

$$\Phi_m(\cdot) = \sum_{i=1}^{N} (\varphi_m f)(\xi_i \cdot) \tag{11.1.1.7}$$

where N depends only on m and U_0, and $\{\xi_1, \ldots, \xi_N\}$ is the finite set of elements of $F^\times$ comprising the intersection $F^\times \cap (U_0^{-1} K_{m+1})$, the finiteness being ensured by the compactness of the closure of $U_0^{-1} K_{m+1}$, which holds since $(\overline{U}_0)^{-1} K_{m+1}$ is compact by [**DE**, Lemma 1.1.3(d)]. It follows that Φ_m is, over $U_0 \cap Z$, a finite sum of Borel-measurable functions (since the left-translation action by $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$ is continuous). Here we are using the foregoing applied to the Schwartz-Bruhat functions $f_k(\xi_i \cdot)$ in place of f_k, and to $f(\xi_i \cdot)$ in place of f, to deduce that each $f(\xi_i \cdot)$, for $1 \leq i \leq N$, is also Borel-measurable over Z.

Being U_0 arbitrary, as an open subset of $\mathrm{GL}_1(\mathbf{A}_F)$ with compact closure, and being $\mathrm{GL}_1(\mathbf{A}_F)$ locally compact Hausdorff, it follows that Φ_m is Borel-measurable over Z, because there exists a covering $\{U_{0,i}\}_{i \in I}$ by open subsets of $\mathrm{GL}_1(\mathbf{A}_F)$ with compact closure, such that for each $i \in I$, Φ_m is Borel-measurable over $U_{0,i} \cap Z$.

By Dominated Convergence, applied to the sequence $\{\varphi_m f\}_{m \geq 0} \subset L^1(F^\times g, \mu_g)$, dominated by $\xi g \mapsto |f(\xi g)|$, we deduce that $\Phi_m \to \Phi_F(f)$ pointwise over Z, and hence $\Phi_F(f)$ is Borel-measurable over Z (endowed with the subspace topology inherited from $\mathrm{GL}_1(\mathbf{A}_F)$, and hence the induced σ-algebra of Borel sets) by Lemma 11.1.1.1 (see also [**Ru-2**, p.15,Corollaries,(a)]). Since Z is a Borel set of $\mathrm{GL}_1(\mathbf{A}_F)$, the extension by zero of $\Phi_F(f)$ to a complex-valued function on $\mathrm{GL}_1(\mathbf{A}_F)$ is also Borel-measurable.

(2) : Since $|\Phi_F(f_k)| \leq J_F(\alpha)$ over Z by (11.1.1.11), we have $\Phi_F(f) \in L^r(Z, \eta|_Z)$ by assumption (1).

By (11.1.1.11) and Dominated Convergence, applied, for every $g \in Z$, to $\{f_k\}_{k \geq 0} \subset L^1(F^\times g, \mu_g)$, the limit of the sequence $\{\Phi_F(f_k)\}_{k \geq 0}$ in the $L^r(F \times g, \mu_g)$-norm topology, is $\Phi_F(f)$.

This means that $\Phi_F(f_k) \to \Phi_F(f)$ pointwise over Z.

By (11.1.1.11) and the assumption that $J_F(\alpha) \in L^r(Z, \eta|_Z)$, we have, by another application of Dominated Convergence, applied to the sequence of functions $\{\Phi_F(f_k)\}_{k\geq 0} \subset L^r(Z, \eta|_Z)$ whose pointwise limit is $\Phi_F(f)$, and which are all dominated by $J_F(\alpha)$ over Z, that $\Phi_F(f_k) \to \Phi_F(f)$ in $L^r(Z, \eta|_Z)$. □

We now call $T := \tau(\mathbf{R}_{>0})$, recalling that we have chosen a continuous homomorphic section τ to the idelic norm. T is a closed subgroup of $\mathrm{GL}_1(\mathbf{A}_F)$ (see Lemma A.2.2.9(1)).

Corollary 11.1.1.4. *We let $\{f_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{A}_F)$ be a sequence of Schwartz-Bruhat functions on $\mathbf{A}_F$. Assume:*

(1) for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$, $f(g) := \lim_{k\to+\infty} f_k(g)$ exists.
(2) for each $g \in \mathrm{GL}_1(\mathbf{A}_F)$, there exists a function $\alpha_g : F^\times g \to \mathbf{R}_{\geq 0}$ and a fractional ideal $I_g \subset F$, such that
  (a) The series $\sum_{\xi\in I_g\cap F^\times} \alpha_g(\xi)$ is absolutely convergent.
  (b) We have, for every $g \in T \cdot E_F^1$, every $\xi \in F^\times$ and every integer $k \geq 0$

$$\sum_{\xi\in F^\times} |f_k(\xi g)| \leq \sum_{\xi\in I_g\cap F^\times} \alpha_g(\xi). \tag{11.1.1.8}$$

Then, for each $g \in T \cdot E_F^1$, the series

$$\sum_{\xi\in F^\times} f(\xi g) \tag{11.1.1.9}$$

is absolutely convergent, the limit $\lim_{k\to+\infty} \Phi_F(f_k)(g)$ exists, and we have, as functions over C_F

$$\Phi_F(f) = \lim_{k\to+\infty} \Phi_F(f_k) \tag{11.1.1.10}$$

pointwise over C_F.

We note that the upper bound to the series in (11.1.1.8) from Corollary 11.1.1.4(2) depends on $g \in C_F$.

Proof. We apply Proposition 11.1.1.2 to the Borel set $Z := \{g\}$, and $\eta := \delta_{\{g\}}$, the Dirac measure on $\mathrm{GL}_1(\mathbf{A}_F)$ concentrated at $\{g\}$. By (2.a) and (2.b), conditions (1) and (2) of Proposition 11.1.1.2 are satisfied with $r = 1$, and hence the series in (11.1.1.12) is absolutely convergent, and (11.1.1.13) holds with respect to the norm topology on $L^1(\{g\}, \delta_{\{g\}})$, which is to say that $\Phi_F(f_k) \to \Phi_F(f)$ pointwise over $\mathrm{GL}_1(\mathbf{A}_F)$. Being each function $\Phi_F(f_k)$ and $\Phi_F(f)$, invariant under the left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$, the identity in (11.1.1.13) holds as functions over C_F. □

We recall that we have chosen a continuous homomorphic section $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ to the idelic norm, and that such choice defines a closed subgroup $T := \tau(\mathbf{R}_{>0}) \subset \mathrm{GL}_1(\mathbf{A}_F)$, and provides the isomorphism of abelian topological groups

$$\theta_{F,\tau} : \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_F)^1 \xrightarrow{\simeq} T \cdot \mathrm{GL}_1(\mathbf{A}_F)^1 = \mathrm{GL}_1(\mathbf{A}_F)$$

from (A.2.2.12).

We observe that the subset $T \cdot E_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)$ is a Borel set, with E_F^1 the idelic fundamental domain of (A.2.3.14). Indeed, T can be written as an increasing countable union of compact subsets, and E_F^1 is relatively compact, presenting $T \cdot E_F^1$ as an increasing countable union of relatively compact subsets with non-empty interior within $\mathrm{GL}_1(\mathbf{A}_F)$ (see Proposition A.2.3.10(1)), each of which is therefore a Borel set.

Corollary 11.1.1.5. *We let $\{f_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{A}_F)$ be a sequence of Schwartz-Bruhat functions on $\mathbf{A}_F$. Assume:*

(1) for every $g \in \mathrm{GL}_1(\mathbf{A}_F)$, $f(g) := \lim_{k\to+\infty} f_k(g)$ exists.

(2) *There exists a Borel-measurable function* $\alpha : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{\geq 0}$ *and a fractional ideal* $I \subset F$ *such that:*

(a) *For every* $g \in \mathrm{GL}_1(\mathbf{A}_F)$, *the series* $\sum_{\xi \in I \cap F^\times} \alpha(\xi g)$ *is absolutely convergent, and its sum* $J_F(\alpha)(g)$ *is, as a function of* $g \in \mathrm{GL}_1(\mathbf{A}_F)$, *Borel-measurable and an element of* $L^1(T \cdot E_F^1, (\delta_x \otimes \mu_1))$ *for every* $x \in T$.

(b) *We have, for every integer* $k \geq 0$

$$\sum_{\xi \in F^\times} |f_k(\xi g)| \leq J_F(\alpha)(g) \tag{11.1.1.11}$$

over $T \cdot E_F^1$.

Then, for every character $\omega \in \widehat{C_F^1}$, *and each* $g \in \mathrm{GL}_1(\mathbf{A}_F)$, *the integral*

$$\Phi_{F,\omega}(f)(g) := \int_{E_F^1} \sum_{\xi \in F^\times} f(\xi g h)\omega(gh)\,\mathrm{d}\mu_1(h) \tag{11.1.1.12}$$

is absolutely convergent, the limit $\lim_{k \to +\infty} \Phi_{F,\omega}(f_k)(g)$ *exists, and we have, as functions over* C_F/C_F^1

$$\Phi_{F,\omega}(f) = \lim_{k \to +\infty} \Phi_{F,\omega}(f_k) \tag{11.1.1.13}$$

pointwise over C_F/C_F^1.

Proof. We apply Proposition 11.1.1.2 to the choice $Z = T \cdot E_F^1$ and, for each $x \in T$, to $\eta := \delta_x \otimes \mu_1$. Since $|c(gh)| = |c(g)|$ for every $h \in E_F^1$ as per (11.1.1.12), since $E_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1$, then the integral defining $\Phi_{F,\omega}(f)(g)$ in (11.1.1.12) is absolutely convergent by Proposition 11.1.1.2(2), since we have

$$\int_{E_F^1} \sum_{\xi \in F^\times} f(\xi g h)c(gh)\,\mathrm{d}\mu_1(h) = c(x) \int_{E_F^1} \sum_{\xi \in F^\times} f(\xi x g_0)c(g_0)\,\mathrm{d}\mu_1(g_0) = c(x) \int_{T \cdot E_F^1} \Phi_F(f)(g')c(g'x^{-1})\,\mathrm{d}(\delta_x \otimes \mu_1)(g') \tag{11.1.1.14}$$

writing $g = xg_0$ for unique $x \in T$ and $g_0 \in \mathrm{GL}_1(\mathbf{A}_F)$, for then renaming $g_0 h$ as g_0 by $\mathrm{GL}_1(\mathbf{A}_F)^1$-invariance of μ_1. We may assume g_0 is in E_F^1, since $\mathrm{GL}_1(\mathbf{A}_F)^1 = \bigcup_{\xi \in F^\times} \xi E_F^1$ by Proposition A.2.3.10, the union being disjoint.

Since $|c(g'x^{-1})| = 1$, Proposition 11.1.1.2(2) implies that $\Phi_F(f) \in L^1(T \cdot E_F^1, (\delta_x \otimes \mu_1))$ and hence the integral in the rightmost side of (11.1.1.14) is absolutely convergent.

Since we have, by Proposition D.1.2.4 [**Bou-II**, INT VII.52,§3,Thm.4(c)]

$$\Phi_{F,\omega}(f_k)(g) = \int_{E_F^1} \Phi_F(f_k)(gh)\omega(gh)\,\mathrm{d}\mu_1(h)$$

as E_F^1 is a fundamental domain for the continuous action of $F^\times$ by left translation on $\mathrm{GL}_1(\mathbf{A}_F)^1$, we have

$$\Phi_{F,\omega}(f)(x) - \Phi_{F,\omega}(f_k)(x) = \omega(x) \int_{E_F^1} (\Phi_F(f_k)(g') - \Phi_F(f)(g'))\omega(g'x^{-1})\,\mathrm{d}(\delta_x \otimes \mu_1)(g')$$

and hence

$$\|\Phi_{F,\omega}(f) - \Phi_{F,\omega}(f_k)\|_{L^1(T_{v_0}, \delta_x)} \leq |c(x)|\,\|\Phi_F(f) - \Phi_F(f_k)\|_{L^1(T \cdot E_F^1, (\delta_x \otimes \mu_1))} \tag{11.1.1.15}$$

then Proposition 11.1.1.2(2) implies that $\Phi_{F,\omega}(f_k) \to \Phi_{F,\omega}(f)$ pointwise over T.

Since both $\Phi_F(f)$ and $\Phi_F(f_k)$, for every integer $k \geq 0$, are $F^\times$-invariant functions over $\mathrm{GL}_1(\mathbf{A}_F)$, and the $L^1(E, \mu_1)$-norm of $\Phi_F(f)(x\cdot) - \Phi_F(f_k)(x\cdot)$ is preserved under left-translation by $\mathrm{GL}_1(\mathbf{A}_F)^1$, (11.1.1.15) implies that $\Phi_{F,\omega}(f_k) \to \Phi_{F,\omega}(f)$ pointwise over $\mathrm{GL}_1(\mathbf{A}_F) = T \cdot \mathrm{GL}_1(\mathbf{A}_F)^1$, and since both $\Phi_{F,\omega}(f_k)$ and $\Phi_{F,\omega}(f)$ are $F^\times$-invariant, we deduce that the mode of convergence in (11.1.1.13) is pointwise as functions over C_F. By C_F^1-invariance of each $\Phi_{F,\omega}(f_k)$ and $\Phi_{F,\omega}(f)$, the mode of convergence is also pointwise over C_F/C_F^1. □

11.1.2. S-local carvings. We are ready to establish the main result of this section.

For F a number field, $\chi \in \widehat{C_F}$ a continuous character, S a finite set of places of F containing all the Archimedean places, and such that χ is unramified outside S, and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$ (which implies $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$ for all finite sets of places of F containing S), we introduce the following notation.

We define the countable set:

$$I_S := \{S' \subset \Sigma_F : S' \text{ is finite, } S \cap S' = \varnothing\} \tag{11.1.2.1}$$

partially ordered by inclusion. We denote by i the arbitrary element of I_S, and write S_i for the finite set of places of F corresponding to $i \in I_S$.

THEOREM 11.1.2.1 (Existence of S-local carvings on the 0-line). [3] *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character, S a finite set of places of F containing all the Archimedean places, and such that χ is unramified outside S. Assume S is large enough so as to have* $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let s, Λ be real numbers, $\Lambda > 0$. Assume:

$$s \geq 2\,[F : \mathbf{Q}]. \tag{11.1.2.2}$$

We define, for every $i \in I_S$ and for $\rho \in \{\rho_{\Lambda,+}, \rho_\Lambda\}$:

$$W^{s,2}_{0,\rho}\, P_{S_i} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.3}$$

to be the closure in $W^{s,2}(C_F)(\chi|_{C^1_F})$ of $M_\rho\, P_{S_i} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \subset W^{s,2}(C_F)(\chi|_{C^1_F})$.

Then:

(1) The set-theoretic limit:

$$\lim_{k\geq 0} W^{s,2}_{0,\rho}\, P_{S_k} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \subset W^{s,2}(C_F)(\chi|_{C^1_F}) \tag{11.1.2.4}$$

for $\{S_k\}_{k\geq 0} \subset I_S$ a choice of cofinal sequence, exists, it is a closed linear subspace of $M_\rho W^{s,2}(C_F)(\chi|_{C^1_F})$, and it is independent of the choice of cofinal sequence in I_S.

(2) Let $i, i' \in I_S$ be distinct, with $S_i \subset S_{i'}$ and such that there exists a prime number p such that no place $v \in S_i$ divides p, and at least one place in $S_{i'}$ does. Then we have the inclusions of closed linear subspaces:

$$W^{s,2}_0 P_{S_{i'}} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \subset W^{s,2}_0 P_{S_i} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.5}$$

and:

$$W^{s,2}_{0,\rho} P_{S_{i'}} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)) \subset W^{s,2}_{0,\rho} P_{S_i} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F}, \mathcal{S}_0(\mathbf{A}_F)). \tag{11.1.2.6}$$

The inclusion in (11.1.2.5) *is strict, and for any non-decreasing sequence $\{\Lambda_k\}_{k\geq 0} \subset \mathbf{R}_{>0}$ such that $\lim_{k\geq 0} \Lambda_k = +\infty$, there exists a subsequence $\{\Lambda_{k_m}\}_{m\geq 0}$ such that, for every integer $m \geq 0$ and for the choice $\Lambda = \Lambda_{k_m}$, the inclusion in* (11.1.2.6) *is strict.*

(3) The closed subspace from (11.1.2.4) *is the closure in $M_\rho W^{s,2}(C_F)(\chi|_{C^1_F})$ of the linear subspace generated by the functions of $g \in C_F$:*

$$\rho(g) \sum_{\xi \in F^\times} (f \otimes \prod_{v \notin S} 1_{\mathcal{O}^\times_{F_v}})(\xi g) \tag{11.1.2.7}$$

for $f \in \mathcal{S}(\mathbf{A}_{F,S}) \cap \ker(\delta_0) \cap \ker(\widehat{\delta_0})$.

REMARK 11.1.2.2. We recall that a cofinal sequence $\{S_k\}_{k\geq 0} \subset I_S$ consists of the datum of a sequence $\{S_k\}_{k\geq 0} \subset I_S$, i.e. with each S_k a finite subset of finite places of F, such that $S_k \subset S_{k+1}$ for each integer $k \geq 0$, and $\bigcup_{k\geq 0} S_k = \Sigma_F - S$.

[3] Return here.

PROOF. (1) : By Proposition 11.2.0.1, each PV_{S_i} is continuous and with closed range as a linear endomorphism of $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$.

In particular, the closure within $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ of $\mathrm{PV}_{S'}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$, for any $S'\in I_S$, is exactly the closed subspace $\mathrm{PV}_{S'}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$.

The topology on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ induced by

(11.1.2.8)

$$M_\rho:\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset W^{s,2}(C_F)(\chi|_{C_F^1})\to M_\rho W^{s,2}(C_F)(\chi|_{C_F^1})$$

is induced by the seminorm:

$$\|\phi\|_{s,2,\rho}:=\left(\int_{C_F^{\geq 1/\Lambda}}|\phi(g)|^2(1+|\log|g|_F|^2)^{s/2}\,\mathrm{d}\overline{\mu}(g)\right)^{1/2}. \tag{11.1.2.9}$$

To see this, it is enough to observe that since the topology on $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ is induced by the $\|\cdot\|_{s,2}$-norm on $W^{s,2}(C_F)(\chi|_{C_F^1})$, then the pull-back of the norm $\|\cdot\|_{s,2}$ along M_ρ from (11.1.2.8) is a seminorm $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$, which generates a locally convex topology whose separated completion is closed and dense within $M_\rho W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$, and hence agrees with it.

Such pull-back seminorm is computed as:

$$\phi\mapsto\|\rho\cdot\phi\|_{s,2}=\left(\int_{C_F}|\rho(g)\phi(g)|^2(1+|\log|g|_F|^2)^{s/2}\,\mathrm{d}\overline{\mu}(g)\right)^{1/2}$$

which gives the formula for $\|\cdot\|_{s,2,\rho}$ in (11.1.2.9), since the indicator function ρ of $C_F^{>1/\Lambda}$, respectively of $C_F^{(1/\Lambda,\Lambda)}$, is its own square.

Since $\{g\in C_F:|g|_F=1/\Lambda,\Lambda\}$ has $\overline{\mu}$-measure zero, indeed the formula in (11.1.2.9) for $\|\cdot\|_{s,2,\rho}$ computes the pull-back of $\|\cdot\|_{s,2}$ on $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ along M_ρ from (11.1.2.8).

Since $\|\cdot\|_{s,2,\rho}$ is a continuous seminorm on $W^{s,2}(C_F)(\chi|_{C_F^1})$, the seminormed topology induced by $\|\cdot\|_{s,2,\rho}$ on $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ is coarser than the $\|\cdot\|_{s,2}$-norm topology.

It follows that the following inclusion:

$$M_\rho W_0^{s,2}\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset W_{0,\rho}^{s,2}\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.10}$$

holds.

The left side in (11.1.2.10) (which, note, equals $M_\rho\mathrm{PV}_{S_i}W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ by continuity of PV_{S_i}) contains $M_\rho P_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$ and it is already closed, since M_ρ has closed range by [**Cnw-a**, Ch.II,§3,Prop.3.2(b),p.37], being an idempotent operator.

It follows that we have, as closed subspaces of $W^{s,2}(C_F)(\chi|_{C_F^1})$:

$$M_\rho W_0^{s,2}\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))=W_{0,\rho}^{s,2}\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)). \tag{11.1.2.11}$$

We now observe that if $S_i\subset S_{i'}$ for $i,i'\in I_S$, then we have:

$$W_{0,\rho}^{s,2}\mathrm{PV}_{S_{i'}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset W_{0,\rho}^{s,2}P_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.12}$$

since $\mathrm{PV}_{S_{i'}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))\subset\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$, this latter inclusion being a consequence of Theorem 6.3.2.2.

We let $\{S_k\}_{k\geq 0}\subset I_S$ be any cofinal sequence. Then

$$\bigcap_{k\geq 0}W_{0,\rho}^{s,2}\mathrm{PV}_{S_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.13}$$

is a closed linear subspace of $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))$.

For $\{S'_m\}_{m\geq 0}$ another cofinal sequence in I_S, we have, for every integer $k \geq 0$, that there exists a large enough integer $m_k \geq 0$ such that S'_m contains S_k for every $m \geq m_k$.

In particular, by (11.1.2.12), the following inclusion holds:

$$\bigcap_{m\geq 0} W_{0,\rho}^{s,2}\mathrm{PV}_{S'_m}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \subset \bigcap_{k\geq 0} W_{0,\rho}^{s,2}\mathrm{PV}_{S_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)). \tag{11.1.2.14}$$

Upon exchanging the roles of $\{S_k\}_{k\geq 0}$ and $\{S'_m\}_{m\geq 0}$, we deduce that the reverse inclusion holds, and hence:

$$\bigcap_{m\geq 0} W_{0,\rho}^{s,2}\mathrm{PV}_{S'_m}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) = \bigcap_{k\geq 0} W_{0,\rho}^{s,2}\mathrm{PV}_{S_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)). \tag{11.1.2.15}$$

as closed subspaces of $W^{s,2}(C_F)(\chi|_{C_F^1})$. Formation of the common such closed subspace therefore does not depend on the choice of cofinal sequence in I_S, but only on I_S itself.

(2) : [4]

We now prove that the inclusion in (11.1.2.5) is strict. By Theorem 11.2.0.2, the strong dual of the orthogonal in $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ of $W_0^{s,2}\,\mathrm{PV}_{S_{i'}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ is non-trivial, the Theorem exhibiting a Schauder basis (given by the functionals η_{ε_p} as ε_p varies over the zeros of $(1-q_v^{-s-it_v})^{-1}$ for $v \in S_{i'}$ of residue characteristic p), and since $S_i \subset S_{i'}$ is a strict inclusion, as i, i' are distinct elements of I_S, and there exists a prime number p divided by a place in $S_{i'}$ and by no place in S_i, the functionals η_{ε_p} separate the points of $W_0^{s,2}\,\mathrm{PV}_{S_{i'}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ from the points of $W_0^{s,2}\,\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

All such functionals exist and provide elements of the Schauder bases from Theorem 11.2.0.2, because $s \geq 2\,[F:\mathbf{Q}]$.

We now prove that for any non-decreasing sequence $\{\Lambda_k\}_{k\geq 0} \subset \mathbf{R}_{>0}$ such that $\lim_{k\to+\infty}\Lambda_k = +\infty$, there exists a subsequence $\{\Lambda_{k_m}\}_{m\geq 0}$ such that for each choice $\Lambda = \Lambda_{k_m}$, the inclusion in (11.1.2.6) is also strict.

If not, then we have:

$$W_{0,\Lambda_k,+}^{s,2}\mathrm{PV}_{S_{i'}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) = W_{0,\Lambda_k,+}^{s,2}\mathrm{PV}_{S_i}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.1.2.16}$$

for every integer $k \geq 0$.

[5]

Since $\lim_{k\to+\infty}\Lambda_k = +\infty$, the sequence of operators $\{M_{\rho_{\Lambda_k,+}}\}_{k\geq 0}$ on $W^{s,2}(C_F)(\chi|_{C_F^1})$ tends to the identity with respect to the strong operator topology on $\mathscr{B}(W^{s,2}(C_F)(\chi|_{C_F^1}))$ by [**Cnw-a**, Ch.I,§2,Cor.2.11,p.11], forcing the inclusion in (11.1.2.5) to not be strict, a contradiction.

(3) : We will now be using Proposition 11.1.1.2. We choose τ a continuous homomorphic section to the idelic norm on C_F, and define $Z := \tau(\mathbf{R}_{>0}) \cdot E_F^1$, with E_F^1 the idelic fundamental domain in $\mathrm{GL}_1(\mathbf{A}_F)^1$ for the discrete left translation action of $F^\times$ (see (A.2.3.14)).

The subset $Z \subset \mathrm{GL}_1(\mathbf{A}_F)$ is Borel with non-empty interior, and its $F^\times$-translates cover $\mathrm{GL}_1(\mathbf{A}_F)$ by Theorem A.2.3.10.

We choose $\eta := 1_{(1/\Lambda,+\infty)} \cdot (1+|\log|\cdot|_F|^2)^{s/2}\,\mu$ as positive Radon measure on $\mathrm{GL}_1(\mathbf{A}_F)$, and choose $r = 2$ in the statement of Proposition 11.1.1.2.

We choose $f \in \mathcal{S}_0(\mathbf{A}_F)$ of the form $f = f_\infty \prod_{v\in\Sigma_{F,f}} f_v$ for $f_\infty \in \mathcal{S}(\mathbf{A}_{F,S_\infty})$ and $f_v \in \mathcal{S}(F_v)$ for every $v \in \Sigma_{F,f}$, with $f_v = 1_{\mathcal{O}_{F_v}}$ for all but finitely many finite places v of F all contained in S. We assume that $f_S := f_\infty \otimes \prod_{v\in S\cap\Sigma_{F,f}} f_v \in \mathcal{S}(\mathbf{A}_{F,S})$ satisfies $f_S(0) = \widehat{f_S}(0) = 0$.

Note that Schwartz-Bruhat functions of this form $\mathbf{C}$-linearly span a subspace W within $\mathcal{S}_0(\mathbf{A}_F)$ which is linearly mapped onto a dense subspace in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ by our assumptions on S and by the Fifth Structure Theorem (see Theorem 6.2.1.1).

[4]Return here.
[5]Return here.

There exists a fractional ideal I of F depending only on f through the finitely many finite places v of F such that $f_v \neq 1_{\mathcal{O}_{F_v}}$, and through f_∞, such that, as $1_{\mathcal{O}_{F_v}}$ is replaces with $1_{\mathcal{O}_{F_v}^\times}$ for each $v \notin S$, the new fractional ideal I' corresponding to the resulting Schwartz-Bruhat function satisfies $I' \subset I$ strictly. We choose α in Proposition 11.1.1.2 to be the pre-composition of f_∞ with the continuous homomorphic projection $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

We choose $\{S_k\}_{k\geq 0}$ a cofinal sequence in I_S, and define $f_k := P_{S_k}(f)$. For every $g \in \tau(\mathbf{R}_{>0}) \cdot E_F^1$, the limit $\lim_{k\to+\infty} f_k(g)$ exists, and its value is exactly:

$$(f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})(g).$$

Proposition 11.1.1.2 therefore applies, and we deduce that the sequence $\{\Phi_F(f_k)\}_{k\geq 0}$ converges in $L^2(\tau(\mathbf{R}_{>0}) \cdot E_F^1, 1_{(1/\Lambda,+\infty)} \cdot (1+|\log|\cdot|_F|^2)^{s/2}\mu)$ to the function $\sum_{\xi\in F^\times} 1_{(1/\Lambda,+\infty)}(g)(f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})(\xi g)$ of $g \in C_F$.

This is equivalent to saying that the function

$$\sum_{\xi\in F^\times} 1_{(1/\Lambda,+\infty)}(\xi g)(f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})(\xi g) = 1_{(1/\Lambda,+\infty)}(g) \sum_{\xi\in F^\times} (f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})(\xi g)$$

of $g \in C_F$ is an element of $R_{\Lambda,S} W_0^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

We have shown that, upon calling $\delta_0, \widehat{\delta_0} \in \mathcal{S}(\mathbf{A}_{F,S})'_\beta$ the evaluation at $x=0$ functional and its S-adelic Fourier transform, then as $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$ varies over $\ker(\delta_0) \cap \ker(\widehat{\delta_0})$, the functions $1_{(1/\Lambda,+\infty)} \cdot (f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})$ are contained in $R_{\Lambda,S} W_0^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

Such functions linearly generate a subspace $W_{S,\Lambda} \subset R_{\Lambda,S} W_0^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, whose closure $\overline{W}_{S,\Lambda}$ in $W^{s,2}(C_F)(\chi|_{C_F^1})$ contains the subspace $W_{S,\Lambda,0}$ linearly generated by the functions of $g \in C_F$ of the form:

$$1_{(1/\Lambda,+\infty)} \cdot (f_S \otimes \prod_{v\notin S} 1_{\mathcal{O}_{F_v}^\times})$$

for $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$ satisfying only the one vanishing condition $\widehat{f_S}(0) = 0$. If the closure $\overline{W}_{S,\Lambda,0}$ of $W_{S,\Lambda,0}$ in $W^{s,2}(C_F)(\chi|_{C_F^1})$ is not $\overline{W}_{S,\Lambda}$ itself, then the kernel of the continuous linear quotient $\overline{W}_{S,\Lambda} \to \overline{W}_{S,\Lambda}/\overline{W}_{S,\Lambda,0}$ provides a continuous extension of $\widehat{\delta_0}$ to the completion of $\mathcal{S}(\mathbf{A}_{F,S})$ with respect to the L^2-norm topology induced by the Radon measure $1_{(1/\Lambda,+\infty)} \cdot (1+|\log|\cdot|_{F,S}|^2)^{s/2}\mu_S$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})$. However, $\widehat{\delta_0}$ is discontinuous with respect to such topology on $\mathcal{S}(\mathbf{A}_{F,S})$, because multiplication by $|\cdot|_{F,S}$ is a discontinuous operator with respect to this topology, since $|\cdot|_{F,S}$ grows polynomially as $g \in \{g \in \mathrm{GL}_1(\mathbf{A}_{F,S}) : |g|_{F,S} \geq 1/\Lambda\}$ satisfies $|g|_{F,S} \to +\infty$.

We deduce that we have $\overline{W}_{S,\Lambda,0} = \overline{W}_{S,\Lambda}$. Since $\overline{W}_{S,\Lambda}$ is exactly $R_{\Lambda,S} W_0^{s,2}\, \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, we deduce that (3) holds. □

6

11.2. The pieces of the puzzle: Paley-Wiener levels and Orbital Markushevich Bases

We recalled in (10.2.1.3) that we denoted the pairing $\langle\cdot,\cdot\rangle_{s,w}$ from (9.1.1.11) by:

$$\langle\cdot,\cdot\rangle_{s,2} : W^{s,2}(C_F) \times W^{-s,2}(C_F) \to \mathbf{C} \tag{11.2.0.1}$$

for which we refer the reader to (10.2.1.3), where $\langle\cdot,\cdot\rangle_{s,2}$ is assigned by:

$$\langle\phi,\psi\rangle_{s,2} := \int_{C_F} \phi(g)\psi(g)\,\mathrm{d}\overline{\mu}(g).$$

[6]Return here.

We recall that for a finite set of finite places S_0 of F, we defined the operator:

$$\mathrm{PV}_{S_0} : \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to \mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.0.2}$$

for which we refer the reader to (5.3.1.10).

In Lemma 5.3.1.5, we proved that PV_{S_0} is a linear topological embedding with closed range, and Theorem 6.3.2.2 describes its range.

Formation of PV_{S_0} is continuous with respect to the Sobolev topologies induced by the norms $\|\cdot\|_{W_a^{s,2}}$ on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, and so PV_{S_0} extends to a unique continuous linear endomorphism on $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ for every pair of real numbers a, s with $s \geq 0$, and it remains a linear topological embedding with closed range, which is given by the norm-completion of $\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$.

We will therefore write, with no potential for ambiguity:

$$W_a^{s,2}\,\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) = \mathrm{PV}_{S_0} W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)). \tag{11.2.0.3}$$

We collect these properties of the operators PV_{S_0} into the following:

PROPOSITION 11.2.0.1. *Let F be a number field, S_0 a finite set of places of F. Let $s \geq 0$ be a real number. Then $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset W_0^{s,2}(C_F)$ is dense.*

The operator PV_{S_0} on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ defined in (5.3.1.10) satisfies the following properties:

(1) *It is continuous with respect to the topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ induced by the norm $\|\cdot\|_{W_0^{s,2}}$ from (10.1.1.5).*

(2) *Assume $s \geq 2|S_0|$. Its extension by continuity to a linear map:*

$$\mathrm{PV}_{S_0} : W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \to W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.0.4}$$

is continuous with closed range, and compatible with passage to C_F^1-isotypes.

PROOF. The continuous injective linear operator $\mathcal{S}_{\mathrm{E}}(C_F) \to W^{s,2}(C_F)$ is compatible with the duality $\langle \mathcal{S}_{\mathrm{E}}(C_F), \mathcal{S}_{\mathrm{E}}(C_F)'_\beta \rangle$ on the source, and the duality $\langle \cdot, \cdot \rangle_{s,2}$ on the target.

By Theorem 10.2.1.1 applied with $a = 0$, and by [**CF**, Ch.VIII,§2,Thm.2,p.212], the orthogonal of the closure of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ in $W^{s,2}(C_F)$ with respect to $\langle \cdot, \cdot \rangle_{s,2}$ is trivial. By [**NB**, Thm.8.9.2(b)], $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is dense in $W^{s,2}(C_F)$ with respect to the weakened topology $\sigma(W^{s,2}(C_F), W^{-s,2}(C_F))$. By [**Jar**, Ch.8,§8.2,Prop.5,p.149], it is dense.

(1) : Since PV_{S_0} from (5.3.1.9) is the finite composition of the operators P_v from (5.3.1.1), all continuous endomorphisms of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ compatible with passage to C_F^1-isotypes, and with closed range by Lemma 5.3.1.5, we may and do assume $S_0 = \{v\}$, for a single finite place v of F.

In this case and by (5.3.1.1), we have:

$$P_v(\Phi_F(f)) = \Phi_F(P_v(f))$$

which is in turn the function on C_F obtained from the global adelic Schwartz-Bruhat function $f(x) - f(\pi_v^{-1}x)$, as a function of $x \in \mathbf{A}_F$, for π_v any choice of uniformizer at v.

It is therefore enough to show that the operator $f \mapsto \rho_F(\pi_v, f)$, with ρ_F the continuous left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$ on $\mathcal{S}(\mathbf{A}_F)$, descends to a continuous endomorphism of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, compatible with passage to C_F^1-isotypes, and continuous with respect to the normed topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ defined by the norm $\|\cdot\|_{W_a^{s,2}}$. Since $\pi_v^{\mathbf{Z}}$ acts on $\mathcal{S}(\mathbf{A}_F)$ through the continuous left-translation action of the closed subgroup $F_v^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$ through the insertion continuous group homomorphism with closed range into the v-indexed "coordinate" of $\mathrm{GL}_1(\mathbf{A}_F)$, the assertions follow by continuity of the left-translation action of $\mathrm{GL}_1(\mathbf{A}_F)$ with respect to the topology induced by the norm $\|\cdot\|_{W_a^{s,2}}$.

(2) : Since PV_{S_0} is the finite composition of the continuous operators P_v from (5.3.1.1), obtained as continuous extensions to $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ from (1), and since each P_v is compatible with passage

to C_F^1-isotypes, then so is PV_{S_0}, and it is enough to check that each P_v has closed range. This follows from continuity of the linear functionals:

$$\phi \mapsto \int_{C_F} \phi(g)\chi(g)|g|_F^{\varepsilon}(\log|g|_F)^m \,\mathrm{d}\overline{\mu}(g)$$

for $0 \leq m \leq |S_0| - 1$, as ε ranges over the zeros of $\prod_{v\in S_0} L_v(\chi_v|\cdot|_v^s)^{-1}$. Indeed, the range of PV_{S_0} is the intersection of their kernels. To see this, we use denseness of $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ in $W^{s,2}(C_F)$, which was shown in the foregoing, to replace $W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ with $W^{s,2}(C_F)$, and PV_{S_0} with the finite composition of the operators $P_v : \phi \mapsto \phi(x) - \phi(\pi_v^{-1}x)$ for each $v \in S_0$, for then proceeding as in the proof of the same assertion for the functionals D_t^m from (10.2.1.9) from the proof of Theorem 10.2.1.1, using the Approximation Principle from Theorem 9.3.2.1. □

We record here below a result that is analogous to Theorem 10.2.1.1, identifying a Markushevich basis for the orthogonal to

$$\mathrm{PV}_{S_0}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F)) \subset W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F) = W^{s,2}(C_F)$$

where $\mathrm{PV}_{S_0}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ is the closed subspace defined in (11.2.0.3), and orthogonality is defined with respect to the duality pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (9.1.1.11), whose definition we recalled in (10.2.1.3).

THEOREM 11.2.0.2 (Global Orbital Markushevich Bases on the 0-line). *Let F be a number field, $s \geq 0$ a real number. Let S_0 be a finite set of finite places of F, $\chi \in \widehat{C_F}$ a continuous character, unramified at all the places of S_0. The functions, for each rational prime p such that some $v \in S_0$ has residue characteristic p:*

$$\eta_{\chi,\varepsilon}(g) := \chi(g)|g|_F^{\varepsilon}(\log|g|_F)^m \tag{11.2.0.5}$$

such that:

1. *ε is a zero of the entire function $\prod_{v\in S_0} L_v(\chi_v|\cdot|_v^s)^{-1} = \prod_{v|p, v\in S_0}(1 - q_v^{-s-it_v})$, for $t_v \in \mathbf{R}$ the unique real number contained in $[0, 2\pi/\log(q_v))$ with $q_v := |\pi_v|_v^{-1}$, such that $\chi|\cdot|_F^{-it_v} \in (\pi_v^{\mathbf{Z}})^\perp \subset \widehat{C_F}$, of multiplicity $N(\varepsilon)$.*
2. *$m \geq 0$ is an integer satisfying $m \leq N_s(\varepsilon) - 1$*

with:

$$N_s(\varepsilon) = \min\left\{N(\varepsilon), \left[\frac{s-1}{2} + 1\right]\right\} \tag{11.2.0.6}$$

are elements of $W^{-s,2}(C_F)(\overline{\chi}|_{C_F^1})$. If $s \geq 2|S_0|$, then any enumeration is a Markushevich basis for the orthogonal

$$\mathrm{PV}_{S_0}W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$$

in $W^{-s,2}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (10.2.1.3), endowed with the subspace topology of the strong topology.

REMARK 11.2.0.3. We note that, in light of Lemma 4.2.1.3(2), there is no ambiguity as to how to topologize the orthogonal $\mathrm{PV}_{S_0}W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$ in $W^{-s,2}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (10.2.1.3), since the subspace topology of the strong topology is equivalent to the strong topology on the continuous dual of the quotient $W^{s,2}(C_F)(\chi|_{C_F^1})/W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$ endowed with the quotient topology.

TERMINOLOGY 11.2.0.1 (Orbital Markushevich Bases on the 0-line). We refer to the Markushevich Bases identified in Theorem 11.2.0.2 for $\mathrm{PV}_{S_0}W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$ endowed with the subspace topology of the strong topology on $W^{s,2}(C_F)(\chi|_{C_F^1})'_\beta$, as the *Orbital Markushevich Bases on the 0-line*.

PROOF. We write $\omega := \chi|_{C_F^1} \in \widehat{C_F^1}$. We choose a continuous homomorphic section τ to the idelic norm, and identify C_F/C_F^1 with $\mathbf{R}_{>0}$ as a Hausdorff locally compact abelian group by means of $|\cdot|_F$. The following diagram:

$$\begin{array}{ccccc} \mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) & \times & \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta & \longrightarrow & \mathbf{C} \\ \big\downarrow & & \big\uparrow & & \big\| \\ W_a^{s,2}\,\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F)) & \times & W^{-s,2}(C_F)(\omega) & \longrightarrow & \mathbf{C} \end{array} \tag{11.2.0.7}$$

is commutative, where the top horizontal arrow is the duality pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}),\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})'_\beta}$, noting that by Theorem 6.3.2.2, $\mathrm{PV}_{S_0}\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}_0(\mathbf{A}_F))$ is closed as a subspace of $\mathcal{S}_{\mathrm{E}}(C_F)(\omega)$.

By commutativity of (11.2.0.7), the argument is formally analogous to the argument for Theorem 10.2.1.1, upon replacing Theorem 6.3.2.1 with Theorem 6.3.2.2 to deduce that the set of continuous functionals on $W^{s,2}(C_F)(\omega)$ represented under the isometry T_s from (10.2.1.3) by the elements of $W^{-s,2}(C_F)(\overline{\omega})$ from (11.2.0.5) is topologically free, for then using the Bounded Approximation Principle of Lemma 9.3.2.1 as in the proof of Theorem 10.2.1.1 to deduce the required denseness properties for the linear subspaces linearly generated by the set of elements from (11.2.0.5) in $W^{-s,2}(C_F)(\overline{\omega})$, and its bi-orthogonal set in $W^{s,2}(C_F)(\omega)$. □

In analogy with Proposition 10.2.1.3 and Corollary 10.2.1.4, we record here the following properties of the Orbital Markushevich basis on the 0-line identified in Theorem 11.2.0.2, for the orthogonal

$$\mathrm{PV}_{S_0}W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^\perp$$

in $W^{-s,2}(C_F)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{s,2}$ from (10.2.1.3).

We further refer the reader to Terminology 10.2.1.2 for the terminology on Markushevich bases in force throughout the remainder of this subsection.

PROPOSITION 11.2.0.4. *Let F be a number field, $s > 1$ a real number. Let S_0 be a finite set of finite places of F, $\chi \in \widehat{C_F}$ a continuous character, unramified at all the places of S_0.*

There exists a real number $C > 0$ such that the functions:

$$\eta_{\chi,\varepsilon,m}(g) := \chi(g)|g|_F^{\varepsilon}(\log|g|_F)^m \tag{11.2.0.8}$$

as ε ranges over the zeros of the entire function $\prod_{v\in S_0} L_v(\chi_v|\cdot|_v^s)^{-1} = \prod_{v|p, v\in S_0}(1 - q_v^{-s-it_v})$, with $0 \leq m \leq N(\varepsilon) - 1$, where $N_s(\varepsilon)$ is the function defined in (11.2.0.6)*, all satisfy:*

$$\{\chi(g)|g|_F^{\varepsilon}(\log|g|_F)^m\}_{\varepsilon,m} \subset B_C := \{\phi \in W^{-s,2}(C_F)(\chi|_{C_F^1}) : 1/C \leq \|\phi\|_{-s,2} \leq C\}. \tag{11.2.0.9}$$

In particular, the set of functions from (11.2.0.8) *is not contained in the open ball in $W^{-s,2}(C_F)(\chi|_{C_F^1})$ of radius $1/C$.*

PROOF. If $s > 1$, then:

$$\|\eta_{\chi,\varepsilon,m}\|_{-s,2} = \left(\int_{C_F} |\log|g|_F|^{2m}(1 + |\log|g|_F|^2)^{-s/2}\,\mathrm{d}\overline{\mu}(g)\right)^{1/2}$$

by Proposition 9.1.2.5, and so $\|\eta_{\chi,\varepsilon,m}\|_{-s,2}$ can only attain finitely many values, none of which is zero. There therefore exists a real number $C > 0$ such that $1/C$ is a strict lower bound for their minimum, and C is a strict upper bound for their maximum. □

In analogy with Corollary 10.2.1.4, we have the following:

COROLLARY 11.2.0.5. *Let F be a number field, $s > 1$ a real number. Let S_0 be a finite set of finite places of F, $\chi \in \widehat{C_F}$ a continuous character, unramified at all the places of S_0.*

Assume $s \geq 2|S_0|$.

Any enumeration of the set of functions:

$$\eta_{\chi,\varepsilon,m}(g) := \chi(g)|g|_F^{\varepsilon}(\log|g|_F)^m \tag{11.2.0.10}$$

as ε ranges over the zeros of the entire function $\prod_{v\in S_0} L_v(\chi_v|\cdot|_v^s)^{-1} = \prod_{v|p,v\in S_0}(1-q_v^{-s-it_v})$, with $0 \leq m \leq N(\varepsilon)-1$, where $N_s(\varepsilon)$ is the function defined in (11.2.0.6)*, is a normalized Markushevich basis for the orthogonal*

$$\mathrm{PV}_{S_0} W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^{\perp}$$

in $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))'_\beta = W^{s,2}(C_F)(\chi|_{C_F^1})'_\beta$, topologized with the subspace topology of the strong topology.

Proof. There exists a real number $C > 0$ such that the set of functions $\{\eta_{\chi,\varepsilon,m}\}_{\varepsilon,m}$ is contained in the collar B_C from (11.2.0.9) of Proposition 11.2.0.4.

In particular, as per Proposition 11.2.0.4 combined with Theorem 11.2.0.2, any enumeration of $\{\eta_{\chi,\varepsilon,m}\}_{\varepsilon_p,m}$, a normalized Markushevich basis for $\mathrm{PV}_{S_0} W_0^{s,2}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))^{\perp}$, the orthogonal being formed with respect to the duality $\langle\cdot,\cdot\rangle_{s,2}$ from (10.2.1.3), and topologized with the subspace topology inherited from the norm topology on $W^{-s,2}(C_F)(\overline{\chi}|_{C_F^1})$, which is equivalent to the strong topology on $W^{s,2}(C_F)(\chi|_{C_F^1})'_\beta$ under the isometry T_s from (9.1.2.14). □

11.2.1. Paley-Wiener Levels. Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $\varphi \in \mathcal{D}(C_F)$ be a test function.

In this subsection we analyze a class of large closed subspaces of $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, within which the trace of $P_a^{\mathrm{geom}} \circ A(\varphi)$ is concentrated, i.e. the trace

$$\mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi))$$

for ρ a (Λ,ε)-cutoff, is mostly concentrated, as $\Lambda \to +\infty$ in the regime $\varepsilon\Lambda = o(1)$.

The main result of this subsection, Theorem 11.2.1.2, indeed shows that the restriction of the nuclear operator $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ to such growing family of closed subspaces of $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$, tends to $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ in the sense of nuclear operators, therefore reducing the calculation of the trace of $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$ to the trace of each of the restrictions of $P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)$. The calculation will, in turn, be possible upon further assuming $a = 1/2$ and $s \geq 2[F:\mathbf{Q}]$, showing which is the subject of the remainder of this Chapter and the next.

Definition 11.2.1.1 (Absolute Paley-Wiener Levels). Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let χ_S be the unique continuous character of $J_{F,S}$ obtained by descending by $\widehat{\mathcal{O}}_{F,\Sigma_F-S}^{\times}$-invariance the restriction of χ to $C_{F,S}$.

Let $\Lambda_0 > 0$ be a real number. Let s, a be real numbers, $s \geq 0$.

We define:

(1) the closed subspaces:

$$W_a^{s,2}\,\mathrm{PW}_{\Lambda_0}(\chi_S|_{J_{F,S}^1}, \mathcal{S}_0(\mathbf{A}_{F,S})) \subset W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J_{F,S}^1}, \mathcal{S}_0(\mathbf{A}_{F,S})) \tag{11.2.1.1}$$

and:

$$W_a^{s,2}\,\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J_{F,S'}^1}, \mathcal{S}(\mathbf{A}_{F,S'})) \subset W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.1.2}$$

to be the closure of $|\cdot|_{F,S}^a \mathrm{PW}_{\Lambda_0}(\chi_S|_{J_{F,S}^1}, \mathcal{S}_0(\mathbf{A}_{F,S}))$ in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J_{F,S}^1}, \mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively of $|\cdot|_F^a \mathrm{PW}_{S,X,0}(\chi_{S'}|_{J_{F,S'}^1}, \mathcal{S}(\mathbf{A}_{F,S'}))$ in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$.

We refer to $W_a^{s,2}\,\mathrm{PW}_{\Lambda_0}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively $W_a^{s,2}\,\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'}))$ as the *absolute Paley-Wiener Level of time* Λ_0 in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$.

(2) The continuous operator:

$$P^{\mathrm{PW}}_{S,\Lambda_0,a}: W^{s,2}(C_F)(\chi|_{C^1_F}) \to W_a^{s,2}\,\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'})) \tag{11.2.1.3}$$

to be the orthogonal projector onto the closed subspace $W_a^{s,2}\,\mathrm{HS}_{\Lambda_0}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F)) \subset W^{s,2}(C_F)(\chi|_{C^1_F})$.

We refer to $P^{\mathrm{PW}}_{\Lambda_0,a}$ as the *Paley-Wiener truncation at time* Λ_0.

(3) For any test function $\varphi \in \mathcal{D}(C_F)$ supported over $C_F^{\geq 1}$, we refer to $P_a^{\mathrm{geom}} \circ A(\varphi) \circ P^{\mathrm{PW}}_{S,\Lambda_0,a}$ as the *absolute Paley-Wiener condensation of* $P_a^{\mathrm{geom}} \circ A(\varphi)$ *at time* Λ_0.

Theorem 11.2.1.2 (Absolute Paley-Wiener Condensation of the Geometric Trace - strong convergence). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $\varphi \in \mathcal{D}(C_F)$ be a test function. Assume φ is supported over $C_F^{\geq 1}$. Let $\Lambda_0 > 0$ be a real number. Let s, a be real numbers, $s \geq 0$.*

We choose a finite set S of places of F containing all the Archimedean places and all the finitely many finite places at which χ is ramified. We assume S is so large so as to have $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Then:

(1) $P^{\mathrm{PW}}_{S,\Lambda_0,a} \to P_a^{\mathrm{geom}}$ *with respect to the strong operator topology, as* $\Lambda_0 \to +\infty$.

(2) *For every (Λ,ε)-cutoff ρ, we have:*

$$P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi) \circ P^{\mathrm{PW}}_{S,\Lambda_0,a} \to P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi) \tag{11.2.1.4}$$

in the sense of nuclear operators on $W^{s,2}(C_F)(\chi|_{C^1_F})$. In particular, we have:

$$\lim_{\Lambda_0 \to +\infty} \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi) \circ P^{\mathrm{PW}}_{S,\Lambda_0,a}) = \mathrm{Trace}\,(P_{\rho,a}^{\mathrm{geom}} \circ A(\varphi)). \tag{11.2.1.5}$$

Proof. (1) : By Proposition 7.3.2.2(1) and Corollary C.3.5.5(1), we indeed have $P^{\mathrm{PW}}_{\Lambda_0,a} \to P_a^{\mathrm{geom}}$ with respect to the strong operator topology, since the directed union of the $W_a^{s,2}\,\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'}))$ in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ is sequentially dense, being $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ sequentially dense in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$, and being $\mathrm{PW}_{S,X,0}(\chi_{S'}|_{J^1_{F,S'}},\mathcal{S}(\mathbf{A}_{F,S'}))$ sequentially dense in $\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ by Proposition 7.3.2.2(1).

(2) : Both assertions follow from Corollary C.3.5.5(2) and (3). □

The following is an analog of Definition 11.2.1.1:

Definition 11.2.1.3 (Invariant Paley-Wiener Levels). Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let S be a finite set of places of F containing all the Archimedean places. Assume $S_\chi \subset S$ and $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let χ_S be the unique continuous character of $J_{F,S}$ obtained by descending by $\widehat{\mathcal{O}}^\times_{F,\Sigma_F - S}$-invariance the restriction of χ to $C_{F,S}$.

Let $\Lambda_1 > 0$ be a real number. Let s, a be real numbers, $s \geq 0$.

We define:

(1) the closed subspaces:

$$W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{\Lambda_1}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S})) \subset W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S})) \tag{11.2.1.6}$$

and:

$$W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{S,\Lambda_1}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F)) \subset W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.1.7}$$

to be the closure of $|\cdot|^a_{F,S}\mathrm{PW}^{\mathrm{inv}}_{\Lambda_1}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$ in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively of $|\cdot|^a_F\mathrm{PW}^{\mathrm{inv}}_{S,\Lambda_1}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$.

We refer to $W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{\Lambda_1}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively $W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{S,\Lambda_1}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ as the *invariant Paley-Wiener Level of time* Λ_1 in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$, respectively in $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$.

(2) The continuous operator:

$$Q^{\mathrm{PW}}_{S,\Lambda_1,a}:W^{s,2}(C_F)(\chi|_{C^1_F})\to W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{S,\Lambda_1}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.1.8}$$

to be the orthogonal projector onto the closed subspace $W_a^{s,2}\,\mathrm{PW}^{\mathrm{inv}}_{S,\Lambda_1}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))\subset W^{s,2}(C_F)(\chi|_{C^1_F})$. We refer to $Q^{\mathrm{PW}}_{S,\Lambda_1,a}$ as the *invariant Paley-Wiener truncation at time* Λ_1.

(3) For any test function $\varphi\in\mathcal{D}(C_F)$ supported over $C_F^{\geq 1}$, we refer to $P_a^{\mathrm{geom}}\circ A(\varphi)\circ Q^{\mathrm{PW}}_{S,\Lambda_1,a}$ as the *invariant Paley-Wiener condensation of* $P_a^{\mathrm{geom}}\circ A(\varphi)$ *at time* Λ_1.

The following is an analog of Theorem 11.2.1.2:

THEOREM 11.2.1.4 (Invariant Paley-Wiener Condensation of the Geometric Trace - strong convergence). *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let $\varphi\in\mathcal{D}(C_F)$ be a test function. Assume φ is supported over $C_F^{\geq 1}$. Let $\Lambda_1>0$ be a real number. Let s,a be real numbers, $s\geq 0$.*

We choose a finite set S of places of F containing all the Archimedean places and all the finitely many finite places at which χ is ramified. We assume S is so large so as to have $\mathrm{Pic}(\mathcal{O}_{F,S})=1$.

Then:

(1) $Q^{\mathrm{PW}}_{S,\Lambda_1,a}\to P_a^{\mathrm{geom}}$ *with respect to the strong operator topology, as* $\Lambda_1\to+\infty$.

(2) *For every (Λ,ε)-cutoff ρ, we have:*

$$P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi)\circ Q^{\mathrm{PW}}_{S,\Lambda_1,a}\to P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi) \tag{11.2.1.9}$$

in the sense of nuclear operators on $W^{s,2}(C_F)(\chi|_{C^1_F})$. In particular, we have:

$$\lim_{\Lambda_1\to+\infty}\mathrm{Trace}\,(P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi)\circ Q^{\mathrm{PW}}_{S,\Lambda_1,a})=\mathrm{Trace}\,(P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi)). \tag{11.2.1.10}$$

PROOF. Since, by (7.3.2.20) from Proposition 7.3.2.4(4), we have $P^{\mathrm{PW}}_{S,\Lambda_1,a}\leq Q^{\mathrm{PW}}_{S,\Lambda_1^{|S|},a}$, both assertions follow from Theorem 11.2.1.2. □

11.2.2. Orbital Markushevich Bases on the Paley-Wiener Levels. Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let p a prime number, and S_p the finite set of all finite places of F whose residue characteristic is p. Assume that χ is unramified at every place of F contained in S_p.

We let $\varphi\in\mathcal{D}(C_F)$ be a test function.

We discuss here an analog of Theorem 10.3.1.1 for the operator induced by $A(\varphi)$ on $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ for $a=0$, on the subquotients:

$$\mathscr{E}^s_{0,S_p}(\chi|_{C^1_F}):=W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_p}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$$

for S_p the finite set of finite places of F comprising all the finite places v dividing the fixed prime number p.

Recall that $A(\varphi)$ is the regularized action by continuous left-translation of C_F on $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ defined in Definition 9.3.1.1, i.e. the action of the test function φ by convolution, and PV_{S_p} is Weil's Principal Value Operator (see §5.3.1, and specifically Definition 5.3.1.2 and Definition 5.3.1.4).

The continuous convolution $\mathcal{D}(C_F)$-module structure on $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ is inherited by the closed subspace $\mathrm{PV}_{S_p}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$, which was, recall, defined in (11.2.0.3) in light of closedness of the range of the continuous linear operator PV_{S_p} on $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))$ from (11.2.0.2).

We define, for any real number $s\geq 0$:

$$\mathscr{E}^s_{0,S_p}(\chi|_{C^1_F}):=W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{S_p}W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C^1_F},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.2.2.1}$$

endowed with the quotient topology. $\mathscr{E}^s_{0,S_p}(\chi|_{C^1_F})$ is therefore a separable Hilbert space.

We further refer the reader to (7.4.0.2), observing that $\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})$ is the completion of $\mathscr{E}_{S_p}(\chi|_{C_F^1})$ with respect to the quotient of the normed topology on $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$; see also Theorem 7.4.2.1.

THEOREM 11.2.2.1 (Orbital Nuclearity on the 0-line). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $s > 1$ be a real number. Assume:*

$$s \geq 2[F : \mathbf{Q}]. \tag{11.2.2.2}$$

Let p be a prime number and let S_p be the set of those finite places of F whose residue characteristic is p. Assume χ is unramified at every finite place of F contained in S_p.

The $\mathcal{D}(C_F)$-module structure:

$$A : \mathcal{D}(C_F) \to \mathscr{L}_\beta(\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})) \tag{11.2.2.3}$$

is valued in the Banach space of nuclear operators $S_1(\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1}))$ on $\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})$, and is continuous with respect to the Hausdorff locally convex topology on the source, and the trace-norm topology on the target.

In particular, composition with the trace functional on $S_1(\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1}))$ is a distribution on C_F, whose value at $\varphi \in \mathcal{D}(C_F)$ is:

$$\mathrm{Trace}_{s,p}\, A(\varphi) = \sum_{\varepsilon_p} N_s(\varepsilon_p) \int_{C_F} \varphi(g)\chi(g)|g|_F^{\varepsilon_p}\, \mathrm{d}\overline{\mu}(g) \tag{11.2.2.4}$$

where N_s is the counting function defined in (11.2.0.6)*, i.e.*

$$N_s(\varepsilon_p) = \min\left\{N(\varepsilon_p), \left[\frac{s-1}{2} + 1\right]\right\} \tag{11.2.2.5}$$

where $[x]$ is the largest integer $N \geq 0$ satisfying $N \leq x$, and ε_p varies over the zeros of the entire function $\prod_{v\in S_p} L_v(\chi|\cdot|_F^s)^{-1}$, where $L_v(\chi|\cdot|_F^s)$ is the local Euler factor at v of the Hecke L-function $L(s,\chi)$.

PROOF. Since $W^{s,2}(C_F)$ and $W_0^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ are reflexive Banach spaces, being Hilbert spaces, then by [**Tre**, p.484] $A(\varphi)$ is nuclear if and only if its adjoint $A(\varphi)'$, a continuous linear endomorphism of $(\mathscr{E}^s_{0,S_p})'_\beta$, is nuclear. Its trace-norm is computed by $\|A(\varphi)\|_{\mathrm{Tr}} = \|A(\varphi)'\|_{\mathrm{Tr}}$.

We choose an enumeration for the elements:

$$B := \{\eta_{\chi,\varepsilon_p,m}\}_{(\varepsilon_p,m)} \subset \mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})'_\beta$$

which, by Theorem 11.2.0.2 is a Markushevich basis for $\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})'_\beta$. By Corollary 11.2.0.5, $B \subset \mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})'_\beta$ is bounded, and the bi-orthogonal set $B' := \{\psi_{\chi,\varepsilon_p,m}\}_{(\varepsilon_p,m)}$ is also bounded.

We compute:

$$\rho'_F(g, \eta_{\chi,\varepsilon_p,m})(h) = \chi(gh)|gh|_F^{\varepsilon_p}(\log|gh|_F)^m \tag{11.2.2.6}$$

$$= \chi(g)\chi(h)|g|_F^{\varepsilon_p}|h|_F^{\varepsilon_p}(\log|g|_F + \log|h|_F)^m \tag{11.2.2.7}$$

$$= \chi(g)\chi(h)|g|_F^{\varepsilon_p}|h|_F^{\varepsilon_p} \sum_{0\leq k\leq m} \binom{m}{k} (\log|g|_F)^k(\log|h|_F)^{m-k} \tag{11.2.2.8}$$

$$= \chi(g)|g|_F^{\varepsilon_p} \sum_{0\leq k\leq m} \binom{m}{k} (\log|g|_F)^k \left\{\chi(h)|h|_F^{\varepsilon_p}(\log|h|_F)^{m-k}\right\} \tag{11.2.2.9}$$

$$= \chi(g)|g|_F^{\varepsilon_p} \sum_{0\leq k\leq m} \binom{m}{k} (\log|g|_F)^k \eta_{\chi,\varepsilon_p,m-k}(h) \tag{11.2.2.10}$$

Upon integrating over $g \in C_F$ the test function $\varphi \in \mathcal{D}(C_F)$, we obtain a single linear endomorphism of $W_{\varepsilon_p} := \bigoplus_{0\leq m\leq N_s(\varepsilon_p)-1} \mathbf{C}\eta_{\chi,\varepsilon_p,m}$ whose matrix with respect to the basis $\{\eta_{\chi,\varepsilon_p,m}\}$ is triangular, and of the same form of the transpose of (10.3.1.17) from the proof of Theorem 10.3.1.1(1).

Its trace is:

$$N_s(\varepsilon_p)\int_{C_F}\chi(g)\varphi(g)|g|_F^{\varepsilon_p}\,\mathrm{d}\overline{\mu}(g). \tag{11.2.2.11}$$

The series:

$$A(\varphi)'=\sum_{(\varepsilon_p,m)}\langle A(\varphi)'\eta_{\chi,\varepsilon_p,m},\cdot\rangle_{s,2}\psi_{\chi,\varepsilon_p,m} \tag{11.2.2.12}$$

converges in $S_p(\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})'_\beta)$ for every real number $p\geq 1$ by the same argument as in the proof of Theorem 10.3.1.1(1), leading to the expression in (11.2.2.4) by continuity of the trace functional on $S_1(\mathscr{E}^s_{0,S_p}(\chi|_{C_F^1})$, combined with the calculation of the value in (11.2.2.11) for the trace of the finite rank partial sums of the series (11.2.2.12). □

Remark 11.2.2.2. Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let $s>1$ be a real number. Assume:

$$s\geq 2[F:\mathbf{Q}]. \tag{11.2.2.13}$$

Let p be a prime number such that χ is unramified at every finite place of F whose residue characteristic is p.

We observe that $\mathrm{Trace}_{s,p}A(\varphi)$, computed in Theorem 11.2.2.1, is independent of s (so long as s satisfies (11.2.2.13). Indeed, $N_s(\varepsilon_p)=N(\varepsilon_p)$ for all such s, since $N(\varepsilon_p)\leq[F:\mathbf{Q}]$ for all primes p.

11.3. Orbital Kernels and Orbital Traces

11.3.1. Calculation of the Orbital Kernels by global to S-local degeneration on the orbital basis elements.

11.3.1.1. *Divisorial stratification of the ideles.* The next step now is to split $k_{\mathrm{geom},I,\varphi}(x,y)$ into an infinite sum of pieces indexed by the places of F:

$$k_{\mathrm{geom},I,\varphi}(x,y)=k_{S,\varphi}(x,y)+\sum_{p\nmid S}k_{p,\varphi}(x,y)$$

and let the pieces $k_{p,\varphi}(x,y)$ act on certain subspaces of $W_I^{b,2}(|\cdot|_F^{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)))$ to understand them.

This subsection is aimed at conveying the main ideas on how to accomplish this, i.e. how to split the geometric kernel.

We start by identifying an interesting stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$, which is found only on $\mathrm{GL}_1(\mathbf{A}_F)$ as opposed to its quotient C_F, since it is defined by leveraging crucially the restricted product nature of $\mathrm{GL}_1(\mathbf{A}_F)$.

This stratification will adapt itself nicely to the structure of the Schwartz-Bruhat functions representing the global adelic θ-functions in

$$\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)).$$

Recall that there exists a finite set of finite places S of F satisfying the following conditions:

(1) S contains all the ramified places of F and all the Archimedean places.
(2) S contains all the places of F at which χ is ramified.
(3) $\mathrm{Pic}(\mathcal{O}_{F,S})=1$.

such that any such θ-function ϕ can be realized as being of the form, as a function of $x\in C_F$:

$$\phi(x)=\Phi_{F,\chi|_{C_F^1}}(f)(x)$$

where

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}} \tag{11.3.1.1}$$

for some $f_S \in \mathcal{S}(\mathbf{A}_{F,S})$.

This is the last of the five Structure Theorems, proved at the end of Chapter 5 of this work.

The stratification of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ that we are about to define will weld itself perfectly well with the non-Archimedean "transversal" (i.e. tensor component) $\bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$ which all the global Schwartz-Bruhat functions on $\mathbf{A}_F$ representing the $\chi|_{C_F^1}$-homogeneous global θ-functions can be made to share in common.

The guiding principle is simple: since $\mathcal{O}_{F_v}^\times \subset F_v^\times$ is compact and open, hence so it is in F_v, and we have

$$1_{\mathcal{O}_{F_v}^\times} \cdot 1_{\mathcal{O}_{F_v}} = 1_{\mathcal{O}_{F_v}^\times}$$

as Schwartz-Bruhat functions on F_v. Under the local Mellin transform, $Z_v(1_{\mathcal{O}_{F_v}^\times}, c)$ and $Z_v(1_{\mathcal{O}_{F_v}}, c)$ are related by "Weil's Identity":

$$\mu_v(\mathcal{O}_{F_v}^\times) \cdot Z_v(1_{\mathcal{O}_{F_v}^\times}, \chi_v|\cdot|_v^s) = L_v(\chi_v|\cdot|_v^s)^{-1} Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$$

as meromorphic functions on $\Omega(F_v^\times)(\chi_v)$.

Multiplication by $1_{\mathcal{O}_{F_v}^\times}$ of $1_{\mathcal{O}_{F_v}}$ has the effect of replacing the local Z-function $Z_v(1_{\mathcal{O}_{F_v}}, \chi_v|\cdot|_v^s)$ with the product by the reciprocal local Euler factor $L_v(\chi_v|\cdot|_v^s)$.

Note also that the infinite product of the volumes $\mu_v(\mathcal{O}_{F_v}^\times)$ over all the finite places v of F is a finite product (these volumes are 1 for all but finitely many v's) and therefore converges to the square root of the absolute different.

We define

$$\rho_v := \bigotimes_{w \in S} 1_{F_w} \otimes \bigotimes_{w \notin S, w \neq v} 1_{F_v} \otimes 1_{\mathcal{O}_{F_v}^\times}.$$

This is a smooth function on $\mathrm{GL}_1(\mathbf{A}_F)$ (since $\mathcal{O}_{F_v}^\times \subset F_v$ is a compact open subset).

We define, for any finite subset T of finite places of F such that $S \cap T = \varnothing$:

$$\rho_T := \prod_{v \in T} \rho_v. \tag{11.3.1.2}$$

For any increasing chain of finite subsets T_k of finite places of F, as $k \geq 0$ varies over the nonnegative integers, and such that $T_{k+1} - T_k$ contains only one place, then the functions $\{\rho_{T_k}\}_{k \geq 0}$ define on $\mathrm{GL}_1(\mathbf{A}_F)$ a stratification by open and closed subsets, which we call the *divisorial stratification*.

Each stratum is:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : \rho_{T_k}(g) = 1\} = \{g \in \mathrm{GL}_1(\mathbf{A}_F)_{T_k} : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k\}$$

and the complement to each stratum into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{T_k} - \mathrm{GL}_1(\mathbf{A}_F)_{T_{k+1}} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in T_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in T_{k+1} - T_k\}.$$

At last, we will term the *primary divisorial stratification* the stratification $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ of the global ideles $\mathrm{GL}_1(\mathbf{A}_F)$ where we assume:

(1) S is saturated (it contains all the finite places of F of a given residue characteristic, if it contains one of them).
(2) Each finite set of finite places P_k is saturated, and obtained as a union of finitely many of the T_m's.
(3) $\{P_k\}_{k \geq 0}$ is the finest (under the refinement partial ordering) among such stratifications.

More directly, we define P_k to be the pre-image under the map $|\mathrm{Spec}(\mathcal{O}_F)| \to |\mathrm{Spec}(\mathbf{Z})|$ of the subset T_k, and therefore the stratum $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$ of the kth divisorial stratum on $\mathrm{GL}_1(\mathbf{A_Q})$. Explicitly, we have

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} = \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in \bigcup_{p \in T_k} S_p\}$$

where T_k denotes, here, the finite set of the first k rational primes. In other words, $\mathrm{GL}_1(\mathbf{A}_F)_{P_k}$ is the pre-image of the kth stratum $\mathrm{GL}_1(\mathbf{A_Q})_{T_k}$ of the divisorial stratification on $\mathrm{GL}_1(\mathbf{A_Q})$ under the norm map $\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A_Q})$.

The complement to each stratum of $\{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}\}_{k \geq 0}$ into the next is of the following form:

$$\mathrm{GL}_1(\mathbf{A}_F)_{P_k} - \mathrm{GL}_1(\mathbf{A}_F)_{P_{k+1}} = \bigcup_{Q \subset P_{k+1} - P_k} \{g \in \mathrm{GL}_1(\mathbf{A}_F) : g_v \in \mathcal{O}_{F_v}^\times \text{ for all } v \in P_k, g_v \in \bigcup_{m \in \mathbf{Z}, m \neq 0} \pi_v^m \mathcal{O}_{F_v}^\times \text{ for } v \in Q\}$$

where Q varies over all nonempty finite subsets of $P_{k+1} - P_k$.

The primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$ adapts itself nicely to the Schwartz-Bruhat functions f on $\mathbf{A}_F$ of the form in (11.3.1.1). Indeed, upon calling:

$$\alpha_k := 1_{\mathrm{GL}_1(\mathbf{A}_F)_{P_k}}$$

the indicator function of the kth stratum of the primary divisorial stratification on $\mathrm{GL}_1(\mathbf{A}_F)$, we have

$$\alpha_k \cdot (f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}) = f_S \otimes \bigotimes_{v \in S - P_k} 1_{\mathcal{O}_{F_v}} \otimes \bigotimes_{v \in P_k} 1_{\mathcal{O}_{F_v}^\times}$$

and hence multiplication by α_k replaces the Zeta function of f with the product of $Z(f, \chi|\cdot|_F^s)$ by the finitely many reciprocal Euler factors of $L(s, \chi)$ at the non-Archimedean places v of F indexed by P_k (the reciprocal of the product of the volumes of the local units is 1 because S is now assumed to contain all the ramified places of F!).

We call such multiplication process "restriction" from one stratum to the next, of the primary divisorial filtration.

In particular, and this is the main observation of this subsection: *the $(k+1)$st iteration of the restriction to the strata of the primary divisorial filtration is obtained from the kth, on the forward side of the Mellin transform, by multiplication by*

$$\prod_{v \in S_{p_{k+1}}} L_v(\chi_v|\cdot|_v^s)^{-1}.$$

Now recall the main idea: "degenerate" a global adelic θ-function into an S-local one, by representing it as the image of a global Schwartz-Bruhat function

$$f = f_S \otimes \bigotimes_{v \notin S} 1_{\mathcal{O}_{F_v}}$$

and then replacing each $1_{\mathcal{O}_{F_v}}$ by $1_{\mathcal{O}_{F_v}^\times}$. Here S is large enough (it satisfies the assumptions we stated so far throughout this subsection, and it is fixed). Note that replacing $1_{\mathcal{O}_{F_v}}$ by $1_{\mathcal{O}_{F_v}^\times}$ has the effect of sending f to the function of $x \in \mathbf{A}_F$:

$$x \mapsto f(x) - f(\pi_v^{-1} x)$$

with π_v any choice of uniformizer at the non-Archimedean place v. We repackage this operation into an operator PV_v, which turns out to descend to the space of global adelic θ-functions.

We denote by PV_{P_k} the composition of the PV_v as v varies over the places in P_k.

If we let f (equivalently f_S) vary, we obtain that all the resulting global adelic θ-functions restricted to the kth stratum are contained in

$$\mathrm{PV}_{P_k} \mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1}, \mathcal{S}_0(\mathbf{A}_F))$$

and all the global adelic θ-functions restricted to the $(k+1)$st stratum are contained in the yet smaller and closed subspace

$$\mathrm{PV}_{S_{p_{k+1}}}\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) = \mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)).$$

But now the subquotient:

$$\mathrm{PV}_{P_k}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F))/\mathrm{PV}_{P_{k+1}}\mathrm{H}_{0,\mathrm{cont}}(\chi|_{C_F^1},\mathcal{S}_0(\mathbf{A}_F)) \tag{11.3.1.3}$$

carries a Schauder basis that is analogous to the exponential system from the toy example (see Example 1.8.1.1)!

(1) Using the indicator functions α_k, $k \geq 0$ of the strata of the divisorial filtration on the global ideles, we chop the kernel function $k_{\mathrm{geom},I,\varphi}$ along the strata of the primary divisorial filtration on $\mathrm{GL}_1(\mathbf{A}_F)$. Consider the successive differences

$$k_{p_k}(x,y) := (\alpha_{k+1}-\alpha_k)k_{\mathrm{geom},I,\varphi}(x,y) \tag{11.3.1.4}$$

(this will be what we called $k_{p_k,\varphi}(x,y)$) and write them down explicitly by letting them act on (lifts of the) Schauder basis elements on the subquotients (11.3.1.3), upon incorporating the effect of multiplication by $|\cdot|_F^{1/2}$ (here I am omitting several items. The subquotients used to this effect incorporate a support condition on the Fourier transform of the comprising global Schwartz-Bruhat functions f. They carry such Schauder bases because they are all realized as copies of twisted forms of the exponential system itself, and we can always arrange such support condition to hold (by compactness of the quotients $C_F/\pi_v^{\mathbf{Z}}$ for each $v \notin S$, in a nutshell). The version of the subquotients from (11.3.1.3) incorporating such support condition is termed *Paley-Wiener Levels* in the paper. The support condition aims at making multiplication by $|\cdot|_F^a$ for any $a > 0$, continuous on the closure of the subspace generated by the Schwartz-Bruhat functions f satisfying such support condition on their Fourier transform. We are going to omit all of this from this overview. These elements are not difficult, and they are quite natural: they only make for a very long story).

(2) Smoothen the "partial kernels" $k_{p_k,\varphi}$ along the diagonal (using a version of the Hardy-Littlewood Maximal Theorem) and compute their own mass along the diagonal, by using their explicit form as a limit of finite rank operators, obtained by using the basis.

The answer we get is:

$$\sum_{\varepsilon_p} N(\chi|\cdot|_F^{\varepsilon_p}) \int_{C_F} \varphi(g)\chi(g)|g|_F^{\varepsilon_p - 1/2}\,\mathrm{d}\overline{\mu}(g)$$

where the sum is over the complex numbers ε_p such that $\chi|\cdot|_F^{\varepsilon_p}$ is a zero of the entire function $\prod_{v\in S_p} L_v^{-1}$ (we call $N(\chi|\cdot|_F^{\varepsilon_p})$ its multiplicity) with $\chi|\cdot|_F^{\varepsilon_p}$ contained in the connected component $\Omega(C_F)(\chi|_{C_F^1})$.

Once again (as in the toy example) the Poisson Summation Formula also gives us that such trace equals:

$$\sum_{v\in S_p} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \sum_{k\in\mathbf{Z}} \varphi(\pi_v^k)$$

and so if φ is supported strictly over $C_F^{(1,|\pi_v^{-1}|_v)}$ then this gives zero!

If φ is fixed and compactly supported over $(1,+\infty)$, then outside of finitely many places v, indeed φ will be supported on $C_F^{(1,|\pi_v^{-1}|_v)}$ because $|\pi_v^{-1}|_v = q_v$ grows to $+\infty$ as the residue characteristic

of p grows, and so for all but finitely many v, the compact support of φ will be contained in the relatively compact regions $C_F^{(1,|\pi_v^{-1}|_v)}$.

In other words, for all but finitely many finite places v of F, the "partial kernels" $k_{p,\varphi}$ contribute zero to the mass of the geometric kernel.

11.3.2. Calculation of the Orbital Traces.

THEOREM 11.3.2.1 (Orbital Traces on the Paley-Wiener Levels). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let $\varphi \in \mathcal{D}(C_F)$ be a test function. Let S be a finite set of places of F containing all the Archimedean places, all the places at which F is ramified over $\mathbf{Q}$, all the places at which χ is ramified, and such that φ is invariant under the continuous left-translation action of the image in C_F of the closed subgroup $\prod_{v\notin S}\mathcal{O}_{F_v}^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$.*

Assume $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$.

Let $a = 1/2$, $s > 1$ a real number, $I = [-\alpha, \alpha]$ a compact interval, with $\alpha \geq 0$ a real number.

Assume φ is supported over $C_F^{>1}$.

Let $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ be a choice of continuous homomorphic section to the idelic norm.

Then:

$$\sum_{\xi\in F^\times}\int_{g\in E_F^1}((\alpha_{k+1}-\alpha_k)k^\Delta_{\mathrm{geom},a,I,s,\varphi})(\tau(t),\xi g\tau(t))\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t} = \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\sum_{k\in\mathbf{Z}}\varphi(\pi_v^k) \tag{11.3.2.1}$$

where the Δ-subscript denotes the indented kernel of $k_{\mathrm{geom},a,I,s,\varphi}$ along the piece-wise constant indentation given by the function:

$$u_0(x,y) := \sum_{\xi\in F^\times}\int_{g\in E_F^1} 1_{\tau(\mathbf{R}_{>0}}(\xi g x)\otimes 1_{\tau(\mathbf{R}_{>0}}(\xi g y). \tag{11.3.2.2}$$

PROOF. We observe that $k^\Delta_{\mathrm{geom},a,I,s,\varphi}(\tau(t),\xi g\tau(t))$ vanishes unless $\xi \in F^\times$ maps to a uniformizer in $F_v^\times$ for some $v \in P_{k+1} - P_k$. We denote the set of such ξ's as Ξ_k. In particular, the subset $\{\xi g\tau(t), t \in \mathbf{R}_{>0}, g \in E_F^1, \xi \in \Xi_k\}$ satisfies:

$$\{\xi g\tau(t), t \in \mathbf{R}_{>0}, g \in E_F^1, \xi \in \Xi_k\} \subset \mathrm{GL}_1(\mathbf{A}_{F,S'}) \times \prod_{v\notin S'}\mathcal{O}_{F_v}^\times$$

for S' large enough containing S and $P_{k+1} - P_k$.

We call $\beta_{S'}$ the indicator function of $\mathrm{GL}_1(\mathbf{A}_{F,S'}) \times \prod_{v\notin S'}\mathcal{O}_{F_v}^\times$ within $\mathrm{GL}_1(\mathbf{A}_F)$. Multiplication of $(\alpha_{k+1}-\alpha_k)k^\Delta_{\mathrm{geom},a,I,s,\varphi}$ by the indicator function α_k (as a function of the second named variable) for $k \geq k_0$ for a suitable large enough integer $k_0 \geq 0$ such that the support of α_k contains $\mathrm{GL}_1(\mathbf{A}_{F,S'}) \times \prod_{v\notin S'}\mathcal{O}_{F_v}^\times$, preserves the value of the left side of (11.3.2.1). We let ρ be a smooth (ε,Λ)-cutoff and multiply $(\alpha_{k+1}-\alpha_k)k^\Delta_{\mathrm{geom},a,I,s,\varphi}$ by ρ as a function of the first named variable.

Then the process of multiplication by α_k in the second named variable converges within $W_I^s(C_F)$ by Theorem 11.1.2.1, and by Proposition 12.2.2.2 the value:

$$\sum_{\xi\in F^\times}\int_{g\in E_F^1}\rho(\tau(t))((\alpha_{k+1}-\alpha_k)k^\Delta_{\mathrm{geom},a,I,s,\varphi})(\tau(t),\xi g\tau(t))\alpha_{k_1}(\xi g\tau(t))\,\mathrm{d}\mu_1(g)\,\frac{\mathrm{d}t}{t} \tag{11.3.2.3}$$

equals the value of the trace of the nuclear operator on $W_I^s(C_F)$ which is the unique continuous extension that the kernel $\rho(\tau(t))((\alpha_{k+1}-\alpha_k)k^\Delta_{\mathrm{geom},a,I,s,\varphi})(\tau(t),\xi g\tau(t))\alpha_{k_1}(\xi g\tau(t))$ induces, by defining a continuous integral operator $\mathcal{S}(\mathbf{A}_F) \to W_I^s(C_F)$ valued in the range of M_ρ.

As $k_1 \to +\infty$ the effect of this latter on the arbitrary Schwartz-Bruhat function matches the effect of convolution on the resulting S'-local Schwartz-Bruhat function, by invariance under of $\prod_{v\notin S}\mathcal{O}_{F_v}^\times$ of φ, and with S' determined by k as in the foregoing, so that the indicator function of $\mathrm{GL}_1(\mathbf{A}_{F,S'}) \times \prod_{v\notin S'}\mathcal{O}_{F_v}^\times$ is the

limit of the α_{k_1}, so long as the Schwartz-Bruhat function is supported over the kth stratum of the divisorial filtration.

The images under $\Phi_{F,\chi|_{C_F^1}}$ of such Schwartz-Bruhat functions comprise of the finite union of the orbital Schauder bases at the places $v \in P_{k+1} - P_k$, and the limit process as $k_1 \to +\infty$ turns them, after global inversion by Theorem 2.2.3.2, into the corresponding S'-local orbital Schauder basis. We can choose these elements to belong to the Paley-Wiener level at time $\lambda' > 0$ large enough, and the same method used in the estimates from Theorem 10.3.2.1 shows that such operator is the limit of the finite rank kernels from (11.2.2.12) from the proof of Theorem 11.2.2.1.

This gives us its trace, and hence the value of:

(11.3.2.4)
$$\sum_{\xi \in F^\times} \int_{g \in E_F^1} \rho(\tau(t))((\alpha_{k+1} - \alpha_k) k^{\Delta}_{\mathrm{geom},a,I,s,\varphi})(\tau(t), \xi g \tau(t)) \beta_{S'}(\xi g \tau(t)) 1_{g \in \mathrm{GL}_1(\mathbf{A}_{F,S'})^{> \lambda'}}(\xi g \tau(t)) \, \mathrm{d}\mu_1(g) \, \frac{\mathrm{d}t}{t}$$

which, as λ' is large enough and for $\Lambda \to +\infty$ in the regime $\varepsilon \Lambda = o(1)$, attains the limit value:

(11.3.2.5)
$$\sum_{\varepsilon_p} N(\chi | \cdot |_F^{\varepsilon_p}) \int_{C_F} \varphi(g) \chi(g) |g|_F^{\varepsilon_p - 1/2} \, \mathrm{d}\overline{\mu}(g)$$

which is the right side of (7.4.2.1) from Theorem 7.4.2.1. The Theorem follows from the Poisson Summation Formula (see Proposition B.4.1.2). □

CHAPTER 12

Trace formula: Geometric Side

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12.1. Smoothenings along the diagonal

12.1.1. Localization along the global diagonal. In this subsection we discuss how to use the kernel functions $k^{\mathrm{geom}}_{\varphi,a,s}(x,y)$, $k^{\mathrm{spec}}_{\varphi,a,s}$, and their products with (Λ,ε) cutoffs functions $\rho(x)$ on C_F in the sense of Definition 9.3.3.1, to compute the traces of $P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi)$ as $\rho\to 1$ (as $\Lambda\to+\infty$ while having $\varepsilon\Lambda=o(1)$), and of $P^{\mathrm{spec}}_{a}\circ A(\varphi)$.

This requires a discussion of how to make sense of each of the above kernel functions along the diagonal in $C_F\times C_F$, a null set with respect to the product measure $\overline{\mu}\otimes\overline{\mu}$.

To this end, we will be using the analog for the idele class group C_F (and later for its S-local subgroup $C_{F,S}$ and this latter's quotient $J_{F,S}$) of the Differentiation Theorem of Lebesgue, together with continuity properties of the Haar integral of $\overline{\mu}$-absolutely integrable functions over C_F.

This rests on an analog of the Hardy-Littlewood Maximal Theorem on Maximal functions, for which we refer the reader to Appendix A (see Theorem A.5.3.7 and Theorem A.5.4.1).

We begin with some results of a general nature, on Hilbert-Schmidt kernels on $L^2(G,\lambda)$, where G is a Hausdorff locally compact second countable abelian group, and λ is a Radon measure on G subordinate to a choice of Haar measure μ_G, i.e. such that there exists a locally μ_G-integrable function $f: G\to\mathbf{R}_{\geq 0}$ satisfying $f\,\mu_G=\lambda$ as Radon measures on G.

Definition 12.1.1.1. Let G be a Hausdorff locally compact abelian group. Assume G is second countable. Let $\{U_k\}_{k\geq 0}$ be a D'-sequence for G in the sense of Definition A.5.2.1.

Let $\phi\in L^1_{\mathrm{loc}}(G,\mu_G)$ be a locally μ_G-integrable complex-valued function on G. We define the *smoothening* of ϕ to be the function defined as:

$$\widetilde{\phi}(x):=\lim_{k\to+\infty}\frac{1}{\mu_G(U_k)}\int_{xU_k}\phi(y)\,\mathrm{d}\mu_G(y) \tag{12.1.1.1}$$

at each point $x\in G$ at which the limit in (12.1.1.1) exists and is finite, and zero elsewhere.

We call a point $x\in G$ at which the limit in (12.1.1.1) exists and is finite, a *regular point* for ϕ.

We recall that a D-sequence for a Hausdorff locally compact abelian group G, with μ_G a choice of Haar measure on G, is the datum of a sequence $\{U_k\}_{k\geq 0}$ where for each integer $k\geq 0$, $U_k\subset G$ is a μ_G-measurable subset, $U_{k+1}\subset U_k$, the μ_G-measure of each U_k is finite and nonzero, there exists a real number $C>0$ such that $\mu_G(U_kU_k^{-1})<C\,\mu_G(U_k)$ for every $k\geq 0$.

We say $\{U_k\}_{k\geq 0}$ is a Borel D-sequence, respectively an open, symmetric D-sequence, if and only if each of its members is a Borel set, respectively an open subset, respectively a symmetric subset (i.e. stable under the inversion homomorphism on G).

We say $\{U_k\}_{k\geq 0}$ is a D'-sequence, if it is a Borel D-sequence satisfying the following additional property: for every unit-neighborhood V of G, there exists a large enough integer $k_0\geq 0$ such that $U_{k_0}\subset V$. In particular, there exists $k_0\geq 0$ such that for every $k\geq k_0$, U_k has compact closure.

For instance, for $G=\mathbf{R}^N$, the two sequences $\{U_k:=B(0,r_k)\}_{k\geq 0}$ and $\{U'_k:=Q_{r_k}\}$, for $\{r_k\}_{k\geq 0}$ a strictly decreasing sequence of real numbers with $r_0>0$, and with $B(0,r)$ the closed ball centered at 0 with radius r, and Q_r the cube in $\mathbf{R}^N$ of the form $[-r,r]^N$, are both symmetric D'-sequences.

The Hausdorff locally compact abelian groups under consideration throughout this work (global ideles, S-local ideles, their class groups, their closed subgroups of norm-one elements, their class groups of norm-one elements) are all equipped with symmetric D'-sequences. These are constructed in Appendix A,§A.5.2; see Definition A.5.2.1 and Definition A.5.2.18.

The presence of a D'-sequence in G enables the definition of a corresponding maximal function analogous to the Hardy-Littlewood maximal function on Euclidean spaces, and analogs of the Maximal Theorem and of Lebesgue's Differentiation Theorem both hold. We refer the reader to [**HR-2**, Ch.XI,§44], and to Theorem A.5.3.7 and Theorem A.5.4.1.

In particular, maximal functions, the Maximal Theorem and Lebesgue's Differentiation Theorem, are available on the idelic groups and groups of idele classes being considered throughout this Chapter.

For locally integrable functions on $G \times G$, if $\{U_k\}_{k\geq 0}$ is a D'-sequence for G, we will be using the D'-sequence $\{U_k \times U_k\}_{k\geq 0}$ for $G \times G$ to define smoothenings (see Lemma A.5.2.5(1), which proves that indeed $\{U_k \times U_k\}_{k\geq 0}$ is a D'-sequence for $G \times G$).

The following Theorem explains the role of maximal functions and of the analogs of the Maximal Theorem of Hardy-Littlewood, and of the Differentiation Theorem of Lebesgue.

THEOREM 12.1.1.2. *Let G be a Hausdorff locally compact abelian group. Assume G is second countable. Let μ_G be a choice of Haar measure on G. Let $\{U_k\}_{k\geq 0}$ be a D'-sequence for G in the sense of Definition A.5.2.1.*

Let λ be a Radon measure on G such that there exists a locally μ_G-integrable function $f : G \to \mathbf{R}_{\geq 0}$ satisfying $\lambda = f\,\mu_G$ as Radon measures on G. Assume f is continuous and nowhere vanishing on G.

Let $T \in \mathscr{L}_\beta(L^2(G,\lambda))$ be a nuclear operator, and $k \in L^2(G \times G, \lambda \otimes \lambda)$ a choice of representative for the Hilbert-Schmidt kernel of T.

Then, μ_G-almost all points $(x,x) \in G \times G$ are regular for k.

(1) *The function $\widetilde{k}(x,y)$ is a representative for the Hilbert-Schmidt kernel of T, and it is defined at μ_G-almost all points of the diagonal in $G \times G$. The function $\widetilde{k}(x,x)$ of $x \in G$ is absolutely λ-integrable. We have:*

$$\int_G \widetilde{k}(x,x)\,\mathrm{d}\lambda(x) = \mathrm{Trace}(T). \tag{12.1.1.2}$$

(2) *Let $\rho \in C_c(G)$, and M_ρ the corresponding multiplication operator on $L^2(G,\lambda)$. The smoothening of $\rho(x)k(x,y)$ is $\rho(x)\widetilde{k}(x,y)$, and both functions are representatives for the Hilbert-Schmidt kernel of the operator $M_\rho \circ T$ on $L^2(G,\lambda)$. If $\rho \in L^\infty(G,\lambda)$, then the function $\rho(x)\widetilde{k}(x,y)$ is a representative for the Hilbert-Schmidt kernel of the operator $M_\rho \circ T$, and we have:*

$$\int_G \rho(x)\widetilde{k}(x,x)\,\mathrm{d}\lambda(x) = \mathrm{Trace}(M_\rho \circ T). \tag{12.1.1.3}$$

(3) *For each integer $m \geq 0$, let $T_m \in \mathscr{L}_\beta(L^2(G,\lambda))$ be a finite rank operator, such that the sequence $\{T_m\}_{m\geq 0}$ converges in $S_p(L^2(G,\lambda))$ for $p = 1, 2$, to T. Let $k_m \in L^2(G \times G, \lambda \otimes \lambda)$ be a choice of representative for the Hilbert-Schmidt kernel of T_m.*

Then $k_m \to k$ and $\widetilde{k}_m \to \widetilde{k}$ in $L^2(G \times G, \lambda \otimes \lambda)$, and $\widetilde{k}_m(x,x) \to \widetilde{k}(x,x)$ weakly in $L^1(G,\lambda)$.

(4) *Assume that there exists $\{\phi_m\}_{m\geq 0}$ a simple, normalized Schauder basis for $L^2(G,\lambda)$. Let $\{\theta_m\}_{m\geq 0}$ be the dual basis with respect to the duality pairing:*

$$\langle \cdot, \cdot \rangle_\lambda : L^2(G,\lambda) \times L^2(G,\lambda^\vee) \to \mathbf{C} \tag{12.1.1.4}$$

where $\lambda^\vee$ is the Radon measure $(1/f)\,\mu_G$, defined by:

$$\langle \phi, \psi \rangle_\lambda := \int_G \phi(x)\psi(x)\,\mathrm{d}\mu_G(x). \tag{12.1.1.5}$$

We define:

$$k_N(x,y) := \sum_{0\leq m\leq N} \frac{(T'\theta_m)(y)}{f(y)}\,\phi_m(x) \tag{12.1.1.6}$$

where $T' \in \mathscr{L}_\beta(L^2(G,\lambda^\vee)$ is the adjoint of T with respect to the duality $\langle L^2(G,\lambda), L^2(G,\lambda^\vee)\rangle_\lambda$.

Then $k_N \to k$ and $\widetilde{k}_N \to \widetilde{k}$ in $L^2(G \times G, \lambda \otimes \lambda)$, and $\widetilde{k}_N(x,x) \to \widetilde{k}(x,x)$ weakly in $L^1(G,\lambda)$.

PROOF. (1) : T can be written as $T = A_1 - A_2 + i(A_3 - A_4)$ where A_i is nuclear and positive, for $1 \leq i \leq 4$. Since formation of the smoothening of locally μ_G-integrable functions on $G \times G$ with respect to the

D'-sequence $\{U_k \times U_k\}_{k\geq 0}$ is $\mathbf{C}$-linear, we may rename each A_i as T and assume T is nuclear and positive. By [**Cnw-a**, Ch.II,§2,Prop.2.12,p.33], T is self-adjoint. Since T is nuclear, T is compact.

By [**Cnw-a**, Ch.II,§5,Cor.5.4,p.47], we have $T = \sum_{m\geq 0} \sigma_m(T)\langle \phi_m, \cdot\rangle_{L^2(G,\lambda)}\, \phi_m$, the convergence being with respect to the strong operator topology (see also Theorem C.3.6.1 applied to $E = F = L^2(G,\lambda)$), where the singular values of T are listed decreasingly and counted with multiplicity.

Upon calling P_m the orthogonal projector onto $\ker(T - \sigma_m(T)\mathrm{Id})$, then we may write $T = \sum_{m\geq 0} \sigma_m(T) P_m$, where the singular values of T are now listed decreasingly and counted once for each integer $m \geq 0$. Each P_m is self-adjoint by [**Cnw-a**, Ch.II,§2,Prop.2.12,p.33] and [**Cnw-a**, Ch.II,§3,Prop.3.3,p.37], and hence by Proposition C.3.6.5 (see also Corollary C.3.5.5), $P_m T P_m \to T$ in $S_p(L^2(G,\lambda))$ for $p = 1, 2$.

The Hilbert-Schmidt kernel of $P_m T P_m$ is represented by the function

$$k_m(x,y) := \sigma_m(T) \sum_{1\leq r\leq N_m} \overline{\phi}_{m,r}(y)\phi_{m,r}(x) \tag{12.1.1.7}$$

where $N_m = \dim(\ker(T - \sigma_m(T)\mathrm{Id}))$ and $\{\phi_{m,r}\}_{1\leq r\leq N_m}$ is an orthonormal basis for $\ker(T - \sigma_m(T)\mathrm{Id})$.

Since $S_2(L^2(G,\lambda))$ is isometrically isomorphic to $L^2(G\times G, \lambda\otimes\lambda)$, and $P_m T P_m$ is sent to the class of $k_m(x,y)$ under such isometry, we deduce that the sequence of finite sums $\sum_{0\leq m\leq N} k_m(x,y)$ converges in $L^2(G\times G, \lambda\otimes\lambda)$.

We write:

$$\psi_{m,r} := \phi_{m,r}\cdot (f)^{1/2} \tag{12.1.1.8}$$

and observe that $\psi_{m,r} \in L^2(G,\mu_G)$ for every $m \geq 0$ and every $1 \leq r \leq N_m$. Note that being f continuous, real-valued and nonnegative, its square-root is well defined and single valued.

We define:

$$k'_m(x,y) := \sigma_m(T) \sum_{1\leq r\leq N_m} \overline{\psi}_{m,r}(y)\psi_{m,r}(x) \tag{12.1.1.9}$$

and observe that we have $k'_m \in L^2(G\times G, \mu_G\otimes\mu_G)$. The linear operator $\phi \mapsto \phi\cdot(f)^{1/2}$ is a surjective isometry $J_f : L^2(G,\lambda) \to L^2(G,\mu_G)$. It is indeed surjective, since its inverse is given by $\psi \mapsto \psi\cdot(f)^{-1/2}$, being f nowhere vanishing over G.

The operator $J_f \circ (P_m T P_m) \circ J_f^{-1}$ is Hilbert-Schmidt and nuclear (in fact, of finite rank N_m) on $L^2(G,\mu_G)$, with kernel represented by the function $k'_m(x,y)$. The operator $J_f \circ T \circ J_f^{-1}$ is Hilbert-Schmidt and nuclear, with kernel:

$$k'(x,y) := f(x)^{1/2} k(x,y) f(y)^{1/2}. \tag{12.1.1.10}$$

Indeed, for any $\phi \in L^2(G,\mu_G)$, we have:

$$\int_G k'(x,y)\phi(y)\,\mathrm{d}\mu_G(y) = f(x)^{1/2}\int_G k(x,y) f(y)^{1/2}\phi(y)\,\mathrm{d}\mu_G(y) = f(x)^{1/2}\int_G k(x,y) f(y)^{-1/2}\phi(y)\,\mathrm{d}\lambda(y) \tag{12.1.1.11}$$

for μ_G-almost all $x \in G$, and the rightmost side of (12.1.1.11) reads $J_f(T J_f^{-1}(\phi))$.

Since f is continuous, as is $1/f$, then by Corollary A.5.4.3 we may rename $\psi_{m,r}$ as $\phi_{m,r}$, and assume $f = 1$ identically over G. We are reduced to the case $\lambda = \mu_G$.

We now proceed as in the proof of [**Bris-a**, Thm.3.1,p.1185]. Since T is nuclear and $\mathrm{Trace}(P_m T P_m) \to \mathrm{Trace}(T)$ by Proposition C.3.6.5, we deduce that for μ_G-almost all $x \in G$, the sequence of finite sums:

$$\sum_{1\leq m\leq N} \sigma_m(T) \sum_{1\leq r\leq N_m} |\phi_{m,r}(x)|^2 \tag{12.1.1.12}$$

converges as $N \to +\infty$. Upon replacing the classical Hardy-Littlewood Maximal Theorem in $\mathbf{R}^n$ with Theorem A.5.3.7, using the Maximal function $M(\cdot)$ defined in (A.5.3.16), and applying the Theorem for $p = 2$ (i.e.

using the strong $(2,2)$ inequality from (A.5.3.18), the argument for (1) is now identical to that provided in [**Bris-a**, pp.1185-1186] for [**Bris-a**, Thm.3.1], using the Differentiation Theorem of Lebesgue in its form from Theorem A.5.4.1 so as to find the set Y in loc.cit. with $\mu_G(G-Y)=0$ to carry out the rest of the argument.

(2) : We first assume $\rho \in C_c(G)$. Since $\rho(x)k(x,y)$ is a representative function for the Hilbert-Schmidt kernel of $M_\rho \circ T$ on $L^2(G,\lambda)$, and $\widetilde{k}(x,y)$ equals $k(x,y)$ on $(\mu_G \otimes \mu_G)$-almost every point, we have $k(x,y)=\widetilde{k}(x,y)$ also $(\lambda\otimes\lambda)$-almost everywhere, since λ and μ_G have the same null sets (indeed, both λ and μ_G are σ-finite Radon measures on G since G is second countable, and f is the Radon-Nikodym derivative of λ with respect to μ_G. Since it is nowhere vanishing, λ and μ_G are equivalent: a subset $A\subset G$ is μ_G-measurable if and only if it is λ-measurable, and $\mu_G(A)=\int_A f(x)\,\mu_G(x)=0$ if and only if $1_A(x)f(x)=0$ for μ_G-almost all $x\in A$, which forces $\mu_G(A)=0$).

It follows that we have:

$$\rho(x)\int_G \widetilde{k}(x,y)\phi(y)\,\mathrm{d}\lambda(y)=\rho(x)\int_G k(x,y)\phi(y)\,\mathrm{d}\lambda(y)=(M_\rho\circ T\phi)(x)$$

for λ-almost all $x\in G$, which shows that $\rho(x)\widetilde{k}(x,y)$ is another representative function for the Hilbert-Schmidt kernel of $M_\rho\circ T$. We write $k_\rho(x,y):=\rho(x)k(x,y)$, and compute $\widetilde{k}_\rho(x,y)$. Upon showing that we have, for μ_G-almost all $x\in G$, $\widetilde{k}_\rho(x,x)=\rho(x)\widetilde{k}(x,x)$, then the same identity holds for λ-almost all $x\in G$, and then (12.1.1.3) follows from (12.1.1.2) of (1).

We have, for $(\mu_G\otimes\mu_G)$-almost every $(x,y)\in G$ (in fact, for any regular point $(x,y)\in G\times G$ for k):

$$\widetilde{k}_\rho(x,y)=\lim_{k\to+\infty}\frac{1}{\mu_G(U_k)^2}\int_{xU_k\times yU_k}\rho(x')k'(x',y')\,\mathrm{d}(\mu_G\otimes\mu_G)(x',y')=\rho(x)\widetilde{k}(x,y) \tag{12.1.1.13}$$

using Lemma A.5.2.5(1) to deduce that $\{U_k\times U_k\}_{k\geq 0}$ is a D'-sequence for $G\times G$. Indeed, by Corollary A.5.4.3, the limit in (12.1.1.13) exists and is finite exactly at the points at which the limit of the sequence $\{\mu_G(U_k)^{-2}\int_{xU_k\times yU_k}k'(x',y')\,\mathrm{d}(\mu_G\otimes\mu_G)(x',y')\}_{k\geq 0}$ does, and its value is $\rho(x)\widetilde{k}(x,y)$.

Let $\rho\in L^\infty(G,\lambda)$ be arbitrary, and by [**Fol**, Thm.2.10(b)], we choose a sequence $\{\rho_m\}_{m\geq 0}\subset A(G)$, the $\mathbf{C}$-algebra of simple functions on G with complex coefficients, such that $\rho_m\to\rho$ in $L^\infty(G,\lambda)$. Then $M_{\rho_m}\to M_\rho$ with respect to the strong operator topology on $\mathscr{L}(L^2(G,\lambda))$ (in fact, with respect to the operator-norm topology, by [**Cnw-a**, Ch.II,§1,Thm.1.5,p.28]), and hence $M_{\rho_m}\circ T\to M_\rho\circ T$ in $S_p(L^2(G,\lambda))$ for $p=1,2$ by Lemma C.3.4.4.

We may therefore assume $\rho=1_E$ for E a λ-measurable subset of G. We now let $\{K_n\}_{n\geq 0}$ be an exhaustion by compact subsets of G, and call $E_n:=E\cap K_n$. Then $1_{E_n}\to 1_E$ pointwise λ-almost everywhere, and hence so does $1_{E_n}(x)\widetilde{k}(x,x)$. It follows that $M_{1_{E_n}}\to M_{1_E}$ as $n\to+\infty$ with respect to the strong operator topology on $\mathscr{L}(L^2(G,\lambda))$, and by Lemma C.3.4.4 we have $M_{1_{E_n}}\circ T\to M_{1_E}\circ T$ in $S_p(L^2(G,\lambda))$ for $p=1,2$.

By Dominated Convergence, we may assume $\rho=1_{E_n}$, the indicator function of a λ-measurable subset of G of finite λ-measure. There now exists a sequence $\{\rho_m\}_{m\geq 0}\subset C_c(G)$ such that ρ_m is real valued, nonnegative and bounded above by 1 for every integer $m\geq 0$, and $\rho_m\to\rho$ pointwise λ-almost everywhere, and in fact in $L^p(G,\lambda)$ for every (finite) $p\geq 1$.

By the foregoing, we have:

$$\int_G \rho_m(x)\widetilde{k}(x,x)\,\mathrm{d}\lambda(x)=\mathrm{Trace}(M_{\rho_m}\circ T) \tag{12.1.1.14}$$

for every integer $m\geq 0$, and since $\widetilde{k}$ is λ-absolutely integrable over G, we have:

$$\int_G \rho_m(x)\widetilde{k}(x,x)\,\mathrm{d}\lambda(x)\to\int_G \rho(x)\widetilde{k}(x,x)\,\mathrm{d}\lambda(x) \tag{12.1.1.15}$$

as $m\to+\infty$, by Dominated Convergence.

We deduce that (12.1.1.3) holds.

(3) : Since $S_2(L^2(G,\lambda))$ is isometrically isomorphic to $L^2(G\times G,\lambda\otimes\lambda)$, the isometry sending T_m to the class of k_m and T to the class of k, we deduce that $T_m\to T$ implies (and is implied by) $k_m\to k$ in $L^2(G\times G,\lambda\otimes\lambda)$, and since, $(\lambda\otimes\lambda)$-almost everywhere on $G\times G$, we have $k_m=\widetilde{k}_m$ and $k=\widetilde{k}$, we deduce that we have $\widetilde{k}_m\to\widetilde{k}$ in $L^2(G\times G,\lambda\otimes\lambda)$.

We let $\rho\in L^\infty(G,\lambda)$ be a λ-essentially bounded, equivalently μ_G-essentially bounded, measurable complex valued function on G, and $M_\rho\in\mathscr{L}_\beta(L^2(G,\lambda))$ the resulting multiplication operator.

By Lemma C.3.4.4, $M_\rho\circ T_m\to M_\rho\circ T$ in $S_p(L^2(G,\lambda))$ for $p=1,2$, and by (2), we deduce from (12.1.1.3) that we have, as $m\to+\infty$:

$$\int_G\rho(x)\widetilde{k}_m(x,x)\,\mathrm{d}\lambda(x)=\mathrm{Trace}(M_\rho\circ T_m)\to\mathrm{Trace}(M_\rho\circ T)=\int_G\rho(x)\widetilde{k}(x,x)\,\mathrm{d}\lambda(x). \tag{12.1.1.16}$$

This means that $\widetilde{k}_m(x,x)\to\widetilde{k}(x,x)$ weakly in $L^1(G,\lambda)$.

(4) : We call $P_N:L^2(G,\lambda)\to L^2(G,\lambda)$ the Nth basis projector:

$$P_N:=\sum_{0\leq m\leq N}\langle\theta_m,\cdot\rangle_\lambda\,\phi_m \tag{12.1.1.17}$$

and observe that we have, upon defining $T_N:=P_N\circ T$:

$$T_N(\phi)(x)=\int_G k_N(x,y)\phi(y)\,\mathrm{d}\lambda(y) \tag{12.1.1.18}$$

for λ-almost all $x\in G$. Indeed, we have:

(12.1.1.19)

$$T_N(\phi)(x)=\sum_{0\leq m\leq N}\langle\theta_m,T\phi\rangle_\lambda\,\phi_m(x)=\sum_{0\leq m\leq N}\langle T'\theta_m,\phi\rangle_\lambda\,\phi_m=\sum_{0\leq m\leq N}\int_G(T'\theta_m)(y)\phi(y)\,\mathrm{d}\lambda(y)\,\phi_m(x)$$

for λ-almost all $x\in G$. We deduce that (12.1.1.18) holds.

Since $\{\phi_m\}_{m\geq0}$ is a simple, normalized Schauder basis, then $\{P_N\}_{N\geq0}\subset\mathscr{L}_\beta(L^2(G,\lambda))$ is a bounded subset, and $P_N\to\mathrm{Id}$ with respect to the strong operator topology. By Proposition C.3.6.5, $T_N\to T$ in $S_p(L^2(G,\lambda))$ for $p=1,2$. (4) therefore follows from (3). □

By Theorem A.5.4.1, $\widetilde{\phi}$ agrees with ϕ on a μ_G-measurable subset $\mathcal{L}_\phi$ of G whose complement is μ_G-negligible. $\mathcal{L}_\phi$ is termed the *Lebesgue set of* ϕ (see Remark A.5.4.2). By Theorem A.5.4.1(2), the subset of G of the points of continuity of ϕ (i.e. those points $x\in G$ such that there exists a neighborhood $V\subset G$ of x in G such that $\phi|_V$ is continuous) is included in $\mathcal{L}_\phi$.

The set of points of G at which the limit in (12.1.1.1) defining the smoothening of a locally μ_G-integrable function ϕ exists, may be larger than the Lebesgue set $\mathcal{L}_\phi$. We will be applying the smoothening construction to representative functions of the Hilbert-Schmidt kernel of a nuclear operator on an L^2-space of G, and while the Lebesgue set of such function may not contain the diagonal of $G\times G$ (which is a null set with respect to the product measure $\mu_G\otimes\mu_G$), the smoothening may still be defined over the diagonal, and it is under the assumptions of Theorem A.5.4.1.

THEOREM 12.1.1.3 (Localization along the global diagonal). *Let F be a number field, a,s,Λ real numbers, $s\geq3$ and $a,\Lambda,\varepsilon>0$. Assume $0<1/\Lambda+\varepsilon<\Lambda-\varepsilon$. Let $\varphi\in\mathcal{D}(C_F)$ be a test function.*

1. *For $\overline{\mu}$-almost all $x\in C_F$, the point $(x,x)\in C_F\times C_F$ is a regular point for the functions $k^{\mathrm{geom}}_{\varphi,a,s}$, $k^{\mathrm{spec}}_{\varphi,a,s}$. For every $x\in C_F$, $(x,x)\in C_F\times C_F$ is a regular point for the function $\varphi(xy^{-1})$.*
2. *Let $\rho\in\mathcal{D}(C_F)$ be a (Λ,ε)-cutoff. We have:*

$$\mathrm{Trace}\,(P^{\mathrm{geom}}_{\rho,a}\circ A(\varphi))=\int_{C_F}\rho(x)\widetilde{k}^{\mathrm{geom}}_{\varphi,a,s}(x,x)\,\mathrm{d}\overline{\mu}(x) \tag{12.1.1.20}$$

and:

$$\text{Trace}\,(P_a^{\text{spec}} \circ A(\varphi)) = \int_{C_F} \widetilde{k}^{\text{spec}}_{\varphi,a,s}(x,x)\,\mathrm{d}\overline{\mu}(x). \tag{12.1.1.21}$$

(3) *Assume $\varphi(1) = 0$. Then $\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)$ is absolutely $\overline{\mu}$-integrable over C_F, and we have:*

$$\lim_{\Lambda\to+\infty,\varepsilon\Lambda=o(1)} \text{Trace}\,(P^{\text{geom}}_{\rho,a} \circ A(\varphi)) = \int_{C_F} \widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)\,\mathrm{d}\overline{\mu}(x). \tag{12.1.1.22}$$

Proof. (1) : By Theorem 12.1.1.2, Theorem 10.4.3.4 and Theorem 10.4.2.1, $\overline{\mu}$-almost all points of the diagonal in $C_F \times C_F$ is a regular point for $k^{\text{spec}}_{\varphi,a,s}$. Since $\varphi(xy^{-1})$ is continuous over $C_F \times C_F$, by Theorem A.5.4.1(2) every point of $C_F \times C_F$ is a regular point, and hence $\overline{\mu}$-almost all points of the diagonal in $C_F \times C_F$ is regular for $k^{\text{geom}}_{\varphi,a,s} = \varphi(xy^{-1}) - k^{\text{spec}}_{\varphi,a,s}$.

(2) : By Theorem 12.1.1.2(1) and Theorem 10.4.3.4, $\widetilde{k}^{\text{spec}}_{\varphi,a,s}$ is a representative function for the Hilbert-Schmidt kernel of $P_a^{\text{spec}} \circ A(\varphi)$, since $k^{\text{spec}}_{\varphi,a,s}$.

Since ρ is continuous, so is $\rho(x)\varphi(xy^{-1})$ as a function on $C_F \times C_F$, and hence, by Theorem 10.4.3.4, $\rho(x)\widetilde{k}^{\text{geom}}_{\varphi,a,s}$ and $\rho(x)\widetilde{k}^{\text{spec}}_{\varphi,a,s}$ are representative functions for the Hilbert-Schmidt kernels of $P^{\text{geom}}_{\rho,a} \circ A(\varphi)$ and $P^{\text{spec}}_{\rho,a} \circ A(\varphi)$, respectively.

By (12.1.1.2) from Theorem 12.1.1.2, (12.1.1.20) and (12.1.1.21) hold.

(3) : The function $\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)$ is the sum of two measurable functions of $x \in C_F$, and it is, as such, measurable. If $\varphi(1) = 0$, then we have:

$$\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x) = -\widetilde{k}^{\text{spec}}_{\varphi,a,s}(x,x) \tag{12.1.1.23}$$

and by Theorem 10.4.3.1(4), the rightmost side of (12.1.1.23) is absolutely $\overline{\mu}$-integrable.

Since $\rho \to 1$ as $\Lambda \to +\infty$ in the regime $\varepsilon\Lambda = o(1)$, as continuous uniformly bounded (by 1) functions on C_F, then $\rho(x)\,\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x) \to \widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)$ pointwise $\overline{\mu}$-almost everywhere over C_F. Since $|\rho(x)\,\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)| \leq |\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)|$ for $\overline{\mu}$-almost all $x \in C_F$, by Dominated Convergence we deduce that we have:

$$\lim_{\Lambda\to+\infty,\varepsilon\Lambda=o(1)} \int_{C_F} \rho(x)\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)\,\mathrm{d}\overline{\mu}(x) = \int_{C_F} \widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)\,\mathrm{d}\overline{\mu}(x). \tag{12.1.1.24}$$

By (10.4.3.32) and Theorem 12.1.1.2(1), the leftmost side of (12.1.1.24) equals

$$\lim_{\Lambda\to+\infty,\varepsilon\Lambda=o(1)} \text{Trace}\,(P^{\text{geom}}_{\rho,a} \circ A(\varphi))$$

and so (10.4.3.31) holds. □

Remark 12.1.1.4. We note that, if $\varphi(1) = 0$, then, by Theorem 10.4.3.4(4), $\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,x)$ is absolutely $\overline{\mu}$-integrable, but this does not make the operator $P_a^{\text{geom}} \circ A(\varphi)$ nuclear, since $(1 + |\log| \cdot |_F|^2)^{-s/2}\, k^{\text{spec}}_{\varphi,a,s}(x,y)$, while measurable over $C_F \times C_F$, is not square-integrable with respect to $((1 + |\log| \cdot |_F|^2)^{s/2}\,\overline{\mu})^{\otimes 2}$.

As such, the induced integral operator is not Hilbert-Schmidt, and hence it cannot be nuclear.

The regular representation of C_F on $W^{s,2}(C_F)$ induces operators $A(\varphi)$ that are not nuclear nor Hilbert-Schmidt, even in the special case $\varphi(1) = 0$. Responsible for this is the non-compactness of the idele class group.

This leads to the necessity of truncating C_F into growing compact subsets with non-empty interior, i.e. $C_F^{[1/\Lambda,\Lambda]}$, since the resulting kernel functions $1_{[1/\Lambda,\Lambda]}(x)\,k^{\text{geom}}_{\varphi,a,s}(x,y)$ are indeed the kernels of nuclear operators.

The aim is to use the truncated kernel functions $1_{[1/\Lambda,\Lambda]}(x)\,k^{\text{geom}}_{\varphi,a,s}(x,y)$ and their operator-theoretic interpretation, to compute the integral of $k^{\text{geom}}_{\varphi,a,s}(x,y)$ itself along the diagonal, by approximation.

We will be using the smoothened kernels $\rho(x)\,\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,y)$, with ρ a (Λ,ε)-cutoff in the sense of Definition 9.3.3.1, with a view towards the main trace calculation. Passage from $1_{[1/\Lambda,\Lambda]}(x)\,k^{\text{geom}}_{\varphi,a,s}(x,y)$ to $\rho(x)\,\widetilde{k}^{\text{geom}}_{\varphi,a,s}(x,y)$ is harmless for the purposes of computing the value of the integral of $\lim_{\Lambda\to+\infty} \text{Trace}\,(P_a^{\text{geom}} \circ A(\varphi))$, since

the respective smoothened kernels have masses along the diagonal which are asymptotically equal as we let $\rho \to 1$ (as a consequence of letting $\Lambda \to +\infty$ in the regime $\varepsilon\Lambda = o(1)$).

This is an example of an integral operator $(P_a^{\mathrm{geom}} \circ A(\varphi))$ whose integral kernel $(k^{\mathrm{geom}}_{\varphi,a,s}(x,y))$ can be made sense of along the diagonal (within $C_F \times C_F$), and is absolutely integrable there, without this being a sufficient condition for nuclearity.

For a simple example of how integrability along the diagonal of a piece-wise continuous integral kernel of an integral Hilbert-Schmidt operator on an L^2-space does not ensure nuclearity of the associated integral operator, we further refer the reader to [**Bris-b**, Example 3.2,p.1186].

12.2. Lebesgue Differentiation along the diagonal

In this section we analyze the Hilbert-Schmidt kernel of $P^{\mathrm{geom}}_{\Lambda,a} \circ A(\varphi)$ and its trace, as $\Lambda \to +\infty$, with the aim of reducing the computation of the limit value of the latter to separate simpler tasks, amounting to trace calculations for component kernels.

We recall that throughout this Chapter, for $(X, \mathcal{A}, \mu)$ a measure space, we will be assuming X is a Hausdorff locally compact second countable topological space, the σ-algebra $\mathcal{A}$ contains the Borel σ-algebra $\mathcal{B}(X)$, μ is a complete positive Radon measure on $\mathcal{A}$, and, for simplicity, that $\mathcal{A}$ is the initial σ-algebra containing $\mathcal{B}(X)$ and such that μ is complete on $\mathcal{A}$. In particular, for any $N \subset X$ such that $\mu(N) = 0$, N is measurable.

Recall that any Radon measure on such X is regular (see [**Fol**, Thm.7.8]).

We will be abbreviating $L^p(X,\mu)$ for $L^p(X, \mathcal{A}, \mu)$, for every real number $p \geq 1$.

Under these assumptions, for any element $\phi \in L^p(X,\mu)$, there exists a Borel measurable function $\phi_0 : X \to \mathbf{C}$ such that $\phi = \phi_0$ away from a subset N of X such that $\mu(N) = 0$.

Any element in $L^p(X,\mu)$ is, recall, an equivalence class of Lebesgue measurable functions on X, the equivalence relation being equality away from μ-null subsets of X.

In light of [**Fol**, Prop.7.9], $L^p(X,\mu)$ is also obtained as the norm-completion of $C_c(X)$ with respect to the seminormed topology induced by the L^p-seminorm (in fact a norm) on $C_c(X)$, i.e. $\|f\|_p := \left(\int_X |f(x)|^p \, \mathrm{d}\mu(x)\right)^{1/p}$.

Both perspectives on the elements of $L^p(X,\mu)$ will be used below.

In light of [**Fol**, Prop.7.1] (see also [**Bou-I**, Ch.III,§1,n.3,(4),INT III.7]; see also [**Bou-I**, Ch.III,§1,n.5,Thm.1,INT III.12]) and [**Fol**, Thm.7.2], under the assumptions being made on X, the approaches to measure theory and integration contained in [**Bou-I, Bou-II**] and from classical measure theory (see [**Fol**, Ch.1,§1.3] and [**Fol**, Ch.7], and the related literature referred to throughout this work) are equivalent.

12.2.1. Indentations. In this section we provide the instruments to construct a kernel function:

$$K^{\mathrm{geom}}_{\varphi,a,s} : \mathrm{GL}_1(\mathbf{A}_F) \times \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{C} \tag{12.2.1.1}$$

that "lifts" the integral kernel $k^{\mathrm{geom}}_{\varphi,a,s}$ of the continuous linear operator $A(\varphi)$ on $W_a^{s,2}\,\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$, for then deconstructing it into an S-local part and an error term, which we are both piece-wise going to interpret as operators (on other Hilbert spaces, each, respectively).

We begin by recording a construction from [**Bou-II**, Ch.VII,§2,n.4,Prop.8,INT VII.37], and we discuss some consequences and complements.

LEMMA 12.2.1.1. *Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H. We denote by $\pi : G \to G/H$ the continuous quotient homomorphism. There exists a function $u_0 : G \to \mathbf{C}$ satisfying the following conditions:*

(1) *u_0 is continuous, real-valued and nonnegative.*
(2) *Let K be an arbitrary compact subset of G. Then the intersection of the support of u_0 with $\pi^{-1}(\pi(K))$ is compact.*

(3) *For all $x \in G$, the integral:*

$$\int_H u_0(h^{-1}x)\,\mathrm{d}\mu_H(h) \tag{12.2.1.2}$$

converges, and upon denoting u_0^H its value function on G, we have that u_0^H is continuous on G, invariant under the continuous left-translation action by H on G, and identically 1 as a function on G/H.

For every continuous function $u \in C(G/H)$, we have that for all $x \in G$, the integral

$$\int_H (u_0 \cdot (u \circ \pi))(h^{-1}x)\,\mathrm{d}\mu_H(h) \tag{12.2.1.3}$$

converges, and its value is $u(\pi(x))$.

Proof. (1) : Since G is second countable and $H \subset G$ is closed, π is open and G/H is Hausdorff locally compact, and hence second countable. As such, it is paracompact. We choose a locally finite countable open cover $\{U_k\}_{k\geq 0}$ of G/H, and consider the induced open cover $\{\pi^{-1}U_k\}_{k\geq 0}$ on G, also locally finite. We choose a partition of unity $\{\rho_k\}_{k\geq 0}$ on G/H subordinate to $\{U_k\}_{k\geq 0}$, calling $S_k \subset U_k$ the compact support of ρ_k for each integer $k \geq 0$.

For each integer $k \geq 0$, there exists, by [**DE**, Lemma 1.5.1], a continuous compactly supported function ρ'_k on G whose support S'_k satisfies $S'_k \subset \pi^{-1}(U_k)$, and $S_k \subset \pi(S'_k)$ (this last containment follows from the proof of [**DE**, Lemma 1.5.1]).

The sum $f := \sum_{k\geq 0} \rho'_k$ converges pointwise on G to a continuous real-valued nonnegative function that is not identically zero on any orbit of the continuous left-translation action of H on G. Indeed, since $\{U_k\}_{k\geq 0}$ is locally finite, for each $x \in G$ there exists an open neighborhood U_x of x in G that intersects non-trivially at most a finite number of the U_k's, say $U_{k_1}, \ldots, U_{k_N}$, and hence for every $y \in U_x$ we have $\rho'_k(x) \neq 0$ only if $k \in \{k_1, \ldots, k_N\}$. Over U_x, the sum defining f is therefore finite.

We consider f^H the function on G defined, for each $x \in G$, by the integral:

$$f^H(x) := \int_H f(h^{-1}x)\,\mathrm{d}\mu_H(h). \tag{12.2.1.4}$$

Let K_0 be an arbitrary compact subset of G. The intersection of the support K of f with $\pi^{-1}(\pi(K_0))$ is contained in $K \cap \bigcup_{1\leq i\leq N} \pi^{-1}(\pi(S'_{k_i}))$ where $k_1, \ldots, k_N$ are the finitely many integers such that the compact subset $\pi(K_0)$ of G/H intersects $\pi(S'_k)$ non-trivially. Indeed, there exist only finitely many such integers, for $\pi(K_0)$ intersects U_k, and hence $\pi(S'_k) \subset U_k$, non-trivially for at most finitely many integers k.

Since $K \cap \bigcup_{1\leq i\leq N} \pi^{-1}(\pi(S'_{k_i})) \subset \bigcup_{1\leq i\leq N} S'_{k_i}$, a compact subset, we deduce that the intersection of the support of f with $\pi^{-1}(\pi(K_0))$ is compact, for the arbitrary compact subset $K_0 \subset G$. The assumptions of [**Bou-II**, Ch.VII,§2,n.1,Prop.1(a),(b),INT VII.27] (applied to $\chi = 1$) are therefore satisfied by f, and the integral defining f^H converges to a continuous function on G, invariant under the continuous left-translation action by H.

We define $u_0 := f/f^H$. The function u_0 satisfies (1), (2) and (3). Since, for any $u \in C(G/H)$, the continuous function $u \circ \pi$ is invariant under the continuous left-translation action of H on G, then the integral in (12.2.1.4) indeed converges to a continuous function on G by [**Bou-II**, Ch.VII,§2,n.1,Prop.1(a),(b),INT VII.27] (applied for $\chi = 1$), and its value is $u(\pi(x)) \int_H u_0(h^{-1}x)\,\mathrm{d}\mu_H(h) = u(\pi(x))$.

We further refer the reader to the proof of [**Bou-II**, Ch.VII,§2,n.4,Corollary,INT VII.38]. □

Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H. We denote by $\pi : G \to G/H$ the continuous quotient homomorphism.

We denote by:

$$\Phi_H : C_c(G) \to C_c(G/H) \tag{12.2.1.5}$$

the linear operator sending $u \in C_c(G)$ to $\Phi_H(u)(x) := \int_H u(h^{-1}x)\,\mathrm{d}\mu_H(h)$, a continuous function of $x \in G$, invariant under the continuous left-translation action by H on G, inducing a unique continuous function denoted again by $\Phi_H(u)$ on G/H, whose support is, therefore, compact (being contained in the image under π of $\pi^{-1}(\pi(K))$, for K the support of u, and such image is $\pi(K)$, compact).

COROLLARY 12.2.1.2. *Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H. We denote by $\pi : G \to G/H$ the continuous quotient homomorphism.*

Let λ be a positive Radon measure on G/H.

(1) *There exists a unique positive measure λ' on G satisfying the following normalization condition:*

$$\int_{G/H} \Phi_H(u)(x)\,\mathrm{d}\lambda(x) = \int_G u(g)\,\mathrm{d}\lambda'(g) \tag{12.2.1.6}$$

for all $u \in C_c(G)$. λ' is H-invariant.

(2) *The linear operator Φ_H from* (12.2.1.5) *extends by continuity to a continuous surjective operator:*

$$\Phi_H : L^1(G, \lambda') \to L^1(G/H, \lambda). \tag{12.2.1.7}$$

The assignment

$$\Psi_{u_0} : u \mapsto u_0 \cdot (u \circ \pi) \tag{12.2.1.8}$$

for a choice of function u_0 as provided by Lemma 12.2.1.1, is an isometric right-inverse $\Psi_{u_0} : L^1(G/H, \lambda) \to L^1(G, \lambda')$ to Φ_H.

PROOF. (1) : This is [**Bou-II**, Ch.VII,§2,n.2,Prop.4,INT VII.31], applied to $X = G$, $\lambda' = \lambda^\sharp$, in which case the measure λ' is indeed H-invariant. [**Bou-II**, Ch.VII,§2,n.2,Prop.4,INT VII.31] provides a measure λ', i.e. a continuous linear form on $C_c(G)$, where $C_c(G)$ is equipped with the locally convex inductive limit topology induced by the inclusions $C(G, K) \subset C_c(G)$, as $K \subset G$ ranges over all non-empty compact subsets, and $C(G, K)$ is the linear subspace of continuous functions on G compactly supported within K, endowed with the topology of uniform convergence.

λ' is a positive functional on $C_c(G)$ (i.e. forgetting its continuity), and by [**Fol**, Prop.7.1] its positivity and linearity only imply its continuity on $C_c(G)$ with respect to its locally convex inductive limit topology, by [**Bou-I**, Ch.III,§1,n.3,(4),INT III.7] (see also [**Bou-I**, Ch.III,§1,n.5,Thm.1,INT III.12]). By [**Fol**, Thm.7.2], λ' corresponds to a unique Radon measure on G in the sense of classical measure theory (see [**Fol**, Ch.1,§1.3] and [**Fol**, Ch.7]), which we again denote by λ'.

By [**Bou-I**, Ch.III,§1,n.8,INT III.16], it is enough to show that there exists a real number $M \geq 0$ such that for any $u \in C_c(G)$, we have $|\lambda'(u)| \leq M\|u\|_u$.

(2) : If $u \in C_c(G)$, then so is $|u|$, and since $|\Phi_H(u)| \leq \Phi_H(|u|)$ as continuous functions on G/H, we deduce from (12.2.1.6) that Φ_H is continuous with respect to the normed topology on $C_c(G)$ induced by the $L^1(G, \lambda')$-norm on the source, and the normed topology on $C_c(G/H)$ induced by the $L^1(G/H, \lambda)$-norm on the target. This provides the continuous extension of Φ_H in (12.2.1.7).

The assignment of Ψ_{u_0} on compactly supported continuous functions on G/H is isometric with respect to the L^1-norm topologies on source and target by (12.2.1.6) and Lemma 12.2.1.1. It therefore extends by continuity to an isometric embedding $L^1(G/H, \lambda) \to L^1(G, \lambda')$, since $C_c(G/H) \subset L^1(G/H, \lambda)$ is norm-dense by [**Fol**, Prop.7.9]. The right-inverse condition holds because it holds on compactly supported continuous functions by Lemma 12.2.1.1. □

DEFINITION 12.2.1.3. We let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H. We define an *H-indentation on G* to be a continuous real-valued function $u_0 : G \to \mathbf{R}_{\geq 0}$ satisfying the conditions in Lemma 12.2.1.1.

For any positive Radon measure λ on G/H, we denote by $\Psi_{u_0} : L^1(G/H, \lambda) \to L^1(G, \lambda')$ the continuous right section to Φ_H provided by Corollary 12.2.1.2.

We recall the following general situation and related terminology:

REMARK 12.2.1.4. We let X be a Hausdorff locally compact topological space, G a Hausdorff locally compact abelian group. Assume consider a continuous action:

$$\theta : G \times X \to X \tag{12.2.1.9}$$

We assume that the action θ of G on X is *proper*, by which we mean, following [**Bou-GT**, Ch.III,§4,n.1,Def.1,p.250], that the continuous map:

$$G \times X \to X \times X$$

sending (g, x) to $(x, \theta(g, x))$, is proper (see [**Bou-GT**, Ch.I,§10,n.1,Def.1,p.97]).

By [**Bou-GT**, Ch.III,§4,n.2,Prop.3,p.253], the quotient space X/G is Hausdorff.

If the action θ of G on X is, in addition, free, then, by [**Bou-GT**, Ch.III,§4,n.4,Corollary,p.257] to [**Bou-GT**, Ch.III,§4,n.4,Prop.8,256], the quotient map:

$$\pi : X \to X/G \tag{12.2.1.10}$$

is open.

In this case, if X is second countable, so is X/G.

We assume two Radon measures given, one, μ_X, on X, another, a choice of Haar measure μ_G on G. We assume μ_X is G-invariant, and denote by $\overline{\mu}$ the unique Radon measure on X/G induced by these choices, so that the following integral formula holds:

$$\int_X \phi(x)\, \mathrm{d}\mu_X(x) = \int_{X/G} \left(\int_G \phi(\theta(g, x))\, \mathrm{d}\mu_G(g) \right) \mathrm{d}\overline{\mu}(\pi(x)). \tag{12.2.1.11}$$

In the special case X is itself a Hausdorff locally compact abelian group, and G is a closed subgroup, with θ the action by continuous left-translation, the existence and uniqueness of $\overline{\mu}$, upon choosing μ_X to be a Haar measure on X, is by Theorem D.1.2.1. For the general case, we refer the reader to [**Bou-II**, Ch.VII,§2,n.2,Prop.4,INT VII.31].

Assume that there exists a non-empty connected subset $D \subset X$ with non-empty interior, such that

$$\bigcup_{g \in G} \theta(g, D) = X, \quad D \cap \theta(g, D) = \varnothing \tag{12.2.1.12}$$

the last condition being assumed to hold for every $g \in G$ with $g \neq 1$.

Assume that for every $g \in G$, with $g \neq 1$, the intersection:

$$\overline{D} \cap \theta(g, \overline{D}) \tag{12.2.1.13}$$

has $\overline{\mu}$-measure zero.

We will be referring to such D as a *fundamental domain* for the continuous G-action on X.

The above general situation and the respective terminology will be specialized shortly to the cases $X = \mathrm{GL}_1(\mathbf{A}_F)$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and $G = F^\times$, respectively $G = \mathcal{O}_{F,S}^\times$.

12.2.2. Regular points on the diagonal and indentations. We need the following "two-variable" complements around H-indentations on G.

SITUATION 12.2.2.1. Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, μ_H a choice of Haar measure on H.

Assume H is discrete, and μ_H is normalized so as to have $\mu_H(\{1\}) = 1$ (i.e. μ_H is the counting measure, by [**DE**, Prop.1.4.4(c)]).

Let λ be a positive Radon measure on G/H, and λ' the unique positive Radon measure on G provided by Corollary 12.2.1.2(1).

We denote by $\pi : G \to G/H$ the continuous quotient homomorphism. Assume there exists a fundamental domain D for the continuous left-translation action by H on G.

Assume that $\{U_k\}_{k\geq 0}$ is a D'-sequence for G such that $U_k \subset D$, and U_k is a relatively compact open symmetric unit-neighborhood for all integers $k \geq 0$. Then, upon defining $\overline{U}_k := \pi(U_k)$ for each integer $k \geq 0$, $\{\overline{U}_k\}_{k\geq 0}$ is a D'-sequence for G/H.

We define smoothenings over $G \times G$ and $G/H \times G/H$ using the D'-sequences $\{U_k \times U_k\}_{k\geq 0}$ for $G \times G$ and $\{\overline{U}_k \times \overline{U}_k\}_{k\geq 0}$ for $G/H \times G/H$.

Let λ be a positive Radon measure on G/H, and λ' the unique positive Radon measure on G provided by Corollary 12.2.1.2(1).

PROPOSITION 12.2.2.2. *Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a discrete subgroup. Assume μ_G is a choice of Haar measure on G, μ_H is the counting measure on H, and $\mu_{G/H}$ is the unique Haar measure on G/H determined by μ_H and μ_G as per Theorem D.1.2.1.*

Assume that G, H satisfy the conditions in Situation 12.2.2.1.

Let u_0 be an H-indentation on G. Then $u_0 \otimes u_0 \in C(G) \otimes C(G) \subset C(G \times G)$, a continuous function on $G \times G$, satisfies the following properties:

(1) *The integral:*

$$\int_H u_0(h^{-1}x)u_0(h^{-1}y)\,\mathrm{d}\mu_H(h) \tag{12.2.2.1}$$

converges pointwise everywhere on $G \times G$. The limit function $u_0^{\Delta}(x, y)$ is defined everywhere, continuous on $G \times G$, real-valued and nonnegative, and invariant under the continuous diagonal action of H on $G \times G$.

(2) *For all $\mu_{G/H}$-measurable, locally $\mu_{G/H}$-integrable functions $u : G/H \times G/H \to \mathbf{C}$, the assignment:*

$$\Psi_{u_0}^{\Delta} : u \mapsto u_0^{\Delta} \cdot (u \circ (\pi \times \pi)) \tag{12.2.2.2}$$

is valued in measurable complex-valued functions over $G \times G$. If $\mu_{G/H}$-almost every point $(x, x) \in G/H \times G/H$ is regular for u, then $\widetilde{u}(x, x)$ is $\mu_{G/H}$-measurable, defined $\mu_{G/H}$-almost everywhere. If it is absolutely $\mu_{G/H}$-integrable as a function on G/H, then the functions $\Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x, x)$ and $\Psi_{u_0}^{\Delta}(\widetilde{u})(x, h^{-1}x)$, defined for all $h \in H$, μ_G-almost all $x \in G$, and $(\mu_H \otimes \mu_G)$-measurable as functions of $(x, h) \in G \times H$, satisfy the following condition: the following two-fold integrals

$$\int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x, x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x), \quad \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x, x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{12.2.2.3}$$

are both absolutely convergent, and the following identities hold:

$$\int_{G/H} \widetilde{u}(x, x)\,\mathrm{d}\mu_{G/H}(x) = \int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x, x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x) \tag{12.2.2.4}$$

$$= \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(h^{-1}x, x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{12.2.2.5}$$

$$= \int_H \int_D \Psi_{u_0}^{\Delta}(\widetilde{u})(x, h^{-1}x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{12.2.2.6}$$

$$= \int_D \int_H \Psi_{u_0}^{\Delta}(\widetilde{u})(x, h^{-1}x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x). \tag{12.2.2.7}$$

(3) *Let $\rho \in C_c(G/H \times G/H)$. Then:*

$$\Psi_{u_0}^{\Delta}(\rho \cdot u)(x, y) = (\rho \circ (\pi \times \pi)) \cdot \Psi_{u_0}^{\Delta}(u)(x, y) \tag{12.2.2.8}$$

for $(\mu_G \otimes \mu_G)$-almost every $(x, y) \in G \times G$.

(4) *We have, for $(\mu_G \otimes \mu_G)$-almost all $(x,y) \in G \times G$:*

$$\Psi^{\Delta}_{u_0}(\widetilde{u})(x,y) = \lim_{k \to +\infty} \frac{1}{\mu_G(U_k)^2} \int_{xU_k \times yU_k} \Psi^{\Delta}_{u_0}(u)(y_1,y_2)\, \mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2). \tag{12.2.2.9}$$

The point $(x,y) \in G \times G$ is regular for $\Psi^{\Delta}_{u_0}(u)$ if and only if $(\pi(x),\pi(y)) \in G/H \times G/H$ is regular for u.

REMARK 12.2.2.3. For a complex-valued function $f : X \to \mathbf{C}$ on a Hausdorff locally compact second countable topological space X, and a Radon measure μ on X, Lebesgue measurability for f (upon endowing X with the completion of the Borel σ-algebra, i.e. with the σ-algebra of μ^*-measurable sets, promoting μ to a complete measure; see [**Fol**, Thm.1.11] and [**Bou-I**, Ch.IV,§5,n.1,Cor.1,INT IV.60]) and μ-measurability in the sense of [**Bou-I**, Ch.IV,§5,n.1,Def.2,INT IV.60], are equivalent concepts. μ-measurability of a map $f : X \to T$ for T a Hausdorff locally compact topological space, is defined in [**Bou-I**, Ch.IV,§5,n.1,Def.1,INT IV.59], and will be used only for f continuous, surjective and open, in which case it holds by [**Bou-I**, Ch.IV,§5,n.2,Cor.4,INT IV.62].

PROOF. (1) : Since u_0 is an H-indentation on G, then, upon calling S the support of u_0 in G, for any compact subset $K \subset G$ we have, by Lemma 12.2.1.1(2), that the intersection $S \cap \pi^{-1}(\pi(K))$ is compact.

The function $u_1(x,y) := u_0(x)u_0(y)$ on $G \times G$ is real valued, nonnegative, and continuous. Moreover, its support is contained in $S \times S \subset G \times G$. Let K be a compact subset of $G \times G$. There exists a compact subset $K' \subset G$ such that $K \subset K' \times K'$. Indeed, we define K_1, respectively K_2, to be the image of K in G under the first projection, respectively the second projection. Then $K \subset K_1 \times K_2$, and upon defining $K' := K_1 \cup K_2$, we have $K \subset K' \times K'$.

The intersection:

$$(S \times S) \cap \{(x,y) \in G \times G : (hx,hy) \in K \text{ for some } h \in H\} = (S \times S) \cap \bigcup_{h \in H} (h,h).K \tag{12.2.2.10}$$

is therefore compact, since the rightmost side in (12.2.2.10) is contained within:

$$(S \times S) \cap (\pi^{-1}(\pi(K')) \times \pi^{-1}(\pi(K'))) \subset (S \cap \pi^{-1}(\pi(K'))) \times (S \cap \pi^{-1}(\pi(K'))) \tag{12.2.2.11}$$

and since $S \cap \pi^{-1}(\pi(K'))$ is compact, so is the intersection in (12.2.2.10).

The integral in (12.2.2.1) therefore converges by [**Bou-II**, Ch.VII,§2,n.1,Prop.1(a),INT VII.27] for every $(x,y) \in G \times G$, defining the function $u_0^{\Delta}(x,y)$ everywhere over $G \times G$. By [**Bou-II**, Ch.VII,§2,n.1,Prop.1(b),INT VII.27], u_0^{Δ} is continuous over $G \times G$ and invariant under the continuous diagonal action of H on $G \times G$. It is real valued and nonnegative, since u_0 is.

(2) : For every $(\mu_{G/H} \otimes \mu_{G/H})$-measurable complex-valued function $u : G/H \times G/H$, we have that the composition $u \circ (\pi \times \pi)$ is also $(\mu_G \otimes \mu_G)$measurable, as a function on $G \times G$, by [**Bou-II**, Ch.VII,§2,n.3,Prop.6(b),INT VII.34]. We note that u may not be everywhere defined over $G/H \times G/H$.

By [**Bou-II**, Ch.VII,§2,n.3,Prop.6(a),INT VII.33], the pre-image under $\pi \times \pi$ of a $(\mu_{G/H} \otimes \mu_{G/H})$-negligible subset is $(\mu_G \otimes \mu_G)$-negligible, and hence $u \circ (\pi \times \pi)$ is defined $(\mu_G \otimes \mu_G)$-almost everywhere.

By the same token, since $\widetilde{u}(x,x)$ is $\mu_{G/H}$-measurable as a function of $x \in G/H$, which is the case by Theorem A.5.4.1(2), being $\{A_{\overline{U}_k \times \overline{U}_k}(u)(x,x)\}_{k \geq 0}$ a sequence of (uniformly) continuous functions on $G/H \times G/H$ by Theorem A.5.4.1(1), then $u(\pi(x),\pi(x))$ is μ_G-measurable as a function on G.

Indeed, since G is Hausdorff and $H \subset G$ is a closed subgroup, the diagonal maps $\delta_G : G \to G \times G$ and $\delta_{G/H} : G/H \to G/H \times G/H$ are homeomorphisms onto their closed images, and since $\widetilde{u}(x,x) = \widetilde{u} \circ \delta_{G/H}$ is $\mu_{G/H}$-measurable on G/H, its pre-composition with $\pi : G \to G/H$ is μ_G-measurable on G by [**Bou-II**, Ch.VII,§2,n.3,Prop.6(b),INT VII.34].

Pre-composition of $\widetilde{u} \circ \delta_{G/H} \circ \pi$ with δ_G^{-1} gives, set-theoretically at every point $(x',x') \in G \times G$ such that $(\pi(x'),\pi(x'))$ is a point of definition of $\widetilde{u}$, i.e. a regular point for u on $G/H \times G/H$, $\widetilde{u} \circ (\pi \times \pi)$, and this latter function is $(\delta_G)_* \mu_G$-measurable since δ_G is a homeomorphism onto its image. Note that the function

$\widetilde{u} \circ \delta_{G/H} \circ \pi$ is defined μ_G-almost everywhere by [**Bou-II**, Ch.VII,§2,No.3,Prop.6(a),INT VII.33] (see also [**Bou-I**, Ch.V,§6,No.6,Cor.1(a),INT V.72]), i.e. the pre-image under π of the set of points of G/H at which $\widetilde{u} \circ \delta_{G/H}$ is undefined, a $\mu_{G/H}$-null set, is locally μ_G-negligible in G, and since G is second countable, it is μ_G-negligible (see Lemma A.5.3.3).

We will be omitting mention of δ_G and $\delta_{G/H}$ from the notation.

Since u_0^Δ is continuous as a function on $G \times G$, it is $(\mu_G \otimes \mu_G)$-measurable and $u_0^\Delta(h^{-1}x, x)$ and $u_0^\Delta(x, h^{-1}x)$ are both $(\mu_G \otimes \mu_H)$-measurable as functions of $(x, h) \in D \times H$. Finite products of measurable functions are measurable, since finite intersections of measurable subsets are measurable.

The measurability assertions in (2) therefore all hold, and all the integrals in (12.2.2.3) and (12.2.2.4) make sense, allowed to possibly be divergent.

We now prove that (12.2.2.3) holds.

We have:

(12.2.2.12)
$$\int_D \int_H \Psi_{u_0}^\Delta(\widetilde{u})(h^{-1}x, x)\, \mathrm{d}\mu_H(h)\, \mathrm{d}\mu_G(x) = \int_D \int_H u_0^\Delta(h^{-1}x, x) \cdot \widetilde{u}(\pi(x), \pi(x))\, \mathrm{d}\mu_H(h)\, \mathrm{d}\mu_G(x)$$

(12.2.2.13)
$$= \int_D \int_{h \in H} \int_{h' \in H} u_0(h^{-1}(h')^{-1}x) u_0((h')^{-1}x) \widetilde{u}(\pi(x), \pi(x))\, \mathrm{d}\mu_H(h')\, \mathrm{d}\mu_H(h)\, \mathrm{d}\mu_G(x)$$

(12.2.2.14)
$$= \int_D \int_{(h,h') \in H \times H} u_0(h^{-1}(h')^{-1}x) u_0((h')^{-1}x) \widetilde{u}(\pi(x), \pi(x))\, \mathrm{d}\mu_{H \times H}(h, h')\, \mathrm{d}\mu_G(x)$$

where $\mu_{H \times H} := \mu_H \otimes \mu_H$ (note that we are choosing $\mu_{H \times H}$ to be $\mu_H \otimes \mu_H$, likewise as we are choosing $\mu_G \otimes \mu_G$ to be the Haar measure on $G \times G$, so as to also have $\mu_{G/H} \otimes \mu_{G/H}$ as the Haar measure on G/H uniquely determined as per Theorem D.1.2.1. We require the latter two be the Haar measures on $G/H \times G/H$, respectively $G \times G$, so as to use the D'-sequences $\{\overline{U}_k \times \overline{U}_k\}_{k \geq 0}$, respectively $\{U_k \times U_k\}_{k \geq 0}$, to define smoothenings of locally $(\mu_{G/H} \otimes \mu_{G/H})$-integrable functions on $G/H \times G/H$, respectively locally $(\mu_G \otimes \mu_G)$-integrable functions on $G \times G$).

The innermost integral in (12.2.2.14) reads:

(12.2.2.15)
$$\widetilde{u}(\pi(x), \pi(x)) \int_{(h,h') \in H \times H} u_0(h^{-1}(h')^{-1}x) u_0((h')^{-1}x)\, \mathrm{d}\mu_{H \times H}(h, h')$$

which turns, under the "shearing" change of variables $(h, h') \mapsto (h(h')^{-1}, h')$, into:

(12.2.2.16)
$$\widetilde{u}(\pi(x), \pi(x)) \int_{(h,h') \in H \times H} u_0(h^{-1}x) u_0((h')^{-1}x)\, \mathrm{d}\mu_{H \times H}(h(h')^{-1}, h') = \widetilde{u}(\pi(x), \pi(x))$$

where we are using H-invariance of the Haar measure μ_H, and the identity $\int_H u_0(h^{-1}x)\, \mathrm{d}\mu_H(h) = 1$, which holds since u_0 is an H-indentation on G (see Lemma 12.2.1.1(3)). The identity in (12.2.2.16) holds for any $x \in G$ at which $\widetilde{u}(\pi(x), \pi(x))$ is defined, since u_0 is continuous. It follows that it holds μ_G-almost everywhere on G.

We therefore deduce that the last integral in (12.2.2.14) equals:

(12.2.2.17)
$$\int_D \widetilde{u}(\pi(x), \pi(x))\, \mathrm{d}\mu_G(x).$$

By [**Bou-II**, Ch.VII,§2,n.3,Prop.5(c),INT VII.33], the integral in (12.2.2.17) equals $\int_{G/H} \widetilde{u}(x, x)\, \mathrm{d}\mu_{G/H}(x)$.

Indeed, we call, using the notation of loc.cit., $\mathbf{f} := 1_D(x) \cdot |\widetilde{u}(\pi(x), \pi(x))|$, and observe that since D is a fundamental domain, $\mathbf{f}^\flat = |\widetilde{u}|$, and formula [**Bou-II**, Ch.VII,§2,n.3,Prop.5(c),(12),INT VII.33] (which applies by [**Bou-II**, Ch.VII,§2,n.3,Prop.5(d),(12),INT VII.33]) implies that the following identity:

(12.2.2.18)
$$\int_D \widetilde{u}(\pi(x), \pi(x))\, \mathrm{d}\mu_G(x) = \int_{G/H} \widetilde{u}(x, x)\, \mathrm{d}\mu_{G/H}(x)$$

holds. We deduce from (12.2.2.18) that the first identity in (12.2.2.4) holds.

We observe that we have, for every such u and for all $h \in H$ and μ_G-almost all $x \in G$:

(12.2.2.19)
$$|\Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)| = |u_0^{\Delta}(h^{-1}x,x)\cdot(\widetilde{u}(\pi(h^{-1}x),\pi(x))| = u_0^{\Delta}(h^{-1}x,x)\cdot|\widetilde{u}(\pi(h^{-1}x),\pi(x))| = \Psi^{\Delta}_{u_0}(|\widetilde{u}|)(h^{-1}x,x)$$

since u_0, and hence both $u_0 \otimes u_0$ and u_0^{Δ}, are real-valued and nonnegative (the latter because μ_H is a positive Radon measure). By assumption, $|\widetilde{u}|$ is defined $\mu_{G/H}$-almost everywhere on the diagonal in $G/H \times G/H$, $\mu_{G/H}$-measurable and $\mu_{G/H}$-integrable over the diagonal in $G/H \times G/H$.

By the first identity in (12.2.2.4), we deduce that we have:

(12.2.2.20)
$$\int_{G/H} |\widetilde{u}(x,x)|\,\mathrm{d}\mu_{G/H}(x) = \int_D \int_H \Psi^{\Delta}_{u_0}(|\widetilde{u}|)(h^{-1}x,x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x) = \int_D \int_H |\Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)|\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x)$$

and hence the function $\Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)$ of $(x,h) \in D \times H$ is $\mu_H \otimes \mu_G$ absolutely integrable over $D \times H$ by Tonelli's Theorem (see [**Fol**, Thm.2.37(a)]) applied to $f := |\Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)|$. By [**Fol**, Thm.7.27], which applies since G is second countable, and hence so are D and H, and μ_G and μ_H are Radon measures, hence σ-finite, we deduce that the identity:

$$\int_D \int_H \Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_G(x) = \int_H \int_D \Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(h) \tag{12.2.2.21}$$

holds.

By invariance of u_0^{Δ} with respect to the diagonal action of H on $G \times G$, we deduce that we have $\Psi^{\Delta}_{u_0}(\widetilde{u})(h^{-1}x,x) = \Psi^{\Delta}_{u_0}(\widetilde{u})(x,h^{-1}x)$ for all $h \in H$ and μ_G-almost all $x \in G$, and hence the remaining identities in (12.2.2.4) follow from (12.2.2.21).

(3) : Let $\rho \in C_c(G/H \times G/H)$. Then $\rho \circ (\pi \times \pi)$ is a continuous function on $G \times G$, invariant under the continuous left-translation action of $H \times H$ on $G \times G$. For $u : G/H \times G/H \to \mathbf{C}$ a $(\mu_{G/H} \otimes \mu_{G/H})$-measurable and locally $(\mu_{G/H} \otimes \mu_{G/H})$-integrable function, $\rho \cdot u$ is also $(\mu_{G/H} \otimes \mu_{G/H})$-measurable and locally $(\mu_{G/H} \otimes \mu_{G/H})$-integrable function. We have:

$$\Psi^{\Delta}_{u_0}(\rho \cdot u) = u_0^{\Delta} \cdot ((\rho \cdot u) \circ (\pi \times \pi)) = (\rho \circ (\pi \times \pi)) \cdot \Psi^{\Delta}_{u_0}(u)$$

$(\mu_G \times \mu_G)$-almost everywhere.

(4) : Since u_0^{Δ} is continuous, then for all $(x,y) \in G \times G$ which is regular for $u \circ (\pi \times \pi)$ (see Corollary A.5.4.3):

$$\lim_{k \to +\infty} \frac{1}{\mu_G(U_k)^2} \int_{xU_k \times yU_k} u_0^{\Delta}(y_1,y_2)\cdot(u(\pi(y_1),\pi(y_2)))\,\mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2) \tag{12.2.2.22}$$

exists and is finite, and by Corollary A.5.4.3 we have that the value of the limit in (12.2.2.22) is:

$$u_0^{\Delta}(x,y) \lim_{k \to +\infty} \frac{1}{\mu_G(U_k)^2} \int_{xU_k \times yU_k} u(\pi(y_1),\pi(y_2))\,\mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2) \tag{12.2.2.23}$$

and the integral in (12.2.2.23) reads:

(12.2.2.24)
$$\int_{xU_k \times yU_k} u(\pi(y_1),\pi(y_2))\,\mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2) = \int_{\pi(x)\overline{U}_k \times \pi(y)\overline{U}_k} u(\pi(y_1),\pi(y_2))\,\mathrm{d}(\mu_{G/H} \otimes \mu_{G/H})(\pi(y_1),\pi(y_2))$$

and since $U_k \subset D$ for every $k \geq 0$ by assumption, another application of [**Bou-II**, Ch.VII,§2,n.3,Prop.5(c),(12),INT VII.33] gives $\mu_G(U_k) = \mu_{G/H}(\overline{U}_k)$, and hence:

$$\lim_{k \to +\infty} \frac{1}{\mu_G(U_k)^2} \int_{xU_k \times yU_k} u(\pi(y_1),\pi(y_2))\,\mathrm{d}(\mu_G \otimes \mu_G)(y_1,y_2) = \widetilde{u}(\pi(x),\pi(y)). \tag{12.2.2.25}$$

We deduce that $(x,y) \in G \times G$ is regular for $\Psi^{\Delta}_{u_0}(u)$ if and only if $(\pi(x),\pi(y))$ is regular for u on $G/H \times G/H$. Since the pre-image under $\pi \times \pi$ of a $(\mu_{G/H} \otimes \mu_{G/H})$-null set is a $(\mu_G \otimes \mu_G)$-null set, since it is locally $(\mu_G \otimes \mu_G)$-negligible by [**Bou-II**, Ch.VII,§2,No.3,Prop.6(a),INT VII.33] (see also [**Bou-I**, Ch.V,§6,No.6,Cor.1(a),INT V.72]), and hence negligible by Lemma A.5.3.3, since G is second countable, we deduce that (12.2.2.9) holds $(\mu_G \otimes \mu_G)$-almost everywhere. □

PROPOSITION 12.2.2.4. *Let G be a second countable Hausdorff locally compact abelian group, $H \subset G$ a discrete subgroup, μ_H the counting measure on H, μ_G a choice of Haar measure on G, $\mu_{G/H}$ the unique Haar measure on G/H determined by the choices of μ_H and μ_G as per Theorem D.1.2.1.*

Assume that G, H satisfy the conditions in Situation 12.2.2.1.

Let $G_k \subset G$ be an exhaustive $\mathbf{N}$-indexed filtration of G by open subgroups (i.e. $G_k \subset G$ is an open, hence closed, subgroup, such that $G_k \subset G_{k+1}$ for every integer $k \geq 0$, and satisfying $\bigcup_{k\geq 0} G_k = G$).

For each integer $k \geq 0$, we define $H_k := H \cap G_k$ the induced filtration on H.

Assume:

(1) *there exists a fundamental domain D for the continuous action of H on G.*
(2) *$D \subset G_0$.*
(3) *D is a fundamental domain within G_k of the continuous action of H_k on G_k.*
(4) *For every integer $k \geq 0$, G_{k+1}/G_k is free of rank 1.*

We denote by α_k the indicator function of G_k in G, and by β_k the indicator function of the clopen subset $G - G_k$.

Let $u : G/H \times G/H \to \mathbf{C}$ be a measurable function, locally $(\mu_{G/H} \otimes \mu_{G/H})$-integrable, such that $\mu_{G/H}$-almost every point $(x,x) \in G/H \times G/H$ is regular for u, and $\widetilde{u}(x,x)$ is absolutely $\mu_{G/H}$-integrable.

Then, there exists an H-indentation u_0 on G such that $\alpha_k \cdot u_0$ is an H_k-indentation on G_k for every $k \geq 0$, and, writing:

$$u_k(x,y) := \Psi^{\Delta}_{u_0}(\widetilde{u})(x,y)\beta_k(y), (x,y) \in G \times G \tag{12.2.2.26}$$

we have:

$$\lim_{k\to+\infty} \int_D \int_H u_k(x, h^{-1}x)\, \mathrm{d}\mu_H(h)\, \mathrm{d}\mu_G(x) = 0. \tag{12.2.2.27}$$

PROOF. Note that since $G_k \subset G$ is an open subgroup, it is closed by [**DE**, Lemma 1.1.7(c)]. In particular, G/G_k, and hence G_{k+1}/G_k, is a discrete group with respect to the quotient topology.

We note that the induced group homomorphism:

$$H_{k+1}/H_k \to G_{k+1}/G_k \tag{12.2.2.28}$$

is continuous upon endowing H_{k+1}/H_k with the quotient topology. Since $H \subset G$ is closed, $H_k \subset G_k$ is closed, and hence so is $H_k \subset H_{k+1}$. Since $G_k \subset G_{k+1}$ is open, so is $H_k \subset H_{k+1}$, since each H_k is endowed with the subspace topology inherited from G.

In particular, H_{k+1}/H_k is also discrete, and since $H_{k+1} \cap G_k = H_k$, then H_{k+1}/H_k free, being a subgroup of a free abelian group of finite rank. It is therefore either trivial or of rank 1, in which case the homomorphism in (12.2.2.28) is an isomorphism.

Let $g \in G_{k+1}$ be of the form $h \cdot x$ for unique $x \in D$ and $h \in H_{k+1}$. Since $D \subset G_k$, the homomorphism in (12.2.2.28) is surjective, and hence an isomorphism.

We now consider the continuous homomorphisms:

$$G_k/H_k \to G/H \tag{12.2.2.29}$$

for each integer $k \geq 0$, all injective since $H \cap G_k = H_k$, and open since $G_k \subset G$ is an open subgroup, and hence $G_k \cdot H \subset G$ is open by [**DE**, Lemma 1.1.3(e)].

The cokernel of the injective continuous open group homomorphism from (12.2.2.29) is discrete when endowed with the quotient topology, and it trivial, since $D \subset G_k$, and hence $G = D \cdot H \subset G_k \cdot H$.

The quotient group:

$$G/G_k \tag{12.2.2.30}$$

is discrete and free. Indeed, it is the direct limit topological group (see [**DE**, Ch.1,§1.8,p.29]; see also [**DE**, Prop.1.8.6]) of the open and closed subgroups G_m/G_k for $m \geq k$: open and closed since the pre-image of G_m/G_k in G is G_m, since $G_k \subset G_m$ for every integer $m \geq k$, and $G_m \subset G$ is open and closed. The transition maps $G_m/G_k \to G_{m+1}/G_k$ are injective, their source is free of rank $m-k$, and their cokernel is free of rank 1, and hence G_{m+1}/G_k splits as the product of G_m/G_k by the free group G_{m+1}/G_m of rank 1.

Upon therefore choosing a $\mathbf{Z}$-basis $\{\xi_k, \ldots, \xi_{m+k}\}$ for G_m/G_k, and an element ξ_{m+k+1} of G_{m+1} that is a $\mathbf{Z}$-basis for G_{m+1}/G_m, then the image of $\{\xi_k, \ldots, \xi_{m+k+1}\}$ is a $\mathbf{Z}$-basis for G_{m+1}/G_k, and hence $\bigcup_{m\geq k}\{\xi_k, \ldots, \xi_{m+k}\}$ is a $\mathbf{Z}$-basis for G/G_k.

Let $u_1 : G_0 \to \mathbf{C}$ be an H_0-indentation on G_0.

We denote by ε_k the function on $\xi_k^{\mathbf{Z}}$ defined to be 1 on ξ_k^0 and zero elsewhere, we define $u_0 := u_1 \otimes \bigotimes_{k\geq 1} \varepsilon_k$. u_0 is a complex-valued function on G, continuous, since it is the tensor product of a continuous function on G_0 and a continuous function on $\prod_{k\geq 1} \xi_k^{\mathbf{Z}}$, pulled back to G along the isomorphism of Hausdorff locally compact abelian groups $G \simeq G_0 \times \prod_{k\geq 1} \xi_k^{\mathbf{Z}}$.

G_0 as an open subgroup of G is isomorphic to $G_0 \times 1$ under such isomorphism, and hence u_0 is realized as the extension by zero of u_1 to G.

u_0 is real-valued and nonnegative, since u_1 is. Let $K \subset G$ be an arbitrary compact subset. Then, upon denoting by $\pi : G \to G/H$ the continuous quotient homomorphism, and by S the support of u_0, we deduce that the intersection $S \cap \pi^{-1}(\pi(K))$ is compact, since it is contained in the intersection $S \cap \pi^{-1}(\pi(K \cap G_0))$, and $K \cap G_0$ is compact, being $G_0 \subset G$ closed.

Moreover, we have, for every $x \in G$:

$$\int_H u_0(h^{-1}x)\,\mathrm{d}\mu_H(h) = \int_{H/H_0} \bigotimes_{k\geq 1} \varepsilon_k((h')^{-1}\overline{x}) \int_{H_0} u_1(h^{-1}x)\,\mathrm{d}\mu_{H_0}(h)\,\mathrm{d}\mu_{H/H_0}(h') \tag{12.2.2.31}$$

where $\overline{x}$ is the image of x in G/G_0. We have:

$$\int_{H/H_0} \bigotimes_{k\geq 1} \varepsilon_k((h')^{-1}\overline{x})\,\mathrm{d}\mu_{H/H_0}(h') = 1, \quad \int_{H_0} u_1(h^{-1}x)\,\mathrm{d}\mu_{H_0}(h) = 1$$

and hence the integral in (12.2.2.31) equals 1.

The function u_0 is therefore an H-indentation on G, and by its design the function $\alpha_k \cdot u_0$ is an H_k-indentation on G_k for every integer $k \geq 0$.

The function $f(x,h) := \Psi_{u_0}^{\Delta}(\widetilde{u})(x, h^{-1}x)$ of $(x,h) \in D \times H$ is absolutely $(\mu_G \otimes \mu_H)$-integrable by Proposition 12.2.2.2(2).

We observe that $f(x,h)\beta_k(h^{-1}x) = f(x,h) \cdot 1_{S_k(x)}(h)$ for $S_k(x) \subset H$ the subset, for each $x \in D$, of those $h \in H$ such that $h^{-1}x \in G - G_k$, then we deduce that $f_k(x,h) := f(x,h) \cdot 1_{S_k(x)}(h)$ tends to zero pointwise $(\mu_G \otimes \mu_H)$-almost everywhere on $D \times H$.

Indeed, for each fixed $x \in D$ and $h \in H$, the point $h^{-1}x$ is contained in at most finitely many of the subsets $G - G_k$, since $\bigcup_{k\geq 0} G_k = G$ and $G_k \subset G_{k+1}$. Indeed, there exists $k \geq 0$ large enough such that $h^{-1}x \in G_k$, in which case $h^{-1}x$ is contained in G_m for every integer $m \geq k$, and hence $h^{-1}x \notin G - G_m$ for every $m \geq k$, i.e. $1_{S_m(x)}(h) = 0$ for all $m \geq k$. This shows that $f_m(x,h) = 0$ for all $m \geq k$, where k depends on the point (x,h), i.e. $f_k \to 0$ pointwise on $D \times H$, and hence pointwise $(\mu_G \otimes \mu_H)$-almost everywhere on $D \times H$.

Since $|f_k| \leq |f|$ for every integer $k \geq 0$, then, by Dominated Convergence, (12.2.2.26) holds. □

12.3. Putting the pieces together: global to S-local deconstruction of the geometric kernel

[1]

12.4. S-local trace calculation

In this section we compute the part of the trace formula that is delegated to the S-local function spaces discussed. The calculations are analogous to calculations performed by Connes (see [**Co**, §VII,Thm.4,p.31]) and reprised by R. Meyer (see [**Me1**, §3]). The calculation presented below exhibits small variations on the same theme.

Most importantly, the role of the S-adelic Fourier transform, whose job is to turn the integral calculation of interest into a simpler one coming down to a volume calculation on the S-idele class group $J_{F,S}$, was identified by Connes in his 1998 paper (see [**Co**, §VII,Thm.4,p.31]).

12.4.1. S-local trace on the Paley-Wiener Levels.

THEOREM 12.4.1.1 (S-local Trace calculation). *Let F be a number field, S a finite set of places containing all the Archimedean places.*

Let $\varphi \in \mathcal{D}(C_F)$. Assume $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$, and that S contains all the ramified places of F. Assume that φ is $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F - S}$-invariant.

Then:

(12.4.1.1)
$$\frac{1}{\overline{\mu}_{S,1}(J^1_{F,S})}\int_{J_{F,S}} k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}(x,x)\,\mathrm{d}\overline{\mu}_S(x) = \sum_{v\in S_\infty,\ v\ \text{real}} \frac{1}{2}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v)$$

(12.4.1.2)
$$+ \sum_{v\in S_\infty,\ v\ \text{complex}} \frac{1}{2\pi}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v)$$

(12.4.1.3)
$$+ \sum_{v\in S_f} \frac{\log(q_v)}{\mu_v(\mathcal{O}^\times_{F_v})}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|_v}\,\mathrm{d}\mu_v(g_v) + \varphi(1)\log(\lambda\lambda') + o(1)$$

where $q_v := |\pi_v|_v^{-1}$, for $v \in S_f$ and π_v any choice of uniformizer at v of the non-Archimedean local field F_v, and as $\lambda, \lambda' \to +\infty$.

Note how the volumes of all the local unit groups feature at the denominators in the formula from (12.4.1.4), as they should, them being stabilizers, featuring as inertias of the quotient stacks arising as the periodic and fixed orbital supports of the Schauder bases constructed in Chapter 7, and responsible for the local traces in Theorem 7.4.2.1 and Theorem 7.4.3.2.

The measures μ_v for v real, $\frac{1}{2\pi}\mu_v$, and $\frac{\log(q_v)}{\mu_v(\mathcal{O}^\times_{F_v})}\mu_v$ for v non-Archimedean, happen to be exactly the canonical Haar measures on the modulated groups $F_v^\times$ (see Remark B.4.4.2).

Note also that $\overline{\mu}_{S,1}(J^1_{F,S})^{-1}\cdot\overline{\mu}_S$ is the canonical measure on the $\mathbf{R}_{>0}$-modulated group $J_{F,S}$.

PROOF. We recall that for $\lambda, \varepsilon > 0$ two real numbers, we defined smooth (λ,ε)-cutoff functions $\rho(x)$ in Definition 9.3.3.1.[2]

We rename as φ the unique smooth compactly supported function on $J_{F,S}$ obtained by composing φ with the isomorphism $C_{F,S} \to C_F$ (an isomorphism because, by assumption, $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$) and descending the

[1] Include section.

[2] Extend Definition 9.3.3.1 so as to allow cutoffs whose support is over $(0,\lambda]$ and not only over $[1/\lambda,\lambda]$. Return here and change the choice of smooth (λ,ε)-cutoff into a smoothening of $1_{[1/\lambda'',\lambda]}$ with λ'' not tied to λ. Then study the convergence of the integral in (12.4.1.23), with $\log|w|_{F,S}$ replaced by $\log\max\{\frac{|w|_{F,S}}{\lambda'}, \frac{1}{\lambda''}\}$, as $\lambda', \lambda'' \to +\infty$.

composition to $J_{F,S}$ by $\widehat{\mathcal{O}}^{\times}_{F,\Sigma_F-S}$-invariance. We assume, without loss of generality, that φ is $\chi|_{C^1_F}$-homogeneous. Its action by convolution on $W^{s,2}(C_F)$ and its closed subspace $W^{s,2}_{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(F^{\times},\mathcal{S}_0(\mathbf{A}_F))$ and its own closed invariant subspaces, factors through each $\chi|_{C^1_F}$-isotype, respectively.

Acting by convolution on the invariant Paley-Wiener level

$$W^{s,2}\mathrm{PW}^{\mathrm{inv}}_{S,\lambda'}(\chi_S|_{J^1_{F,S}},\mathcal{S}_0(\mathbf{A}_{F,S}))$$

and composing the resulting convolution action by the truncation operator $M_{\rho_{\lambda-}}$ (where, recall [3], $\rho_{\lambda-}$ is notation intended to recall that ρ is supported over $(0,\lambda]$), we obtain a nuclear operator. We compute its trace below.

Since ρ is a smooth (λ,ε)-cutoff on C_F, and hence on $J_{F,S}$, then the operator $R_\rho A(\varphi)$ on

$$W^{s,2}_{1/2}\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^{\times}_{F,S},\mathcal{S}_0(\mathbf{A}_{F,S}))$$

factors through the image under the multiplication operator by $|\cdot|^{1/2}_{F,S}$ (within $\mathcal{S}_{(0,+\infty)}(J_{F,S})\times\mathcal{S}_{(-\infty,1)}(J_{F,S})$) of the image under the S-local θ-function operator $\Phi_{F,S}$, of $\mathcal{S}_{00}(\mathbf{A}_{F,S})$.

Note that this is, in turn, a copy, linearly and topologically, of $|\cdot|^{1/2}_{F,S}\,\mathcal{S}_{\mathrm{E}}(J_{F,S})$, by Proposition 5.3.4.4[4]).

By Lemma C.3.2.2, it is enough to compute the trace of $R_\rho A(\varphi)$ on

$$E:=|\cdot|^{1/2}_{F,S}\,\mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}^{\times}_{F,S},\mathcal{S}_{00}(\mathbf{A}_{F,S})).$$

This, we do as follows, following the arguments first presented in [**Co**, §VII,Thm.4,p.31] and revisited in [**Me1**, §3.4,p.19].

We choose $\widetilde{\varphi}\in\mathcal{D}(\mathrm{GL}_1(\mathbf{A}_{F,S}))$ such that, as continuous functions of $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, invariant under the continuous left-translation action of $\mathcal{O}^{\times}_{F,S}$, we have

$$\sum_{\xi\in\mathcal{O}^{\times}_{F,S}}\widetilde{\varphi}(\xi g)=\varphi(g). \tag{12.4.1.4}$$

The existence of $\widetilde{\varphi}$ is ensured by discreteness of $\mathcal{O}^{\times}_{F,S}$ in $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and compactness of the support of $\widetilde{\varphi}$ (see, for example, the proof of Lemma 4.2.1.5(2)).

The composition of R_ρ and $A(\varphi)$ on E can thus be analyzed by studying the composition of the following operator on $\mathcal{S}_{00}(\mathbf{A}_{F,S})$:

$$A_{\widetilde{\varphi}}:f\mapsto\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})}\widetilde{\varphi}(g)(|g^{-1}x|^{1/2}_{F,S}f(g^{-1}x))\,\mathrm{d}\mu_S(g)$$

which is valued in E, followed by R_ρ.

Indeed, $R_\rho A_{\widetilde{\varphi}}$ factors through $R_\rho A(\varphi)$, and the trace of the latter can be computed using the recipe from Proposition 12.2.2.2 from §12.2.2 and the integral kernel of the former. The main difficulty of the calculation being performed in this argument, is computing the integral kernel of $R_\rho A_{\widetilde{\varphi}}$. This, we now do as follows.

We begin by observing that the operator:

$$A_{\widetilde{\varphi}}:\mathcal{S}_{00}(\mathbf{A}_{F,S})\to E$$

has a Schwartz kernel which can be described explicitly. Our goal is three-fold:

(I) write down a candidate for the Schwartz kernel of $A_{\widetilde{\varphi}}$ and then be show that it is indeed the Schwartz kernel of $A_{\widetilde{\varphi}}$.

(II) Write down a candidate for the Schwartz kernel of $R_\rho A_{\widetilde{\varphi}}$ and then show that it is indeed the Schwartz kernel of $R_\rho A_{\widetilde{\varphi}}$.

[3]See previous footnote.

[4]Return here after the Proposition is updated.

(III) Use the recipe from Proposition 12.2.2.2 from §12.2.2 to compute the trace of $R_\rho A(\varphi)$, and deduce the formula from (12.4.1.4) from it, by identifying the integral kernel of $R_\rho A(\varphi)$ with $k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}$.

The first two points will be carried out using the integral in the sense of Pettis, and topological properties of the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}_{00}(\mathbf{A}_{F,S})$ and E. The third and last will use the S-adelic Fourier transform on the source, to compute the integral of the smoothening of the resulting integral kernel along the diagonal.

Step I: We denote, for each $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ and $x \in \mathbf{A}_{F,S}$, by

$$\delta_{y=g^{-1}x} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathbf{C}$$

the continuous linear functional defined by sending $f \in \mathcal{S}(\mathbf{A}_{F,S})$ (as a function of $y \in \mathbf{A}_{F,S}$) to $f(g^{-1}x)$.

We now integrate the continuous function $g \mapsto \delta_{y=g^{-1}x}$ to obtain

$$K := \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2} \widetilde{\varphi}(g) \delta_{y=g^{-1}x} \, \mathrm{d}\mu_S(g) \tag{12.4.1.5}$$

and since $\widetilde{\varphi}$ is continuous (in fact, smooth) and compactly supported, then by Theorem C.4.3.5 the integral converges in the sense of Pettis (i.e. it is valued in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$, which, note, is complete, hence quasi-complete, so the Theorem applies).

We rename $\delta_{y=g^{-1}x}$ as $\delta(y - g^{-1}x)$ to ease the notation, and note that, as $x \in \mathbf{A}_{F,S}$ varies, this is a continuous function of $(g, x) \in \mathrm{GL}_1(\mathbf{A}_{F,S}) \times \mathbf{A}_{F,S}$ values in $\mathcal{S}(\mathbf{A}_{F,S})'_\beta$.

The integral from (12.4.1.5) is a continuous function $\mathbf{A}_{F,S} \to \mathcal{S}(\mathbf{A}_{F,S})'_\beta$, and in fact an element of $\mathcal{S}(\mathbf{A}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})'_\beta)$, which is in turn linearly and topologically isomorphic, by Theorem C.2.4.19, to

$$\mathcal{S}(\mathbf{A}_{F,S}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(\mathbf{A}_{F,S})'_\beta \simeq \mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_{F,S})). \tag{12.4.1.6}$$

Under the composition linear topological isomorphism $\mathcal{S}(\mathbf{A}_{F,S}, \mathcal{S}(\mathbf{A}_{F,S})'_\beta) \simeq \mathscr{L}_\beta(\mathcal{S}(\mathbf{A}_{F,S}))$, K is sent to a continuous linear endomorphism, which we now compute. We have, for any $f \in \mathcal{S}(\mathbf{A}_{F,S})$:

(12.4.1.7)

$$(Kf)(x) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2} \widetilde{\varphi}(g) \langle f, \delta(y - g^{-1}x) \rangle_{\mathcal{S}(\mathbf{A}_{F,S}), \mathcal{S}(\mathbf{A}_{F,S})'} \, \mathrm{d}\mu_S(g) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2} \widetilde{\varphi}(g) f(g^{-1}x) \, \mathrm{d}\mu_S(g).$$

We deduce that (the image of) K as an element of $\mathcal{S}_{00}(\mathbf{A}_{F,S}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}_{00}(\mathbf{A}_{F,S})'_\beta$ is the Schwartz kernel of $A_{\widetilde{\varphi}}$. Note that K is contained in the linear and topological subspace

$$\mathcal{S}_{00}(\mathbf{A}_{F,S}) \mathbin{\widehat{\otimes}}_\pi \mathrm{H}_{0,\mathrm{cont}}(\mathcal{O}_{F,S}^\times, \mathcal{S}_{00}(\mathbf{A}_{F,S}))'_\beta.$$

This carries out (I).

Step II: We now identify the Schwartz kernel of the operator $R_\rho A_{\widetilde{\varphi}}$. Recall that this is the composition of three operators:

- $\widehat{P}_{\rho,\lambda'} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}(\mathbf{A}_{F,S})$, which is defined as the S-adelic Fourier transform, multiplied by the choice of smooth (λ', ε)-cutoff ρ given by a choice of ε-smoothening of the indicator function of the subset
$$\{x \in \mathbf{A}_{F,S} : |x|_{F,S} \leq \lambda'\}$$
further followed by the S-adelic Fourier transform again, composed by the linear topological automorphism induced by negation (i.e. multiplication by -1) on $\mathbf{A}_{F,S}$.
- The operator $A_{\widetilde{\varphi}}$.
- $P_{\rho,\lambda} : \mathcal{S}(\mathbf{A}_{F,S}) \to \mathcal{S}_{00}(\mathbf{A}_{F,S})$ given by multiplication by the choice of smooth (λ, ε)-cutoff ρ given by a choice of ε-smoothening of the indicator function of the subset
$$\{x \in \mathbf{A}_{F,S} : |x|_{F,S} \leq \lambda\}.$$

Equivalently, $R_\rho A_{\widetilde{\varphi}}$ is the operator

$$R_\rho A_{\widetilde{\varphi}} := P_{\rho,\lambda} \circ A_{\widetilde{\varphi}} \circ \widehat{P}_{\rho,\lambda'}.$$

We compute

(12.4.1.8)

$$(A_{\widetilde{\varphi}} \circ \widehat{P}_{\rho,\lambda'})(f) = A_{\widetilde{\varphi}} \circ \mathcal{F}_S^{-1}(\rho_{\lambda',-}(\cdot)\widehat{f}(\cdot))$$

(12.4.1.9)

$$= \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2} \widetilde{\varphi}(g) \int_{\mathbf{A}_{F,S}} \rho_{\lambda',-}(x)\widehat{f}(x)\psi_{S,\mathrm{std}}(xg^{-1}y)\,\mathrm{d}\nu_S(x)\,\mathrm{d}\mu_S(g)$$

(12.4.1.10)

$$= \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2} \widetilde{\varphi}(g) \langle \delta(u - g^{-1}y), \mathcal{F}_{S,x}^{-1}(\rho_{\lambda',-}(x)\widehat{f}(x))(u)\rangle_{\mathcal{S}(\mathbf{A}_{F,S})',\mathcal{S}(\mathbf{A}_{F,S})}\,\mathrm{d}\mu_S(g).$$

We now look into the operator sending $f \in \mathcal{S}(\mathbf{A}_{F,S})$ to $\widehat{P}_{\rho,\lambda'}(f) = \mathcal{F}_S^{-1}(\rho_{\lambda',-} \cdot \widehat{f})$, where $\mathcal{F}_S^{-1}$ is the inverse Fourier transform. By the Convolution Theorem, the Schwartz kernel of $\widehat{P}_{\rho,\lambda'}$ is

$$k_{\rho,\lambda'}(x,y) = \widehat{\rho_{\lambda',-}}(x-y)$$

as a function of $(x,y) \in \mathbf{A}_{F,S}$ To see this, we compute

$$\int_{\mathbf{A}_{F,S}} k_{\rho,\lambda'}(x,y)f(y)\,\mathrm{d}\nu_S(y) = \int_{\mathbf{A}_{F,S}} \widehat{\rho_{\lambda',-}}(x-y)f(y)\,\mathrm{d}\nu_S(y) = \mathcal{F}_S^{-1}(\rho_{\lambda',-}\widehat{f}).$$

To compute the kernel of $P_{\rho,\lambda} \circ A_{\widetilde{\varphi}} \circ \widehat{P}_{\rho,\lambda'}$ (noting that $P_{\rho,\lambda}$ is the multiplication operator by the chosen smooth (λ,ε)-cutoff $\rho_{\lambda,-}(x)$) we compute the distributional convolution of K and $k_{\rho,\lambda'}$, which is in turn computed using the Fourier transform.

We obtain

$$k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}(x,y) = \rho_{\lambda,-}(x) \int_{\mathbf{A}_{F,S}} K(x,z)k_{\rho,\lambda'}(z,y)\,\mathrm{d}\nu_S(z).$$

This finishes carrying out **Step II**.

Step III: Using the Hardy-Littlewood Maximal Theorem from Theorem A.5.3.7 and the fact that $k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}$ is the integral kernel of a nuclear operator, we rename as $k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}$ the smoothening of $k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}$ along the diagonal, and use Parseval's Formula to compute, for each $\xi \in \mathcal{O}_{F,S}^\times$, the integral

(12.4.1.11)

$$\int_{E_{S,0}} k^{\mathrm{PW}}_{S,\varphi,\lambda,\lambda'}(x,\xi x)\,\mathrm{d}\nu_S(x)$$

and observe that it equals

(12.4.1.12)

$$\int_{E_{S,0}} \rho_{\lambda,-}(x) \int_{\mathbf{A}_{F,S}} \int_{\mathbf{A}_{F,S}} K(x+z,\xi x)\overline{\psi_{S,\mathrm{std}}(zu)}\,\mathrm{d}\nu_S(z) \int_{\mathbf{A}_{F,S}} k_{\rho,\lambda'}(x+z,x)\overline{\psi_{S,\mathrm{std}}(zu)}\,\mathrm{d}\nu_S(z)\,\mathrm{d}\nu_S(u)\,\mathrm{d}\nu_S(x).$$

We have

(12.4.1.13)

$$\int_{\mathbf{A}_{F,S}} K(x+z,\xi x)\overline{\psi_{S,\mathrm{std}}(zu)}\,\mathrm{d}\nu_S(z) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2}\widetilde{\varphi}(g) \int_{\mathbf{A}_{F,S}} \delta(x+z-g^{-1}\xi x)\overline{\psi_{S,\mathrm{std}}(zu)}\,\mathrm{d}\nu_S(z)\,\mathrm{d}\mu_S(g)$$

(12.4.1.14)

$$= \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2}\widetilde{\varphi}(g)\overline{\psi_{S,\mathrm{std}}((g^{-1}\xi-1)xu)}\,\mathrm{d}\mu_S(g)$$

and

(12.4.1.15)

$$\int_{\mathbf{A}_{F,S}} k_{\rho,\lambda'}(x+z)\overline{\psi_{S,\mathrm{std}}(zu)}\,\mathrm{d}\nu_S(z) = \rho_{\lambda',-}(u).$$

We now further analyze the integral

(12.4.1.16)
$$\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{-1/2}\widetilde{\varphi}(g)\overline{\psi_{S,\mathrm{std}}((g^{-1}\xi-1)xu)}\,\mathrm{d}\mu_S(g) = \int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |g|_{F,S}^{1/2}\widetilde{\varphi}(g^{-1})\overline{\psi_{S,\mathrm{std}}((g\xi-1)xu)}\,\mathrm{d}\mu_S(g)$$

where we used invariance of the S-local multiplicative Tamagawa measure μ_S under inversion.

We note that we have

$$\mu_S = c_S|\cdot|_{F,S}^{-1}\nu_S$$

as Radon measures on $\mathbf{A}_{F,S}$. The positive real number $c_S > 0$ is given by:

(12.4.1.17)
$$c_S := \prod_{v\in S_f}\frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}.$$

We call $z := g\xi - 1$, and hence $g = \xi^{-1}(1+z)$.

The integral from (12.4.1.16) reads as follows:

(12.4.1.18)
$$c_S\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})} |\xi^{-1}(1+z)|_{F,S}^{-1/2}\widetilde{\varphi}(\xi(1+z)^{-1})\overline{\psi_{S,\mathrm{std}}(zxu)}\,\mathrm{d}\nu_S(z)$$

We call $\eta(x) = |x|_{F,S}^{1/2}\widetilde{\varphi}(x^{-1})$, $x\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, and the integral in (12.4.1.18) reads:

(12.4.1.19)
$$c_S\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})}\left\{\frac{\eta(\xi^{-1}(1+z))}{|\xi^{-1}(1+z)|_{F,S}}\right\}\overline{\psi_{S,\mathrm{std}}(zxu)}\,\mathrm{d}\nu_S(z).$$

We define:

(12.4.1.20)
$$\eta_\xi(z) := \frac{\eta(\xi^{-1}(1+z))}{|\xi^{-1}(1+z)|_{F,S}}$$

and rewrite (12.4.1.19) as:

(12.4.1.21)
$$c_S\int_{\mathrm{GL}_1(\mathbf{A}_{F,S})}\eta_\xi(z)\overline{\psi_{S,\mathrm{std}}(zxu)}\,\mathrm{d}\nu_S(z) = c_S\widehat{\eta_\xi(xu)}.$$

In light of (12.4.1.12), the integral from (12.4.1.11) equals:

(12.4.1.22)
$$c_S\int_{x\in E_{S,0},|x|_{F,S}\in(0,\lambda],|u|_{F,S}\le\lambda'}\widehat{\eta_\xi(xu)}\mathrm{d}\nu_S(x)\mathrm{d}\nu_S(u).$$

We now perform the change of variables $w = xu$ and obtain

$$\mathrm{d}\nu_S(x)\otimes\mathrm{d}\nu_S(u) = \frac{\mathrm{d}\nu_S(x)}{|x|_{F,S}}\otimes\mathrm{d}\nu_S(w)$$

which gives that (12.4.1.22) in turn equals:

(12.4.1.23)
$$c_S\int_{|w|_{F,S}\le\lambda\lambda'}\int_{x\in E_{S,0},\frac{|w|_{F,S}}{\lambda'}\le|x|_{F,S}\le\lambda}\widehat{\eta_\xi(w)}\mathrm{d}\nu_S(w)\frac{\mathrm{d}\nu_S(x)}{|x|_{F,S}} = \int_{|w|_{F,S}\le\lambda\lambda'}\int_{x\in E_{S,0},\frac{|w|_{F,S}}{\lambda'}\le|x|_{F,S}\le\lambda}\widehat{\eta_\xi(w)}\mathrm{d}\nu_S(w)\mathrm{d}\mu_S(x)$$

(12.4.1.24)
$$= \int_{|w|_{F,S}\le\lambda\lambda'}\widehat{\eta_\xi(w)}\left\{\overline{\mu}_{S,1}(J_{F,S}^1)(\log(\lambda\lambda') - \log|w|_{F,S})\right\}\,\mathrm{d}\nu_S(w).$$

The calculation from [Co, §VII,Thm.4,p.31] now applies, and, using Lemma B.4.4.3, Lemma B.4.4.5 and Lemma B.4.3.2, gives that the sum over $\xi \in \mathcal{O}_{F,S}^{\times}$ of last integral from (12.4.1.23) equals:

$$\overline{\mu}_{S,1}(J^1_{F,S})\varphi(1)\log(\lambda\lambda') + \overline{\mu}_{S,1}(J^1_{F,S}) \sum_{v\in S_\infty,\ v \text{ real}} \frac{1}{2}\int'_{F_v^\times,\alpha_{v,0}} \frac{|g|_v^{1/2}\varphi(g^{-1})}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{12.4.1.25}$$

$$+ \overline{\mu}_{S,1}(J^1_{F,S}) \sum_{v\in S_\infty,\ v \text{ complex}} \frac{1}{2\pi}\int'_{F_v,\alpha_{v,0}} \frac{|g|_v^{1/2}\varphi(g^{-1})}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{12.4.1.26}$$

$$+ \overline{\mu}_{S,1}(J^1_{F,S}) \sum_{v\in S_f} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times,\alpha_{v,0}} \frac{|g|_v^{1/2}\varphi(g^{-1})}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{12.4.1.27}$$

$$+ o(1) \tag{12.4.1.28}$$

as $\lambda, \lambda' \to +\infty$ as $\varepsilon\lambda = o(1)$ and $\varepsilon\lambda' = o(1)$.

We now note that the three finite sums in (12.4.1.25) are equal to the corresponding finite sums with $\varphi(g)$ in place of $\varphi(g^{-1})$, which proves that (12.4.1.4) holds. □

12.5. Trace formula: the Main Theorem of this work

In this section we perform the main trace calculation, i.e. we compute the trace of the nuclear operator:

$$P^{\mathrm{geom}}_{\rho,a} \circ A(\varphi)$$

for any test function $\varphi \in \mathcal{D}(C_F)$ satisfying $\varphi(1) = 0$, in the special case the parameter a is chosen to be $a = 1/2$, and as ρ is a (Λ, ε)-cutoff in the sense of Definition 9.3.3.1, as $\Lambda \to +\infty$ while having $\varepsilon\Lambda = o(1)$.

This is also where the necessity of the choice $a = 1/2$ is explained.

12.5.1. Trace Formula: the statement. The following is the main Theorem of this work. Throughout the remainder of this Chapter, we will be collecting the fruits of the work done so far, and prove it.

THEOREM 12.5.1.1 (Trace Formula). *Let F be a number field. Let $\varphi \in \mathcal{D}(C_F)$ be a test function satisfying $\varphi(C_F^1) = 0$.*

Then, the following identity holds:

$$\sum_{v\in S_\infty,\ v \text{ real}} \frac{1}{2}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v) + \sum_{v\in S_\infty,\ v \text{ complex}} \frac{1}{2\pi}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|}\,\mathrm{d}\mu_v(g_v) \tag{12.5.1.1}$$

$$+ \sum_{v\in\Sigma_{F,f}} \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times} \frac{\varphi(g_v)|g|_v^{1/2}}{|1-g_v|_v}\,\mathrm{d}\mu_v(g_v) \tag{12.5.1.2}$$

$$= M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2}) - \sum_{c,I} N(c)M(\varphi)\left(c|\cdot|_F^{-1/2}\right) \tag{12.5.1.3}$$

where $q_v := |\pi_v|_v^{-1}$, for $v \in \Sigma_{F,f}$ and π_v any choice of uniformizer at v of the non-Archimedean local field F_v, and

$$M(\varphi)(c) := \int_{C_F} \varphi(g)c(g)\,\mathrm{d}\overline{\mu}(g)$$

is the Mellin transform of the test function φ, and $c \in \Omega(C_F)$ varies over the zeros of the Hecke L-function $c \mapsto L(c)$ contained in the interior of the relative vertical strip

$$\Omega(C_F)_I = \{c \in \Omega(C_F) : \sigma(c) \in 1/2 + I\}.$$

The nonnegative integer $N(c)$ is the multiplicity of the zero c of the Hecke L-function.

PROOF. We let S be a finite set of places containing all the Archimedean places, the places of F that ramify over $\mathbf{Q}$, the places at which the character $\chi \in \widehat{C_F}$ is ramified, and such that $\mathrm{Pic}(\mathcal{O}_{F,S}) = 1$. We further assume S is large enough so as to arrange that the support K of φ satisfies $|K|_F \subset (1/T, T)$ with $T < N(\mathfrak{p}_v)$ for every finite place $v \notin S$, with $\mathfrak{p}_v$ the unique prime ideal of $\mathcal{O}_F$ associated to v. Further assume that S is so large that φ is invariant under the left-translation action on $\mathcal{D}(C_F)$ of the image of $\prod_{v \notin S} \mathcal{O}_{F_v}^\times \subset \mathrm{GL}_1(\mathbf{A}_F)$.

We compute the integral with respect to the global Haar measure $\overline{\mu}$ of the smoothened geometric kernel $k := \widetilde{k}^{\mathrm{geom}}_{\varphi,1/2,I,s}$ along the diagonal, denoted $k(x,x)$. By (11.3.1.4), this reads:

$$\int_{C_F} k_S(x,x)\,\mathrm{d}\overline{\mu}(x) + \sum_{p \nmid S} \int_{C_F} k_p(x,x)\,\mathrm{d}\overline{\mu}(x) = \int_{C_F} k(x,x)\,\mathrm{d}\overline{\mu}(x)$$

and by Theorem 11.3.2.1, each $\int_{C_F} k_p(x,x)\,\mathrm{d}\overline{\mu}(x) = 0$ for all prime numbers p that are not the prime residue characteristic of any finite place contained in S.

By Theorem 12.4.1.1, we further have that $\int_{C_F} k_S(x,x)\,\mathrm{d}\overline{\mu}(x)$ equals the left side of (12.5.1.1) upon subtracting $M(\varphi)(|\cdot|_F^{1/2}) + M(\varphi)(|\cdot|_F^{-1/2})$.

For $I = [-\alpha, \alpha]$, we further have:

$$\int_{C_F} k(x,x)\,\mathrm{d}\overline{\mu}(x) = -\sum_{c,I} N(c) M(\varphi)\left(c|\cdot|_F^{-1/2}\right) \tag{12.5.1.4}$$

by design of the global geometric kernel (see Definition 10.4.3.7) and by Theorem 10.4.2.8.

Upon combining (12.5.1.4) with the foregoing, we deduce that the identity in (12.5.1.1) holds. □

CHAPTER 13

Zeros of Hecke L-functions

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13.3. Main Theorems on the location of the zeros 522
13.3.1. The Divisor of the Hecke L-function L on $\Omega(C_F)$ 522

In this Chapter, we show how to deduce information about the zeros of Hecke L-functions of an arbitrary number field F, from the Trace Formula proved in Theorem 12.5.1.1.

13.1. Recollections on the Explicit Formulas

13.2. Holomorphic families of test functions

13.3. Main Theorems on the location of the zeros

Let F be a number field.

We will then be showing how to use the Trace Formula from Theorem 12.5.1.1 to deduce information on the zeros of the Hecke L-functions of F.

13.3.1. The Divisor of the Hecke L-function L on $\Omega(C_F)$. We record here below the statement of the Trace Formula, established in Theorem 12.5.1.1

THEOREM 13.3.1.1 (Trace Formula). *Let F be a number field.*[1]

We define, on the set $\{c \in \Omega(C_F) : L(c) = 0\}$, the counting function $N_{\mathrm{spec}}(c)$ assigned by the formula:

$$N_{\mathrm{spec}}(c) := \min\{N(c), [F : \mathbf{Q}]\} \tag{13.3.1.1}$$

where $N(c)$ is the multiplicity of the zero c of the Hecke L-function $L(c)$.

Let $\varphi \in \mathcal{D}(C_F)$ be a test function. Assume, upon choosing a continuous homomorphic section τ to the idelic norm, that we have, for all integers $0 \leq k \leq [F : \mathbf{Q}] - 1$:

$$\left(\frac{\partial^k}{\partial s^k}\int_{\mathbf{R}_{>0}} \chi(\tau(t))t^s \left(\int_{C_F^1} \varphi(\tau(t)g)\chi(g)\,\mathrm{d}\overline{\mu}_1(g)\right)\frac{\mathrm{d}t}{t}\right)_{s=-ix_0(\chi)} = 0 \tag{13.3.1.2}$$

for all $\chi \in \widehat{C_F}$, with $x_0(\chi) \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0(\chi)}$ as smooth functions on $\mathbf{R}_{>0}$.

Then the sums on each side of the identity:

(13.3.1.3)

$$\sum_{c\in\Omega(C_F):L(c)=0,\sigma(c)=1/2} N_{\mathrm{spec}}(c)\int_{C_F}\varphi(g)c(g)|g|_F^{-1/2}\,\mathrm{d}\overline{\mu}(g) = \int_{C_F}\varphi(g)|g|_F^{-1/2}\,\mathrm{d}\overline{\mu}(g) + \int_{C_F}\varphi(g)|g|_F^{1/2}\,\mathrm{d}\overline{\mu}(g)$$

$$- \sum_{v\in S_{F,\infty}} {\int_{F_v^\times,\alpha_{v,0}}}' \frac{\varphi(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{13.3.1.4}$$

$$- \sum_{v\in S_{F,f}} \log(q_v) {\int_{F_v^\times,\alpha_{v,0}}}' \frac{\varphi(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{13.3.1.5}$$

converge, and their formation is continuous with respect to the subspace topology on the $[F : \mathbf{Q}]$-codimensional linear subspace $A \subset \mathcal{D}(C_F)$ defined by the vanishing conditions from (13.3.1.2)*. They define the same continuous functional on A.*

Upon combining Theorem 13.3.1.1 and Theorem **??**, we obtain the following:

THEOREM 13.3.1.2 (Divisor of $L(s, \chi)$). [2]

[3]

[1] Correct statement.

[2] Insert statement.

[3] State and prove the Main Theorems, also explaining how to deduce them by approximation $\varphi \to e^{-s|\log(t)|}$ as $\mathrm{Re}(s) > 1$.

CHAPTER 14

Complements

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14.1. On the multiplicities of the zeros

In this section we record a few complements, and discuss an important sanity check: we explain how the Multiplicity Bound obtained in Theorem 13.3.1.2, is compatible with expectations from Global Langlands and the Grand Simplicity Hypothesis.

We will then explain how the methods discussed throughout this paper, do not adapt to the case of Hecke L-functions of global function fields of smooth projective geometrically connected curve X over a finite field $\mathbf{F}_q$ of positive characteristic p.

Specifically, we will be discussing which parts extend to this case, and which, most importantly, do not, and why this is to be expected: briefly, a phenomenon caused by the existence of *maximal curves*, leading to counterexamples to the positive characteristic direct analog of the Simplicity Hypothesis (i.e. that the zeros of the L-functions $L(s,\pi)$ attached to cuspidal automorphic representations π of GL_m over $F=\mathbf{F}_q(t)$ for some integer $m\geq 0$, are simple).

14.1.1. Artin-Weil L-functions. We let F be a local or global field, and $\overline{F}$ a fixed choice of separable closure. We denote by W_F any Weil group for F. We recall that a Weil group for F, is a triple

$$(W_F,\varphi,\{r_E\}_E)$$

where W_F is a topological group

$$\varphi: W_F\to \mathrm{Gal}(\overline{F}/F)$$

is a continuous group homomorphism with dense image.

We will be referring to φ as the *reciprocity homomorphism*.

For any finite separable sub-extension E/F contained within the chosen $\overline{F}$, we define $W_E:=\varphi^{-1}(\mathrm{Gal}(\overline{F}/E))$, noting that since φ is continuous, then $W_E\subset W_F$ is an open subgroup of finite index.

If E/F is Galois, then $W_E\subset W_F$ is normal.

Denseness of the image of φ ensures that we have a bijection of finite homogeneous spaces:

$$W_F/W_E\to \mathrm{Gal}(\overline{F}/F)/\mathrm{Gal}(\overline{F}/E) \tag{14.1.1.1}$$

where E/F is a general finite separable extension. In the special case E/F is finite Galois, then this is a group isomorphism.

The collection $\{r_E\}_E$, as E varies over the finite separable extensions of F contained in the chosen $\overline{F}$, is a family of isomorphisms of Hausdorff locally compact abelian groups:

$$r_E: C_E\to W_E^{\mathrm{ab}} \tag{14.1.1.2}$$

where C_E is the idele class group, if F is global, and $C_E=E^\times$, if F is local. For each finite separable extension E/F contained within $\overline{F}$, W_E^{ab} is notation for the Hausdorff locally compact abelian quotient of W_E obtained as the quotient of W_E by the closure W_E^c of the commutator subgroup of W_E.

We will be referring to the isomorphisms of topological groups r_E from (14.1.1.2) as *abelianization isomorphisms*.

The triple $(W_F,\varphi,\{r_E\}_E)$ is assumed to satisfy a collection of axioms, for which we refer the reader to [**Tate**, §1]. Most importantly, the composition of r_E followed by the unique continuous group homomorphism $W_E^{\mathrm{ab}}\to \mathrm{Gal}(\overline{F}/E)^{\mathrm{ab}}$ induced by φ, is assumed to agree with Artin's Reciprocity map Art_E from Class Field Theory.

We define, for any finite Galois extension E/F contained within the chosen $\overline{F}$:

$$W_{E/F}:=W_F/W_E^c \tag{14.1.1.3}$$

equipped with the quotient topology, where W_E^c again denotes the closure of the commutator subgroup of W_E.

We refer the reader to [**Tate**, §1,pp.3,4], to Artin and Tate's treatise on Class Field Theory (see [**AT**, Ch.XV,Thm.7,p.179] and [**AT**, Ch.V,Thm.11,p.186]), and to Weil's own work on the Explicit Formulas (see [**Weil-e**, §1,6] (where Weil groups are termed W-groups) and [**Weil-f**, §II,V]) where Weil groups are constructed and introduces a generalization of Artin L-functions associated to continuous finite dimensional complex representations of W_F, for F a global field.

We further refer the reader to Langlands' notes on Artin L-functions (see [**Langlands**, Ch.20]), where Weil's generalization of Artin L-functions is defined and their basic properties summarized.

The construction of the Weil group W_F is carried out in [**AT**, Ch.XV] and rests upon the formalism of (topological class formations). The group $W_{E/F}$ is a topological extension of $\mathrm{Gal}(E/F)$ by C_E, classified by a class $\alpha_{E/F} \in \mathrm{H}^2(\mathrm{Gal}(E/F), C_E)$, which turns out to agree (granting the existence of a Weil group for the topological class formation [**AT**, Ch.XV,Def.3,p.175]) with the fundamental class of Class Field Theory, or can be defined to be such fundamental class, for then deducing the existence of a Weil group as a projective limit topological group of the extensions classified by $\alpha_{E/F}$, whose properties and functorialities follow from the cohomological formalism discussed in [**AT**, Ch.XIII,XIV].

We collect the most important facts about Weil groups throughout the Proposition below:

PROPOSITION 14.1.1.1. *Let F be either a global field, or a local field. We fix a separable closure $\overline{F}$ of F. Let $(W_F, \varphi, \{r_E\}_E)$ be a Weil group for F. Then:*

(1) *For every finite Galois extension E/F contained in $\overline{F}$, the topological group $W_{E/F}$ is a central extension:*

$$1 \to C_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1 \tag{14.1.1.4}$$

the leftmost group homomorphism being continuous and with closed image, and the rightmost being continuous and open upon endowing $\mathrm{Gal}(E/F)$ with the discrete topology.

(2) *There is a canonical continuous homomorphism:*

$$W_F \to \varprojlim_{E/F} W_{E/F} \tag{14.1.1.5}$$

as E varies over the finite separable sub-extensions of F contained within $\overline{F}$, which is an isomorphism of topological groups.

(3) *If F is global, for any place v of F, upon denoting $\overline{F}_v$ a choice of separable closure of the local field F_v, and for any F-homomorphism $i_v : \overline{F} \to \overline{F}_v$, we define, for each finite extension E/F contained within $\overline{F}$, $E_v := i(E)F_v$. There exists a continuous injective homomorphism:*

$$\theta_v : W_{F_v} \to W_F \tag{14.1.1.6}$$

compatible with reciprocity homomorphisms and abelianization isomorphisms.

(4) *Let $\rho : W_F \to \mathrm{GL}(V)$ be a finite dimensional continuous complex representation of W_F. Then there exists a large enough finite Galois extension E/F contained within $\overline{F}$ such that ρ factors through a unique continuous representation $\rho_E : W_{E/F} \to \mathrm{GL}(V)$.*

(5) *Let $\rho : W_F \to \mathrm{GL}(V)$ be a finite dimensional continuous complex representation of W_F. Assume ρ is irreducible. Then:*

   (a) *If V is not induced by a proper subgroup of W_F, then there exists a continuous quasi-character $c \in \Omega(C_F)$ and a continuous representation $\rho_0 : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V)$ such that $\rho = (\rho_0 \circ \varphi) \otimes c$.*

   (b) *For any finite separable extension E/F contained within $\overline{F}$, the induced representation $\mathrm{Ind}_{W_E}^{W_F}(\rho)$ is semisimple.*

   (c) *If ρ is one-dimensional, then for any finite separable extension E/F contained within $\overline{F}$, the induced representation $\rho' := \mathrm{Ind}_{W_E}^{W_F}(\rho)$ satisfies the following property: for every element $g \in W_F$, $\rho'(g)$ is semisimple.*

PROOF. (1) : This is a consequence of [**AT**, Ch.XV,Thm.7,p.179] and [**AT**, Ch.XV,Def.3,(W3),p.176].

(2) : This is a consequence of [**AT**, Ch.XV,Thm.7,p.179] and [**AT**, Ch.XV,Def.3,(W4),p.176].

(3) : This is [**Tate**, §1.6,Prop.1.6.1,p.8].

(4) : Let $U \subset \mathrm{GL}(V)$ be an open unit-neighborhood, small enough so as to have that no non-trivial subgroup of $\mathrm{GL}(V)$ is contained in U. Then $\rho^{-1}(U)$ is open and by (2) it contains W_E for a large enough finite Galois extension E/F contained within the chosen $\overline{F}$. It follows that $\rho(W_E^c)$ is trivial, and hence ρ factors through $W_{E/F}$.

(5) : We let E/F be a large enough finite Galois extension contained within $\overline{F}$, such that ρ factors through $W_{E/F}$. By [**Weil-e**, Lemma 1], there exists a finite group Γ which is a central extension of $\mathrm{Gal}(E/F)$ by a cyclic group, and a representation $\rho_0 : \Gamma \to \mathrm{GL}(V)$, such that $\rho(W_F) \subset \rho_0(\Gamma) \cdot Z$, for Z the center of $\mathrm{GL}(V)$. It follows that we have $\rho(W_E^{\mathrm{ab}}) \subset Z$. Since $\rho(W_E^{\mathrm{c}}) = 1$ and $\rho(W_E^{\mathrm{ab}}) \subset Z$, we deduce that we have $\rho(W_E) \subset Z$, and hence the composition of ρ followed by the quotient $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ annihilates W_E. The remainder of the argument in [**Tate**, §2,(2.2.3),p.10] applies and proves (a).

Upon calling E' the Galois closure in $\overline{F}$ of the finite separable extension E/F from (b), then E'/F is finite Galois, and $W_{E'} \subset W_E$, with $W_{E'} \subset W_F$ a normal subgroup of finite index. $\mathrm{Ind}_{W_E}^{W_F}(\rho)$ is a W_F-submodule of $\mathrm{Ind}_{W_{E'}}^{W_F}(\rho)$. We rename E' as E and assume E is a finite Galois extension of F contained in $\overline{F}$.

Under this additional assumption, $W_E \subset W_F$ is normal, and hence $\mathrm{Ind}_{W_E}^{W_F}(\rho)$ is semisimple since ρ is irreducible. Part (c) follows from [**Langlands**, Ch.1,Lemma 1.1,p.5] applied to $G = W_F$, $H = W_E$ and $\sigma = \rho'$. □

We now review Weil's generalization of Artin L-functions, and recall their main properties. We will then be using them to illustrate how the multiplicity bound obtained in Theorem 13.3.1.1 from the Trace Formula (see Theorem 13.3.1.2) relates to expectations from the literature, with particular emphasis on Global Langlands, as well as heuristics around pair correlation conjectures and the Grand Simplicity Hypothesis.

Let F be a global field, and E/F a finite Galois extension.

We fix a choice $\overline{F}$ of separable closure, and for each finite place v of E, $\overline{E}_v$ a choice of separable closure of the non-Archimedean local field E_v.

We denote by v_0 the unique finite place of F of which v is an extension to E. Then E_v is a finite Galois extension of F_{v_0} contained in $\overline{E}_v$.

The local Weil group $W_{E_v/F_{v_0}}$ is equipped with a ramification filtration inherited from the local Galois group. We refer the reader to [**Langlands**, Ch.6,p.23], where the upper-numbering filtration is defined, resting on the upper-numbering ramification filtration on $G_{E_v/F_{v_0}} := \mathrm{Gal}(E_v/F_{v_0})$ from [**Se**, Ch.IV,§2,3].

We refer the reader to [**Langlands**, Ch.6] for the definition and properties of $W^u_{E_v/F_{v_0}}$, for each real number $u \geq -1$, a closed normal subgroup (by [**Langlands**, §6,Lemma 6.11], [**Langlands**, §6,Lemma 6.12] and [**Langlands**, Ch.6,Lemma 6.13], which establishes stability under the group operation and normality, while $W^u_{E_v/F_{v_0}}$ being closed follows from the definition of the function $v_{E_v/F_{v_0}}$ from the discussion preceding [**Langlands**, Ch.6,Lemma 6.11]).

Since $E_v^\times \subset W_{E_v/F_{v_0}}$ is a normal subgroup of finite index, then if $\rho : W_{E_v/F_{v_0}} \to \mathrm{GL}(V)$ is a finite dimensional continuous complex representation of $W_{E_v/F_{v_0}}$ such that every element of $W_{E_v/F_{v_0}}$ maps to a semisimple element of $\mathrm{GL}(V)$, it is semisimple.

It is a finite direct sum of 1-dimensional continuous representations of $W_{E_v/F_{v_0}}$, since it is a continuous $W_{E_v/F_{v_0}}$-submodule of the induced representation $\mathrm{Ind}_{E_v^\times}^{W_{E_v/F_{v_0}}}(V_0)$, where V_0 is the restriction to $E_v^\times$ of the continuous representation V of $W_{E_v/F_{v_0}}$. We further refer the reader to [**Langlands**, Ch.1,Lemma 1.1,p.5].

We will, throughout the rest of this subsection and following both [**Langlands**, Ch.1] and [**Weil-f**, §VI,p.31], be assuming that continuous finite dimensional complex representations V of local and global Weil groups map each element to a semisimple element of $\mathrm{GL}(V)$.

Upon denoting by V^0 the linear subspace of those elements of V fixed by the normal closed subgroup $W^0_{E_v/F_{v_0}} \subset W_{E_v/F_{v_0}}$ (which, in case ρ factors through $\mathrm{Gal}(E_v/F_{v_0})$, recovers the subspace of inertial invariants), V^0 inherits a structure of continuous finite dimensional complex representation ρ_0 of $W_{E_v/F_{v_0}}$ from V.

It is therefore itself a finite direct sum of 1-dimensional continuous complex representations of $W_{E_v/F_{v_0}}$, i.e., by (14.1.1.2), we have:

$$r_E^*\rho_0 \simeq \bigoplus_{1\leq k\leq \dim(V)} c_{v,k}$$

for continuous quasi-characters $c_{v,k}$ of $E_v^\times$. By design of V^0, all such quasi-characters are unramified.

We define the *local Artin-Weil L-function* to be the meromorphic function:

$$L_v(s,\rho) := \prod_{1\leq k\leq \dim(V)} \frac{1}{1-c_{v,k}(\pi_v)|\pi_v|_v^s} \tag{14.1.1.7}$$

holomorphic for $\mathrm{Re}(s) > 0$.

If v is an Archimedean place of E, then, upon denoting by v_0 the unique Archimedean place of F of which v is an extension to E, and E_v, F_{v_0} the respective completions, then W_{E_v/F_v} is isomorphic, as a Hausdorff locally compact topological group, to either $\mathbf{C}^\times$, or a central extension of μ_2 by $\mathbf{C}^\times$.

In either case, finite dimensional continuous representations of W_{E_v/F_v} and their local Artin-Weil L-functions are defined explicitly (we refer the reader to [**Langlands**, Ch.20,p.212]).

Global Artin-Weil L-functions are then defined as follows. Let $\rho : W_{E/F} \to \mathrm{GL}(V)$ be a finite dimensional continuous complex representation. We denote by ρ_v the continuous finite dimensional complex representation of $W_{E_v/F_{v_0}}$ for each place v of E, obtained by composition with θ_v from (14.1.1.6) of Proposition 14.1.1.1(3).

Over the half-plane $\mathbf{C}_{>A}$ for large enough $A > 1$, the product:

$$L(s,\rho) := \prod_{v\in\Sigma_E} L_v(s,\rho_v) \tag{14.1.1.8}$$

converges absolutely and uniformly (see [**Langlands**, Ch.20,p.213]).

If $\rho : W_F \to \mathrm{GL}(V)$ is a continuous finite dimensional representation of W_F, there exists, in light of Proposition 14.1.1.1(4), a large enough finite Galois extension E/F contained within $\overline{F}$, such that ρ factors through $\rho_E : W_{E/F} \to \mathrm{GL}(V)$. We define

$$L(s,\rho) := L(s,\rho_E) \tag{14.1.1.9}$$

and observe that this definition is independent of the choice of such E (i.e. formation of $L(s,\rho_E)$, for E/F finite Galois, is stable under inflation).

For $\rho^\vee$ the contragradient representation on $V^* = V'$, the global Artin-Weil L-function $L(s,\rho)$ satisfies the functional equation:

$$L(s,\rho) = \epsilon(s,\rho)\, L(1-s,\rho^\vee). \tag{14.1.1.10}$$

This is proved in [**Langlands**, Ch.21] (see [**Langlands**, Introduction,Thm.B,p.vii], where (14.1.1.10) is stated, after automorphic normalization); see also [**Tate**, §3,Thm.3.4.1,p.14] for the existence of the ϵ-factors, and [**Tate**, §3,Thm.3.5.3,p.16] for the functional equation.

The function $\epsilon(s,\rho)$ is an inexplicit nowhere vanishing entire function on the complex s-plane, defined as the product:

$$\epsilon(s,\rho) := \prod_{v\in\Sigma_F} \epsilon_v(s,\rho_v)$$

where each $\epsilon_v(s,\rho_v)$ is 1 with the exception of at most finitely many places v of F, and further depends on a choice of local additive character ψ_v of F_v. The existence, uniqueness and properties of each $\epsilon_v(s,\rho_v)$ is recalled in [**Langlands**, Introduction,Thm.A,p.vii].

The functional equation in (14.1.1.10) implies that $L(s,\rho)$ extends to a meromorphic function on the complex s-plane.

If ρ is 1-dimensional, then ρ factors through W_F^{ab}, and hence $r_F^*\rho$ is a continuous quasi-character c_ρ of C_F. We have:

$$L(s,\rho) = L(c_\rho|\cdot|_F^s) \tag{14.1.1.11}$$

as meromorphic functions on the complex s-plane.

For $\rho = \rho_1 \oplus \rho_2$, with ρ_1, ρ_2 two continuous finite dimensional complex representations of W_F, we have:

$$L(s,\rho) = L(s,\rho_1)\,L(s,\rho_2) \tag{14.1.1.12}$$

as meromorphic functions on the complex s-plane.

If $F'/E/F$ are two finite separable extensions contained within $\overline{F}$, with F'/E and F'/F Galois, then we may induce a continuous finite dimensional complex representation $\rho : W_{F'/E} \to \mathrm{GL}(V)$ to $\mathrm{Ind}_{W_{F'/E}}^{W_{F'/F}}(\rho) : W_{F'/F} \to \mathrm{GL}(V)$, and we have:

$$L(s,\rho) = L(s, \mathrm{Ind}_{W_{F'/E}}^{W_{F'/F}}(\rho)) \tag{14.1.1.13}$$

as meromorphic functions on the complex s-plane, i.e. formation of global Artin-Weil L-functions is inductive (the same property holds for local Artin-Weil L-functions, and implies (14.1.1.13)).

We refer the reader to [**Weil-f**, §VI,p.32].

In the special case of ρ is Galois type, i.e. of the form $|\cdot|_F^z\rho_0$ for some $z \in \mathbf{C}$ and for $\rho_0 : W_F \to \mathrm{GL}(V)$ of the form $\rho_1 \circ \varphi$ with ρ_1 a finite dimensional complex Galois representation, i.e. an Artin representation, then $L(s,\rho) = L(s+z,\rho_1)$ where $L(s,\rho_1)$ is the Artin L-function of ρ_0, and the properties recalled in (14.1.1.11), (14.1.1.12) and (14.1.1.13), including the functional equation from (14.1.1.10), extend those of the Artin L-function $L(\rho_1, s)$ (we further refer the reader to [**Art-2**], and to [**Murty**, §1] for more on such properties of Artin L-functions).

We further refer the reader to [**Tate**, §3] for a discussion of Artin-Weil L-functions, and to [**Tate**, §3,Thm.3.5.3,p.16] for the meromorphic continuation and functional equation of Artin-Weil L-functions, for a discussion of Artin-Weil L-functions. Specifically, to [**Tate**, §3,(3.6),p.17] for the difference between Langlands' ϵ-factors and Deligne's (see [**Del**]).

The L-functions defined in this subsection agree with the generalized Artin L-functions defined by Weil, for whose properties we further refer the reader to [**Weil-f**, §VI,p.25].

14.1.2. Expected Multiplicity Bound. Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.[1]

The Trace Formula (Theorem 13.3.1.1) reveals constraints on the multiplicity of the non-trivial zeros of the Hecke L-function $L(s,\chi)$, although not all of them. Indeed, Theorem 13.3.1.2, a consequence of the Trace Formula combined with the Positivity Criterion (in its form from Theorem **??**), asserts that if $\rho \in \mathbf{C}$ is a non-trivial zero of $L(s,\chi)$, then $\rho \in 1/2 + i\mathbf{R}$, and if, in addition, $\rho \notin \mathbf{R}$, i.e. $\rho \neq 1/2$, then

$$N(\rho) \leq [F : \mathbf{Q}]$$

where $N : \mathbf{C} \to \mathbf{N}$ is the multiplicity function of the divisor of $L(s,\chi)$.

In particular, no information is provided on the value of $N(c)$ when c is the central zero. This is due to the assumption on the test functions $\varphi \in \mathcal{D}(C_F)$ on which the Trace Formula holds, that they should vanish, together with their holomorphic derivatives of order up to $[F : \mathbf{Q}] - 1$, at all the central zeros.

Recall that by *central zero* we mean the unique point of each connected component of the Riemann surface $\Omega(C_F)$, that is of the form $|\cdot|_F^{1/2}c_0$, where c_0 is the unique point of each connected component allowed to

[1] Correct this subsection.

occur as a pole of multiplicity higher than 1 of the partial Euler products $\prod_{v\in S} L_v(\chi_v|\cdot|_v^s)$ for any finite set S of places of F containing all the Archimedean places.

Upon fixing a continuous homomorphic section τ to the idelic norm, and calling $x_0 \in \mathbf{R}$ the unique real number such that $\chi(\tau(t)) = t^{ix_0}$ as smooth functions on $\mathbf{R}_{>0}$, then the central zero in the connected component $\Omega(C_F)(\chi|_{C_F^1})$ parametrized according to the holomorphic parametrization

$$\chi|\cdot|_F^s,\ s\in\mathbf{C}$$

is parametrized by the complex number $s = 1/2 - ix_0$.

We could of course renormalize our very definition of Hecke L-function $L(s,\chi)$ to ensure that this point is located at $s = 1/2$, which we have not done throughout this work (such normalization is a generalization of the normalization customarily in force in discussions from the literature concerning Dirichlet L-functions and, more generally, the L-functions associated to cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A_Q})$. It is referred to as unitary normalization).

This one point of each connected component of $\Omega(C_F)$ therefore remains mysterious: understanding its multiplicity is beyond the reach of the methods of this work.

We will show in this subsection by means of examples, that in fact it does not obey the multiplicity bound of $[F:\mathbf{Q}]$ which applies to the other non-trivial zeros. As we do so, we will also show how the multiplicity bound of $[F:\mathbf{Q}]$ on the non-central non-trivial zeros, is compatible with expectations from Global Langlands and the Grand Simplicity Hypothesis, i.e. it follows as a consequence of a combination of conjectures on automorphic L-functions for $\mathrm{GL}_n(\mathbf{A_Q})$ for $n \geq 1$.

The following two conjectures can be found in the existing literature on automorphic L-functions. We write $L(s,\pi)$ for the completed and unitarily normalized L-function of a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A_Q})$. In particular, the line of symmetry of the functional equation for $L(s,\pi)$ is $\mathrm{Re}(s) = 1/2$, and the central zero of $L(s,\pi)$ is $s = 1/2$, with multiplicity $N \geq 0$.

Conjecture 14.1.2.1 (GRH and GSH). Let $n \geq 1$ be an integer, and π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A_Q})$.

Then:

(1) (GRH - Grand Riemann Hypothesis) All the zeros of the L-function $L(s,\pi)$ are located on $1/2 + i\mathbf{R}$.
(2) (GSH - Grand Simplicity Hypothesis) With the sole possible exception of $s = 1/2$, all the zeros of $L(s,\pi)$ are simple.

Our main references on the Grand Simplicity Hypothesis are the papers by Rudnick-Sarnak and Rubinstein-Sarnak (see [**Rud-Sa**] and [**Ru-Sa**, §5,p.195]). The methods contained in the former lead to the observation that the zeros of pairs of (unitarily normalized) completed L-functions of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A_Q})$ (for possibly distinct integers $n \geq 1$) are, in a precise sense, uncorrelated, with the possible exception of central zeros, if any.

In the latter (see [**Ru-Sa**, §5,p.195]) expectations on the non existence of relations of algebraic origin on the ordinates of the zeros of the L-functions of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A_Q})$ are discussed, with the exception of those concerning the existence of central zeros.

To illustrate the phenomenon concerning central zeros, we refer the reader to [**Arm**], where an example is constructed of an Artin representation ρ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ whose Artin L-function $L(s,\rho)$ vanishes at $s = 1/2$, leading to an example of a number field F whose Dedekind zeta function $\zeta_F(s)$ vanishes at $s = 1/2$ with multiplicity higher than 1. The existence of the central zero at $s = 1/2$ is caused by the sign of the root number in the functional equation for the Artin L-functions in the Artin-Takagi factorization of $\zeta_F(s)$.

Serre produced an example of another number field, a quaternionic Galois extension of $\mathbf{Q}$, with $\zeta_F(1/2) = 0$ (see [**Ng**, Ch.2,§2.5,p.23]).

Central zeros with high multiplicity are, however, expected to not exceed the multiplicity bound of Theorem 13.3.1.2, even though the multiplicity bound of Theorem 13.3.1.2 does not apply to them.

A conjecture of Serre in fact predicts that the Artin L-functions of irreducible representations of the Galois group of $\mathbf{Q}$ have only simple zeros (central zeros being no exception), and expects that a central zero exist if and only if the character of the Artin representation is real and the root number is -1. We refer the reader to [**Goss**, Ch.8,§8.24,Conjecture 8.24.1,p.324] for the statement of Serre's conjecture.

Some evidence in support of such conjecture of Serre can be found in [**Omar**, §6].

The question of whether Hecke L-functions $L(s,\chi)$ with a central zero whose multiplicity is higher than the degree of the number field, exist, has an affirmative answer. Before exhibiting explicit examples from the theory of CM elliptic curves, we will be showing that the multiplicity bound obtained in Theorem 13.3.1.2 and which applies to all other non-trivial zeros of Hecke L-functions apart from central zeros, is compatible with expectations from the literature, and specifically with Conjecture 14.1.2.1, and the Langlands Program.

Let now $\overline{\mathbf{Q}}$ be a fixed choice of separable closure of $\mathbf{Q}$. We choose, up to isomorphism, a Weil group $W_{\mathbf{Q}}$ for $\mathbf{Q}$ relative to the choice $\overline{\mathbf{Q}}$, and refer to it as *the* Weil group of $\mathbf{Q}$.

We further record here a form of Global Langlands for continuous finite dimensional complex representations of the Weil group $W_{\mathbf{Q}}$ of the field of rational numbers.

Conjecture 14.1.2.2 (Global Langlands for $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$ - Weil representations). Let $n \geq 1$ be an integer.

Let $\rho : W_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C})$ be a continuous finite dimensional complex representation of the Weil group $W_{\mathbf{Q}}$. Assume:

(1) ρ is irreducible.
(2) For each $g \in W_{\mathbf{Q}}$, $\rho(g)$ is semisimple.

Then there exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$ such that:

$$L(s,\rho) = L(s,\pi) \tag{14.1.2.1}$$

where $L(s,\rho)$ is the Artin-Weil L-function of ρ, normalized as in the discussion between [**Langlands**, Introduction,Thm.A,p.vii] and [**Langlands**, Introduction,Thm.B,p.vii], and $L(s,\pi)$ is the completed and unitarily normalized automorphic L-function of π.

We recall from the previous subsection that Artin-Weil L-functions are not assumed to be unitarily normalized throughout this work.

The L-functions arising from the Hasse-Weil L-functions of elliptic curves will instead have $\mathrm{Re}(s) = 1$ as the line of symmetry of their functional equations, as opposed to $\mathrm{Re}(s) = 1/2$, in order for this work to adhere with conventions from the literature on Hasse-Weil L-functions of elliptic curves (which will serve as our source of examples at the end of this subsection).

We will therefore be mentioning explicitly when such Hasse-Weil L-functions are being unitarily normalized.

Both Conjecture 14.1.2.1 and Conjecture 14.1.2.2 serve the sole purpose of instructing us on the expectations from the literature concerning the multiplicity of the zeros of Artin-Weil L-functions, for the purposes of testing the multiplicity bound obtained in Theorem 13.3.1.2: such multiplicities are unaffected by unitary normalization.

We will now perform the desired test on the multiplicity bound obtained in Theorem 13.3.1.2.

Assuming Conjecture 14.1.2.1 and Conjecture 14.1.2.2, we will be proving that indeed the bound obtained in Theorem 13.3.1.2 holds as stated, for all non-trivial zeros of Hecke L-functions of the number field F, with the sole possible exception of the central zero.

We will then be showing by means of examples, constructed using the theory of CM elliptic curves, that the central zero indeed does not obey such bound.

PROPOSITION 14.1.2.3 (Multiplicity Bound, on Global Langlands, GRH and GSH). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character.*

Assume that Conjecture 14.1.2.1 and Conjecture 14.1.2.2 hold. Let $\rho \in \mathbf{C}$ be a zero of $L(s,\chi)$, of multiplicity $N(\rho)$.

Then:

(1) $\rho \in 1/2 + i\mathbf{R}$.
(2) *If ρ is not the central zero, then $N(\rho) \leq [F : \mathbf{Q}]$.*

REMARK 14.1.2.4. Proposition 14.1.2.3 shows that the Multiplicity Bound obtained in Theorem 13.3.1.2 for the non-central non-trivial zeros of Hecke L-functions, is consistent with known expectations from the literature: specifically, the combination of expectations from Global Langlands, and the Grand Simplicity Hypothesis.

PROOF. We choose a separable closure $\overline{F}$ of F, and call F' the Galois closure of F in $\overline{F}$. $\chi \in \widehat{C_F}$ inflates to a continuous character of $C_{F'}$ by composition with the continuous group homomorphism $i : C_{F'} \to C_F$. Since F'/F is a finite Galois extension, we have $L(s,\chi) = L(s,\chi')$ for $\chi' = \chi \circ i$ (see [**Weil-f**, §VI,p.32] applied to $K = k = F$, and $K' = F'$, with $\varphi = \chi$, $\varphi' = \chi'$).

In particular, χ' as a continuous one-dimensional complex representation of W_F, factors through $W_{F'/F}$. We will omit reference to this fact from the notation below.

We have, by (14.1.1.13):

$$L(s,\chi) = L(s,\chi') = L(s, \mathrm{Ind}_{W_{F'/F}}^{W_{F'/\mathbf{Q}}}(\chi')). \tag{14.1.2.2}$$

By Proposition 14.1.1.1(5) parts (b) and (c) combined with (14.1.1.12), we have:

$$L(s,\chi) = \prod_{\lambda} L(s,\lambda)^{n_\lambda} \tag{14.1.2.3}$$

as λ varies over the irreducible sub-representations of $W_{\mathbf{Q}}$ in $V := \mathrm{Ind}_{W_{F'/F}}^{W_{F'/\mathbf{Q}}}(\chi')$, and n_λ is the multiplicity of λ in V (we further refer the reader to Remark 7.1.2.4).

By Conjecture 14.1.2.2, there exists, for each such λ, and upon denoting by d_λ the dimension of λ, a cuspidal automorphic representation π_λ of $\mathrm{GL}_{d_\lambda}(\mathbf{A}_{\mathbf{Q}})$ such that $L(s,\lambda) = L(s,\pi_\lambda)$ as meromorphic functions on the complex s-plane.

By Conjecture 14.1.2.1, the zeros of $L(s,\pi_\lambda)$, with the sole possible exception of the central zero, are simple and located on $1/2 + i\mathbf{R}$.

Since χ' is one-dimensional, then:

$$\dim \mathrm{Ind}_{W_{F'/F}}^{W_{F'/\mathbf{Q}}}(\chi') = [F : \mathbf{Q}]$$

being the finite homogeneous space $W_{F'/\mathbf{Q}}/W_{F'/F}$ bijective to $\mathrm{Gal}(F'/\mathbf{Q})/\mathrm{Gal}(F'/F)$, which is in turn a finite set of $[F : \mathbf{Q}]$ elements (see (14.1.1.1)).

It follows that, with the sole possible exception of the central zero, the zeros ρ of $L(s,\chi)$ are also located on $1/2 + i\mathbf{R}$, and satisfy the multiplicity bound $N(\rho) \leq [F : \mathbf{Q}]$. □

We will now prove the following:

PROPOSITION 14.1.2.5 (Abelian central zeros with high multiplicity). *Assume that for E_d the Mordell curve of affine equation $Y^2 = X^3 + d$ over $\mathbf{Q}$, we have:*

$$\mathrm{rk}\, E_d(F) \leq \mathrm{ord}_{s=1} L(E_F/F, s) \tag{14.1.2.4}$$

where $F := \mathbf{Q}(\sqrt{-3}) = \mathbf{Q}(\zeta_3)$, $L(E_F/F,s)$ is the uncompleted, non-normalized, Hasse-Weil L-function of E_F, the base change of E to F.

Then there exists an example of:

(1) *A continuous character* $\chi \in \widehat{C_F}$*, algebraic and of infinite order.*
(2) *A cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbf{A_Q})$.

such that $\rho := \mathrm{Ind}_{W_F}^{W_\mathbf{Q}}(\chi)$ *is an irreducible continuous* 2*-dimensional complex representation of* $W_\mathbf{Q}$*, satisfying:*

$$L(s,\chi) = L(s,\rho) = L(s,\pi) \tag{14.1.2.5}$$

the automorphic L*-function being unitarily normalized, and such that the order of vanishing of* $L(s,\chi)$ *at* $s = 1/2$ *exceeds* $2 = [F : \mathbf{Q}]$.

Remark 14.1.2.6. The assumption in (14.1.2.4) follows from the BSD Conjecture for E_d. The existence of the examples provided by Proposition 14.1.2.5 is therefore reliant on the BSD Conjecture, or at least on the inequality from (14.1.2.4), for the Mordell curve E_{16D_1} with D_1 as defined in [**Quer**, §1,p.215].

Proof. Let E_d be the Mordell curve over $\mathbf{Q}$ given by the affine equation $f(X,Y) = 0$ for $f \in \mathbf{Z}[X,Y]$ defined as:

$$f(X,Y) := Y^2 - X^3 - d. \tag{14.1.2.6}$$

By [**Quer**, §2,Thm.2,p.216], there exists $d \in \mathbf{Z}$, $d < 0$, such that $\mathrm{rk}\, E_d(\mathbf{Q}) = 12$. By (14.1.2.4), the order of vanishing of $L(E_F/F,s)$ at $s = 1$ is at least $\mathrm{rk}\, E_d(F) \geq \mathrm{rk}\, E_d(\mathbf{Q}) = 12$.

We choose $d := 16D_1$, for D_1 defined in [**Quer**, §1,p.215].

The elliptic curve $E := E_d$ acquires complex multiplication over the imaginary quadratic extension $F := \mathbf{Q}(\sqrt{-3}) = \mathbf{Q}(\zeta_3)$, by the ring of integers $\mathcal{O}_F = \mathbf{Z}[\zeta_3]$. By Deuring's Theorem (see [**Silverman**, Ch.10,Thm.10.5,p.175]), there exists an algebraic Hecke character $\psi \in \widehat{C_F}$ of exponent $-1/2$, such that:

$$L(E_F/F,s) = L_f(\psi|\cdot|_F^s)L_f(\overline{\psi}|\cdot|_F^s) \tag{14.1.2.7}$$

where $L(E/F,s)$ is the (uncompleted and non-normalized) Hasse-Weil L-function of the elliptic curve E_F obtained from E by base change to F, and where $L_f(\psi|\cdot|_F^s)$ is the finite Hecke L-function $L_f(s-1/2, |\cdot|_F^{1/2}\psi)$, and similarly for $L_f(\overline{\psi}|\cdot|_F^s)$.

We define $\chi := |\cdot|_F^{1/2}\psi$, a continuous character of C_F. χ is a unitary algebraic Hecke character of F of infinite order, and the Hecke L-function $L(s,\chi)$ vanishes at $s = 1/2$ if and only if $L_f(\psi|\cdot|_F) = 0$, with matching orders of vanishing at $s = 1/2$, respectively $s = 1$.

Since $\dim \mathrm{Ind}_{W_F}^{W_\mathbf{Q}}(\chi) = \dim \mathrm{Ind}_{W_F}^{W_\mathbf{Q}}(\overline{\chi}) = 2$, then either $L(s,\chi)$ or $L(s,\overline{\chi})$ vanishes at $s = 1/2$ with order of vanishing exceeding 2, since the sum of the respective orders of vanishing must be at least 12.

Without loss of generality, we assume that this is so for $L(s,\chi)$. We call $\rho := \mathrm{Ind}_{W_F}^{W_\mathbf{Q}}(\chi)$. This is a continuous irreducible 2-dimensional complex representation of $W_\mathbf{Q}$. The desired automorphic representation π of $\mathrm{GL}_2(\mathbf{A_Q})$ is given by the automorphic induction of χ (see [**JL**, §12,Thm.12.2,p.197]), and it is cuspidal since $\chi \neq \overline{\chi}$ (see [**JL**, §12,Prop.12.1,p.197]). □

Remark 14.1.2.7. An example of a Mordell curve E_d, for $d \in \mathbf{Z}$, $d < 0$, of Mordell-Weil rank at least 17, was announced by Elkies in February 2016.

Remark 14.1.2.8. We note that, under the assumption of (14.1.2.4), itself a consequence of the BSD Conjecture for Mordell elliptic curves over $\mathbf{Q}$, Proposition 14.1.2.5 implies that if we replace the Galois group of $\mathbf{Q}$ with the Weil group $W_\mathbf{Q}$, the analog of Serre's Conjecture from [**Goss**, Ch.8,§8.24,Conjecture 8.24.1,p.324] does not hold.

Let F be a number field, and $\overline{F}$ a fixed choice of separable closure.

The reader will note that the Multiplicity Bound from Theorem 13.3.1.2 obtained on the generic zeros of Hecke L-functions, is a priori not sharp.

Indeed, instead of $[F:\mathbf{Q}]$, the method behind the trace calculation leading to Theorem 13.3.1.1 and hence to Theorem 13.3.1.2 through the Positivity Criterion of Weil, in its form of Theorem **??**, it should rather be $m_F(\chi)$ supremum of the integers $m_p(\chi)$, as p varies over all prime numbers, and where m_p is the highest multiplicity of the poles of the meromorphic functions:

$$\prod_{v\in S_p} L_v(\chi_v|\cdot|_v^s)$$

as $\chi\in\widehat{C_F}$ is a fixed continuous character, and $L_v(\chi_v|\cdot|_v^s)$ is the local Euler factor at the non-Archimedean place v. Recall that S_p is the finite set of finite places of F extending the p-adic absolute value on $\mathbf{Q}$.

The integer $m_F(\chi)$ satisfies $m_F(\chi)\leq[F:\mathbf{Q}]$ by Theorem 7.1.3.2, but it may be strictly smaller.

In fact, it likely is, since, proceeding heuristically by comparison with a form of Chebotarev (for which we refer the reader to [**Moroz**, Appendix 1,Thm.2,p.68] and [**KleMo**, Thm.4]), a reasonable expectation, under the assumption of the holomorphy of the Artin-Weil L-functions of non-trivial irreducible finite dimensional continuous complex representations of $W_{\mathbf{Q}}$, for the probability that a prime number p below $x>0$ will be such that m_p attains the maximum allowed value of $[F:\mathbf{Q}]$, is:

$$\frac{\mu_{S_\infty,1}(U)}{[F:\mathbf{Q}]}\int_2^x\frac{\mathrm{d}y}{\log(y)}$$

where $U\subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1/\mathcal{O}_F^\times$ is the image in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1/\mathcal{O}_F^\times$ of the maximal closed subset of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ on which the restriction of χ to $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is trivial, and μ_{S_∞} is the multiplicative Tamagawa measure on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, with $\mu_{S_\infty,1}$ the Haar measure on the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ obtained by choosing a continuous homomorphic section to the S_∞-idelic norm.

However, $\mu_{S_\infty,1}(U)=0$ for most Hecke characters χ. U will have non-empty interior for χ of finite order, and more generally for Hecke characters of Galois type (i.e. of the form $\chi|\cdot|_F^{ix_0}$ for some $x_0\in\mathbf{R}$, possibly nonzero, and χ of finite order). I know of no other example where $\mu_{S_\infty,1}(U)$ is nonzero.

For Hecke characters χ of Galois type, indeed $m_F(\chi)=[F:\mathbf{Q}]$, and so the choice of Sobolev degree $s=2[F:\mathbf{Q}]$ made to perform the trace calculation leading to Theorem 13.3.1.1, is forced.

A similar argument as in Proposition 14.1.2.3, using the proof of Proposition 14.1.1.1(5.a), shows that $m_F(\chi)$ will always be larger than the maximum of the dimensions of the irreducible sub-representations of $W_{\mathbf{Q}}$ occurring as the irreducible constituents of $\mathrm{Ind}_{W_{F'/F}}^{W_{F'/\mathbf{Q}}}(\chi')$, for χ' the inflation of χ to the Galois closure of F' in $\overline{F}$, multiplied by the multiplicity with which such irreducible constituent occurs.

This is also compatible with heuristics on the non-correlation of the zeros of the Artin-Weil L-functions of such constituents, with the possible exception of central zeros (we refer the reader to [**Ru-Sa**, §5]). If we are to trust the Grand Simplicity Hypothesis, then the multiplicity of the zeros of $L(s,\chi)$, with the exception of a possible central zero, should be bounded above by the maximum of the multiplicities of the irreducible constituents of $\mathrm{Ind}_{W_{F'/F}}^{W_{F'/\mathbf{Q}}}(\chi')$, for χ' the inflation of χ to the Galois closure of F' in $\overline{F}$, and this integer is therefore lower than $m_F(\chi)$.

Computing $m_F(\chi)$ explicitly appears to be arduous, and choosing the Sobolev degree $s=2[F:\mathbf{Q}]$ for the purposes of performing the trace calculation leading to Theorem 13.3.1.1, done for simplicity and practicality, appears to be a natural course of action, and leads to the Multiplicity Bound of $[F:\mathbf{Q}]$ on generic zeros (i.e. non-central zeros).

The question of whether the multiplicity bound of $[F:\mathbf{Q}]$ on generic zeros can be improved to an explicit bound that is lower than $[F:\mathbf{Q}]$, is an important one. It appears to remain at a strictly deeper level than the methods of this work, and of its higher rank analogs, which also lead to weaker bounds than those predicted by the Grand Simplicity Hypothesis.

14.2. The case of global function fields of positive characteristic

In this subsection, we answer the question of whether the Trace Formula from Theorem 13.3.1.1 holds when F is a global function field of positive prime characteristic p, and whether it leads to a novel approach to GRH for the Hecke L-functions of global fields of positive characteristic.

The short answer is "no" to both parts of the question.

The trace calculation cannot be performed within the Sobolev ambient that is responsible for the constraints on the zeros contributing to the spectral side (on their location and multiplicity).

Let p be a prime number, and q a positive integer power of p, i.e. $q = p^d$ for some integer $d \geq 1$. Let X be a smooth projective geometrically connected curve over $k := \mathbf{F}_q$, and F its function field, $F := k(X)$.

We will be referring to F as a *global function field*.

While the methods from Chapters 2,3,4 and 5 adapt to the case of such F, as do the main results from [**CF**, Ch.XV], instrumental to these Chapters (see [**RV**, Ch.7,§7.2,p.264]), it is Chapter 6 and Chapter 7 that are unavailable for such F. It is not possible to perform the necessary trace calculations and within a Sobolev ambient, as is done for F a number field.

The reason can be traced directly to the Euler product of the zeta function of X:

$$\zeta(X,s) = \prod_{x \in |X|} \frac{1}{1 - |k(x)|^{-s}} \tag{14.2.0.1}$$

where the product is indexed by the closed points of X, and $k(x)$ is the residue field of X at the closed point $x \in X$, itself a finite field of size $|k(x)|$.

The size $|k(x)|$ of the residue field $k(x)$ is an integer divisible by q for *all* closed points $x \in X$, and hence the Euler factors of $\zeta(X,s)$ have infinitely many poles in common. This is the key difference with the case of F a number field: for any closed point $x \in \mathrm{Spec}(\mathcal{O}_F)$, only finitely many closed points $y \in \mathrm{Spec}(\mathcal{O}_F)$ have the same residue characteristic, i.e. are in the same fibre over $x_p : \mathrm{Spec}(\mathbf{F}_p) \to \mathrm{Spec}(\mathbf{Z})$.

Let $f : X \to \mathbf{P}^1_{\mathbf{F}_q}$ be a non constant morphism, hence finite flat surjective.

This amounts to choosing a structure on F of finite field extension of $\mathbf{F}_q(t)$, i.e. a field embedding $\iota = f^* : \mathbf{F}_q(t) \subset F$.

Since f is finite, it has finite fibres. However, correcting the superposition of the poles of the Euler factors of $\zeta(X,s)$ as is done for Euler factors over distinct prime numbers in the number field case, by assuming the vanishing of the test function along with finitely many of its derivatives, cannot be done for F a global function field, since such vanishing condition would have to hold for the holomorphic derivatives of arbitrarily high order of the Mellin transform of the test function, an entire function on the Riemann surface $\Omega(C_F)$ (whose connected components are, for F a global function field, non simply connected).

Such correction enables the deconstruction of the global geometric kernel, and it is at this early stage that the trace calculation cannot be carried out any further: only finitely many of the infinitely many functionals (obtained as derivatives of point-evaluation functionals supported at points of $\Omega(C_F)$, with unbounded order) required for the method to adapt, remain continuous on $W^{s,2}(C_F)$, shifting the burden of correcting the discrepancy between the S_1- and S_2-limits of the analogs of the finite rank operators from Theorem 11.2.2.1, onto the test function, forcing the necessity of vanishing conditions to hold for its Mellin transform and its derivatives of unbounded order. A necessity that cannot be fulfilled, since no nonzero holomorphic function exists on any connected non-empty complex domain, which vanishes at a point, with infinite order of vanishing.

In short, the Theorem on Polar Multiplicities (see Theorem 7.1.3.2) does not hold, when F is a global function field.

What is more, is that not only Euler factors have infinitely many poles in common, but they repeat themselves as Euler factors infinitely many times. As many as the number of closed points $x \in X$ whose residue field has fixed degree $m \geq 0$ over $\mathbf{F}_q$.

This number is estimated from the Weil bound and it is:

$$\frac{q^m}{m} + O_X(q^{m/2}/m)$$

as $m \to +\infty$ (see [**TsV**, Ch.2.3,(2.3.2),p.177] combined with the inequality $mB_m \leq N_m - N_1$, with B_m the number of closed points of X of degree m, and $N_m := |X(\mathbf{F}_{q^m})|$).

The implied constant depends on the curve through its genus.

An analog of the trace calculation underlying Theorem 13.3.1.2 can indeed be made, although the topology on $\mathrm{H}_{0,\mathrm{cont}}(F^\times, \mathcal{S}_0(\mathbf{A}_F))$ and $\mathcal{S}_{\mathrm{E}}(C_F)$ that serves the desired purpose is non-normable, rendering the kernels of the continuous operators on each Hausdorff locally convex topological $\mathbf{C}$-vector space only *generalized* functions, as opposed to measurable functions on the idele classes.

This in turn leads us to employing topologies that are fine enough to render infinite compositions of the Principal Value Operators of Weil (see (5.3.1.5)), i.e. indexed by infinitely many places of F, convergent with respect to the topology of uniform convergence on bounded subsets. These serve as a replacement the operation of multiplying a measurable kernel function $k(x,y)$ on $\mathrm{GL}_1(\mathbf{A}_F) \times C_F$ by an indicator function supported over $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \prod_{v \notin S} \mathcal{O}_{F_v}) \times C_F$ to restrict $k(x,y)$ to smaller loci of the product.

The desired convergence of such countably infinite compositions only holds with respect to topologies that are fine enough to promote multiplication by $|\cdot|_F^a$ for $a \in [0, 1+\varepsilon]$ to a continuous and continously invertible endomorphism on the resulting completion of $\mathcal{S}_{\mathrm{E}}(C_F)$ for all small enough $\varepsilon > 0$, to access the absolute convergence of the Euler product of Hecke L-functions over the half plane of absolute convergence. We refer the reader to the proof of [**Me**, §5,Lemma 5.3,p.39,40], for a similar assertion and connection to convergence of operator-theoretic Euler products.

We are lead to a trace formula where the spectral side involves no non-trivial constraints on the non-trivial zeros of $\zeta(X,s)$: if we did not know GRH for $\zeta(X,s)$ as an input.

In other words, the resulting trace formula by itself would not inform us on the location nor on the multiplicity of such zeros. It would recover Weil's own Explicit Formulas only.

This has been accomplished by R. Meyer in [**Me**, §5], and a trace calculation along the lines being alluded to in the foregoing would be close to the one performed in [**Me**, §5,Thm.5.8,p.43], leading to a representation-theoretic proof of the Explicit Formulas only.

Not only does the method of trace calculation not adapt, for F a global function field, but the Multiplicity Bound obtained in Theorem 11.2.2.1 for F a number field, does not hold in its analogous formulation.

Indeed, if it did, then it would be so for any choice of non-constant morphism $f : X \to \mathbf{P}^1_{\mathbf{F}_q}$, and hence it would imply that the multiplicity of the Frobenius eigenvalues on $\mathrm{H}^1_{\text{ét}}(X_{k^{\mathrm{sep}}}, \overline{\mathbf{Q}}_\ell)$, the geometric ℓ-adic cohomology of X, for ℓ a prime invertible on k^{sep}, a separable closure of $k = \mathbf{F}_q$ would be bounded above by the gonality of the curve X. This is most definitely not the case, due to additional symmetries of X that may arise in positive characteristic: maximal curves provide counterexamples to this assertion.

Maximal curves over $k = \mathbf{F}_{q^2}$ are such that the characteristic polynomial of geometric Frobenius on $\mathrm{H}^1_{\text{ét}}(X_{k^{\mathrm{sep}}}, \overline{\mathbf{Q}}_\ell)$ is $P_1(X) \in \mathbf{Z}[X]$, with:

$$P_1(X) = (X+q)^{2g}$$

a consequence of the relative Frobenius acting on the geometric Jacobian

$$J(X) = \mathrm{Pic}^0_{X_{k^{\mathrm{sep}}}/k^{\mathrm{sep}}}$$

of X as multiplication by $-q$.

It follows that while the multiplicity of each Frobenius eigenvalue is $2g$, the gonality of the curve is strictly lower, being equal to $g+1$ (for g the genus of X). We refer the reader to [**FGT**], for more on maximal curves (including examples and a partial classification in terms of Hermitian curves).

We further note that already over $\mathbf{F}_q(t)$, the analog of Serre's conjecture from [**Goss**, Ch.8,§8.24,Conjecture 8.24.1] on simplicity of the zeros of Artin L-functions of irreducible representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ does not hold.

Indeed, for $q = p^2$, we let E be a supersingular elliptic curve over $\mathbf{F}_q$, a double cover $f : E \to \mathbf{P}^1_{\mathbf{F}_q}$ of the projective line. By Geometric Class Field Theory, the associated quadratic character χ_f of $\mathbf{F}_q(t)$ is such that $L(s, \chi_f)$ has a double zero.

14.3. The higher rank case

For π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, the bound on generic zeros to which we are led by the $n > 1$ version of these methods is:

$$n[F : \mathbf{Q}].$$

In particular, for $F = \mathbf{Q}$, the bound is the rank n itself, which shows how the Grand Simplicity Hypothesis remains beyond the reach of the methods developed throughout this work, and its version GL_n.

We will now share a few elements of the version of these methods for $\mathrm{GL}_n(\mathbf{A}_F)$, i.e. to analyze the divisor of the completed, unitarily normalized, L-function of a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, to highlight what changes from the case $n = 1$.

[2]

[2]Say a few words on GL_n. Add that the version for GL_n is the subject of ongoing work with Brian Conrad.

APPENDIX A

Preparations from Algebraic Number Theory

In this appendix we review some classic results in algebraic number theory, encompassing compactness of the norm-1 subgroup of the idele class group C_F of a number field F, along with the construction of normalized Haar measures on $\mathrm{GL}_1(\mathbf{A}_F)$, C_F, and their closed subgroups of norm-1 elements, as well as the construction of fundamental domains for the F-action on $\mathbf{A}_F$ and the $F^\times$-action on $\mathrm{GL}_1(\mathbf{A}_F)^1$. We also review an adelic version of the theory of Hardy-Littlewood maximal functions for use in our work with measurable adelic kernel functions.

For F a number field, we denote by:

(1) Σ_F the set of places of F: equivalence classes of non-trivial absolute values (in the sense of [**CF**, Ch. II, §13-14]).
(2) S_∞ the set of Archimedean places of F, r_1 the number of real places, r_2 the number of complex places.
(3) $S_{F,f}$ or S_f the set of non-Archimedean places of F. We refer to its elements as the *finite* places.

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A.1. Topology of $\mathbf{A}_F$, $\mathbf{A}_{F,S}$ and $\mathrm{GL}_1(\mathbf{A}_F)$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, and their class groups

In this section we review some basic notation and definitions related to the adele ring of a number field F and the idele group $\mathrm{GL}_1(\mathbf{A}_F)$. We will henceforth adopt the terminology and conventions from [**CF**, Ch. II]. In particular, the square of the complex Euclidean norm is an absolute value according to our conventions. Its equivalence class contains the complex Euclidean norm. We moreover note that a place v of F is required to be non-trivial. Proofs for assertions below are given in many references (such as [**CF**] and [**RV**]), so they are generally not given here.

A.1.1. The adeles $\mathbf{A}_F$ as a topological ring. Let F be a number field. In this subsection we recall the construction of the adele ring $\mathbf{A}_F$ and its topological ring structure.

A.1.1.1. *Recollections on restricted products.* We recall the definition of restricted product of a collection of pairs (G_i, K_i), indexed by a set I, where G_i is a Hausdorff locally compact abelian group, and $K_i \subset G_i$ is a compact open subgroup.

Definition A.1.1.1. Let I_0 be a *finite* set, I_1 a set, and write $I := I_0 \times I_1$. Let $\{G_i\}_{i\in I}$ be an I-indexed collection of Hausdorff locally compact abelian groups, and $\{K_i\}_{i\in I}$ an I-indexed collection of subgroups $K_i \subset G_i$ for each $i \in I$, where $K_i = 1$ for every $i \in I_0$, and K_i is compact open in G_i for every $i \in I_1$.

We define

$$\widehat{\prod_{i\in I}}(G_i, K_i) \subset \prod_{i\in I} G_i \tag{A.1.1.1}$$

as the subset of those I-indexed tuples $(x_i) \in \prod_{i\in I} G_i$ such that $x_i \in K_i$ for all but finitely many $i \in I$. We topologize $\widehat{\prod}_{i\in I}(G_i, K_i)$ by defining a neighborhood base at 1 to be given by the subsets of the form $\prod_{i\in I} U_i$, where U_i is a neighborhood of 1 in G_i for every $i \in I$, and $U_i = K_i$ for all but finitely many $i \in I$.

We refer to $\widehat{\prod}_{i\in I}(G_i, K_i)$ as the *restricted product* of the collection $\{(G_i, K_i)\}_{i\in I}$.

We write $\mathbf{A}_F$ for the adele ring of the number field F.

Recall that, upon writing v for the arbitrary place of F, this is the topological ring defined as the restricted direct product

$$\mathbf{A}_F := \widehat{\prod_{v}}(F_v, R_v) \tag{A.1.1.2}$$

of the local fields F_v, each a finite extension of either $\mathbf{R}, \mathbf{C}$, or $\mathbf{Q}_p$ for a unique prime number p, and where R_v is the valuation ring $\mathcal{O}_{\mathfrak{p}}$ in case $v = v_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, and $R_v = F_v$ for v Archimedean.

We denote by

$$\iota : F \to \mathbf{A}_F \tag{A.1.1.3}$$

the ring homomorphism sending $\xi \in F$ to the adele whose local component at the place $v \in \Sigma_F$ is the image of ξ under the completion map $\iota_v : F \to F_v$. We call ι the *diagonal embedding* of F in $\mathbf{A}_F$.

We refer the reader to [**RV**, Ch. 5, §5.1], [**DE**, Ch. 13], [**CF**, Ch. II, §13-14], and Tate's thesis [**CF**, Ch. XV, §3, §4.1] for more on restricted products and the adeles of a number field.

We recall the following description of the topology on the restricted product of Hausdorff locally compact abelian groups:

Lemma A.1.1.2. *Let I_0 be a finite set, I_1 a set, and $I := I_0 \times I_1$. Let $\{(G_i, K_i)\}_{i\in I}$ be an I-indexed collection of pairs (G_i, K_i) of Hausdorff locally compact abelian groups, where $K_i = 1$ for all $i \in I_0$ and $K_i \subset G_i$ is compact open for all $i \in I_1$. Then*

(1) *The restricted product $\widehat{\prod}_{i\in I}(G_i, K_i)$ is a Hausdorff locally compact abelian group.*

(2) *Let $\{K_i'\}_{i\in I}$ be an I-indexed collection of subgroups $K_i' \subset G_i$, with $K_i' = 1$ for all $i \in I_0$, $K_i' \subset G_i$ compact open for all $i \in I_1$, and $K_i' = K_i$ for all but finitely many $i \in I$. Then the restricted product $\widehat{\prod}_{i\in I}(G_i, K_i')$ is canonically isomorphic, as a Hausdorff locally compact abelian group, to the restricted product $\widehat{\prod}_{i\in I}(G_i, K_i)$.*

(3) *A subset $H \subset \widehat{\prod}_{i\in I}(G_i, K_i)$ has compact closure if and only if there exists an I-indexed collection $\{H_i\}_{i\in I}$ with $H_i \subset G_i$ a compact subset for every $i \in I$, and $H_i = K_i$ for all but finitely many $i \in I$, and we have $H \subset \prod_{i\in I} H_i$.*

(4) *For any partition $I = J_1 \coprod J_2$ with J_1 a finite subset, the canonical map*

$$\prod_{j\in J_1} G_i \times \widehat{\prod_{j\in J_2}} (G_j, K_j) \to \widehat{\prod_{i\in I}}(G_i, K_i) \tag{A.1.1.4}$$

is an isomorphism of Hausdorff locally compact abelian groups.

(5) *For any partition $I = J_1 \coprod J_2$ with J_1 a finite subset, the subset*

$$\prod_{j\in J_1} U_i \times \prod_{j\in J_2} K_j \subset \widehat{\prod_{i\in I}}(G_i, K_i) \tag{A.1.1.5}$$

with $U_i \subset G_i$ an open subset, is open.

Definition A.1.1.3. Let F be a number field. We define:

(1) $\mathbf{A}_{F,S_\infty}$ as the topological ring $\mathbf{R} \otimes_{\mathbf{Q}} F = \prod_{v|\infty} F_v$ topologized with the unique $\mathbf{R}$-linear topology induced by the Euclidean metric topology on $\mathbf{R}$.

(2) $\mathbf{A}_F^f$ as the topological ring obtained as the restricted product of the pairs $(F_v, \mathcal{O}_v)$ as v ranges over all the finite places of F. We refer to $\mathbf{A}_F^f$ as the topological ring of the *finite adeles*.

The topological ring $\mathbf{A}_F$ can be equivalently described as the product topological ring

$$\mathbf{A}_{F,S_\infty} \times \mathbf{A}_F^f \tag{A.1.1.6}$$

topologized with the product topology. This is a consequence of Lemma A.1.1.2(4) with $J_1 = S_\infty$ and $J_2 = \Sigma_{F,f}$ the respective sets of Archimedean and non-Archimedean places of F.

A.1.2. The adeles as an abelian topological group. As an abelian topological group, the additive group underlying the topological ring $\mathbf{A}_F^f$ is the inductive limit of the abelian topological groups $\prod_{v\nmid\infty} H_v$ where $H_v \subset F_v$ is a compact open subgroup for every non-trivial finite place v, and $H_v = \mathcal{O}_v$ for all but finitely many such v.

Each $\prod_{v\nmid\infty} H_v$ is in turn contained in a compact open subgroup of $\mathbf{A}_F^f$ of the form $(1/N)\widehat{\mathcal{O}}_F$ for some integer $N \geq 1$, where $\widehat{\mathcal{O}}_F$ is the completion of $\mathcal{O}_F$ as a discrete additive abelian group with respect to the topology generated by the family of proper subgroups $\{I\mathcal{O}_F\}$ as I varies over the proper ideals of the ring $\mathcal{O}_F$.

A.1.2.1. *Finiteness results on $\mathbf{A}_F$.* We record below certain results on the topology of $\mathbf{A}_F$, its class group $\mathbf{A}_F/F$, and S-local analogs.

Lemma A.1.2.1. *Let F be a number field. The following short exact sequence of abelian groups*

$$0 \to F \xrightarrow{\iota} \mathbf{A}_F \to \mathbf{A}_F/F \to 0 \tag{A.1.2.1}$$

where the first homomorphism is the diagonal embedding induced by the completion field homomorphism $\iota_v : F \to F_v$ for every place v of F, is topologically exact in the strict sense, i.e. ι is a closed homomorphism upon endowing F with the discrete topology. Moreover

(1) *The image of ι is discrete.*

(2) *The quotient topological group $\mathbf{A}_F/F$ is compact Hausdorff.*

(3) *The abelian group $\mathbf{A}_F/F$ is uniquely divisible, and the canonical continuous homomorphism of Hausdorff locally compact abelian group $\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F \to \mathbf{A}_F$ induces an isomorphism of Hausdorff locally compact abelian groups*

$$(\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F)/\mathcal{O}_F \to \mathbf{A}_F/F. \tag{A.1.2.2}$$

Formation of the sequence in (A.1.2.1) *and of the isomorphism in* (A.1.2.2) *is functorial with respect to field homomorphisms of finite extensions F'/F of the number field F.*

A.1.3. The idele group $\mathrm{GL}_1(\mathbf{A}_F)$. In this subsection we record basic properties of the idelic topology on the group $\mathrm{GL}_1(\mathbf{A}_F)$.

A.1.3.1. *The idelic topology on* $\mathrm{GL}_1(\mathbf{A}_F)$. Central to number theory is the group of units $\mathbf{A}_F^\times$ of $\mathbf{A}_F$. The subspace topology on $\mathbf{A}_F^\times$ does not render, however, $\mathbf{A}_F^\times$ a topological group, insofar as inversion is not continuous. Moreover, $\mathbf{A}_F^\times$ is not open in $\mathbf{A}_F$. The group $\mathbf{A}_F^\times$ is therefore re-topologized:

Definition A.1.3.1. Let F be a number field. We define the *idelic topology* on the unit group $\mathbf{A}_F^\times$ of the adele ring $\mathbf{A}_F$ to be the topology induced by the anti-diagonal map

$$\mathbf{A}_F^\times \to \mathbf{A}_F \times \mathbf{A}_F \tag{A.1.3.1}$$

sending x to (x, x^{-1}). We denote by $\mathrm{GL}_1(\mathbf{A}_F)$ the group $\mathbf{A}_F^\times$ endowed with the idelic topology, and refer to it as the *idele group* of F.

We refer the reader to [**CF**, Ch. II, §16] and to [**DE**, §13.3], for more on the ideles of a number field. We will be recalling further facts on adeles and ideles as the need arises.

Remark A.1.3.2. The task of topologizing the adelic points of algebraic groups can be carried out following Weil's approach in [**Weil-c**] (see, specifically, [**Weil-c**, Ch. III]) by means of a process of "adelization", or Grothendieck's, by topologizing the adelic points of the underlying scheme. We refer the reader to [**Conrad**, §2] for an account of the methods and history around the task of topologizing the adelic points of an algebraic group, of which the first special non-trivial case of $(\mathrm{GL}_1)_F$ leads to the ideles $\mathrm{GL}_1(\mathbf{A}_F)$.

One more piece of information about $\mathrm{GL}_1(\mathbf{A}_F)$ is that there exists a canonical homomorphism of abelian topological groups

$$|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0} \tag{A.1.3.2}$$

defined as

$$|g|_F := \prod_{v \in \Sigma_F} |g_v|_v,\ g \in \mathrm{GL}_1(\mathbf{A}_F) \tag{A.1.3.3}$$

where $|\cdot|_v$ is the normalized absolute value on F_v contained in $v \in \Sigma_F$.

The assignment of $|g|_F$ in (A.1.3.3) converges since $g_v \in \mathcal{O}_v^\times$ for all but finitely many v.

We define the subgroup

$$\mathrm{GL}_1(\mathbf{A}_F)^1 := \ker(|\cdot|_F) \subset \mathrm{GL}_1(\mathbf{A}_F) \tag{A.1.3.4}$$

of ideles of "norm 1".

An important feature of the idelic topology on $\mathrm{GL}_1(\mathbf{A}_F)$ is that, contrary to the subspace topology on $\mathbf{A}_F^\times$, it promotes $\mathbf{A}_F^\times$ to a Hausdorff locally compact abelian group.

Lemma A.1.3.3. *Let F be a number field. The idele group $\mathrm{GL}_1(\mathbf{A}_F)$ is a Hausdorff locally compact abelian group with respect to the idelic topology, and isomorphic to the restricted product*

$$\widehat{\prod_{v \in \Sigma_F}} (F_v^\times, \mathcal{O}_{F_v}^\times).$$

The idelic norm $|\cdot|_F$ *from* (A.1.3.2) *is a continuous homomorphism of Hausdorff locally compact abelian groups, and* $\mathrm{GL}_1(\mathbf{A}_F)^1 \subset \mathrm{GL}_1(\mathbf{A}_F)$ *is a closed subgroup.*

A.1.3.2. *Discreteness of the* $F^\times$*-action on* $\mathrm{GL}_1(\mathbf{A}_F)$. In this subsection we recall basic results on the continuous left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)$, and its orbits.

LEMMA A.1.3.4. *Let* F *be a number field. The following short exact sequence*

$$1 \to F^\times \xrightarrow{\iota} \mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_F)/F^\times \to 1 \tag{A.1.3.5}$$

where the first homomorphism is the diagonal embedding induced by the completion field homomorphism $\iota_v : F \to F_v$ *for every place* v *of* F*, is topologically exact in the strict sense, i.e.* ι *is a closed homomorphism upon endowing* $F^\times$ *with the discrete topology. Formation of the sequence in* (A.1.3.5) *is functorial with respect to field homomorphisms of finite extensions of* F'/F *of the number field* F*.*

(1) *The image of* ι *is discrete.*
(2) *The quotient topological group* $\mathrm{GL}_1(\mathbf{A}_F)/F^\times$ *is Hausdorff locally compact.*
(3) *The idelic norm* $|\cdot|_F$ *factors through the quotient* $\mathrm{GL}_1(\mathbf{A}_F)/F^\times$*. The induced continuous surjective homomorphism*

$$|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F)/F^\times \to \mathbf{R}_{>0} \tag{A.1.3.6}$$

admits a continuous homomorphic section.

Some explicit construction of the asserted section in (3) are given in §A.2.2.

A.1.4. The idele class group C_F**.** In this subsection we recall the construction and main results on the class group C_F of the idele group, and their consequences.

A.1.4.1. *Finiteness results on the idele class group.*

DEFINITION A.1.4.1. Let F be a number field. We define

$$C_F := \mathrm{GL}_1(\mathbf{A}_F)/F^\times \tag{A.1.4.1}$$

as the quotient Hausdorff locally compact abelian group. We refer to C_F as the *idele class group* of F.

As per Lemma A.1.3.4(3), the idelic norm $|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$ descends to a continuous homomorphism of Hausdorff locally compact abelian groups

$$|\cdot|_F : C_F \to \mathbf{R}_{>0} \tag{A.1.4.2}$$

whose kernel we denote C_F^1, a compact Hausdorff subgroup by Lemma A.1.3.4(3).

REMARK A.1.4.2. Let F be a number field. The strict exact sequence of continuous homomorphisms of Hausdorff locally compact abelian groups from (A.1.3.5) induces the following short exact sequence

$$1 \to F^\times \xrightarrow{\iota} \mathrm{GL}_1(\mathbf{A}_F)^1 \to C_F^1 \to 1 \tag{A.1.4.3}$$

which is topologically exact in the strict sense, i.e. ι is a closed homomorphism upon endowing $F^\times$ with the discrete topology. In particular, C_F^1 is isomorphic, as a Hausdorff locally abelian group, to $\mathrm{GL}_1(\mathbf{A}_F)^1/F^\times$.

We recall the following fundamental result about the Hausdorff locally compact abelian group C_F and its closed subgroup C_F^1.

THEOREM A.1.4.3. *Let* F *be a number field.*

(1) *The closed subgroup* $C_F^1 \subset C_F$ *is compact Hausdorff.*

(2) *The unit group $\mathcal{O}_F^\times$ is finitely generated, and its maximal free subquotient has rank $r_1 + r_2 - 1$.*
(3) *The class group $\mathrm{Cl}(\mathcal{O}_F)$ of fractional ideal classes (equal to the group $\mathrm{Pic}(\mathcal{O}_F)$ of divisor classes) is finite.*

A.1.5. S-adeles, S-ideles and their class groups. We let F be a number field, S a finite set of places of F containing all of the Archimedean places. We denote by S_f the subset of all the finite places of F contained in S, i.e. $S_f := \Sigma_{F,f} \cap S$. We study in this subsection S-local analogs of $\mathbf{A}_F$ and $\mathrm{GL}_1(\mathbf{A}_F)$, and their class groups $\mathbf{A}_F/F$ and C_F.

A.1.5.1. *S-adeles and S-ideles.* We denote by $\mathbf{A}_{F,S}$ the finite product topological ring:

$$\mathbf{A}_{F,S} := \prod_{v \in S} F_v \tag{A.1.5.1}$$

topologized with the product topology, and by

$$\mathrm{GL}_1(\mathbf{A}_{F,S}) = \prod_{v \in S} F_v^\times \tag{A.1.5.2}$$

the group of units of $\mathbf{A}_{F,S}$, topologized with the subspace topology. Clearly this topology on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ agrees with the product topology upon endowing each $F_v^\times \subset F_v$ with the subspace topology. The group $\mathrm{GL}_1(\mathbf{A}_{F,S})$ is thereby promoted to a Hausdorff locally compact topological group.

We refer to $\mathbf{A}_{F,S}$ as the *S-adele ring* of F, and to $\mathrm{GL}_1(\mathbf{A}_{F,S})$ as the *S-idele group* of F. The ring of S-integers in F is

$$\mathcal{O}_{F,S} = \{\xi \in F : \iota_v(\xi) \in \mathcal{O}_{F_v} \text{ for all } v \notin S\}.$$

We denote by

$$\iota_S : \mathcal{O}_{F,S} \to \mathbf{A}_{F,S} \tag{A.1.5.3}$$

the natural inclusion.

A.1.5.2. *The ring of S-adeles $\mathbf{A}_{F,S}$ and its class group $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$.* We have the following S-local analog of Lemma A.1.2.1. Recall that we denote by S_f the subset of all the finite places of F contained in S, i.e. $S_f := \Sigma_{F,f} \cap S$.

For every $v \in S_f$, we denote by $\mathfrak{p}_v \subset \mathcal{O}_F$ the prime ideal of $\mathcal{O}_F$ corresponding to the finite place $v \in S$.

Lemma A.1.5.1. *Let F be a number field, S a finite set of places containing the Archimedean places of F. The following short exact sequence of abelian groups*

$$0 \to \mathcal{O}_{F,S} \xrightarrow{\iota_S} \mathbf{A}_{F,S} \to \mathbf{A}_{F,S}/\mathcal{O}_{F,S} \to 0 \tag{A.1.5.4}$$

where the first homomorphism is the diagonal embedding induced by the completion field homomorphism $\iota_v : F \to F_v$ for every place $v \in S$ of F, is topologically exact in the strict sense, i.e. ι_S is a closed homomorphism upon endowing $\mathcal{O}_{F,S}$ with the discrete topology. Moreover

(1) *The image of ι_S is discrete.*
(2) *The quotient topological group $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ is compact Hausdorff.*
(3) *The continuous quotient homomorphism*

$$(\mathbf{A}_{F,S} \times \prod_{v \in \Sigma_F - S} \mathcal{O}_{F_v})/\mathcal{O}_{F,S} \to \mathbf{A}_{F,S}/\mathcal{O}_{F,S}$$

is a proper continuous quotient homomorphism of Hausdorff locally compact abelian groups, whose source is isomorphic to $\mathbf{A}_F/F$ under the continuous injective homomorphism induced by $\mathbf{A}_{F,S} \times \prod_{v \in \Sigma_F - S} \mathcal{O}_{F_v} \to \mathbf{A}_F$. In particular, $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ is naturally a $\mathbf{Q}$-vector space.

A.1.5.3. *The topological group of S-ideles $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and its class group $C_{F,S}$.* We define the following S-local analog of the idelic norm from (A.1.3.2):

$$|\cdot|_{F,S} : \mathrm{GL}_1(\mathbf{A}_{F,S}) \to \mathbf{R}_{>0} \tag{A.1.5.5}$$

by the assignment:

$$|g|_{F,S} := \prod_{v\in S} |g_v|_v \tag{A.1.5.6}$$

for every $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$.

We refer to $|\cdot|_{F,S}$ as the *S-idelic norm*. We define

$$C_{F,S} := \mathrm{GL}_1(\mathbf{A}_{F,S} \times \prod_{v\notin S} \mathcal{O}_{F_v})/\mathcal{O}^\times_{F,S} \tag{A.1.5.7}$$

and refer to it as the *S-idele class group*. We note that by Lemma A.1.5.1(1), $C_{F,S}$ is a Hausdorff locally compact abelian group. Moreover, we have $|\mathcal{O}^\times_{F,S}|_{F,S} = 1$, and hence the S-idelic norm descends uniquely to a continuous surjective homomorphism of Hausdorff locally compact abelian groups:

$$|\cdot|_{F,S} : C_{F,S} \to \mathbf{R}_{>0}. \tag{A.1.5.8}$$

Denote by $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, respectively $C^1_{F,S}$, the kernel of $|\cdot|_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})$, respectively on $C_{F,S}$, so $C^1_{F,S} \simeq (\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times \prod_{v\notin S} \mathcal{O}^\times_{F_v})/\mathcal{O}^\times_{F,S}$ under the canonical quotient map $\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \prod_{v\notin S} \mathcal{O}^\times_{F_v} \to C_{F,S}$ since $|\mathcal{O}^\times_{F,S}|_{F,S} = |\mathcal{O}^\times_{F,S}|_F = 1$ (the first copy of $\mathcal{O}_{F,S}$ is diagonally embedded into $\mathbf{A}_{F,S}$, and the second is diagonally embedded into $\mathbf{A}_{F,S} \times \prod_{v\notin S} \mathcal{O}_{F_v} \subset \mathbf{A}_F$).

We have the following analog of Theorem A.1.4.3:

THEOREM A.1.5.2. *Let F be a number field, S a finite set of places containing all the Archimedean places of F.*

(1) *The closed subgroup $C^1_{F,S} \subset C_{F,S}$ is compact Hausdorff.*
(2) *The unit group $\mathcal{O}^\times_{F,S}$ is finitely generated, and its maximal free subquotient has rank $|S| - 1$.*
(3) *The abelian group $\mathrm{Pic}(\mathcal{O}_{F,S})$ is finite.*

From the classical description

$$\widehat{\prod}_{v\notin S}(F_v^\times, \mathcal{O}^\times_{F_v})/F^\times \cdot \prod_{v\notin S} \mathcal{O}^\times_{F_v} \simeq \mathrm{Pic}(\mathcal{O}_{F,S})$$

we get the short exact sequence

$$1 \to \left(\prod\nolimits_{v\in S'_f - S_f} F_v^\times/\mathcal{O}^\times_{F_v}\right)/\mathcal{O}^\times_{F,S'} \to \mathrm{Pic}(\mathcal{O}_{F,S}) \to \mathrm{Pic}(\mathcal{O}_{F,S'}) \to 0 \tag{A.1.5.9}$$

for $S' \supset S$.

REMARK A.1.5.3 (Uniformization and the Archimedean places). We note that by choosing S' large enough i so as to arrange $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$, we deduce from exactness of (A.1.5.9) the isomorphism of finite abelian groups:

$$\left(\prod_{v\in S'_f - S_f} F_v^\times/\mathcal{O}^\times_{F_v}\right)/\mathcal{O}^\times_{F,S'} \to \mathrm{Pic}(\mathcal{O}_{F,S}).$$

In particular, for $S = S_\infty$ and S' large enough so as to arrange for $\mathrm{Pic}(\mathcal{O}_{F,S'}) = 1$ to hold, and upon renaming S' as S, we have the isomorphism of finite abelian groups

$$\left(\prod_{v\in S_f} F_v^\times/\mathcal{O}_{F_v}^\times\right)/\mathcal{O}_{F,S}^\times \to \mathrm{Pic}(\mathcal{O}_F). \tag{A.1.5.10}$$

The isomorphism from (A.1.5.10) is compatible with enlargement of S, and with the natural isomorphism

$$\mathrm{GL}_1(\mathbf{A}_F)^1/F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1\times\widehat{\mathcal{O}}_F^\times)\to\mathrm{Pic}(\mathcal{O}_F).$$

A.2. Orbital structures on adeles and ideles

In this section, we examine adelic and idelic quotients explicitly, constructing relatively compact fundamental domains for the additive F-translation action of F on $\mathbf{A}_F$, and the multiplicative translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$.

While the topics are covered in detail within the literature (for which our main references will be [**RV**, Ch. 5, §5.1], [**CF**, Ch. XV, §4.1, §4.3]), we will be highlighting certain structural, measure-theoretic and topological properties of these domains, to be used later throughout this work.

A.2.1. Fundamental domain of the additive F-translation action on $\mathbf{A}_F$. In light of Lemma A.1.2.1, the diagonal image of F in $\mathbf{A}_F$ is discrete and co-compact. We recall how to construct a fundamental domain for the action by additive translation of F on $\mathbf{A}_F$, i.e. a relatively compact subset $D \subset \mathbf{A}_F$ with non-empty interior, such that $F + D = \mathbf{A}_F$, and $(\xi + D) \cap D = \varnothing$ for every $\xi \in F^\times$.

A.2.1.1. *The adelic fundamental domain F.* In this subsection we describe the compact Hausdorff quotient $\mathbf{A}_F/F$ by means of a fundamental domain within $\mathbf{A}_F$ of the continuous left-translation action by F.

Definition A.2.1.1. Let F be a number field.

We fix a $\mathbf{Z}$-basis $\{a_1,\dots,a_d\} \subset \mathcal{O}_F$, with $d := [F : \mathbf{Q}]$ and enumerate the infinite places of F as $v_1,\dots,v_{r_1+r_2}$. We denote by $D_{F,\infty} \subset \mathbf{A}_{F,S_\infty}$ the half-open unit cube

$$D_{F,\infty} := \{x \in \mathbf{A}_{F,S_\infty} : x = \sum_{1\le i\le r_1+r_2} x_i a_i,\ |x_i|_{v_i} \in [0,1)\}. \tag{A.2.1.1}$$

We define

$$D_F := D_{F,\infty}\times\widehat{\mathcal{O}}_F \subset \mathbf{A}_F \tag{A.2.1.2}$$

and refer to it as the *adelic fundamental domain*.

Lemma A.2.1.2. *Let F be a number field.*

(1) *The additive topological group $\mathbf{A}_F$ satisfies*

$$\mathbf{A}_F = F + D_F \tag{A.2.1.3}$$

where the right side of (A.2.1.3) *is the union within $\mathbf{A}_F$ of the additive F-translates of the adelic fundamental domain D_F from* (A.2.1.2).

(2) *Upon denoting by $\overline{D}_F$ the closure of D_F in $\mathbf{A}_F$, then for any compact subset $K \subset \mathbf{A}_F$, there exist at most finitely many elements $\xi \in F$ such that*

$$(\xi + \overline{D}_F) \cap K \neq \varnothing. \tag{A.2.1.4}$$

We illustrate the example $F = \mathbf{Q}$.

Example A.2.1.3. The additive topological group $\mathbf{A}_\mathbf{Q}$ satisfies

$$\mathbf{A}_\mathbf{Q} = \mathbf{Q} + D \tag{A.2.1.5}$$

where the right side of (A.2.1.5) is the union within $\mathbf{A_Q}$ of the additive $\mathbf{Q}$-translates of the subset

$$D := [0,1) \times \widehat{\mathbf{Z}}. \tag{A.2.1.6}$$

In particular, $\mathbf{A_Q}$ is the union of countably many pre-compact subsets. Moreover, the union in (A.2.1.5) is disjoint.

A.2.2. Continuous homomorphic sections to the idelic norm. Recall from Lemma A.1.3.3 and Lemma A.1.3.4 that the idelic norm

$$|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$$

is a continuous homomorphism of abelian topological groups, and it descends to the idele class group C_F since $|F^\times|_F = 1$ (Product Formula).

In this subsection we present a few constructions of continuous homomorphic sections to the idelic norm; i.e., continuous homomorphisms

$$s : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$$

such that $|\cdot|_F \circ s$ is the identity on $\mathbf{R}_{>0}$. We note that any continuous homomorphic section s to $|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$ also induces a continuous homomorphic section to the idelic norm on C_F, by composition with the continuous quotient $\mathrm{GL}_1(\mathbf{A}_F) \to C_F$.

A.2.2.1. *The canonical section.* The topological groups $\mathrm{GL}_1(\mathbf{A_Q})$ and $C_\mathbf{Q}$ can be described explicitly. We have an isomorphism of topological groups

$$\theta : \mathbf{R}^\times \times \mathbf{Q}_{>0} \times \widehat{\mathbf{Z}}^\times \xrightarrow{\simeq} \mathrm{GL}_1(\mathbf{A_Q}) \tag{A.2.2.1}$$

given explicitly as $\theta(t,q,h) := tqh$, for $t \in \mathbf{R}^\times$, $q \in \mathbf{Q}_{>0}$, $h \in \mathrm{GL}_1(\widehat{\mathbf{Z}})$. It satisfies:

$$|\theta(t,q,h)|_\mathbf{Q} = |t|_\mathbf{R} \tag{A.2.2.2}$$

where $|\cdot|_\mathbf{R}$ is the Euclidean absolute value on $\mathbf{R}^\times$, which is to say that the idelic norm on $\mathrm{GL}_1(\mathbf{A_Q})$ corresponds on the left side of (A.2.2.1) to the composition of the projection onto $\mathbf{R}^\times$, followed by the Euclidean absolute value.

This induces an isomorphism of Hausdorff locally compact abelian groups

$$\theta : \mathbf{R}_{>0} \times \widehat{\mathbf{Z}}^\times \xrightarrow{\simeq} C_\mathbf{Q} \tag{A.2.2.3}$$

Lemma A.2.2.1. *The function*

$$s_\mathbf{Q} : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A_Q}) \tag{A.2.2.4}$$

defined as $s_\mathbf{Q}(t) := \theta(t,1,1)$, is a canonical continuous section to the idelic norm $|\cdot|_\mathbf{Q} : C_\mathbf{Q} \to \mathbf{R}_{>0}$.

Lemma A.2.2.2. *Let F be a number field. The norm homomorphism renders the following diagram of continuous homomorphisms of Hausdorff locally compact abelian groups*

$$\begin{array}{ccc} \mathrm{GL}_1(\mathbf{A}_F) & \xrightarrow{|\cdot|_F} & \mathbf{R}_{>0} \\ {\scriptstyle N_{F/\mathbf{Q}}}\downarrow & & \| \\ \mathrm{GL}_1(\mathbf{A_Q}) & \xrightarrow[|\cdot|_\mathbf{Q}]{} & \mathbf{R}_{>0} \end{array} \tag{A.2.2.5}$$

commutative, the rightmost vertical arrow being the identity.

Moreover, the canonical closed continuous homomorphism

$$i_F : \mathrm{GL}_1(\mathbf{A_Q}) \to \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.2.6}$$

satisfies

$$N_{F/\mathbf{Q}} \circ i_F = (\cdot)^{[F:\mathbf{Q}]} \tag{A.2.2.7}$$

as continuous endomorphisms of $\mathrm{GL}_1(\mathbf{A}_\mathbf{Q})$*, where the right side in* (A.2.2.7) *is the endomorphism* $g \mapsto g^{[F:\mathbf{Q}]}$*.*

Definition A.2.2.3. Let F be a number field. We define the function

$$s_F : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.2.8}$$

as

$$s_F := i_F \circ s_\mathbf{Q} \circ (\cdot)^{1/[F:\mathbf{Q}]}. \tag{A.2.2.9}$$

We refer to s_F as the *canonical section* to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$.

Lemma A.2.2.4. *Let F be a number field. The function s_F from* (A.2.2.8) *is a continuous homomorphic section to the idelic norm on* $\mathrm{GL}_1(\mathbf{A}_F)$*. Explicitly, $s_F(t)$ is the idele whose v-indexed component, for each Archimedean place v, is $t^{1/[F:\mathbf{Q}]}$, and for each non-Archimedean place v is* 1*.*

A.2.2.2. *The section along $F_{v_0}^\times$, for v_0 a choice of Archimedean place.* In the case $F = \mathbf{Q}$, the canonical isomorphism θ from (A.2.2.3) provides a canonical continuous homomorphic section $s_\mathbf{Q}$ to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_\mathbf{Q})$ (see Lemma A.2.2.1).

When F is an arbitrary number field other than $\mathbf{Q}$ and an imaginary quadratic field, the number of Archimedean places of F is higher than 1. Each Archimedean place $v_0 \in S_\infty$ induces a continuous section s_{F,v_0} to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$:

Definition A.2.2.5. Let F be a number field. We define the function

$$s_{F,v_0} : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.2.10}$$

via insertion into $F_{v_0}^\times$ preceded by $t \mapsto t^{1/[F_{v_0}:\mathbf{R}]}$ on $\mathbf{R}_{>0}$.

Lemma A.2.2.6. *Let F be a number field. The map s_{F,v_0} from* (A.2.2.10) *is a continuous homomorphic section to the idelic norm on* $\mathrm{GL}_1(\mathbf{A}_F)$*.*

Proof. Indeed $\iota_{v_0}^{-1} : \mathbf{R}_{>0} \to T_{v_0}$ is an isomorphism of Hausdorff locally compact abelian groups, and i_{v_0} is a closed injective continuous homomorphism. It follows that s_{F,v_0} is a continuous homomorphism. Moreover, for every $t \in \mathbf{R}_{>0}$, we have $|s_{F,v_0}(t)|_F = |i_{v_0}(\iota_{v_0}^{-1}(t))|_F = |\iota_{v_0}^{-1}(t)|_{v_0}$. If v_0 is real, then $|\iota_{v_0}^{-1}(t)|_{v_0} = |t|_\mathbf{R} = t$. If v_0 is complex, then $|\iota_{v_0}^{-1}(t)|_{v_0} = |t^{1/2}|_\mathbf{C} = t$. □

A.2.2.3. *The diagonal section into* $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$*.* We give one more construction of a section.

Definition A.2.2.7. Let F be a number field. We define the function

$$s_{F,\delta} : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.2.11}$$

by sending the real number $t \in \mathbf{R}_{>0}$ to the idele $s_{F,\delta}(t)$ whose v-indexed component, for every non-Archimedean place v of F, is 1, and for every Archimedean place v, it is $t^{1/N_v(r_1+r_2)}$ with $N_v = 1$ if v is real, and $N_v = 2$ if v is complex. We refer to $s_{F,\delta}$ as the *diagonal section* to the idelic norm on $\mathrm{GL}_1(\mathbf{A}_F)$.

Lemma A.2.2.8. *Let F be a number field. The function $s_{F,\delta}$ is a continuous homomorphic section to the idelic norm.*

Lemma A.2.2.9. *Let F be a number field. For any continuous section s to the idelic norm on* $\mathrm{GL}_1(\mathbf{A}_F)$*, the continuous homomorphism*

$$\theta_{F,s} : \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.2.12}$$

defined as $\theta_{F,s}(t, g_0) := s(t)g_0$, is an isomorphism of Hausdorff locally compact abelian groups.

A.2.3. Fundamental domain of the multiplicative $F^\times$-translation action on $\mathrm{GL}_1(\mathbf{A}_F)^1$. In this subsection we describe a fundamental domain for the continuous left translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$, i.e. a relatively compact subset $E \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ with non-empty interior, such that $F^\times E = \mathrm{GL}_1(\mathbf{A}_F)^1$, and $\xi E \cap E = \varnothing$ for every $\xi \in F^\times - \{1\}$.

A.2.3.1. *The logarithm homomorphism.* In this subsection we recall the structure of the orbit space of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ under the action of the unit group $\mathcal{O}_F^\times$.

Definition A.2.3.1. Let F be a number field. We define the continuous homomorphism

$$|\cdot|_\infty : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}_{>0} \tag{A.2.3.1}$$

to be $|\cdot|_{F,S_\infty}$ and the closed subgroup

$$\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 := \{g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) : |g_\infty|_\infty = 1\} \tag{A.2.3.2}$$

of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

We define

$$\mathscr{L} : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}^{S_\infty} \tag{A.2.3.3}$$

as the map

$$\mathscr{L}(g_\infty) := (\log|g_v|_v)_{v\in S_\infty} : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}^{S_\infty} \tag{A.2.3.4}$$

which is a surjective continuous homomorphism of abelian topological groups. The following is elementary.

Lemma A.2.3.2. *Let F be a number field. Upon defining the hyperplane*

$$H_{\log} := \{(x_v)_{v\in S_\infty} \in \mathbf{R}^{S_\infty} : \sum_{v\in S_\infty} x_v = 0\} \subset \mathbf{R}^{S_\infty} \tag{A.2.3.5}$$

the map $\mathscr{L}$ from (A.2.3.3) *satisfies the following conditions:*

(1) *$\mathscr{L}$ is a continuous group homomorphism, and $\mathscr{L}(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1) \subset H_{\log}$.*
(2) *$\mathscr{L}$ is a smooth submersion whose kernel is isomorphic, as a Lie group, to the compact Lie group $(\mu_2)^{r_1} \times \mathrm{U}(1)^{r_2}$.*
(3) *$\mathscr{L}$ is a proper smooth submersion.*

Definition A.2.3.3. Let F be a number field. We define the *regulator homomorphism* to be the restriction

$$\mathrm{Reg} : \mathcal{O}_F^\times \to H_{\log} \tag{A.2.3.6}$$

of $\mathscr{L}$ from (A.2.3.3) to $\mathcal{O}_F^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.

Lemma A.2.3.4. *Let F be a number field. The regulator homomorphism satisfies the following conditions:*

(1) *$\mathrm{Reg}(\mathcal{O}_F^\times)$ is a discrete and co-compact closed subgroup of $H_{\log}$.*
(2) *The kernel* $\ker(\mathrm{Reg})$ *is the torsion subgroup of $\mathcal{O}_F^\times$, which is the finite cyclic group μ_F of size w_F of the roots of unity contained in F.*
(3) *$\mathcal{O}_F^\times$ is a finitely generated abelian group, and $\mathcal{O}_F^\times/\mu_F$ is finite free of rank $r_1 + r_2 - 1$.*

A.2.3.2. *The idelic fundamental domain E_F^1.* Upon denoting

$$\{\varepsilon_1, \ldots, \varepsilon_{r_1+r_2-1}\} \tag{A.2.3.7}$$

a $\mathbf{Z}$-basis for the finite free $\mathbf{Z}$-module $\mathcal{O}_F^\times/\mu_F$, for any $g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ there exist unique real numbers $\{x_i\}_{1\le i\le r_1+r_2-1}$ such that

$$\mathscr{L}(g) = \sum_{i=1}^{r_1+r_2-1} x_i \mathscr{L}(\varepsilon_i) \tag{A.2.3.8}$$

and a fundamental domain for the left translation action of $\mathscr{L}(\mathcal{O}_F^\times)$ on $\mathscr{L}(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1) \simeq \mathbf{R}^{r_1+r_2-1}$ is given by the parallelotope

$$P = \left\{ \sum_{i=1}^{r_1+r_2-1} x_i \mathscr{L}(\varepsilon_i) : (x_i) \in Q \right\} \tag{A.2.3.9}$$

where Q is the hypercube

$$Q := \{(x_i) \in \mathbf{R}^{r_1+r_2-1} : 0 \le x_i < 1\} \subset \mathbf{R}^{r_1+r_2-1}. \tag{A.2.3.10}$$

Lemma A.2.3.5. *Let F be a hyperplane. The subset $P \subset H_{\log}$ satisfies the following conditions:*

(1) *P is relatively compact with non-empty interior within $H_{\log}$.*
(2) *$\mathrm{Reg}(\mathcal{O}_F^\times)P = H_{\log}$, and for every $\xi \in \mathrm{Reg}(\mathcal{O}_F^\times)$, $\xi \neq 0$, we have*

$$(\xi + P) \cap P = \varnothing. \tag{A.2.3.11}$$

The natural map

$$\mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{Pic}(\mathcal{O}_F)$$

sends $\mathrm{GL}_1(\mathbf{A}_F)^1$ onto $\mathrm{Pic}(\mathcal{O}_F)$, so there exist ideles

$$g_1 := 1, \ldots, g_{h_F} \in \mathrm{GL}_1(\mathbf{A}_F)^1 \tag{A.2.3.12}$$

such that their images represent each of the h_F elements of $\mathrm{Pic}(\mathcal{O}_F)$.

Definition A.2.3.6. Let F be a number field, v_0 a choice of Archimedean place of F. We define

$$E_0 := \{g \in \mathscr{L}^{-1}(P) : \arg(\iota_{v_0}(g_{v_0})) \in [0, 2\pi/w_F)\} \subset \mathscr{L}^{-1}(P) \tag{A.2.3.13}$$

and

$$E_F^1 := \bigcup_{i=1}^{h_F} g_i(E_0 \times \widehat{\mathcal{O}}_F^\times) \subset \mathrm{GL}_1(\mathbf{A}_F)^1 \subset \mathrm{GL}_1(\mathbf{A}_F) \tag{A.2.3.14}$$

(If v_0 is real, then $\arg(g_{v_0}) \in [0, 2\pi/w_F)$ is equivalent to requiring $\iota_{v_0}(g_0)$ is positive, where $\iota_{v_0} : F_{v_0} \simeq \mathbf{R}$ is the unique topological field isomorphism.) We endow E_F^1 with the subspace topology and refer to E_F^1 as the *idelic fundamental domain.*

Remark A.2.3.7. The subset E_F^1 from (A.2.3.14) is the same as defined in [**CF**, Ch. XV, Def. 4.3.2].

We write each idele g_i from (A.2.3.12) as

$$g_i = (g_{i,\infty}, g_{i,f}) \tag{A.2.3.15}$$

with $g_{i,\infty} = \mathrm{pr}_\infty(g_i) \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and $g_{i,f} = \mathrm{pr}_f(g_i) \in \mathrm{GL}_1(\mathbf{A}_F^f)$, where $\mathrm{pr}_\infty : \mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and $\mathrm{pr}_f : \mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{GL}_1(\mathbf{A}_F^f)$ are, respectively, the continuous group homomorphisms induced by the first projection $\mathbf{A}_F \to \mathbf{A}_{F,S_\infty}$ and the second projection $\mathbf{A}_F \to \mathbf{A}_F^f$.

The subset $E_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ from (A.2.3.14) is therefore of the form

$$E_F^1 = (E_0 \times \widehat{\mathcal{O}}_F^\times) \cup (g_{2,\infty}E_0 \times g_{2,f}\widehat{\mathcal{O}}_F^\times) \cup \ldots \cup (g_{h_F,\infty}E_0 \times g_{h_F,f}\widehat{\mathcal{O}}_F^\times) \subset \mathrm{GL}_1(\mathbf{A}_F)^1. \tag{A.2.3.16}$$

A.2.3.3. *Topological structure of E_0.* We now focus on E_0 (defined in (A.2.3.13)).

LEMMA A.2.3.8. *The subset $E_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ satisfies the following conditions:*

(1) *E_0 is relatively compact in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ and $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ and its interior within $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ is non-empty.*

(2) *$\mathcal{O}_F^\times E_0 = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, and for every $\xi \in \mathcal{O}_F^\times$, $\xi \neq 1$, we have*

$$\xi E_0 \cap E_0 = \varnothing \tag{A.2.3.17}$$

where $\mathcal{O}_F^\times$ is diagonally embedded into $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$. Moreover, $\mathscr{L}^{-1}(P) = \mu_F E_0$.

(3) *Upon denoting by $\overline{E}_0$ the closure of $E_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, for any compact subset $K \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ there exist at most finitely many elements $\xi \in \mathcal{O}_F^\times$ such that*

$$\xi \overline{E}_0 \cap \overline{E}_0 \neq \varnothing. \tag{A.2.3.18}$$

Due to lack of a reference for a proof in standard books, we give a proof below although the result is very classical.

PROOF. Recall that $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, defined in (A.2.3.1), is the closed subgroup of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \simeq \prod_{v\in S_\infty} F_v^\times$ defined as

$$\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 = \{g_\infty \in \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) : |g_\infty|_\infty = 1\}$$

which is the kernel of the continuous group homomorphism $|\cdot|_\infty := \prod_{v\in S_\infty} |\cdot|_v : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}_{>0}$.

Further recall that the continuous group homomorphism

$$\mathscr{L} : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}^{S_\infty}$$

from (A.2.3.3) is a smooth and surjective.

The parallelotope P from (A.2.3.9) within the hyperplane $H_{\log}$ is relatively compact and has non-empty interior (see Lemma A.2.3.5(1)), so the pre-image $\mathscr{L}^{-1}(P)$ also has non-empty interior within the pre-image $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 = \mathscr{L}^{-1}(H_{\log})$.

We now choose an enumeration of the finite set of Archimedean places of F:

$$S_\infty = \{v_0, v_1, \ldots, v_{r_1+r_2-1}\} \tag{A.2.3.19}$$

where v_0 is a choice of Archimedean place made to define the closed subgroup

$$T_{v_0} := s_{F,v_0}(\mathbf{R}_{>0}) \subset \mathrm{GL}_1(\mathbf{A}_F)$$

induced by the v_0-section s_{F,v_0} to the idelic norm $|\cdot|_F$ (see Definition A.2.2.5).

The choice of the ordering in (A.2.3.19), along with that of isomorphisms of topological fields $\iota_v : F_v \simeq \mathbf{R}$ if v is real, $\iota_v : F_v \simeq \mathbf{C}$ if v is complex (the same which, for $v = v_0$, were used in the definition of s_{F,v_0} in Definition A.2.2.10) induces an isomorphism of $\mathbf{R}$-vector spaces $\mathbf{R}^{S_\infty} \simeq \mathbf{R}^{r_1+r_2}$ (upon choosing within each F_v the $\mathbf{R}$-linear basis given by 1 if v is real, and $1, \iota_v^{-1}(i)$ if v is complex).

The subset $E_0 \subset \mathscr{L}^{-1}(P)$ was defined by the condition

$$\arg(\iota_{v_0}(g_{v_0})) \in [0, 2\pi/w_F)$$

for $g \in \mathscr{L}^{-1}(P)$ in (A.2.3.13) of Definition A.2.3.6. In other words, we have

$$E_0 = \mathscr{L}^{-1}(P) \cap \left(\iota_{v_0}^{-1}(\mathbf{R}_{>0}) \times \prod_{k=1}^{r_1+r_2-1} F_{v_k}^\times\right). \tag{A.2.3.20}$$

If v_0 is complex, then $\arg(\iota_{v_0}(g_{v_0})) \in [0, 2\pi/w_F)$ is equivalent to requiring that $\iota_{v_0}(g_{v_0})$ be a complex number within the sector

$$\mathbf{R}_{>0}\, e^{i[0,2\pi/w_F)} \subset \mathbf{C}.$$

In other words, we have

$$E_0 = \mathscr{L}^{-1}(P) \cap \left(\iota_{v_0}^{-1}(\mathbf{R}_{>0}\, e^{i[0,2\pi/w_F)}) \times \prod_{k=1}^{r_1+r_2-1} F_{v_k}^{\times} \right). \tag{A.2.3.21}$$

(1) : If v_0 is real, we define $U_0 := \iota_{v_0}^{-1}(\mathbf{R}_{>0}) \times \prod_{k=1}^{r_1+r_2-1} F_{v_k}^{\times}$. If v_0 is complex, we define $U_0 := \iota_{v_0}^{-1}(\mathbf{R}_{>0}\, e^{i(0,2\pi/w_F)}) \times \prod_{k=1}^{r_1+r_2-1} F_{v_k}^{\times}$. In either case, $U_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is open and non-empty.

It follows that $U_0 \cap \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ is open in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, and it is non-empty since it contains

$$g_\infty := (\iota_{v_0}^{-1}(\zeta a)\varepsilon,\ \prod_{k=1}^{r_1+r_2-1} \{a_k \varepsilon\})$$

where $\zeta = 1$ if v_0 is real, and $\zeta \in e^{i(0,2\pi/w_F)}$ if v_0 is complex, with $a, a_k \in \mathbf{R}_{>0} - \{1\}$ chosen so as to have $a^{N_0} \prod_{k=1}^{r_1+r_2-1} a_k^{N_k} = 1$ where, for each $0 \leq k \leq r_1 + r_2 - 1$, $N_k = 1$ if v_k is real, and $N_k = 2$ if v_k is complex, and with $\varepsilon \in \mathcal{O}_F^{\times}/\mu_F$.

By Lemma A.2.3.5(2), we may and do choose $\varepsilon \in \mathcal{O}_F^{\times}/\mu_F$ so as to have $\mathscr{L}(g_\infty) \in \mathrm{Int}(P)$, the interior of P in $H_{\log}$.

By Lemma A.2.3.5(1), P has non-empty interior $\mathrm{Int}(P)$ in $H_{\log}$. The pre-image $\mathscr{L}^{-1}(\mathrm{Int}(P))$ is open and non-empty in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 = \mathscr{L}^{-1}(H_{\log})$ since $\mathscr{L}$ maps $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ surjectively and smoothly onto $H_{\log}$. In light of (A.2.3.20) and (A.2.3.21), $\mathscr{L}^{-1}(\mathrm{Int}(P)) \cap (U_0 \cap \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1)$ is contained in E_0, and it is open and non-empty in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ by the foregoing since it contains $\zeta' g_\infty$ for some $\zeta' \in \mu_F$. It follows that E_0 has non-empty interior in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.

By Lemma A.2.3.2(3), $\mathscr{L}$ is a proper smooth submersion, and hence $\mathscr{L}^{-1}(P)$ is relatively compact in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$. Since $E_0 \subset \mathscr{L}^{-1}(P)$, then E_0 is also relatively compact in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$. As $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ is closed in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, E_0 is relatively compact in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.

(2) : Since $\mathscr{L}(\mathcal{O}_F^{\times}) = \mathrm{Reg}(\mathcal{O}_F^{\times})$ tessellates $H_{\log} = \mathscr{L}(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1)$ by copies of P by Lemma A.2.3.5(2), and since $\mathcal{O}_F^{\times} \to \mathscr{L}(\mathcal{O}_F^{\times})$ is surjective with kernel $\mu_F \subset \mathcal{O}_F^{\times}$ by Lemma A.2.3.4(2), then $(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$. Indeed, we have

$$\mathscr{L}((\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P)) = \mathscr{L}(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}(\mathscr{L}^{-1}(P)) = \mathscr{L}(\mathcal{O}_F^{\times})P = \mathrm{Reg}(\mathcal{O}_F^{\times})P = H_{\log}$$

by Lemma A.2.3.5(2) and since $\mathscr{L}^{-1}(P)$ is stable under the action of $\ker(\mathscr{L})$ (and hence of μ_F), then $(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P)$ is stable under the action of $\mathcal{O}_F^{\times} \ker(\mathscr{L})$ and since $\mathcal{O}_F^{\times} \ker(\mathscr{L}) = \mathscr{L}^{-1}(\mathscr{L}(\mathcal{O}_F^{\times}))$, then $(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P)$ contains $\mathscr{L}^{-1}(\mathscr{L}((\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P))) = \mathscr{L}^{-1}(H_{\log}) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, and hence $(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.

Since $\mathcal{O}_F^{\times} \cap \ker(\mathscr{L}) = \mu_F$, the union $(\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P)$ is disjoint, i.e. the finite free abelian group $\mathcal{O}_F^{\times}/\mu_F$ tessellates $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ by copies of $\mathscr{L}^{-1}(P)$.

Since μ_F is cyclic of size w_F, it acts on $F_{v_0}^{\times}$ with fundamental domain $\iota_{v_0}^{-1}(\mathbf{R}_{>0})$ if v_0 is real (in which case $\mu_F = \{\pm 1\}$), and $\iota_{v_0}^{-1}(\mathbf{R}_{>0}\, e^{i[0,2\pi/w_F)})$ as fundamental domain, if v_0 is complex. It follows that we have $\mu_F E_0 = \mathscr{L}^{-1}(P)$, the finite union being disjoint.

We therefore have (since μ_F splits off as a direct factor of $\mathcal{O}_F^{\times}$, being $\mathcal{O}_F^{\times}$ a finitely generated abelian group with torsion group μ_F) $\mathcal{O}_F^{\times} E_0 = (\mathcal{O}_F^{\times}/\mu_F)\mu_F E_0 = (\mathcal{O}_F^{\times}/\mu_F)\mathscr{L}^{-1}(P) = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, the union being disjoint.

(3) : In light of (1) and (2), it is enough to assume $K = \overline{E}_0$, and since μ_F is finite, it is enough to prove the same assertion with $\mathscr{L}^{-1}(P)$ in place of E_0. This in turn reduces to the same assertion for $P \subset H_{\log}$ and its closure, and $\mathrm{Reg}(\mathcal{O}_F^{\times})$ acting on $H_{\log}$. Upon fixing a $\mathbf{Z}$-basis of units for the torsion-free quotient of $\mathcal{O}_F^{\times}$, this in turn reduces to the case of the standard full lattice $\mathbf{Z}^{r_1+r_2-1}$ acting on the Euclidean space $\mathbf{R}^{r_1+r_2-1}$, with P replaced by $[0,1)^{r_1+r_2-1}$. □

A.2.3.4. *Global description of* $\mathrm{GL}_1(\mathbf{A}_F)$. With no mention of the fundamental domains E_F^1 and E_F, we describe, in this subsection, the topological group $\mathrm{GL}_1(\mathbf{A}_F)$. Recall that we write $\widehat{\mathcal{O}}_F^\times$ for $\mathrm{GL}_1(\widehat{\mathcal{O}}_F)$, and not for the closure of $\mathcal{O}_F^\times$ in $\mathrm{GL}_1(\mathbf{A}_F)$.

PROPOSITION A.2.3.9. *Let F be a number field. The following sequences*

$$1 \to \mathcal{O}_F^\times \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times F^\times \times \widehat{\mathcal{O}}_F^\times \to F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \widehat{\mathcal{O}}_F^\times) \to 1 \tag{A.2.3.22}$$

where the unit group $\mathcal{O}_F^\times$ is embedded diagonally as a discrete subgroup, and

$$1 \to F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \widehat{\mathcal{O}}_F^\times) \to \mathrm{GL}_1(\mathbf{A}_F) \to \mathrm{Pic}(\mathcal{O}_F) \to 1 \tag{A.2.3.23}$$

are short exact sequences of Hausdorff locally compact abelian groups, in the strict sense. The short exact sequence in (A.2.3.23) *induces the short exact sequence of Hausdorff locally compact abelian groups*

$$1 \to F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^\times) \to \mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{Pic}(\mathcal{O}_F) \to 1 \tag{A.2.3.24}$$

which is exact in the strict sense.

A.2.3.5. *Topological structure of E_F^1.*

PROPOSITION A.2.3.10. *The subset $E \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ satisfies the following conditions:*

(1) *E_F^1 is relatively compact with non-empty interior in $\mathrm{GL}_1(\mathbf{A}_F)^1$.*
(2) *We have*

$$F^\times E_F^1 = \mathrm{GL}_1(\mathbf{A}_F)^1 \tag{A.2.3.25}$$

and for every $\xi \in F^\times$, $\xi \neq 1$, we have

$$\xi E_F^1 \cap E_F^1 = \varnothing. \tag{A.2.3.26}$$

(3) *Upon denoting by $\overline{E}_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ the closure of E_F^1, for any compact subset $K \subset \mathrm{GL}_1(\mathbf{A}_F)^1$, there exist at most finitely many elements $\xi \in F^\times$ such that:*

$$\xi \overline{E}_F^1 \cap K \neq \varnothing. \tag{A.2.3.27}$$

PROOF. (1) : The subset $E_F^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ was defined upon choosing $g_1 = 1, g_2, \ldots, g_{h_F} \in \mathrm{GL}_1(\mathbf{A}_F)^1$ lifting the h_F elements of $\mathrm{Pic}(\mathcal{O}_F)$ and defining $E_F^1 := (E_0 \times \widehat{\mathcal{O}}_F^\times) \cup g_2(E_0 \times \widehat{\mathcal{O}}_F^\times) \cup \cdots \cup g_{h_F}(E_0 \times \widehat{\mathcal{O}}_F^\times)$. By compactness and openness of $\widehat{\mathcal{O}}_F^\times$ in $\mathrm{GL}_1(\mathbf{A}_F^f)$, E_F^1 is relatively compact, and its interior in $\mathrm{GL}_1(\mathbf{A}_F)^1$ is non-empty since E_0 has non-empty interior in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ by Lemma A.2.3.8(1).

(2) : By Proposition A.2.3.9, we have a strictly exact sequence of continuous homomorphisms of Hausdorff locally compact abelian groups

$$1 \to F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^\times) \to \mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{Pic}(\mathcal{O}_F) \to 1 \tag{A.2.3.28}$$

and hence to see that (A.2.3.25) holds and the union is disjoint, it is enough to check that we have

$$F^\times(E_0 \times \widehat{\mathcal{O}}_F^\times) = F^\times(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \times \widehat{\mathcal{O}}_F^\times)$$

within $\mathrm{GL}_1(\mathbf{A}_F)^1$, which follows from $\mathcal{O}_F^\times E_0 = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, which is given by Lemma A.2.3.8(2). The disjointness condition for non-trivial $F^\times$-translates in (A.2.3.26) follows from (A.2.3.17) of Lemma A.2.3.8(2) upon observing that $\xi(\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^\times) \cap (\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^\times) = \varnothing$ within $\mathrm{GL}_1(\mathbf{A}_F)^1$ if ξ is not a global unit.

(3) : In light of (1) and (2), it is enough to treat the case $K = \overline{E}_F^1$. Since $\mathrm{Pic}(\mathcal{O}_F)$ is finite by Theorem A.1.4.3(3), it is enough to prove, in light of (A.2.3.16), that, upon denoting by $\overline{E}_0$ the closure of $E_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, there exist at most finitely many elements $\xi \in \mathcal{O}_F^\times$ such that $\xi\overline{E}_0 \cap \overline{E}_0 \neq \varnothing$. This is Lemma A.2.3.8(3). □

A.2.3.6. *Fundamental domain E_F for the $F^\times$-action on $\mathrm{GL}_1(\mathbf{A}_F)$.*

LEMMA A.2.3.11. *Let F be a number field. Let s_∞ be any continuous homomorphic section to the continuous homomorphism $|\cdot|_\infty : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}_{>0}$. Then*

$$\mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \to \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$$

defined by $(t, g^1) \mapsto s_\infty(t)g^1$ is an isomorphism of Lie groups.

The following is immediate from Proposition A.2.3.10(2).

LEMMA A.2.3.12. *Let F be a number field. Let s_∞ be any continuous homomorphic section to the continuous homomorphism $|\cdot|_\infty : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \to \mathbf{R}_{>0}$. The natural map*

$$F^\times \times \mathbf{R}_{>0} \times E_F^1 \to \mathrm{GL}_1(\mathbf{A}_F)$$

via $(\xi, t, g_0) \mapsto \xi s_\infty(t) g_0$ is bijective.

A.2.3.7. *Product loci, interiors and compactness.* For each Archimedean place, we denote by

$$H_v := \{x \in \mathbf{A}_{F,S_\infty} : x_v = 0\} \tag{A.2.3.29}$$

the hyperplane of those elements $x \in \mathbf{A}_{F,S_\infty}$ whose coordinate indexed by the Archimedean place v is zero.

LEMMA A.2.3.13. *We fix an Archimedean place v_0 of F, and let $T_{v_0} \subset F_{v_0}^\times \hookrightarrow \mathbf{A}_{F,S_\infty}^\times$ be the canonical copy of $\mathbf{R}_{>0}$. Let $K_{v_0} \subset T_{v_0,\infty}$ be a compact subset.*

(1) *The subset $K_{v_0} \cdot \overline{E}_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is compact. If K_{v_0} has non-empty interior in $T_{v_0,\infty}$, then so does $K_{v_0} \cdot \overline{E}_0$ in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$.*
(2) *We have $(K_{v_0} \cdot \overline{E}_0) \cap (\bigcup_{v \in S_\infty} H_v) = \varnothing$.*
(3) *If K_{v_0} has non-empty interior in $T_{v_0,\infty}$, then so does $K_{v_0} \cdot \overline{E}_0$ in $\mathbf{A}_{F,S_\infty}$.*

PROOF. (1) : Compactness of $K_{v_0} \cdot \overline{E}_0$ follows from Lemma A.2.3.8(1). The interior of $K_{v_0} \cdot \overline{E}_0$ in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ contains the open subset

$$\mathrm{Int}_{T_{v_0,\infty}}(K_{v_0}) \times \mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1}(\overline{E}_0) \subset T_{v_0,\infty} \times \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 = \mathrm{GL}_1(\mathbf{A}_{F,S_\infty}).$$

If $\mathrm{Int}_{T_{v_0,\infty}}(K_{v_0})$ is non-empty then since $\mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1}(\overline{E}_0) = \mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1}(E_0)$ is also non-empty by Lemma A.2.3.8(1), so is $\mathrm{Int}_{T_{v_0,\infty}}(K_{v_0}) \cdot \mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1}(\overline{E}_0) \subset \mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})}(K_{v_0} \cdot \overline{E}_0)$.

(2) : Since $K_{v_0} \subset T_{v_0,\infty}$ and $\overline{E}_0 \subset \mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, we have $K_{v_0} \cdot \overline{E}_0$ is entirely contained within $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$, and the complement of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is $\bigcup_{v \in S_\infty} H_v$.

(3) : Clearly $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty}) \simeq \prod_{v \in S_\infty} F_v^\times$ is open in $\mathbf{A}_{F,S_\infty}$. If $K_{v_0} \subset T_{v_0,\infty}$ is a compact subset with non-empty interior in $T_{v_0,\infty}$, then by (1) we have that the interior of $K_{v_0} \cdot \overline{E}_0$ in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is non-empty, and since it is open, it is open and non-empty also in $\mathbf{A}_{F,S_\infty}$ because $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ is open in $\mathbf{A}_{F,S_\infty}$.The interior

$$\mathrm{Int}_{\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})}(K_{v_0} \cdot \overline{E}_0)$$

formed in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ coincides with the interior $\mathrm{Int}_{\mathbf{A}_{F,S_\infty}}(K_{v_0} \cdot \overline{E}_0)$ formed within $\mathbf{A}_{F,S_\infty}$ by compactness of $K_{v_0}\overline{E}_0$ and openness of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ in $\mathbf{A}_{F,S_\infty}$. □

A.2.4. Fundamental domain for $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ and $C_{F,S}$. In this subsection we construct an S-local analog of the fundamental domain D_F from (A.2.1.2) for the F-action on $\mathbf{A}_F$, i.e. a fundamental domain for the continuous additive translation action by $\mathcal{O}_{F,S}$ on $\mathbf{A}_{F,S}$, and for the $\mathcal{O}_{F,S}^\times$-action on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.

A.2.4.1. *Fundamental domain for the additive $\mathcal{O}_{F,S}$-action on $\mathbf{A}_{F,S}$.* We recall that for F a number field, and S a finite set of places of F containing all the Archimedean places, we denote by S_f the subset of all finite places of F contained in S, i.e. $S_f := S \cap \Sigma_{F,f}$.

DEFINITION A.2.4.1. Let F be a number field, S a finite set of places containing all the Archimedean places of F. We fix a $\mathbf{Z}$-basis $\{a_1, \ldots, a_d\} \subset \mathcal{O}_F$ of the finite free $\mathbf{Z}$-module $\mathcal{O}_F$, with $d := [F : \mathbf{Q}]$ and enumerate the infinite places of F as $v_1, \ldots, v_{r_1+r_2}$. We denote by $D_{F,\infty} \subset \mathbf{A}_{F,S_\infty}$ the half-open unit cube from (A.2.1.1) relative to $\{a_1, \ldots, a_d\}$.

We define $D_{F,S} \subset \mathbf{A}_{F,S}$ to be

$$D_{F,S} := D_{F,\infty} \times \prod_{v \in S \cap \Sigma_{F,f}} \mathcal{O}_{F_v} \tag{A.2.4.1}$$

and refer to $D_{F,S}$ as the *S-adelic fundamental domain.*

LEMMA A.2.4.2. *Let F be a number field, S a finite set of places containing all the Archimedean places of F.*

(1) *The additive topological group $\mathbf{A}_{F,S}$ satisfies*

$$\mathbf{A}_{F,S} = \mathcal{O}_{F,S} + D_{F,S} \tag{A.2.4.2}$$

where the right side of (A.2.4.2) *is the union within $\mathbf{A}_{F,S}$ of the additive $\mathcal{O}_{F,S}$-translates of the adelic fundamental domain $D_{F,S}$ from* (A.2.4.1).

(2) *Upon denoting by $\overline{D}_{F,S}$ the closure of $D_{F,S}$ in $\mathbf{A}_{F,S}$, then for any compact subset $K \subset \mathbf{A}_{F,S}$, there exist at most finitely many elements $\xi \in \mathcal{O}_{F,S}$ such that*

$$(\xi + \overline{D}_{F,S}) \cap K \neq \varnothing. \tag{A.2.4.3}$$

PROOF. (1) : This is a consequence of strong approximation for $\mathbf{A}_F$.

(2) : As in the proof of Lemma A.2.1.2(2), it is enough to check that, upon calling $\overline{D}_{F,\infty}$ the closure of $D_{F,\infty}$ from (A.2.1.1) in $\mathbf{A}_{F,S_\infty}$, for any compact subset $K \subset \mathbf{A}_{F,S_\infty}$ there exist at most finitely many elements $\xi \in \mathcal{O}_F$ such that $(\xi + \overline{D}_{F,\infty}) \cap K \neq \varnothing$. Upon fixing an $\mathbf{R}$-basis for $\mathbf{A}_{F,S_\infty} = F \otimes_{\mathbf{Q}} \mathbf{R}$ induced by an integral $\mathbf{Z}$-basis for $\mathcal{O}_F$, say $\{e_1, \ldots, e_{[F:\mathbf{Q}]}\}$, the $\mathbf{R}$-linear isomorphism $\mathbf{A}_{F,S_\infty} \simeq \mathbf{R}^{[F:\mathbf{Q}]}$ induced by this choice of basis sends $\mathcal{O}_F \subset \mathbf{A}_{F,S_\infty}$ to $\mathbf{Z}^{[F:\mathbf{Q}]} = \sum_{1 \leq i \leq [F:\mathbf{Q}]} \mathbf{Z} e_i$. The subset $D_{F,\infty} \subset \mathbf{A}_{F,S_\infty}$ is sent homeomorphically onto $[0,1)^{[F:\mathbf{Q}]}$, and (2) follows. □

A.2.4.2. *Fundamental domain for the multiplicative $\mathcal{O}_{F,S}^\times$-action on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.* We denote by $w_{F,S}$ the size of the finite torsion subgroup of $\mathcal{O}_{F,S}^\times$. We note that $w_{F,S} = w_F$, since the torsion subgroup of $\mathcal{O}_{F,S}^\times$ is μ_F itself, the finite cyclic group of roots of unity contained in F.

We call $g_1 := 1, \ldots, g_{h_{F,S}}$ a set of representatives within $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times$ of the $h_{F,S}$ ideal classes in the finite abelian group $\mathrm{Pic}(\mathcal{O}_{F,S})$.

DEFINITION A.2.4.3. Let F be a number field, v_0 a choice of Archimedean place of F, S a finite set of places of F containing all the Archimedean places. We denote by S_f the intersection $S \cap \Sigma_{F,f}$. We define

$$E^1_{F,S} := \bigcup_{1 \leq i \leq h_{F,S}} g_i (E_{S,0} \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times) \subset \mathrm{GL}_1(\mathbf{A}_{F,S})^1 \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times = \mathrm{GL}_1(\mathbf{A}_{F,S} \times \prod_{v \notin S} \mathcal{O}_{F_v})^1 \subset \mathrm{GL}_1(\mathbf{A}_F)^1 \tag{A.2.4.4}$$

where $E_{S,0}$ is defined, upon choosing an Archimedean place v_0 of F, as:

$$E_{S,0} := \{g \in \mathscr{L}_S^{-1}(P_S) : \arg(\iota_{v_0}(g_{v_0})) \in [0, 2\pi/w_F)\} \subset \mathscr{L}_S^{-1}(P_S) \tag{A.2.4.5}$$

where w_F is the size of the finite abelian group $\ker(\mathscr{L}_S) \cap \mathcal{O}_{F,S}^\times = \mu_F$, and with:

$$\mathscr{L}_S : \mathrm{GL}_1(\mathbf{A}_{F,S}) \to \mathbf{R}^S \tag{A.2.4.6}$$

defined as:

$$\mathscr{L}_S(g) := (\log|g_v|_v)_{v\in S} \tag{A.2.4.7}$$

and, upon choosing an integral basis $\{\varepsilon_1,\dots,\varepsilon_{|S|-1}\}$ for $\mathcal{O}_{F,S}^\times$, P_S is the parallelotope in the hyperplane $H_S := \{x\in\mathbf{R}^S : \sum_{v\in S} x_v = 0\}$ of $\mathbf{R}^S$ defined as:

$$P_S := \sum_{1\le i\le |S|-1} [0,1)\mathscr{L}_S(\varepsilon_i) \subset H_S. \tag{A.2.4.8}$$

We endow $E^1_{F,S}$ with the subspace topology. We refer to $E^1_{F,S}$ as the *S-idelic fundamental domain*.

The following is an analog of Proposition A.2.3.10, and is proved in a closely analogous manner.

PROPOSITION A.2.4.4. *The subsets $E_{S,0}\subset \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and $E^1_{F,S}\subset \mathrm{GL}_1(\mathbf{A}_F)^1$, satisfy the following conditions:*

(1) *$E_{S,0}$ is relatively compact with non-empty interior in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.*
(2) *We have*

$$\mathcal{O}_{F,S}^\times E_{S,0} = \mathrm{GL}_1(\mathbf{A}_{F,S})^1 \tag{A.2.4.9}$$

and for every $\xi\in\mathcal{O}_{F,S}^\times$, $\xi\neq 1$, we have

$$\xi E_{S,0}\cap E_{S,0} = \varnothing. \tag{A.2.4.10}$$

(3) *Upon denoting by $\overline{E}_{S,0}\subset \mathrm{GL}_1(\mathbf{A}_{F,S})^1$ the closure of $E_{S,0}$, for any compact subset $K\subset \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, there exist at most finitely many elements $\xi\in\mathcal{O}_{F,S}^\times$ such that:*

$$\xi\overline{E}_{S,0}\cap K\neq\varnothing. \tag{A.2.4.11}$$

(4) *$E^1_{F,S}$ is a fundamental domain for the continuous left-translation action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1\times\prod_{v\notin S}\mathcal{O}_{F_v}^\times$.*

REMARK A.2.4.5. For all large enough S we have $\mathrm{Pic}(\mathcal{O}_{F,S})=1$, and so

$$\mathcal{O}_{F,S}^\times\cdot E^1_{F,S} = \mathrm{GL}_1(\mathbf{A}_F)^1$$

since $C_{F,S}=C_F$, compatibly with idelic norms and S-idelic norms.

A.3. Haar measures

In this section we normalize Haar measures on $\mathbf{A}_F$, $\mathrm{GL}_1(\mathbf{A}_F)$ and their S-local analogs $\mathbf{A}_{F,S}$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S})$, along with certain closed subgroups.

The following section will assume knowledge of Haar measures and the Fourier transform on a Hausdorff locally compact abelian group and its basic properties, for which we refer the reader to §D.1.1.

A.3.1. Tamagawa measure on $\mathbf{A}_F$. As an additive abelian topological group, $\mathbf{A}_F$ is Hausdorff and locally compact. As such, it carries a left- and right-invariant Haar measure. Since it is commutative, it is unimodular.

DEFINITION A.3.1.1. Let F be a number field. Denote by ν the unique Haar measure on $\mathbf{A}_F$ satisfying the condition:

$$\overline{\nu}(\mathbf{A}_F/F) = 1 \tag{A.3.1.1}$$

where $\overline{\nu}$ is the Haar measure induced on $\mathbf{A}_F/F$ by ν on $\mathbf{A}_F$ and counting measure on F. We refer to ν as the *Tamagawa measure* on $\mathbf{A}_F$, or *additive Tamagawa measure*. For any finite set S of places of F containing the Archimedean places we define ν_S on $\mathbf{A}_{F,S}=\prod_{v\in S}F_v$ similarly by using $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ in place of $\mathbf{A}_F/F$.

The Tamagawa measure on $\mathbf{A}_F$ can be described explicitly as follows. Recall first that for G a Hausdorff locally compact abelian group isomorphic to the restricted product $\widehat{\prod}_{i\in I}(G_i, K_i)$, with $I = I_0 \times I_1$ with I_0 a finite set, where $K_i = 1$ for all $i \in I_0$ and $K_i \subset G_i$ is a compact open subgroup for every $i \in I_1$, and for μ_{G_i} Haar measures on each G_i such that $\mu_{G_i}(K_i) = 1$ for all but finitely many $i \in I_1$, there exists a unique Haar measure μ_G on G such that for every partition $J_1 \coprod J_2 = I$ with J_1 finite and containing I_0, the restriction of μ_G to the Borel set

$$\prod_{i\in J_1} G_i \times \prod_{i\in J_2} K_i$$

is the Radon product $\bigotimes_{i\in I} \mu_{G_i}$. We refer the reader to [**RV**, Ch.5,Prop.5.5].

Definition A.3.1.2 (Local Additive Tamagawa measures). Let F be a number field, $v \in \Sigma_F$ a place.

(1) If v is a real place, we define ν_v to correspond to $\mathrm{d}x$ under the unique isomorphism of topological fields $F_v \simeq \mathbf{R}$.
(2) If v is complex, we define ν_v to correspond to $|\mathrm{d}z \wedge \mathrm{d}\overline{z}| = 2|\mathrm{d}x \wedge \mathrm{d}y|$ via either of the two isomorphisms of topological fields $F_v \simeq \mathbf{C}$.
(3) If v is non-Archimedean, we define ν_v to be the unique Haar measure on F_v satisfying the condition:

$$\nu_v(\mathcal{O}_{F_v}) = N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2} = |\mathrm{disc}(\mathcal{O}_{F_v}/\mathbf{Z}_p)|_p^{1/2} \tag{A.3.1.2}$$

where p is the residue characteristic of F_v and $N(\mathfrak{a})$ is the p-power generating $N_{\mathcal{O}_{F_v}/\mathbf{Z}_p}(\mathfrak{a})$ for a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_{F_v}$.

We refer to ν_v as the *local additive Tamagawa measure* on F_v.

Lemma A.3.1.3. *Let F be a number field. For every $v \in \Sigma_F$, let ν_v be the local Tamagawa measure. Let S be a finite set of places of F containing the Archimedean places. Then we have*

$$\nu = \bigotimes_{v\in\Sigma_F} \nu_v, \quad \nu_S = \bigotimes_{v\in S} \nu_v \tag{A.3.1.3}$$

as Haar measures $\mathbf{A}_F$ and $\mathbf{A}_{F,S}$ respectively.[1]

Proof. Since $D_F = D_{F,\infty} \times \widehat{\mathcal{O}}_F$ as per (A.2.1.2), the Haar measure $\nu' := \bigotimes_{v\in\Sigma_F} \nu_v$ restricts to the Borel subset D_F of $\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F$ as the Radon product $\nu_\infty \otimes \nu_f$ where $\nu_\infty = \bigotimes_{v\in S_\infty} \nu_v$ and $\nu_f = \bigotimes_{v\in\Sigma_{F,f}} \nu_v$. Since ν' is a Haar measure on $\mathbf{A}_F$, there exists a unique real number $c > 0$ such that $\nu = c\nu'$. Since

$$\nu_f(\widehat{\mathcal{O}}_F) = \prod_{p\in\Pi} \prod_{v|p} N_{F_v/\mathbf{Q}_p}(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2} = N_{F/\mathbf{Q}}(\mathfrak{d}_{F/\mathbf{Q}})^{-1/2}$$

and $\nu_\infty(D_{F,\infty}) = N_{F/\mathbf{Q}}(\mathfrak{d}_{F/\mathbf{Q}})^{1/2} = |\mathrm{disc}(K/\mathbf{Q})|^{1/2}$ by [**CF**, Ch. XV, Lemma 4.1.4], we have $c = 1$ since $\nu(D_F) = \overline{\nu}(\mathbf{A}_F/F) = 1$ via Theorem C.4.4.3 applied to the characteristic function of $D_F \subset \mathbf{A}_F$. □

Lemma A.3.1.4. *Let F be a number field. Let μ be an arbitrary Haar measure on $\mathbf{A}_F$. For any idele $g \in \mathrm{GL}_1(\mathbf{A}_F)$ and any Borel set $B \subset \mathbf{A}_F$, we have*

$$\mu(gB) = |g|_F\, \mu(B). \tag{A.3.1.4}$$

Proof. This is [**CF**, Ch. XV, Lemma 4.1.2]. □

[1] Double-check for the S-local case.

A.3.1.1. *Self-duality of local additive Tamagawa measures.* We now discuss calculations that identify the local Tamagawa measures chosen in Definition A.3.1.2 on each local field F_v, for F a number field, with its own Plancherel measure with respect to an associated choice of non-trivial additive continuous character of F_v.

On any Hausdorff locally compact abelian group, there exists a unique Haar measure up to a nonzero constant multiplier, a positive real number. As recalled in Lemma D.1.1.2, Pontryagin duality determines, associated to a choice of Haar measure μ_G on a Hausdorff locally compact abelian group G, a unique Plancherel measure $\mu_{\widehat{G}}$ on the dual group $\widehat{G}$: the unique Haar measure on $\widehat{G}$ such that the inversion formula of (D.1.1.11) holds as stated with no constant correcting factors.

EXAMPLE A.3.1.5 (Archimedean standard additive characters). For $G = \mathbf{R}$, the Lebesgue measure $\mu_G = \mathrm{d}x$ is self-dual, and it is the local Tamagawa measure chosen for the field of real numbers in Definition A.3.1.2.

Note that the Fourier transform normalization, for $f \in L^1(\mathbf{R})$:

$$\widehat{f}(y) := \int_{\mathbf{R}} f(x) e^{-2\pi i x y}\, \mathrm{d}x$$

is dictated by the fact that the continuous group homomorphism

$$x \mapsto \chi_x(y) = \psi_{r,\mathrm{std}}(xy) := \exp(2\pi i x y)$$

defines a specific isomorphism $\mathbf{R} \simeq \widehat{\mathbf{R}}$.

With this normalization, indeed $\mathrm{d}x$ is the Plancherel measure on $\widehat{\mathbf{R}}$ uniquely determined by the Haar measure $\mathrm{d}x$ on $\mathbf{R}$.

A similar calculation shows that the local Tamagawa measure for the local field $\mathbf{C}$ chosen in Definition A.3.1.2(2) is indeed self-dual, upon choosing the character

$$\psi_{c,\mathrm{std}} : z \mapsto \exp(2\pi i \operatorname{Tr}_{\mathbf{C}/\mathbf{R}}(z)) = \exp(4\pi i \operatorname{Re}(z))$$

to identify $\mathbf{C}$ with $\widehat{\mathbf{C}}$ (see [CF, Ch. XV, §2.2,Thm. 2.2.2]).

We now compute the Fourier transform of some special Schwartz-Bruhat functions on local fields, adeles and ideles.

EXAMPLE A.3.1.6 (Local Fourier transform calculations - I). We compute the Fourier transform of the functions $f = 1_{a+p^N\mathbf{Z}_p}$, for p a prime. We parametrize all continuous characters of $\mathbf{Q}_p$ using the standard character

$$\psi_{\mathrm{std}} : \mathbf{Q}_p \to U(1)$$

obtained as the composition of the quotient $\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p$, followed the natural injective continuous homomorphism $\mathbf{Q}_p/\mathbf{Z}_p = (\mathbf{Q}/\mathbf{Z})[p^\infty] \subset \mathbf{R}/\mathbf{Z}$, followed by $\exp(2\pi i t)$. Then $\chi(g) = \chi_{\mathrm{std}}(cg)$ for a unique $c \in \mathbf{Q}_p$ depending only on $\chi \in \widehat{\mathbf{Q}_p}$. This identifies, as Hausdorff locally compact abelian groups, $\mathbf{Q}_p$ with its dual group. Upon calling $f_0 := 1_{\mathbf{Z}_p}$, then $1_{p^N\mathbf{Z}_p} = f_0(p^{-N}x)$, and $f(x) = f_0(p^{-N}x - p^{-N}a)$. We then have

$$\widehat{f}(x) = p^{-N}\widehat{f}_0(p^N(x-a)) = p^{-N}\psi_{\mathrm{std}}(ax)\cdot 1_{p^{-N}\mathbf{Z}_p}(x), \tag{A.3.1.5}$$

since $\mathbf{Z}_p^\perp = \mathbf{Z}_p$ under $\mathbf{Q}_p \simeq \widehat{\mathbf{Q}}_p$ by ψ_{std}. In particular, for $a = 0$ we have, as functions of $x \in \mathbf{Q}_p$:

$$\widehat{1_{p^N\mathbf{Z}_p}} = p^{-N}\cdot 1_{p^{-N}\mathbf{Z}_p}.$$

As a result, we can compute the Fourier transform of $1_{\mathbf{Z}_p^\times} = 1_{\mathbf{Z}_p} - 1_{p\mathbf{Z}_p}$, which gives

$$\widehat{1_{\mathbf{Z}_p^\times}} = 1_{\mathbf{Z}_p} - p^{-1}\cdot 1_{p^{-1}\mathbf{Z}_p}. \tag{A.3.1.6}$$

EXAMPLE A.3.1.7 (Local Plancherel measures). We exhibit the Plancherel measure for $G = \mathbf{Q}_p$, for p a prime. The choice $\mu_{\mathbf{Q}_p} = \mathrm{d}x$, the standard additive Haar measure, gives the unit ball $\mathbf{Z}_p$ volume 1. Since $\mathbf{Q}_p$

is self-dual via ψ_{std}, the Plancherel measure is a positive real multiple of $\mu_{\mathbf{Q}_p}$. Say $\mu_{\widehat{\mathbf{Q}_p}} = c \cdot \mu_{\mathbf{Q}_p}$ for some unique $c \in \mathbf{R}_{>0}$. It is enough to compute:

$$\widehat{1_{\mathbf{Z}_p}}(y) = \int_{\mathbf{Q}_p} 1_{\mathbf{Z}_p}(x)\overline{\psi_{\mathrm{std}}(xy)}\,\mathrm{d}\mu_{\mathbf{Q}_p}(x)$$

which is $1_{\mathbf{Z}_p}(y)$ by Example A.3.1.6.

By Lemma D.1.1.2(5), this can only be so if $\mu_{\mathbf{Q}_p} = \mu_{\widehat{\mathbf{Q}_p}}$, and so $c = 1$. Note that $\mu_{\mathbf{Q}_p}$ is also the local Tamagawa measure for the local field $\mathbf{Q}_p$ chosen in Definition A.3.1.2(3).

Example A.3.1.8 (Global standard additive character - I). The self-duality of $\mathbf{A}_{\mathbf{Q}}$ is defined by the standard character

$$\psi_{\mathbf{Q},\mathrm{std}} := \prod_{v \in \Sigma_{\mathbf{Q}}} \psi_{v,\mathrm{std}}$$

where $\psi_{v,\mathrm{std}}$ is the standard local character defined in Example A.3.1.6 for v the p-adic absolute value for p a prime, and in Example A.3.1.5 for v real. Upon choosing $\mu_{\mathbf{A}_{\mathbf{Q}}} = \mathrm{d}x := \prod_v \mathrm{d}x_v$ to be the standard additive Haar measure on $\mathbf{A}_{\mathbf{Q}}$, it is the content of [**CF**, Ch. XV, Cor. 3.3.1] that $\mu_{\mathbf{A}_{\mathbf{Q}}}$ is self-dual.

This follows from the fact that the Plancherel measure is a positive real multiple of $\mu_{\mathbf{A}_{\mathbf{Q}}}$, and so upon computing the Fourier transform of $e^{-\pi x_\infty^2} \otimes 1_{\widehat{\mathbf{Z}}}$ to be itself with respect to the choice $\mu_{\mathbf{A}_{\mathbf{Q}}}$ of Haar measure on $\mathbf{A}_{\mathbf{Q}}$, we obtain that such constant multiplier must be 1. Such calculation follows from Example A.3.1.6 combined with [**CF**, Ch.ẊV, §3.3, Lemma 3.3.3]; see also [**CF**, Ch. XV, §3.3, Cor. 3.3.1]. The measure $\mu_{\mathbf{A}_{\mathbf{Q}}}$ is also the additive Tamagawa measure on $\mathbf{A}_{\mathbf{Q}}$ in Definition A.3.1.1 since $\mu_{\mathbf{A}_{\mathbf{Q}}}([0,1) \times \widehat{\mathbf{Z}}) = 1$.

Example A.3.1.9 (Global standard additive character - II). For F an arbitrary number field and $\mathrm{Tr}_{F/\mathbf{Q}} : F \to \mathbf{Q}$ the trace form, we define

$$\psi_{F,\mathrm{std}} := \psi_{\mathbf{Q},\mathrm{std}}(\mathrm{Tr}_{F/\mathbf{Q}}(\cdot)). \tag{A.3.1.7}$$

We note that the additive Tamagawa measure ν chosen on $\mathbf{A}_F$ in Definition A.3.1.1 is self-dual relative to this choice of additive character, by [**CF**, Ch. XV, §4.1, Thm. 4.1.2].

Example A.3.1.10 (Local Fourier transform calculations - II). Let F be a number field, v a non-Archimedean place of F. Let p be the residue characteristic of F_v. Throughout this work, we have normalized the additive character on F_v to be the standard additive character $\psi_{F_v,\mathrm{std}} = \psi_{\mathbf{Q}_p,\mathrm{std}} \circ \mathrm{Tr}_{F_v/\mathbf{Q}_p}$, and relative to this choice, the local additive Tamagawa measure ν_v is self-dual; i.e., the resulting Fourier transform is a unitary automorphism on $L^2(F_v, \nu_v)$.

We recall here the following identity of functions on F_v:

$$\widehat{1_{\mathcal{O}_{F_v}}} = |\mathrm{disc}(\mathcal{O}_{F_v}/\mathbf{Z}_p)|_p^{1/2} 1_{\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}} \tag{A.3.1.8}$$

where $\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}$ is the inverse different. In particular, if $F_v/\mathbf{Q}_p$ is unramified then $1_{\mathcal{O}_{F_v}}$ is an eigenfunction of the local Fourier transform, with eigenvalue 1.

Example A.3.1.11 (S-local dual lattices). We let F be a number field, S a finite set of places of F containing all the Archimedean places and all the ramified places of F over $\mathbf{Q}$. The closed subgroup $\mathcal{O}_{F,S} \subset \mathbf{A}_{F,S}$ is discrete and co-compact by Lemma A.1.5.1(2).

We denote by ψ_v the standard additive character on the local field F_v, for each $v \in S$, constructed in Example A.3.1.8 and Example A.3.1.9, and recall (see Lemma A.3.1.3) that our normalization for the Haar measure ν_S on $\mathbf{A}_{F,S}$ gives the quotient $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$, upon endowing $\mathcal{O}_{F,S}$ with the counting measure, volume 1. This is the additive S-local Tamagawa measure defined in Definition A.3.1.1. For the standard additive

character

$$\psi_{S,\mathrm{std}} = \bigotimes_{v\in S} \psi_v \tag{A.3.1.9}$$

and $x \in \mathbf{A}_{F,S}$ an arbitrary element, $\chi_x(y) := \psi_{S,\mathrm{std}}(xy)$ is a continuous characterof $\mathbf{A}_{F,S}$ and the assignment $x \mapsto \chi_x$ is an isomorphism $\mathbf{A}_{F,S} \to \widehat{\mathbf{A}_{F,S}}$ of Hausdorff locally compact abelian groups. This makes ν_S self-dual (i.e., its own Plancherel measure) due to the same for each (ψ_v, ν_v).

With respect to the duality pairing

$$\langle\cdot,\cdot\rangle_S : \mathbf{A}_{F,S} \times \mathbf{A}_{F,S} \to \mathrm{U}(1) \tag{A.3.1.10}$$

assigned by $\langle x, y\rangle_S := \chi_x(y)$, the closed orthogonal $\Gamma^\perp$ of the closed discrete subgroup $\Gamma := \mathcal{O}_{F,S} \subset \mathbf{A}_{F,S}$ is $\mathcal{O}_{F,S}$ itself, under the stated assumptions on S. To see this, we proceed as follows.

Since Γ is co-compact,[2] the isomorphism $\widehat{\mathbf{A}_{F,S}/\Gamma} \simeq \Gamma^\perp$ from Lemma D.1.4.6(2) yields $\Gamma^\perp$ is discrete. We have $\mathcal{O}_{F,S} \subset \Gamma^\perp$, and hence $\Gamma^\perp/\mathcal{O}_{F,S}$ is a discrete subgroup of $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ that is compact, so it is a finite abelian group. We observe that we have a duality pairing analogous to $\langle\cdot,\cdot\rangle_S$ from (A.3.1.10):

$$\langle\cdot,\cdot\rangle'_S : (\mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v}) \times (\mathbf{A}_{F,S} \times \bigoplus_{v\in\Sigma_F - S} F_v/\mathcal{O}_{F_v}) \to \mathrm{U}(1) \tag{A.3.1.11}$$

defined as:

$$\langle x', y'\rangle'_S := (\psi_{S,\mathrm{std}} \otimes \bigotimes_{v\in\Sigma_F - S} \psi_v)(x'y') = \psi_{S,\mathrm{std}}(x'_S y'_S) \prod_{v\in\Sigma_F - S} \psi_v(x'_v y'_v) \tag{A.3.1.12}$$

where x'_S, respectively y'_S, denotes the projection of x', y' onto $\mathbf{A}_{F,S}$, and similarly x'_v, y'_v denote the projections of x', y' onto $\mathcal{O}_{F_v}$. We note that indeed $\langle\cdot,\cdot\rangle'_S$ is a duality pairing, by [**CF**, Ch. XV, §2.2, Lemma 2.2.3], and because S contains all the ramified primes of F, (so for every $v \in \Sigma_F - S$ and p the prime residue characteristic of the non-Archimedean local field F_v, we have $\mathfrak{d}^{-1}_{F_v/\mathbf{Q}_p} = \mathcal{O}_{F_v}$). We therefore have $\psi_v(x'_v y'_v) = 1$ in (A.3.1.12) for every finite place v of F not contained in S, which gives:

$$\langle x', y'\rangle'_S := \langle x'_S, y'_S\rangle_S \tag{A.3.1.13}$$

for every $x', y' \in \mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v}$.

We denote by $A := (\mathcal{O}_{F,S})^{\perp,\langle\cdot,\cdot\rangle'_S}$ the orthogonal of $\mathcal{O}_{F,S} \subset \mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v}$ with respect to the duality pairing $\langle\cdot,\cdot\rangle'_S$. A similar reasoning as in the foregoing, using compactness of $(\mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v})/\mathcal{O}_{F,S}$, shows that A is discrete and co-compact as a closed subgroup of $\mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v}$, and it contains $\mathcal{O}_{F,S}$ as a subgroup of finite index.

Since $(\mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v})/\mathcal{O}_{F,S}$ is an F-vector space, as it is isomorphic to $\mathbf{A}_F/F$ (see Lemma A.1.5.1(3)), the quotient $A/\mathcal{O}_{F,S}$ is a finite abelian subgroup of a uniquely divisible abelian group (i.e., a $\mathbf{Q}$-vector space), so it is trivial.

The surjective homomorphism $\mathbf{A}_{F,S} \times \prod_{v\in\Sigma_F - S} \mathcal{O}_{F_v} \to \mathbf{A}_{F,S}$ defined by $y' \mapsto y'_S$, induces a homomorphism $A \to \Gamma^\perp$ which is injective and compatible with the finite index subgroups $\mathcal{O}_{F,S}$ of A, and $\Gamma = \mathcal{O}_{F,S}$ of $\Gamma^\perp$.

It follows that $\Gamma^\perp/\mathcal{O}_{F,S}$ is a finite abelian subgroup of $A/\mathcal{O}_{F,S}$, which is trivial, and hence it is itself trivial. We deduce that we have $\mathcal{O}_{F,S} = \Gamma^\perp$.

A.3.2. Tamagawa measure on $\mathrm{GL}_1(\mathbf{A}_F)$. The topological group $\mathrm{GL}_1(\mathbf{A}_F)$ is a Hausdorff locally compact abelian group, so it is unimodular and has a unique Haar measure specified as follows:

Definition A.3.2.1. Let F be a number field. Denote by μ the unique Haar measure on $\mathrm{GL}_1(\mathbf{A}_F)$ such that with counting measure on $F^\times$ the associated Haar measure $\overline{\mu}$ on C_F and $\mathrm{d}t/t$ on $\mathbf{R}_{>0} \overset{|\cdot|_F}{\simeq} C_F/C_F^1$ give

[2]Brian suggested the rest of the original exposition of the argument from here onward should be rewritten, so at least reread the revised version now here to check if now OK.

rise to the measure $\overline{\mu}_1$ on C_F^1 with volume

$$\overline{\mu}_1(C_F^1) = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{|d_F|^{1/2} w_F}, \tag{A.3.2.1}$$

fwhere $d_F \in \mathbf{Z}^+$ is the absolute discriminant of F and $R_F := |\det(\mathscr{L}(\varepsilon_i))_{1\leq i\leq r_1+r_2-1}|$ is the regulator of F. We refer to μ as the *Tamagawa measure* on $\mathrm{GL}_1(\mathbf{A}_F)$, to $\overline{\mu}$ as the *Tamagawa measure* on C_F, and to $\overline{\mu}_1$ as the *Tamagawa measure* on C_F^1.

Definition A.3.2.2. Let F be a number field, $v \in \Sigma_F$ a place. Define a Haar measure on $F_v^\times$ as follows:

(1) If v is Archimedean, we define μ_v to be $|\cdot|_v^{-1}\nu_v|_{F_v^\times}$ (a Haar measure since $|\cdot|_v$ is normalized),
(2) If v is non-Archimedean, we define μ_v to be the unique Haar measure on $F_v^\times$ satisfying the condition:

$$\mu_v(\mathcal{O}_{F_v}^\times) = |\mathrm{disc}(\mathcal{O}_{F_v}/\mathbf{Z}_p)|_p^{1/2}, \tag{A.3.2.2}$$

where p is the residue characteristic of F_v.

We refer to μ_v as the *local multiplicative Tamagawa measure* on $F_v^\times$.

Lemma A.3.2.3. *Let F be a number field. For every $v \in \Sigma_{F,f}$, let μ_v be the local multiplicative Tamagawa measure on $F_v^\times$. We have, as Haar measures on $F_v^\times$:*

$$\mu_v = \frac{q_v}{q_v - 1}\frac{\nu_v}{|\cdot|_v} \tag{A.3.2.3}$$

where q_v is the size of the residue field at v and ν_v is the local additive Tamagawa measure on F_v.

Proof. Since $|\cdot|_v$ is the normalized absolute value on F_v, the measure $|\cdot|_v^{-1}\nu_v$ for ν_v the local additive Tamagawa measure on F_v is a Haar measure on $F_v^\times$. There therefore exists a unique real number $c > 0$ such that $\mu_v = c|\cdot|_v^{-1}\nu_v$. It is enough to compute the measure of $\mathcal{O}_{F_v}^\times$ with respect to each measure and show that we have $c = q_v/(q_v - 1)$.

We have

$$\int_{\mathcal{O}_{F_v}^\times} \frac{\mathrm{d}\nu_v(g_v)}{|g_v|_v} = \int_{\mathcal{O}_{F_v}^\times} \mathrm{d}\nu_v(g_v) = \int_{\mathcal{O}_{F_v}-\mathfrak{m}_{F_v}} \mathrm{d}\nu_v(g_v) = \nu_v(\mathcal{O}_{F_v}) - \nu_v(\mathfrak{m}_{F_v}) = \left(1 - \frac{1}{q_v}\right)\nu_v(\mathcal{O}_{F_v}) \tag{A.3.2.4}$$

since $\mathfrak{m}_{F_v}$ has index q_v in $\mathcal{O}_{F_v}$.

Since ν_v is the local additive Tamagawa measure on F_v, this in turn reads as

$$\frac{q_v - 1}{q_v}\nu_v(\mathcal{O}_{F_v}) = \frac{q_v - 1}{q_v}|\mathrm{disc}(\mathcal{O}_{F_v}/\mathbf{Z}_p)|_p^{1/2} = \frac{q_v - 1}{q_v}\mu_v(\mathcal{O}_{F_v}^\times).$$

This gives $c = q_v/(q_v - 1)$, so (A.3.2.3) holds. □

The measure μ is explicitly described in terms of the μ_v's:

Lemma A.3.2.4. *Let F be a number field. For every $v \in \Sigma_F$, let μ_v be local multiplicative Tamagawa measure on $F_v^\times$. We have, as Haar measures on $\mathrm{GL}_1(\mathbf{A}_F)$,*

$$\mu = \bigotimes_{v\in\Sigma_F} \mu_v. \tag{A.3.2.5}$$

This is [**RV**, Thm. 7.21]. The restricted produce measure makes sense since $\mu_v(\mathcal{O}_{F_v}^\times) = 1$ for all but finitely many v.

Remark A.3.2.5 (On the normalizations for additive character and measure). The Tamagawa measures on $\mathbf{A}_F$ and $\mathrm{GL}_1(\mathbf{A}_F)$ constructed in this section are the same choices of Haar measures made on $\mathbf{A}_F$ and $\mathrm{GL}_1(\mathbf{A}_F)$ made in [**CF**, Ch.XV,p.313,328,334].

The former is also the same choice of normalized Haar measure made in [Co, Appendix I,(34),p.57]. Its purpose is to normalize the adelic Poisson Summation Formula, removing the volume factor (which equals 1, under this choice of normalization; see Theorem 2.2.3.2 as well as [CF, Ch.XV,§4.2,Thm.4.2.1].

They are special cases of general Tamagawa measures on the adelic points of algebraic groups over a number field, for which we refer the reader to [Oes, §3.1,4] (the former corresponding to the invariant differential on $(\mathbf{G}_a)_F$ given by $\mathrm{d}x$, and the latter to $\mathrm{d}x/x$ on $(\mathrm{GL}_1)_F$, x being the standard global parameter on $\mathbf{A}_F^1$).

A.3.3. Haar measures on C_F and C_F^1. We now return to the orbit structure of $\mathrm{GL}_1(\mathbf{A}_F)$ under the left-translation action of the discretely embedded diagonal copy of $F^\times$.

A.3.3.1. *Continuous sections of the idelic norm and Radon product decompositions.* We choose

$$s : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$$

a continuous homomorphic section to the idelic norm $|\cdot|_F : \mathrm{GL}_1(\mathbf{A}_F) \to \mathbf{R}_{>0}$. By Lemma A.2.2.9,

$$\theta_{F,s} : \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_F)^1 \to \mathrm{GL}_1(\mathbf{A}_F)$$

defined as $\theta_{F,s}(t, g_0) = s(t)g_0$ is an isomorphism of Hausdorff locally compact abelian groups. Thus, $(\mathrm{d}t/t) \otimes \mu^1$ corresponds to μ under any such $\theta_{F,s}$. In particular,

$$\mu(s([1,T]) \cdot E_F^1) = \log(T)\overline{\mu}_1(C_F^1)$$

where $\overline{\mu}_1(C_F^1)$ is given by (A.3.2.1).

A.3.4. S-local Tamagawa measures. Let F be a number field, S a finite set of places of F containing all the Archimedean places. The local additive and multiplicative Tamagawa measures on each F_v and $F_v^\times$ can be assembled together into S-local Tamagawa measures. We introduce the relevant definitions and terminology here.

Definition A.3.4.1 (S-local additive and multiplicative Tamagawa measures). Let F be a number field, S a finite set of places of F containing all the Archimedean places. We define:

(1) The Haar measure on $\mathbf{A}_{F,S}$ given by

$$\nu_S = \bigotimes_{v \in S} \nu_v \tag{A.3.4.1}$$

is called the *S-local additive Tamagawa measure on* $\mathbf{A}_{F,S}$, where ν_v is the additive local Tamagawa measure on F_v from Definition A.3.1.2 for each $v \in S$.

(2) The Haar measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ given by

$$\mu_S = \bigotimes_{v \in S} \mu_v \tag{A.3.4.2}$$

is called the *S-local multiplicative Tamagawa measure on* $\mathrm{GL}_1(\mathbf{A}_{F,S})$, where μ_v is the multiplicative local Tamagawa measure on $F_v^\times$ from Definition A.3.2.2 for each $v \in S$.

(3) We define:
   (1) $\mu_{S,1}$ to be the unique Haar measure on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ determined as per Theorem D.1.2.1 by the choices of μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and $\mathrm{d}t/t$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})/\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \overset{|\cdot|_{F,S}}{\simeq} \mathbf{R}_{>0}$,
   (2) $\overline{\mu}_S$ to be the unique Haar measure on $C_{F,S}$ determined as per Theorem D.1.2.1 by the choices μ_S on $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and the counting measure on $\mathcal{O}_{F,S}^\times$,
   (3) $\overline{\mu}_{S,1}$ to be the unique Haar measure on $C_{F,S}^1$ determined as per Theorem D.1.2.1 by the choices $\mu_{S,1}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and the counting measure on $\mathcal{O}_{F,S}^\times$.

REMARK A.3.4.2 (On the normalizations of additive character and measures). We recall that the S-local Tamagawa measure ν_S on $\mathbf{A}_{F,S}$, obtained as the product of the local additive Tamagawa measures ν_v on each of the local fields F_v, is self-dual with respect to the choice of standard additive character ψ_v whose construction is discussed in Example A.3.1.8 and Example A.3.1.9 (see also Remark A.3.2.5), and relative to the resulting Fourier transform.

We caution the reader that while all but finitely may of the non-Archimedean places v of the number field F are such that the local Fourier transform of the Schwartz-Bruhat function $1_{\mathcal{O}_{F_v}}$ with respect to the self-dual ν_v satisfies:

$$\mathcal{F}_v(1_{\mathcal{O}_{F_v}}) = 1_{\mathcal{O}_{F_v}}$$

finitely may of them may not. Namely, for any finite place v with residue characteristic p we have

$$\mathcal{F}_v(1_{\mathcal{O}_{F_v}}) = |\mathrm{disc}(\mathcal{O}_{F_v}/\mathbf{Z}_p)|_p^{1/2} 1_{\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}}$$

as seen in Example A.3.1.10, so this agrees with $1_{\mathcal{O}_{F_v}}$ if and only if v is unramified over p.

The volume of $\mathbf{A}_{F,S}/\mathcal{O}_{F,S}$ with respect to the unique Haar measure induced as per Theorem D.1.2.1 on the compact Hausdorff quotient by the S-local additive Tamagawa measure ν_S on $\mathbf{A}_{F,S}$ and by the counting measure on $\mathcal{O}_{F,S}$, is the prime-to-S_f part of $|\mathrm{disc}(K/\mathbf{Q})|^{1/2}$, similarly to the proof of Lemma A.3.1.3. This is 1 unless S, which already contains all the Archimedean places of F, contains all finite places of F that ramify over $\mathbf{Q}$.

A.4. Dirichlet's S-unit Theorem: estimates by lattice point-counting

In this section we illustrate two applications of Dirichlet's S-unit Theorem (see Theorem A.1.5.2(2)) resting on lattice point-counting techniques that go back to Davenport's paper on a principle of Liouville (see [Davenport-1]). These techniques will enable us to make estimates on the growth of lattice point-counting functions counting the number of S-units contained in certain convex polytopes, with the aim of applying such estimates to the study of the S-local adelic θ-functions through Theorem 2.4.2.1.

A.4.1. Lattice point-counting and Dirichlet's S-Unit Theorem. Theorem 2.4.2.1 rests upon the following estimate, which is obtained with a lattice point-counting argument enabled by Dirichlet's S-Unit Theorem.[3]

THEOREM A.4.1.1 (Estimates on point-counting functions of the lattice of S-units). *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

Define:

(1) $Q \subset \mathbf{A}_{F,S_\infty}$ *the compact hypercube with non-empty interior:*

$$Q := \{x \in \mathbf{A}_{F,S_\infty} : |x_v|_v \leq 1,\ v \in S_\infty\}. \tag{A.4.1.1}$$

(2)

$$u := 1_Q \otimes \bigotimes_{v \in S_f} 1_{\mathcal{O}_{F_v}}, \tag{A.4.1.2}$$

a boundedly supported and bounded (by 1) piece-wise continuous function on $\mathbf{A}_{F,S}$.

There exist real numbers $C > 0$ and $\varepsilon > 0$ such that for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ with $|g|_{F,S} < \varepsilon$ we have:

$$\sum_{\xi \in \mathcal{O}_{F,S}^\times} u(\xi g) \leq C\,|\log|g|_{F,S}|^{|S|-1}. \tag{A.4.1.3}$$

[3]Maybe put in a Remark indicating relation to an assertion in a paper of Connes.

The proof of Theorem A.4.1.1 mirrors a method first devised by Davenport and reprised in [**Schm**, Lemma 1, p. 347], and for which we refer the reader to [**Davenport-1**, §2, Thm., p. 180].

The idea is to estimate a lattice point-counting function by means of a volume of a convex polytope within $\mathbf{R}^S$, which is then computed by slicing it by hyperplane sections: an "approximate Fubini" calculation. This is at the heart of [**Davenport-1**, §2, Thm., p. 180], and applied in the proof of [**Schm**, Lemma 1, p. 347].

PROOF. Define

$$Z := \prod_{v\in S_\infty}(-\infty,0]\times\prod_{v\in S_f}(-\infty,0] \tag{A.4.1.4}$$

and denote by $H_{\log}$ the hyperplane $\{(x_v)_{v\in S}\in\mathbf{R}^S : \sum_{v\in S}x_v=0\}$ and denote by $\mathscr{L}_S : \mathrm{GL}_1(\mathbf{A}_{F,S})\to\mathbf{R}^S$ the logarithm homomorphism, $g\mapsto(\log|g|_v)_{v\in S}$, so $Z=\mathscr{L}_S(\mathrm{GL}_1(\mathbf{A}_{F,S})\cap(Q\times\prod_{v\in S_f}\mathcal{O}_{F_v}))$.

For fixed $g\in\mathrm{GL}_1(\mathbf{A}_{F,S})$, the condition on $\xi\in\mathcal{O}_{F,S}^\times$ that $u(\xi g)\neq 0$ is that $|\xi_v g_v|_v\leq 1$ for all places v of F. We can therefore compute

$$\sum_{\xi\in\mathcal{O}_{F,S}^\times}u(\xi g)=|\mathcal{O}_{F,S}^\times\cap g^{-1}\cdot(Q\times\prod_{v\in S_f}\mathcal{O}_{F_v})|. \tag{A.4.1.5}$$

Since the condition $|\xi_v g_v|_v\leq 1$ intervenes solely through the absolute value at v for each $v\in S$, we can apply $\mathscr{L}_S$ to rewrite:

$$\sum_{\xi\in\mathcal{O}_{F,S}^\times}u(\xi g)=|\mathscr{L}_S(\mathcal{O}_{F,S}^\times)\cap(-\mathscr{L}_S(g)+\prod_{v\in S}(-\infty,0])|=|\mathscr{L}_S(\mathcal{O}_{F,S}^\times)\cap(-\mathscr{L}_S(g)+Z)|. \tag{A.4.1.6}$$

(Note that typically $g\notin\mathcal{O}_{F,S}^\times$, so this count depends very much on g.)

We choose the continuous homomorphic section τ to the S-idelic norm given by the diagonal section $s_{F,\delta}$ from (A.2.2.11). Its image is contained within the closed subgroup $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})$ (as for any continuous homomorphic section). This choice of τ ensures $|\log|\tau(t)_v|_v|\leq|\log(t)|$ for all $t\in\mathbf{R}_{>0}$ since $\tau(t)_v=t^{1/n_v}$ for an integer $n_v\in\{1,2\}$, an inequality that will be useful later.

For the fundamental domain $E_{S,0}$ from (A.2.4.5) for the continuous left-translation action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, the product $\tau(\mathbf{R}_{>0})\cdot E_{S,0}$ is a fundamental domain for the action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})$. Thus, since both sides of the desired bound (A.4.1.3) are $\mathcal{O}_{F,S}^\times$-invariant, for the purposes of proving that bound we can limit g to vary within $\tau(\mathbf{R}_{>0})\cdot E_{S,0}$. Going a step further, recall that $E_{S,0}$ maps onto $\ker(\mathrm{Pic}(\mathcal{O}_F)\to\mathrm{Pic}(\mathcal{O}_{F,S}))$, and that if $g_1,\ldots,g_k$ are representatives in $E_{S,0}$ for the elements of that kernel then

$$\tau(\mathbf{R}_{>0})\cdot E_{S,0}=\coprod g_i\tau(\mathbf{R}_{>0})\cdot(E_{S_\infty,0}\times\widehat{\mathcal{O}}_F^\times). \tag{A.4.1.7}$$

We are going to first prove (A.4.1.3) by working separately with points of the form $g_i g$ for $g\in\tau(\mathbf{R}_{>0})\cdot(E_{S_\infty,0}\times\widehat{\mathcal{O}}_F^\times)$; such g we shall refer to as "basic". In what follows, we fix i (this will yield some constants depending on i, from which we'll arrive at uniform constants at the end).

In light of (A.4.1.6), $\sum_{\xi\in\mathcal{O}_{F,S}^\times}u(\xi g_i g)$ is the number of points of the lattice $\mathscr{L}_S(\mathcal{O}_{F,S}^\times)$ in $H_{\log}$ contained in the $-\mathscr{L}_S(g_i g)$-translate of the box $Z\subset\mathbf{R}^S$. Note that with respect to the standard Euclidean inner product on $\mathbf{R}^S$, $H_{\log}$ is the hyperplane in $\mathbf{R}^S$ orthogonal to the line $\mathbf{R}\cdot(\sum_{v\in S}\mathbf{e}_v)$, where $\{\mathbf{e}_v\}_{v\in S}$ is the standard basis indexed by S.

We write $\mathbf{e}:=\sum_{v\in S}\mathbf{e}_v$ and choose on the line $\mathbf{R}\cdot\mathbf{e}$ the normalized Lebesgue measure $\lambda_{\mathbf{e}}$ giving the segment $[0,1)\cdot\mathbf{e}$ volume 1. By Theorem D.1.2.1 there exists a unique Haar measure λ_0 on $H_{\log}$ such that the quotient measure induced on $\mathbf{R}\mathbf{e}\simeq\mathbf{R}^S/H_{\log}$ by the Lebesgue measure λ on $\mathbf{R}^S$ (normalized so as to give the hypercube $[0,1)^S$ volume 1) and by λ_0 on $H_{\log}$ is $\lambda_{\mathbf{e}}$. We further endow $H_{\log}$ with the inner product induced by the Euclidean inner product on $\mathbf{R}^S$. Note that neither such inner product nor the measures $\lambda_{\mathbf{e}}$ and λ_0 depend on the choice of enumeration of the places in S.

For $\{\varepsilon_k, 1 \leq k \leq |S|-1\} \subset \mathcal{O}_{F,S}^\times$ a set of fundamental S-units, $\{\mathscr{L}_S(\varepsilon_k), 1 \leq k \leq |S|-1\}$ is an $\mathbf{R}$-basis for $H_{\log}$. We denote by $|\det \mathscr{L}_S(\mathcal{O}_{F,S}^\times)|$ the absolute Jacobian of the change of basis matrix from any orthonormal basis on $H_{\log}$ to $\{\mathscr{L}_S(\varepsilon_k), 1 \leq k \leq |S|-1\}$; equivalently, it is the volume of the parallelotope $\sum_{1 \leq k \leq |S|-1} [0,1) \cdot \mathscr{L}_S(\varepsilon_k)$ in $H_{\log}$.

As convenient notation, let

$$N_S(g_i g) := |(-\mathscr{L}_S(g_i g) + Z) \cap \mathscr{L}_S(\mathcal{O}_{F,S}^\times)|$$

(we have seen that this count is $\sum_{\xi \in \mathcal{O}_{F,S}^\times} u(\xi g_i g)$). We note that $(-\mathscr{L}_S(g_i g) + Z) \cap H_{\log}$ is convex, so its intersection with any affine line in $H_{\log}$ is also convex and thus is an interval; in other words, the region $(-\mathscr{L}_S(g_i g) + Z) \cap H_{\log}$ inside $H_{\log}$ is of "class $m = 1$" in the lexicon from [**Schm**, Lemma 1, p. 347]. Before we can apply [**Schm**, Lemma 1, p. 347], we need to prove:

Claim A.4.1.2. There exists a real number $\varepsilon > 0$ such that for every basic $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ with $|g|_{F,S} < \varepsilon$ and every i, the closed convex subset $(-\mathscr{L}_S(g_i g) + Z) \cap H_{\log} \subset H_{\log}$ is contained in the closed ball in $H_{\log}$ of radius $C_0 |\log |g|_{F,S}|$ centered at the origin for some real number $C_0 > 0$ dependent on S but not on g nor i.

We grant the Claim for a moment, and draw some conclusions. The ball in that Claim can be fed into [**Schm**, Lemma 1, p. 347] when its radius is sufficiently large in a manner we need to precisely control. We pick a (very small) real number $0 < \varepsilon < 1$ making $|\log(\varepsilon)|$ large relative to successive minima of the lattice $\mathscr{L}_S(\mathcal{O}_{F,S}^\times)$ in $H_{\log}$ as specified in the technical condition [**Schm**, §2, (2.2), Lemma 1, p. 347], so for all basic $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ that satisfy $|g|_{F,S} < \varepsilon$ (so $|\log |g_i g|_{F,S}| = |\log |g|_{F,S}| > |\log(\varepsilon)|$) we have

$$N_S(g_i g) = \frac{\lambda_0((-\mathscr{L}_S(g_i g) + Z) \cap H_{\log})}{|\det(\mathscr{L}_S(\mathcal{O}_{F,S}^\times))|} + O(|\log |g|_{F,S}|^{|S|-2}) \tag{A.4.1.8}$$

as $|g|_{F,S} \to 0$, where the implied constant has absorbed the constant $C_0 > 0$ from Claim A.4.1.2 and depends only on S, moreover through the geometry of the lattice $\mathscr{L}_S(\mathcal{O}_{F,S}^\times)$ inside $H_{\log}$.

The real number $\varepsilon > 0$ likewise depends only on S, moreover through the successive minima of the lattice $\mathscr{L}_S(\mathcal{O}_{F,S}^\times) \subset H_{\log} \simeq \mathbf{R}^{|S|-1}$, which in turn only depends on $\mathscr{L}_S(\mathcal{O}_{F,S}^\times)$ and the normed structure on the hyperplane $H_{\log}$ inside the Euclidean space $\mathbf{R}^S$. We now prove the Claim.

Proof of Claim A.4.1.2. We are considering only g basic, which we recall means that $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ is constrained to vary over $\tau(\mathbf{R}_{>0}) \cdot (E_{S_\infty,0} \times \widehat{\mathcal{O}}_F^\times)$ from (A.4.1.7). The subset Z from (A.4.1.4) is a closed pointed cone in $\mathbf{R}^S$. Since $H_{\log} \cap Z = \{0\}$, the intersection of $(\mathscr{L}_S(g_i g) + H_{\log}) \cap Z$ is carried onto the intersection of $H_{\log}$ with $-\mathscr{L}_S(g_i g) + Z$ under translation by $-\mathscr{L}_S(g_i g)$ on $\mathbf{R}^S$.

Let's find a real number $C_0 > 0$ depending only on $|S|$ such that $(-\mathscr{L}_S(g_i g) + Z) \cap H_{\log}$ is contained in the closed ball in $\mathbf{R}^S$ centered at the origin and of radius $C_0 |\log |g|_{F,S}| + M_i$ for a constant M_i depending only on the fixed g_i and S. Observe that the vector $\mathscr{L}_S(g_i g)$ has supremum norm bounded above by the maximum of the real numbers $|\log |(g_i)_v g_v|_v|$ as v varies over the finitely many places of F in S, with these real numbers equal to $|\log |g_i|_v|$ for finite v (which is 1 for all but finitely many v) since g is basic: we have $g = \tau(t_0) g_0$ for a unique $t_0 \in \mathbf{R}_{>0}$ and $g_0 \in E_{S_\infty,0} \times \widehat{\mathcal{O}}_F^\times$.

Since $E_{S_\infty,0}$ is relatively compact in $\mathbf{R}^{S_\infty}$, we conclude that $\log(|(g_i g)_w|_w)$ is bounded as g varies over $E_{S_\infty,0} \times \widehat{\mathcal{O}}_F^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,S})$ and w varies over the places in S. In particular, for each $v \in S$, we have $|\log |(g_i g)_v|_v| \leq C_0 \max_{v \in S_\infty} |\log |\tau(t)_v|_v|$ for a real number $C_0 > 0$ and all basic $g = \tau(t_0) g_0$ with real $t_0 > 0$ and $g_0 \in E_{S_\infty,0} \times \widehat{\mathcal{O}}_F^\times$. Since we chose the diagonal section $\tau = s_{F,\delta}$, for every $v \in S_\infty$ we have $|\log |\tau(t)_v|_v| \leq |\log(t)|$ for all $t \in \mathbf{R}_{>0}$. Hence, the sup-norm of $\mathscr{L}_S(g_i g) \in \mathbf{R}^S$ is bounded above by $C_0 |\log(t_0)| + B_i$. where $B_i = \max_{v \in S_f} |\log(|(g_i)_v|)|$. But τ is a section to the idelic norm and $g_0 \in \mathrm{GL}_1(\mathbf{A}_{F,S})^1$, so $|g|_{F,S} = |\tau(t_0|_{F,S} = t_0$, yielding that $\mathscr{L}_S(g_i g)$ has sup-norm as most $C_0 |\log(|g|_{F,S})| + B_i$ as desired.

All norms on $\mathbf{R}^S$ are equivalent, such as the sup-norm and its natural inner product norm, so upon denoting $\|\cdot\|$ the norm induced on $H_{\log}$ by the inner product inherited from $\mathbf{R}^S$ we conclude that $\|\mathscr{L}_S(g_ig)\| \leq C_1|\log|g|_{F,S}| + M_i$ for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ with some real number $C_1 > 0$ dependent on S but not on g and some $M_i > 0$ depending on i and S.

We denote by $B_0(r)$ the closed ball of radius $r > 0$ in $\mathbf{R}^S$ centered at 0. We claim there exists a real number $A_i > 0$ independent of g such that

$$Z \cap (\mathscr{L}_S(g_ig) + H_{\log}) \subset B_0(A_i\|\mathscr{L}_S(g_ig)\|) \subset B_0(A_i(C_1|\log|g|_{F,S}| + M_i)) \tag{A.4.1.9}$$

for all basic g such that $|g|_{F,S} < \varepsilon_i$. The second containment follows from how we made C_1 above. To see the first, we claim more broadly that for a closed polyhedral cone Z in a Euclidean space V with vertex at the origin and a hyperplane H which meets Z in exactly $\{0\}$, for all $v \in V$ the intersection $(v + Z) \cap H$ is contained in the closed ball $B_0(C\|v\|)$ centered at 0 with radius $C\|v\|$ where $C = \max(1/\sin(\theta_i)) \geq 1$ where $0 < \theta_i < \pi$ are the finitely many angles by which H meets the faces of Z (we make the convention $\theta_1 = \pi/2$ if $\dim V = 1$ so that $C = 1$).

The case $\dim V \leq 1$ is trivial (as $H = \{0\}$). In general, we pick any $v \in V$ and $z \in Z$ such that $v + z \in H$ and replace everything with its intersection with the span of v and z: if that span is 1-dimensional then $v + z = 0$ and so there is nothing to do, whereas if the span is 2-dimensional then the two angles made against the line H by the two faces of Z (a single face if Z is a closed half-ray) are among the original angles before the planar slicing.

This reduces us to the case $\dim V = 2$, so H is a line through the origin meeting the closed convex cone Z only in $\{0\}$. By the triangle inequality, via $-v$ translation it is equivalent (at the cost of replacing C with $C + 1$) to now consider $Z \cap (v + H)$ instead. We can also assume that intersection is non-empty (or else there is nothing to do). The parallel translate $v + H$ meets Z in a line segment inside the affine line $v + H$, so an upper bound on the norms of all points in $(v + H) \cap Z$. is the largest norm among the endpoints of that line segment. If $0 < \theta_1, \theta_2 < \pi$ are the angles by which the two faces of the acute cone Z meet the line H at the origin (a single angle when $\dim Z = 1$) then the endpoints of the line segment have respective norms $\|v\|/\sin(\theta_1)$ and $\|v\|/\sin(\theta_2)$. This settles our general claim about putting $(v + Z) \cap H$ inside a closed ball of radius $C\|v\|$ centered at 0.

Using the now-established (A.4.1.9), if we shrink ε_i further to make $C_1x + M_1 \leq 2C_1x$ for all $x > |\log(\varepsilon_i)|$ then

$$Z \cap (\mathscr{L}_S(g_ig) + H_{\log}) \subset B_0(2A_iC_1|\log|g|_{F,S}|)$$

for all basic g such that $|g|_{F,S} < \varepsilon_i$. Translating by $-\mathscr{L}_S(g)$ with Euclidean norm at most $C_1|\log(|g|_{F,S})| + M_i \leq 2C_1|\log(|g|_{F,S})|$ yields that $(-\mathscr{L}_S(g_ig) + Z) \cap H_{\log}$ is contained in a closed ball $B_0(C_3|\log|g|_{F,S}|)$ for the real number $C_3 = 2(\max A_i)C_1 + 2C_1$ that is independent of such g and i. Since $|g_ig|_{F,S} = |g|_{F,S}$, we can rename C_3 as C_0 to finish. □

We have proved the estimate in (A.4.1.8) for g_ig for all i and all g basic such that $|g_ig|_{F,S} < \varepsilon$. We shall show that this in turn implies that (A.4.1.3) holds for g_ig with g basic such that $|g_ig|_{F,S} < \varepsilon$.

By Claim A.4.1.2, there exists $C_0 > 0$ such that

$$(-\mathscr{L}_S(g_ig) + Z) \cap H_{\log} \subset B_0(C_0|\log|g_ig|_{F,S}|) \cap H_{\log}$$

for all basic g with $|g_ig|_{F,S} < \varepsilon$. The λ_0-measure of that ball is $C_1C_0^{|S|-1}|\log|g_ig|_{F,S}|^{|S|-1}$ for some real number $C_1 > 0$ dependent on $|S|$ but not on g nor i. Upon renaming $C_1C_0^{|S|-1}$ as C, the estimate in (A.4.1.3) holds for all g_ig with basic g such that $|g_ig|_{F,S} < \varepsilon$. Since both $\sum_{\xi \in \mathcal{O}_{F,S}^\times} u(\xi g)$, and $|\log|g|_{F,S}|^{|S|-1}$ are $\mathcal{O}_{F,S}^\times$-invariant functions of $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ and the points g_i satisfy $|g_i|_{F,S} = 1$, we deduce that (A.4.1.3) holds for all $g \in \mathrm{GL}_1(\mathbf{A}_{F,S})$ such that $|g|_{F,S} < \varepsilon$. □

A.5. Idelic Hardy-Littlewood Maximal Theorem and Lebesgue Differentiation

In this section we collect analogs of the Maximal Theorem on the Hardy-Littlewood maximal function (see [Fol, Thms.3.17,3.18]) and of Lebesgue's Differentiation Theorem (see [Fol, Thm.3.21]) for absolutely integrable functions on idele and S-idele groups and their quotient groups of idele classes. Our main reference is [HR-2, Ch. XI, pp. 637-672].

A.5.1. Hardy-Littlewood Maximal Theorem on the real line. We begin by recalling the elementary fact that for $f : \mathbf{R} \to \mathbf{R}$ a continuous function and every $x \in \mathbf{R}$, we have

$$f(x) = \lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, \mathrm{d}y. \tag{A.5.1.1}$$

For f discontinuous but measurable and locally integrable over $\mathbf{R}$ with respect to the Lebesgue measure, the identity in (A.5.1.1) remains valid for almost all $x \in \mathbf{R}$. This is the content of the 1-dimensional Differentiation Theorem of Lebesgue, for which we refer the reader to [Fol, Thm.3.18] and [Fol, Thm.3.21].

In this section we discuss a generalization of the Differentiation Theorem, to locally integrable functions over a Hausdorff locally compact abelian group equipped with some additional structure, and with respect to a regular Radon measure.

A key tool to proving such analog of the Differentiation Theorem is an analog for Hausdorff locally compact abelian groups of the Hardy-Littlewood Maximal Theorem, which will be of interest itself and used throughout Chapter 6[4] of this work.

For $f : \mathbf{R} \to \mathbf{C}$ a measurable and absolutely integrable function, we define:

$$(Mf)(x) := \limsup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, \mathrm{d}y \tag{A.5.1.2}$$

and refer to it as the *Hardy-Littlewood Maximal Function of f*. We illustrate the Hardy-Littlewood Maximal Theorem in the special case of the 1-dimensional Euclidean space only, to explain the techniques used in the proof. We will then show how to use it to deduce Lebesgue's Differentiation Theorem.

A.5.1.1. 1-*dimensional Hardy-Littlewood Maximal Theorem.* In the 1-dimensional case for functions over the real line, it asserts the following:

THEOREM A.5.1.1 (Hardy-Littlewood Maximal Theorem over $\mathbf{R}$). *Let $f : \mathbf{R} \to \mathbf{C}$ be a measurable function.*

(1) *If $f \in L^1_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}x)$ is such that $f \log^+|f|) \in L^1_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}x)$, with $\log^+(u) := \max\{\log(u), 0\}$ for $u > 0$, then Mf is measurable and finite almost everywhere.*

(2) *If $f \in L^1(\mathbf{R}, \mathrm{d}x)$ then for any real number $\alpha > 0$, we have:*

$$|\{x \in \mathbf{R} : (Mf)(x) > \alpha\}| \leq \frac{3}{\alpha} \|f\|_{L^1(\mathbf{R}, \mathrm{d}x)}. \tag{A.5.1.3}$$

(3) *For $p > 1$, if $f \in L^p(\mathbf{R}, \mathrm{d}x)$ then there exists a real number $C_p > 0$ independent of f such that*

$$\|Mf\|^p_{L^p(\mathbf{R}, \mathrm{d}x)} \leq \frac{2^p \cdot 3p}{p-1} \|f\|^p_{L^p(\mathbf{R}, \mathrm{d}x)}. \tag{A.5.1.4}$$

PROOF. (1) : We note that Mf is measurable since the function of x given by $\frac{1}{2r} \int_{x-r}^{x+r} |f(x)|\, \mathrm{d}y$ is continuous for every $r > 0$. We let $r > 0$ be an arbitrary real number. Since $f \in L^1_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}x)$, $f \cdot 1_{[-r,r]}$ is absolutely integrable over $\mathbf{R}$. Following the proof of [HR-2, Ch. XI, 44.50(d), p. 670], we choose an arbitrary

[4]Put in labels to refer to Chapters 10-11 instead

real number $0 < \delta < 1$ and have for some real number $C_0 > 0$ independent of f:

(A.5.1.5)

$$\begin{aligned}\int_{[-r,r]} (Mf)(y)\,\mathrm{d}y &= \int_0^{+\infty} |[-r,r] \cap \{x \in \mathbf{R} : (Mf)(x) > y\}|\,\mathrm{d}y \\ &= \int_0^{1/\delta} |[-r,r] \cap \{x \in \mathbf{R} : (Mf)(x) > y\}|\,\mathrm{d}y + \int_{1/\delta}^{+\infty} |[-r,r] \cap \{x \in \mathbf{R} : (Mf)(x) > y\}|\,\mathrm{d}y \\ &\leq \frac{2r}{\delta} + \int_{1/\delta}^{+\infty} \left\{ \frac{C_0}{(1-\delta)y} \int_{[-r,r]} |\{x \in \mathbf{R} : |f(x)| > \delta y\}| \cdot |f(x)|\,\mathrm{d}x \right\} \mathrm{d}y\end{aligned}$$

where we used [**HR-2**, Ch. XI, 44.50(a.ii), p. 669] to obtain the last inequality.

We now apply [**Fol**, Thm. 7.27] to the Lebesgue measure μ on $\mathbf{R}$, $\varphi(y) = \log(y)$, and $E = [-r,r]$, to obtain:

(A.5.1.6)

$$\frac{C_0}{(1-\delta)} \int_{1/\delta}^{+\infty} \left\{ \int_{[-r,r]} |\{x \in \mathbf{R} : |f(x)| > \delta y\}| \cdot |f(x)|\,\mathrm{d}x \right\} \frac{\mathrm{d}y}{y} = \frac{C_0}{(1-\delta)} \int_{x \in [-r,r]} |f(x)| \int_1^{+\infty} |\{y \in \mathbf{R} : |f(x)| > y\}| \frac{\mathrm{d}y}{y}\,\mathrm{d}x$$

(A.5.1.7)

$$= \frac{C_0}{(1-\delta)} \int_{x \in [-r,r]} |f(x)| \log^+ |f(x)|\,\mathrm{d}x$$

which gives the local integrability of Mf over $\mathbf{R}$, and hence its finiteness almost everywhere.

(2) : By (1), Mf is measurable and finite almost everywhere over $\mathbf{R}$. By [**StSha-III**, Ch. 3, §1, Thm. 1.1(iii), p. 101] we deduce that (A.5.1.3) holds.

(3) : We proceed directly as follows. For every real number $\delta \geq 0$, we define $F_\delta(x) := f(x)$ if $|f(x)| > \delta/2$, and $F_\delta(x) := 0$ otherwise. We therefore have $|f(x)| \leq |F_\delta(x)| + \delta/2$ for all $x \in \mathbf{R}$.

We note that F_δ is absolutely integrable for every $\delta > 0$, since, by [**Fol**, Thm. 6.17], we have:

$$|\{x \in \mathbf{R} : |f(x)| > \delta/2\} \leq \frac{2^p}{\delta^p} \|f\|^p_{L^p(\mathbf{R},\mathrm{d}x)}$$

for every $\delta > 0$, and hence $F_\delta(x)$ is nonzero only on a measurable subset of $\mathbf{R}$ of finite Lebesgue measure.

We apply (2) to F_δ for each $\delta > 0$. By (2), we obtain:

(A.5.1.8)

$$|\{x \in \mathbf{R} : (Mf)(x) > \delta\}| \leq |\{x \in \mathbf{R} : (MF_\delta)(x) > \delta/2\}| \leq \frac{2 \cdot 3}{\delta} \int_{\mathbf{R}} |F_\delta(x)|\,\mathrm{d}x = \frac{2 \cdot 3}{\delta} \int_{|f(x)|>\delta/2} |f(x)|\,\mathrm{d}x.$$

By Fubini's Theorem,

(A.5.1.9)

$$\|Mf\|^p_{L^p(\mathbf{R},\mathrm{d}x)} = \int_{\mathbf{R}} |(Mf)(x)|^p\,\mathrm{d}x = p \int_0^{+\infty} y^{p-1} |\{x \in \mathbf{R} : (Mf)(x) > y\}|\,\mathrm{d}y.$$

By combining (A.5.1.9) and (A.5.1.8) we obtain:

(A.5.1.10)

$$\|Mf\|^p_{L^p(\mathbf{R},\mathrm{d}x)} \leq 2 \cdot 3p \int_0^{+\infty} \int_{|f(x)|>y/2}^{+\infty} y^{p-2} |f(x)|\,\mathrm{d}x\,\mathrm{d}y = 2^p \cdot 3p \int_0^{+\infty} \int_{|f(x)|>y} y^{p-2} |f(x)|\,\mathrm{d}x\,\mathrm{d}y$$

and Fubini's Theorem (along with the equality $\int_0^a y^{p-1}\,\mathrm{d}y/y = a^{p-1}/(p-1)$) gives

(A.5.1.11)

$$\int_0^{+\infty} \int_{|f(x)|>y} y^{p-2} |f(x)|\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{p-1} \int_{\mathbf{R}} |f(x)|^{p-1} |f(x)|\,\mathrm{d}x = \frac{1}{p-1} \|f\|^p_{L^p(\mathbf{R},\mathrm{d}x)}.$$

Upon combining (A.5.1.11) and (A.5.1.10) we deduce that (A.5.1.4) holds. □

REMARK A.5.1.2. Since M is a quasi-linear[5] operator on measurable functions on $\mathbf{R}$, we may apply the Interpolation Theorem of Marcinkiewicz (see [**HuWe**, p.996-998]) to deduce (A.5.1.4) from (2), after reducing to the case f is nonnegative and compactly supported.

We refer the reader to [**StSha-III**, Ch. 3, §1, Thm. 1.1, p. 101] for the case of functions on $\mathbf{R}^d$ for every $d \geq 1$ in the case $p = 1$. We further refer the reader to [**Fol**, Thm.3.17] for the maximal theorem in the case $d \geq 1$ and $p = 1$, where the method of proof gives indeed the constant 3^d in place of 3 in (A.5.1.3), resting on the Vitali-type Covering Lemma in [**Fol**, Lemma 3.15] (see also [**StSha-III**, Ch.3, §1, Lemma 1.2, p. 102]).

A similar argument as for (3) in Theorem A.5.1.1 generalizes the case $p > 1$ to the case of d-dimensional real Euclidean spaces. The dependence on the dimension of the constant that appears in the d-dimensional analog of (A.5.1.4) for $p > 1$, can be removed (see [**StStr**, Thm. A], building on previous work of E. Stein on spherical maximal functions).

The inequality in (A.5.1.3) is called *weak* $(1,1)$*-type* while (A.5.1.4) is called *strong* (p,p)*-type*.

A.5.1.2. 1*-dimensional Lebesgue Differentiation Theorem.* Using the weak $(1,1)$-estimate from (A.5.1.3) for the Hardy-Littlewood maximal function, we can prove the following Theorem of Lebesgue:

THEOREM A.5.1.3 (Lebesgue's Differentiation Theorem). *Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}x)$ be a locally integrable function on $\mathbf{R}$. Then, for almost all $x \in \mathbf{R}$, the following identity:*

$$f(x) = \lim_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(y)\,\mathrm{d}y \tag{A.5.1.12}$$

holds.

PROOF. It is enough to treat the case of real-valued f. We define the local-at-x "oscillation" of f, denoted $\omega_f(x)$, as follows. For every $x \in \mathbf{R}$, we define:

$$\omega_f(x) := \left| \limsup_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(y)\,\mathrm{d}y - \liminf_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(y)\,\mathrm{d}y \right|. \tag{A.5.1.13}$$

We fix a real number $r_0 > 0$, large enough so as to have $x \in (-r_0, r_0)$, and consider $F_{r_0} := 1_{[-r_0, r_0]} \cdot f$.

By [**Fol**, Prop. 7.9], for every real number $\varepsilon > 0$, there exists a continuous compactly supported function $\varphi \in C_c(\mathbf{R})$ such that $\|F_{r_0} - \varphi\|_{L^1(\mathbf{R}, \mathrm{d}y)} < \varepsilon$. We call $g := F_{r_0} - \varphi$, and note that we have $\omega_{F_{r_0}}(y) = \omega_g(y)$, since $\omega_\varphi(y) = 0$ for every $y \in \mathbf{R}$ by continuity of φ.

Since $\omega_g \leq 2Mg$, by (A.5.1.3) from Theorem A.5.1.1(2) we have for every $\alpha > 0$ that

$$|\{y \in \mathbf{R} : \omega_{F_{r_0}}(y) > \alpha\}| \leq \frac{2 \cdot 3}{\alpha} \|g\|_{L^1(\mathbf{R}, \mathrm{d}y)} < \frac{2 \cdot 3\varepsilon}{\alpha} \tag{A.5.1.14}$$

and $\varepsilon > 0$ being arbitrary, we deduce that $\{y \in \mathbf{R} : \omega_{F_{r_0}}(y) > \alpha\}$ has Lebesgue measure zero; i.e., $A_r F_{r_0}(x) := \frac{1}{2r} \int_{x-r}^{x+r} F_{r_0}(y)\,\mathrm{d}y$ converges almost everywhere as $r \to 0^+$. To identify the limit it is enough to find a sequence of real numbers $0 < r < r_0$ on which it is possible to do so. We have $\|F_{r_0} - A_r F_{r_0}\|_{L^1(\mathbf{R}, \mathrm{d}y)} \to 0$,[6] and hence there exists such a sequence such that $(A_r F_{r_0})(x) \to F_{r_0}(x)$ for almost all $x \in (-r_0, r_0)$.

We let $x \in (-r_0, r_0)$ be such that $A_r F_{r_0}(x)$ converges as $r \to 0^+$. It follows that $\lim_{r \to 0^+} A_r F_{r_0}(x) = \lim_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(y)\,\mathrm{d}y$, since such limit equals the limit as $r > 0$ tends to zero while varying over the positive reals $r \in (0, r_1)$, with r_1 small enough so as to have $(x - r_1, x + r_1) \subset (-r_0, r_0)$.

We now let $N_k \subset (-k, k)$ be the null set of those points at which (A.5.1.12) does not hold. We have, for every integer $k \geq 1$, $N_k \subset N_{k+1}$, and hence $|\bigcup_{k \geq 1} N_k| = \lim_{k \to +\infty} |N_k| = 0$. We deduce that (A.5.1.12) holds almost everywhere over the real line. □

[5] Means what? Do we care?

[6] Should briefly explain this convergence to 0

A.5.2. D-sequences, D'-sequences, D''-sequences in Hausdorff locally compact abelian groups. We will be recording, throughout this subsection and the next, analogs of both Theorem A.5.1.1 and Theorem A.5.1.3, when $\mathbf{R}$ is replaced by a Hausdorff locally compact abelian group G equipped with a decreasing sequence of Borel measurable subsets satisfying certain axioms.

This is discussed in detail in [**HR-2**, Ch.XI,§44,44.50], the proofs are close in structure and technique to those presented above for Theorem A.5.1.1 and Theorem A.5.1.3, and we will be only providing complementary details to the arguments in loc.cit..

We will then show that the structural assumptions on G are all satisfied by all the Hausdorff locally compact abelian groups of interest to this work, i.e. $\mathrm{GL}_1(\mathbf{A}_F)$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, C_F, $C_{F,S}$ and $J_{F,S}$, and closed subgroups of norm-1 elements, for F a number field, S a finite set of places of F containing all the Archimedean places.

The ability to use analogs of the Hardy-Littlewood maximal function for these choices of G, and the analog of both the Maximal Theorem and Lebesgue's Differentiation Theorem, will be instrumental for us to make sense of certain measurable kernel functions over self-products of the above groups, along their diagonal, and compute their integrals on such diagonals (used to compute some nuclear operator traces).

We refer the reader to [**Bris-a**, **Bris-b**] for related techniques, aimed at computing traces of trace-class integral operators on L^2-spaces of a large class of metric measure spaces.

A.5.2.1. *D-sequences and D'-sequences in Hausdorff locally compact abelian groups.* We begin with a definition, introducing the additional structure on a Hausdorff locally compact abelian group for an analog of the Hardy-Littlewood maximal function to be available, and for analogs of the Hardy-Littlewood Maximal Theorem and Lebesgue's Differentiation Theorem to hold.

Definition A.5.2.1. Let G be a Hausdorff locally compact abelian group. Let λ be a Haar measure on G. Let $\{U_k\}_{k\geq 0}$ be a sequence of λ-measurable subsets in G with $e \in U_k$ for all k. We call $\{U_k\}_{k\geq 0}$ a *D-sequence in G* if $\{U_k\}_{k\geq 0}$ satisfies the following conditions:

(1) (Integrability) For every integer $k \geq 0$, $\lambda(U_k)$ is finite and nonzero.
(2) (Monotonicity) For every integer $k \geq 0$, we have $U_{k+1} \subset U_k$.
(3) (Doubling) There exists a positive real number $C > 0$ such that, for every integer $k \geq 0$, we have:

$$0 < \lambda(U_k U_k^{-1}) < C\,\lambda(U_k). \tag{A.5.2.1}$$

We say a D-sequence $\{U_k\}_{k\geq 0}$ is *open*, respectively *symmetric*, if and only if for each integer $k \geq 0$, U_k is open, respectively symmetric.

We say a D-sequence $\{U_k\}_{k\geq 0}$ is a *D'-sequence in G* if and only if for every neighborhood V of the neutral element in G, there exists an integer $k \geq 0$ such that $U_k \subset V$.

Remark A.5.2.2. The preceding definition can be pushed a bit further: a D'-sequence $\{U_k\}_{k\geq 0}$ is called a *D''-sequence in G* if and only if it is a D'-sequence and there exists a specified sequence $\{V_k\}_{k\geq 0}$ of λ-measurable subsets of G satisfying $V_k V_k^{-1} \subset U_k$ for every integer $k \geq 0$ and a positive real number $C' > 0$ such that, for every integer $k \geq 0$, we have:

$$\lambda(U_k) \leq C'\,\lambda(V_k). \tag{A.5.2.2}$$

One usually writes $\{U_k, V_k\}_{k\geq 0}$ for the datum of a D''-sequence in G. We will not use this concept.

Before we state and prove analogs of the Hardy-Littlewood Maximal Theorem and the Lebesgue Differentiation Theorem, we discuss criteria for a Hausdorff locally compact abelian group to admit a D-sequence, respectively a D'-sequence.

PROPOSITION A.5.2.3. *Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G.*

(1) *If G is first countable and locally profinite, then it admits a D''-sequence of the form $\{U_k, V_k\}_{k\geq 0}$, with, for each integer $k \geq 0$, $U_k = U_k^{-1}$, $V_k = U_k^2$, $C' = 1$, and U_k is a compact open subgroup of G.*

(2) *Let $H \subset G$ be an open subgroup. Let $\{U_k\}_{k\geq 0}$ be a sequence of subsets of H. Then $\{U_k\}_{k\geq 0}$ is a D-sequence, respectively a D'-sequence, for H if and only if it is so for G.*

It is open, respectively symmetric, for H if and only if it is so for G.

PROOF. (1) : By [**DE**, Thm. 4.1.6] and [**DE**, Prop. 4.1.5], G admits a unit-neighborhood base comprised of compact open subgroups. We may assume such base to be countable. Indeed, for every countable unit-neighborhood base B of e we may find for each $U \in B$ an open, hence closed (hence compact open) subgroup $N_U \subset U$. The subset $\{N_U : U \in B\}$ is a countable unit-neighborhood base comprised of compact open subgroups. We may extract a cofinal sequence $\{U_k\}_{k\geq 0}$ from it.

Since U_k is a subgroup for each integer $k \geq 0$, we have $U_k^{-1} = U_k$ and $U_k U_k^{-1} \subset U_k$. The sequence $\{U_k\}_{k\geq 0}$ is therefore an open symmetric D-sequence, with the choice $C = 1$.

Since $\{U_k\}_{k\geq 0}$ is cofinal in a neighborhood base of the neutral element in G, for any neighborhood V of the neutral element in G there exists a large enough integer $k \geq 0$ with $U_k \subset V$. The D-sequence $\{U_k\}_{k\geq 0}$ is therefore a D'-sequence.

(2) : Since $\mu_G|_H$ is a Haar measure for H, for every μ_G-measurable subset $A \subset G$ we see that $A \cap H$ is μ_H-measurable and $\mu_H(A \cap H) = \mu_G(A \cap H)$. In particular, if $A \subset H$ then $\mu_H(A) = \mu_G(A)$. This implies that all the defining conditions on the D-sequence, respectively D'-sequence $\{U_k\}_{k\geq 0}$ for G hold for the same sequence as a D-sequence, respectively D'-sequence for H. □

REMARK A.5.2.4. Let G be a countable discrete abelian group. Then $U_k := \{e\}$ for every integer $k \geq 0$, is an open, symmetric D-sequence, a D'-sequence, and $\{U_k, U_k\}_{k\geq 0}$ is a D''-sequence. Indeed, every Haar measure λ on G is proportional to the counting measure by [**DE**, Prop.1.4.4], the constant of proportionality given by $c := \lambda(\{e\}) \neq 0$.

It follows that, under the above choices, $\lambda(U_k) = c$ for every $k \geq 0$, and $U_{k+1} \subset U_k$. The doubling condition holds with $C = 2$. Since every unit-neighborhood of G contains e, $\{U_k\}_{k\geq 0}$ is a D'-sequence.

Upon choosing $V_k := U_k$ for every integer $k \geq 0$, and $C' = 1$, then $\lambda(U_k) = \lambda(V_k)$, and hence (A.5.2.2) holds too, and $\{U_k, V_k\}_{k\geq 0}$ is a D''-sequence.

LEMMA A.5.2.5. *Let G_1, G_2 be Hausdorff locally compact abelian groups, and let $G := G_1 \times G_2$ be the product topological group. Let μ_1, μ_2 be Haar measures on G_1, respectively G_2. Let $\mu := \mu_1 \otimes \mu_2$ be the product Haar measure on G. Let $\{U_k\}_{k\geq 0}$ be a sequence of μ_1-measurable subsets in G_1, and $\{U'_k\}_{k\geq 0}$ a sequence of μ_2-measurable subsets in G_2. If $\{U_k\}_{k\geq 0}$ is a D-sequence, respectively a D'-sequence, for G_1, and $\{U'_k\}_{k\geq 0}$ is a D-sequence, respectively a D'-sequence, for G_2, then $\{U_k \times U'_k\}_{k\geq 0}$ is a D-sequence, respectively a D'-sequence, for G. For each integer $k \geq 0$, if U_k and U'_k are open, respectively symmetric, then so is $U_k \times U'_k$.*

PROOF. Since, for any two measurable subsets $B_1 \subset G_1$, $B_2 \subset G_2$, we have $\mu(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$, and since the property of being an open subset respectively a symmetric subset is stable under finite products, all defining conditions for $\{U_k \times U'_k\}_{k\geq 0}$ follow from those of $\{U_k\}_{k\geq 0}$, respectively $\{U'_k\}_{k\geq 0}$. □

PROPOSITION A.5.2.6. *Let G be a Hausdorff locally compact abelian group. Assume G is second-countable.*

Let $H \subset G$ be a closed subgroup. Let μ_H be a choice of Haar measure on H, μ_G a choice of Haar measure on G, and $\mu_{G/H}$ the unique Haar measure on G/H induced by the choices μ_G on G and μ_H on H as per Theorem D.1.2.1. Assume H is discrete, and normalize μ_H so as to have $\mu_H(\{1\}) = 1$. Let $q : G \to G/H$ be the continuous quotient homomorphism.

Assume that there exists a relatively compact Borel subset $E \subset G$ whose interior is non-empty and contains the neutral element, and such that the continuous map $q|_E : E \to G/H$ is a bijection.

Let $\{U_k\}_{k\geq 0}$ be a sequence of Borel sets in G/H such that $U_{k+1} \subset U_k$ for every integer $k \geq 0$. Then $\{U_k\}_{k\geq 0}$ is a D-sequence for G/H, respectively a D'-sequence for G/H, if and only if

$$\{q^{-1}(U_k) \cap E\}_{k\geq 0}$$

is a D-sequence, respectively a D'-sequence, for G. If E is symmetric then $\{U_k\}_{k\geq 0}$ is symmetric if and only if $\{q^{-1}(U_k) \cap E\}_{k\geq 0}$ is.

REMARK A.5.2.7. In light of Lemma A.5.2.10 below, if H is discrete then there exists a Borel section to $q : X \to X/H$. Discreteness of H is essential to ensure that such Borel section $F \subset X$ (i.e. a Borel subset F such that $q|_F$ is a continuous bijection) can be used to compute the $\mu_{G/H}$-measure of measurable subsets A in X/H, by computing the μ_G-measure of $q^{-1}(A) \cap F$.

Before proving the Proposition, we record here the following result from [**Kech**, Ch. II, §15, Thm. 15.1] for which we need only an elementary special case:

THEOREM A.5.2.8 (Lusin-Souslin Theorem on Borel embeddings). *Let X, Y be separable and completely metrizable[7] topological spaces, $f : X \to Y$ a continuous map.*

For $B \subset X$ a Borel set, if $f|_B$ is injective then $f(B) \subset Y$ is a Borel set.

We only need this for the quotient map $G \to G/H$ with G a second-countable locally compact topological abelian group and H a discrete subgroup. In such cases, we can pick a fundamental domain $F \subset G$ with $\mu(\partial F) = 0$ (for a Haar measure μ on G) such that $\mathrm{Int}(F) \to G/H$ is a homeomorphism onto an open image who complement in G/H has measure 0. In such cases. the conclusion of Theorem A.5.2.8 is clear when we complete the σ-algebra of Borel sets (i.e., we regard all measure-0 sets as Borel sets).

REMARK A.5.2.9 (Spherical boundedness and second-countability). Let G be a Hausdorff locally compact abelian group. As such, it is a completely regular topological space (see [**Fol**, Cor. 4.33]), i.e. for a point $x \in G$ and a closed subset $A \subset G$ not containing x, there exists a continuous real-valued function f on G such that $f(x) = 1$ and $f(A) = \{0\}$. In making this deduction, the group structure and continuity of the group operations, is not being used.

First countability is in fact a necessary and sufficient condition for a Hausdorff topological group to be metrizable (i.e. the underlying topology, uniformizable in the sense of [**Bou-GT**, Ch. II, §4.1, Def. 1, p. 198], see [**Bou-GT**, Ch. III, §3.1, Def. 1, p. 243], which is to mean that it is induced by a uniform structure, is in fact metrizable, which is to mean it is induced by a metric); see [**Birkhoff**, Theorem, p. 428].

The metric inducing the topology can be chosen to be left-invariant, and a direct construction is given in [**Struble**, Lemma 1, p. 217].

The second axiom of countability for Hausdorff locally compact topological groups G (not necessarily abelian) is characterized by boundedness of the spheres with respect to a left-invariant metric inducing the topology: this is [**Struble**, Theorem, p. 217]. Recall that a subset $B \subset G$ is called *bounded* if for any unit-neighborhood U, there exist finitely many $g_1, \ldots, g_N$ such that $g_1U \cup \ldots \cup g_NU$ contains B (i.e. B is "totally bounded" with respect to the uniform structure on G; we refer the reader to [**Bou-GT**, Ch. II, Exercises, §4,(7), p. 210]).

If G is a Hausdorff locally compact second countable topological group, then there exists a family of open subsets $\{U_r\}_{r\in\mathbf{R}_{>0}}$ such that, for all real numbers $r_1, r_2 > 0$, U_r is bounded and symmetric, and $U_{r_1}U_{r_2} \subset U_{r_1+r_2}$, and such that $\{U_r\}_{r>0}$ is a unit-neighborhood base, and $\bigcup_{r>0} U_r = G$. This is [**Struble**, Lemma 2, p. 218].

We record here the following general existence result for Borel fundamental domains:[8]

[7] Need to record meaning, criteria, and relevant examples of "completely metrizable"

[8] Why is this being recorded? It is never needed in such generality.

LEMMA A.5.2.10. *Let X be a Hausdorff locally compact topological space. Assume X is second countable. Let H be a countable discrete abelian group acting on H properly and freely. Let $q : X \to X/H$ be the continuous quotient.*

Let $E \subset X$ be a Borel subset such that $q|_E : E \to X/H$ is injective. Then there exists a Borel subset $F \subset X$ such that $E \subset F$ and the continuous $q|_F : F \to X/H$ is a bijection.

PROOF. Since X is Hausdorff locally compact and second countable, there exists an exhaustion of X by a sequence of compact subsets $\{K_m\}_{m\geq 0}$. We denote by $q_m : K_m \to X/H$ the composition of the continuous inclusion $K_m \subset X$ followed by the quotient map q. For each integer $m \geq 1$, K_m is metrizable by the Urysohn Metrization Theorem, since it is Hausdorff, compact, second countable.

By [**Bog**, Vol. 2, Ch. 6, Thm. 6.9.7, p. 38] applied to $Y = X/H$, which is Hausdorff since the action of H on X is proper (see [**Bou-GT**, Ch. III, §4.4, Prop. 8(b), p. 256]), and for $f = q_m$ for every integer $m \geq 0$ there exists a Borel subset $B_m \subset K_m$ such that $q_m(B_m) = q_m(K_m)$, with q_m injective on B_m, and the set-theoretic inverse map $q_m^{-1} : q_m(K_m) \to B_m$ is Borel.

It follows that we have $q(\bigcup_{m\geq 0} B_m) = X/H$, since $\bigcup_{m\geq 0} q(K_m) = X/H$. Since H is countable, the set of the B_m's and all of their H-translates is comprised of Borel subsets $B \subset X$ such that $q|_B : B \to X/H$ is injective, and such that X is covered by their union. We define $\mathcal{B}$ to be the set of all the H-translates of the B_m's, and of E. Clearly $\mathcal{B}$ is countable.

We enumerate the elements $\{U_k\}_{k\geq 1} = \mathcal{B}$ setting $U_0 := E$ and $U_{-1} := \varnothing$. We define $U_0' := U_0$ and, for each integer $k \geq 1$:

$$U_k' := U_k \cap (X - (\bigcup_{0\leq i\leq k-1} H \cdot U_i)) \tag{A.5.2.3}$$

and set $F := \bigcup_{k\geq 0} U_k'$. Then F contains E, and it is a Borel subset of X, since each U_k' is.

We have $q(F) = X/H$. Indeed, each U_k is contained in the union $\bigcup_{k\geq 0} H \cdot U_k'$, and hence $q(U_k)$ is contained in $q(\bigcup_{k\geq 0} U_k') = q(F)$ for every integer $k \geq 0$. Since $\bigcup_{k\geq 0} U_k = X$, we deduce $q(F) = X/H$.

For any $h \in H$, $h \neq 1$, we have $h \cdot F \cap F = \varnothing$. Indeed, $(h \cdot U_k') \cap U_{k'}'$ is non-empty only if $k = k'$, in which case, since $q|_{U_k}$ is injective, we must have $h = 1$: if $x \in U_k'$ is such that $y := h \cdot x \in U_k'$, then $q(x) = q(y)$, which forces $y = x$ and hence $h = 1$. Since U_k' and $U_{k'}'$ are disjoint for all pairs of integers (k, k') with $k \neq k'$, we deduce that we have $h \cdot F \cap F = \varnothing$ for all $h \in H$ with $h \neq 1$. It follows that the continuous $q|_F : F \to X/H$ is a bijection. □

PROOF OF PROPOSITION A.5.2.6. We note that since q is continuous, for any Borel subset $B \subset G/H$ clearly $q^{-1}(B)$ is a Borel subset of G.

By [**Klee**, (2.4), p. 487], since G admits a countable unit-neighborhood base, and each member of such base is invariant under all inner automorphisms of G (since G is abelian), G admits an invariant metric d (i.e. both left- and right-invariant).

By [**Klee**, (2.5), p. 487], the invariant metric on G promotes G to a complete metric space (G, d). Indeed, if $\{g_k\}$ is a Cauchy sequence and we consider all k, m large enough so that $g_k g_m^{-1} \in \overline{U}$ for a chosen e-neighborhood U in G. We can arrange that $\overline{U}$ is compact, so convergence of the Cauchy sequence follows.

We deduce that G is completely metrizable,[9] and there exists an invariant complete metric on G. Since G is second-countable by assumption, it is a second-countable completely metrizable space and hence it is separable. In particular, Theorem A.5.2.8 applies with $X = G$ and $Y = G/H$, since the quotient $q : G \to G/H$ is open and hence G/H is also second countable and (as for G) completely metrizable, hence separable and completely metrizable.

By Theorem A.5.2.8, for every subset $B \subset G$, since $q|_{B\cap E}$ is injective (being injective on E itself), then if $B \cap E$ is Borel so is $q(B \cap E)$. If $B = q^{-1}(q(B))$ set-theoretically, then $q(B \cap E) = q(B)$.

[9] what does "completely metrizable" mean?

We now observe that by Lemma A.5.2.10, there exists a Borel subset $F \subset G$ such that $q|_F : F \to G/H$ is a continuous bijection and F contains E. For every measurable subset $A \subset G/H$ with finite $\mu_{G/H}$-measure, the indicator function $\mathbf{f} := 1_{q^{-1}(A)\cap F}$ of $q^{-1}(A) \cap F$ is μ_G-integrable by [**Bou-II**, Ch. VII, §2, No. 3, Prop. 5(d), INT VII.33] applied to $X = G$, $\lambda^\sharp = \mu_G$, $\lambda = \mu_{G/H}$, and $\beta = \mu_H$. Indeed, we have

$$\int_{G/H}\int_H 1_{q^{-1}(A)\cap F}(h^{-1}x)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_{G/H}(q(x)) = \int_{G/H} 1_A(\pi(x))\,\mathrm{d}\mu_{G/H}(\pi(x)) = \mu_{G/H}(A). \tag{A.5.2.4}$$

Here we are using discreteness of H once more: not only through Lemma A.5.2.10, but also through the equality $\mu_H(\{h\}) = 1$. By (A.5.2.4) and Lemma D.1.2.3 we see that indeed $1_{q^{-1}(A)\cap F}$ is μ_G-integrable, and hence [**Bou-II**, Ch. VII, §2, No. 3, Prop.5(c), INT VII.33] applies.

We have $\mathbf{f}^\flat = 1_A$, in the notation from loc.cit., and so by [**Bou-II**, Ch. VII, §2, No. 3, Prop. 5(c),(12), INT VII.33] we have:

$$\mu_G(q^{-1}(A) \cap F) = \mu_{G/H}(A). \tag{A.5.2.5}$$

In particular, upon choosing A contained in $q(E)$, we obtain $q^{-1}(A) \cap F = q^{-1}(A) \cap E$, and so we have:

$$\mu_G(q^{-1}(A) \cap E) = \mu_{G/H}(A). \tag{A.5.2.6}$$

We can now prove the Proposition.

(1) : Let $\{U_k\}_{k\geq 0}$ be a Borel D-sequence for G/H contained in $q(E)$. Then $\{q^{-1}(U_k)\cap E\}_{k\geq 0}$ is a sequence of Borel subsets of G such that $q^{-1}(U_{k+1}) \cap E \subset q^{-1}(U_k) \cap E$ and by (A.5.2.6) we have $\mu_G(q^{-1}(U_k) \cap E) = \mu_{G/H}(U_k)$ for every integer $k \geq 0$. The integrability, monotonicity and doubling conditions for $\{q^{-1}(U_k)\cap E\}_{k\geq 0}$ are therefore satisfied, and hence $\{q^{-1}(U_k) \cap E\}_{k\geq 0}$ is a Borel D-sequence for G.

Conversely, if $\{q^{-1}(U_k) \cap E\}_{k\geq 0}$ is a Borel D-sequence for G, then by Theorem A.5.2.8 each image $q(q^{-1}(U_k) \cap E) = U_k$ is a Borel subset of G/H and by (A.5.2.6) applied to $A = U_k$, we deduce that $\mu_{G/H}(U_k) = \mu_G(q^{-1}(U_k) \cap E)$. Thus, the integrability, monotonicity and doubling conditions for U_k are satisfied, so $\{U_k\}_{k\geq 0}$ is a Borel D-sequence for G/H entirely contained in $q(E)$.

Assume that every unit-neighborhood in G/H contains some U_k, for some integer $k \geq 0$. Let V be an arbitrary unit-neighborhood in G. We want to prove that there exists an integer $k \geq 0$ such that $q^{-1}(U_k) \cap E$ is contained in V. Since V contains an open unit-neighborhood, we may assume V is open.

Then $V \cap \mathrm{Int}(E)$ is also an open unit-neighborhood in G, and hence $q(V \cap E)$ is a unit-neighborhood in G/H since q is open and a group homomorphism. We choose an integer $k \geq 0$ such that $q(V \cap E)$ contains U_k. Since $q|_E$ is injective, we deduce that $V \cap E$ contains $q^{-1}(U_k) \cap E$, and hence we have $q^{-1}(U_k) \cap E \subset V$. We deduce that if $\{U_k\}_{k\geq 0}$ is a D'-sequence for G/H contained in $q(E)$, then $\{q^{-1}(U_k) \cap E\}_{k\geq 0}$ is a D'-sequence for G.

Conversely, assume that $\{q^{-1}(U_k) \cap E\}_{k\geq 0}$ is a D'-sequence for G (so $\{U_k\}$ is a D-sequence for G/H by what has already been done). Let $V \subset G/H$ be an arbitrary unit-neighborhood. Then $q^{-1}(V)$ is a unit-neighborhood in G. There exists an integer $k \geq 0$ such that $q^{-1}(U_k)\cap E \subset q^{-1}(V)$, which implies that we have $q(q^{-1}(U_k) \cap E) \subset V$. Since $q(q^{-1}(U_k) \cap E) = U_k$, we deduce that we have $U_k \subset V$, and hence $\{U_k\}_{k\geq 0}$ is a D'-sequence for G/H.

Assume now that E is symmetric, i.e. $E^{-1} = E$. Since q is a homomorphism, then U_k is symmetric if $q^{-1}(U_k) \cap E$ is, since we have $U_k = q(q^{-1}(U_k) \cap E)$. Assume now that U_k is symmetric. Then so is $q^{-1}(U_k)$, and hence so is the intersection $q^{-1}(U_k) \cap E$.

(2) : For each integer $k \geq 0$, $q^{-1}(V_k)$ is a Borel subset of G. In light of (A.5.2.6), the condition in (A.5.2.2) holds for $\{U_k, V_k\}_{k\geq 0}$ if and only if it holds for $\{q^{-1}(U_k) \cap E, q^{-1}(V_k) \cap E\}_{k\geq 0}$.

Assume $V_k V_k^{-1} \subset U_k$. Then $q^{-1}(V_k V_k^{-1}) \subset q^{-1}(U_k)$. We have $q^{-1}(V_k) \cdot q^{-1}(V_k)^{-1} = q^{-1}(V_k V_k^{-1})$, and hence $q^{-1}(V_k)\cdot q^{-1}(V_k)^{-1}\cap E \subset q^{-1}(V_k V_k^{-1})\cap E$. It follows that we have $q^{-1}(V_k)\cdot q^{-1}(V_k)^{-1}\cap E \subset q^{-1}(U_k)\cap E$.

Since we have $(q^{-1}(V_k)\cap E)\cdot(q^{-1}(V_k)\cap E)^{-1}\subset(q^{-1}(V_k)\cdot q^{-1}(V_k)^{-1})\cap E$ (here we use that E is symmetric), we deduce that $(q^{-1}(V_k)\cap E)\cdot(q^{-1}(V_k)\cap E)^{-1}\subset q^{-1}(U_k)\cap E$ too.

Conversely, if $(q^{-1}(V_k)\cap E)\cdot(q^{-1}(V_k)\cap E)^{-1}\subset q^{-1}(U_k)\cap E$ holds, then so does $q(q^{-1}(V_k)\cap E)\cdot q((q^{-1}(V_k)\cap E)^{-1})\subset q(q^{-1}(U_k)\cap E)$. Since $q(q^{-1}(V_k)\cap E)=V_k$, $q((q^{-1}(V_k)\cap E)^{-1})=V_k^{-1}$, for the settled case of D-sequences and D'-sequences, and $q(q^{-1}(U_k)\cap E)=U_k$, we deduce that we have $V_kV_k^{-1}\subset U_k$.

It follows that $\{U_k,V_k\}_{k\geq 0}$ is a D''-sequence for G/H if and only if $\{q^{-1}(U_k)\cap E,q^{-1}(V_k)\cap E\}_{k\geq 0}$ is a D''-sequence for G. □

Remark A.5.2.11. We highlight that in [**Klee**, (2.4), p. 487] the terminology "countable complete system of neighborhoods" of a point $x\in X$ of a topological space X (which is a topological group in loc.cit.) means a "countable neighborhood base" at x.

A.5.2.2. *The case of S-local and global groups of ideles and idele classes.* We now return to the Hausdorff locally compact abelian groups of interest to this work. Let F be a number field, and let $\tau:\mathbf{R}_{>0}\to\mathrm{GL}_1(\mathbf{A}_F)$ be a choice of continuous section to the idelic norm.

We recall from (A.2.3.14) that we have $E_F^1\subset\mathrm{GL}_1(\mathbf{A}_F)^1$ a relatively compact subset with non-empty interior, which is a fundamental domain for the continuous left-translation action of the discrete subgroup $F^\times\subset\mathrm{GL}_1(\mathbf{A}_F)^1$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$.

The construction of E_F^1 is, recall, as follows: for a fixed choice of Archimedean place v_0, and denoting by ι_{v_0} an isomorphism of topological fields from F_{v_0} to either $\mathbf{R}$ (if v_0 is real) or $\mathbf{C}$ (if v_0 is complex):

$$E_0:=\{g\in\mathscr{L}^{-1}(P):\arg(\iota_{v_0}(g_{v_0}))\in[0,2\pi/w_F)\}\subset\mathscr{L}^{-1}(P) \tag{A.5.2.7}$$

where w_F is the number of roots of unity contained in F (note that if F has a real place, then $w_F=2$), and P is the following relatively compact subset of the Lie group $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$ with non-empty interior:

$$P_0:=\left\{\sum_{i=1}^{r_1+r_2-1}x_i\mathscr{L}(\varepsilon_i):(x_i)\in Q_0\right\} \tag{A.5.2.8}$$

from (A.2.3.9), for $\{\varepsilon_1,\dots,\varepsilon_{r_1+r_2-1}\}$ a $\mathbf{Z}$-linear basis for the maximal torsion-free quotient of the unit group $\mathcal{O}_F^\times$,

$$Q_0:=\{(x_i)\in\mathbf{R}^{r_1+r_2-1}:0\leq x_i<1\}\subset\mathbf{R}^{r_1+r_2-1}, \tag{A.5.2.9}$$

and $\mathscr{L}$ is the logarithm homomorphism $\mathscr{L}((u_v)_{v\in S_\infty})=(\log|u_v|_v)_{v\in S_\infty}$.

We then define:

$$E_F^1:=\bigcup_{i=1}^{h_F}g_i(E_0\times\widehat{\mathcal{O}}_F^\times)\subset\mathrm{GL}_1(\mathbf{A}_F)^1\subset\mathrm{GL}_1(\mathbf{A}_F) \tag{A.5.2.10}$$

where h_F is the class number of F and $\{g_i\}_{1\leq i\leq h_F}$ is a set of representatives in $\mathrm{GL}_1(\mathbf{A}_F)^1$ for the ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$, with $g_1=1$; see (A.2.3.14).

We now adjust the construction of E_F^1, to obtain another fundamental domain, enjoying additional symmetries. We define:

$$P_1:=\left\{\sum_{i=1}^{r_1+r_2-1}x_i\mathscr{L}(\varepsilon_i):(x_i)\in Q_1\right\} \tag{A.5.2.11}$$

with

$$Q_1:=\{(x_i)\in\mathbf{R}^{r_1+r_2-1}:-1/2\leq x_i<1/2\}\subset\mathbf{R}^{r_1+r_2-1} \tag{A.5.2.12}$$

and

$$B_F^1 := E_1 \times \widehat{\mathcal{O}}_F^\times \tag{A.5.2.13}$$

with

$$E_1 := \{g \in \mathscr{L}^{-1}(P_1) : \arg(\iota_{v_0}(g_{v_0})) \in [0, \pi/w_F) \cup [2\pi - \pi/w_F, 2\pi)\} \subset \mathscr{L}^{-1}(P_1) \tag{A.5.2.14}$$

endowed with the subspace topology.

In a similar manner and following Definition A.2.4.3, for S a finite set of places of F containing all the Archimedean places, and upon writing S_f for $S \cap \Sigma_{F,f}$, the finite set of those finite places of F contained in F, we define, upon choosing an Archimedean place v_0 of F:

$$B_{F,S}^1 := \{g \in \mathscr{L}_S^{-1}(P_S) : \arg(\iota_{v_0}(g_{v_0})) \in [0, \pi/w_F) \cup [2\pi - \pi/w_F, 2\pi)\} \subset \mathscr{L}_S^{-1}(P_S) \tag{A.5.2.15}$$

for

$$\mathscr{L}_S : \mathrm{GL}_1(\mathbf{A}_{F,S}) \to \mathbf{R}^S \tag{A.5.2.16}$$

defined as

$$\mathscr{L}_S(g) := (\log|g_v|_v)_{v \in S} \tag{A.5.2.17}$$

and, upon choosing an integral basis $\{\varepsilon_1, \ldots, \varepsilon_{|S|-1}\}$ for $\mathcal{O}_{F,S}^\times$, P_S is the parallelotope in the hyperplane $H_S := \{x \in \mathbf{R}^S : \sum_{v \in S} x_v = 0\}$ of $\mathbf{R}^S$ defined as

$$P_S := \sum_{1 \leq i \leq |S|-1} [-1/2, 1/2)\mathscr{L}_S(\varepsilon_i) \subset H_S. \tag{A.5.2.18}$$

We endow $B_{F,S}^1$ with the subspace topology.

Lemma A.5.2.12. *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

1. *For $g_1 := 1$ and $\{g_i\}_{1 \leq i \leq h_F} \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ a choice of representatives for the h_F ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$, $\bigcup_{1 \leq i \leq h_F} g_i B_F^1$ is a fundamental domain for the continuous left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$.*
2. *$B_{F,S}^1$ is a fundamental domain for the continuous left-translation action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.*
3. *For $g_1 := 1$ and $\{g_i\}_{1 \leq i \leq h_{F,S}} \subset \mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,S})^1$ a choice of representatives for the $h_{F,S}$ ideal classes in $\mathrm{Pic}(\mathcal{O}_{F,S})$, $\bigcup_{1 \leq i \leq h_{F,S}} g_i B_{F,S}^1$ is a fundamental domain for the continuous left-translation action of $\mathcal{O}_{F,S}^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$.*

Proof. (1) : We will be showing that B_F^1 is a fundamental domain within $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F)^1$ for the continuous left-translation action of the global unit group $\mathcal{O}_F^\times$. To this end, we proceed as follows. The kernel of $\mathscr{L}_F := \mathscr{L} \times (\log|\cdot|_v)_{v \in \Sigma_{F,f}} : \mathrm{GL}_1(\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F)^1 \to \mathbf{R}^{S_\infty}$ is the finite group of roots of unity $\mu_F \subset \mathcal{O}_F^\times$, while the image of $\mathscr{L}_F$ is the finite-dimensional Euclidean space $H = \{\sum_{v|\infty} x_v = 0\} \subset \mathbf{R}^{S_\infty}$. We know $\mathscr{L}_F(\mathcal{O}_F^\times/\mu_F)$ is a lattice in H, and $Q_1 \times \{0\} \simeq Q_1$ is a fundamental domain for the additive translation action of $\mathscr{L}_F(\mathcal{O}_F^\times/\mu_F) \simeq \mathscr{L}(\mathcal{O}_F^\times/\mu_F)$ on H.

In order to prove that B_F^1 is a fundamental domain for the continuous left-translation action of $\mathcal{O}_F^\times$ on $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty} \times \widehat{\mathcal{O}}_F)^1$, it remains to note that $e^{i[0,\pi/w_F) \cup i[2\pi - \pi/w_F, 2\pi)}$ is a fundamental domain within $\mathrm{U}(1)$ of the action of the finite cyclic group $\mu_F = e^{\frac{2\pi i \mathbf{Z}}{w_F}}$.

We now let $\{g_1, \ldots, g_{h_F}\}$ be representatives in $\mathrm{GL}_1(\mathbf{A}_F)^1$ for the h_F ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$, with $g_1 = 1$. By (A.2.3.24), the disjoint union $\bigcup_{1 \leq i \leq h_F} g_i B_F^1$ is a fundamental domain for the continuous left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$.

(2) : The same argument provided for (1) shows that (2) holds upon replacing the containment $\mathbf{R}^{S_\infty} \supset H$ with $\mathbf{R}^S \supset H_S$ and noting that $\ker(\mathscr{L}_S) \cap \mathcal{O}_{F,S}^\times = \mu_F$.

(3) : By choosing the g_i's to be representatives of $h_{F,S}$ ideal classes in $\mathrm{Pic}(\mathcal{O}_{F,S})$ in $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$, the subset $\bigcup_{1 \leq i \leq h_{F,S}} g_i B^1_{F,S}$ is a fundamental domain for the continuous left-translation action of $\mathcal{O}^\times_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$, in light of (2) by adapting the method of proof of (1). □

LEMMA A.5.2.13. *Let F be a number field.*

(1) *The interior* $\mathrm{Int}(E_1)$ *of* E_1 *from* (A.5.2.14) *is a symmetric unit-neighborhood in* $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.
(2) *The following sequence of subsets:*

$$U_{S_\infty,k} := \mathscr{L}^{-1}(P_k) \tag{A.5.2.19}$$

with

$$P_k = \{ \sum_{i=1}^{r_1+r_2-1} x_i \mathscr{L}(\varepsilon_i) : -1/(k+1) < x_i < 1/(k+1) \text{ for all } 1 \leq i \leq r_1 + r_2 - 1\} \tag{A.5.2.20}$$

is a symmetric D'-sequence for $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, *and* $\{U_{S_\infty,k}, U_{S_\infty,2k}\}_{k \geq 0}$ *is a D''-sequence for* $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$.

PROOF. (1) : Since $\mathrm{Int}(Q_1)$ from (A.5.2.12) is a symmetric unit-neighborhood in $\mathbf{R}^{r_1+r_2-1}$, then, for $\mathrm{Int}(E_1)$ to be a symmetric unit-neighborhood in $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1$, it is enough for the subset[10]

$$U := \{z \in \mathbf{C}^\times : |z| \in e^{(-1/4,1/4)}, \arg(z) \in (0, \pi/t) \cup (2\pi - \pi/t, 2\pi)\}$$

to be a symmetric unit-neighborhood[11] of $1 \in \mathbf{C}^\times$ for $t \geq 2$.

Indeed, if $z \in U$, then $|z^{-1}| \in e^{(-1/4,1/4)}$ also holds. Since $\arg(z^{-1}) = -\arg(z) \bmod 2\pi$, and for any $\delta \in [0, 2\pi)$ we have that $\delta \in (0, \pi/t) \bmod 2\pi$ if and only if $-\delta \in (2\pi - \pi/t, 2\pi) \bmod 2\pi$, then U is a symmetric unit-neighborhood of 1 in $\mathbf{C}^\times$.

(2) : The logarithm homomorphism $\mathscr{L}$ is injective on each P_k, and $\mathscr{L}(P_k) \subset \mathscr{L}(P_1)$. It is therefore enough to show that[12] $\{Q_k\}_{k \geq 0}$ is a symmetric D'-sequence for $\mathbf{R}^{r_1+r_2-1}$, and $\{Q_k, Q_{2k}\}_{k \geq 0}$ is a D''-sequence for $\mathbf{R}^{r_1+r_2-1}$, which is indeed the case since Q_k is an open cube in $\mathbf{R}^{r_1+r_2-1}$ whose baricenter is the origin, and every Haar measure on $\mathbf{R}^{r_1+r_2-1}$ is a positive multiple of the Lebesgue measure. □

LEMMA A.5.2.14. *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

(1) *Let $\{B_k\}_{k \geq 0}$ be a sequence of compact open subgroups in $\widehat{\mathcal{O}}^\times_{F,S}$ such that $B_{k+1} \subset B_k$ for every integer $k \geq 0$, and $\{B_k\}_{k \geq 0}$ is a unit-neighborhood base in $\widehat{\mathcal{O}}^\times_{F,S}$. Then the sequence $\{U_{S,k}\}_{k \geq 0}$ of subsets in $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$:*

$$U_{S,k} := U_{S_\infty,k} \times B_k \tag{A.5.2.21}$$

is a symmetric D'-sequence for $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.
(2) *For each integer $k \geq 0$ and for $\overline{U}_{S,k}$ the image in $J^1_{F,S}$ of $U_{S,k}$ under the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S})^1 \to J^1_{F,S}$, the sequence $\{\overline{U}_{S,k}\}_{k \geq 0}$ is a symmetric D'-sequence for $J^1_{F,S}$.*

PROOF. (1) holds by Lemma A.5.2.3(2), combined with Lemma A.5.2.3(1) and Lemma A.5.2.5.

(2) : Since $B^1_{F,S}$ is a fundamental domain for the continuous left-translation action of $\mathcal{O}^\times_{F,S}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and for every integer $k \geq 0$ we have $U_{S,k} \subset B^1_{F,S}$, then (2) holds by Proposition A.5.2.6. □

LEMMA A.5.2.15. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. We let $\{B_k\}_{k \geq 0}$ be a choice of a sequence of compact open subgroups of $\widehat{\mathcal{O}}^\times_{F,S}$ such that $B_{k+1} \subset B_k$ for*

[10] Why use $e^{(-1/4,1/4)}$ below rather than $(1/c, c)$ for general $c > 1$?
[11] This misses 1 (which has argument 0)!
[12] Need to define Q_k somehow akin to P_k's

every integer $k \geq 0$ such that $\{B_k\}_{k\geq 0}$ is a unit-neighborhood base in $\widehat{\mathcal{O}}_{F,S}^{\times}$, we let $\{U_{S,k}\}_{k\geq 0}$ be the sequence of open subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^{\times}$ defined in (A.5.2.21).

We let $\tau_S : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_{F,S})$ be a choice of continuous homomorphic section to the S-idelic norm.

The sequence:

$$\{U_k := \tau_S((1-1/k, 1+1/k)) \cdot U_{S,k}\}_{k\geq 0} \tag{A.5.2.22}$$

is a symmetric open D'-sequence for $\mathrm{GL}_1(\mathbf{A}_{F,S})$. For $\overline{U}_k$ the image of U_k in $J_{F,S}$ for each integer $k \geq 0$, $\{\overline{U}_k\}_{k\geq 0}$ is a symmetric open D'-sequence for $J_{F,S}$.

Proof. The Lemma follows upon combining Lemma A.5.2.5 with Lemma A.5.2.14. The assertions on $\{\overline{U}_k\}_{k\geq 0}$ follow from Lemma A.5.2.12(2) and Proposition A.5.2.6. □

Lemma A.5.2.16. *Let F be a number field, S a finite set of places of F containing all the Archimedean places.*

(1) *Let $\{B_k\}_{k\geq 0}$ be a sequence of compact open subgroups in $\widehat{\mathcal{O}}_F^{\times}$ such that $B_{k+1} \subset B_k$ for every integer $k \geq 0$, and $\{B_k\}_{k\geq 0}$ is a unit-neighborhood base in $\widehat{\mathcal{O}}_F^{\times}$. Then the sequence $\{U_{F,k}\}_{k\geq 0}$ of subsets in $\mathrm{GL}_1(\mathbf{A}_F)^1$:*

$$U_{F,k} := U_{S_\infty,k} \times B_k \tag{A.5.2.23}$$

is a symmetric D'-sequence for $\mathrm{GL}_1(\mathbf{A}_F)^1$ and for $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$.

(2) *For each integer $k \geq 0$ and for $\overline{U}_{F,k}$ the image in C_F^1 of $U_{F,k}$ under the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_F)^1 \to C_F^1$, the sequence $\{\overline{U}_{F,k}\}_{k\geq 0}$ is a symmetric D'-sequence for C_F^1.*

(3) *For each integer $k \geq 0$ and for $\overline{U}_{F,S,k}$ the image in $C_{F,S}^1$ of $U_{F,k}$ under the continuous quotient homomorphism $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1 \to C_{F,S}^1$, the sequence $\{\overline{U}_{F,S,k}\}_{k\geq 0}$ is a symmetric D'-sequence for $C_{F,S}^1$.*

Proof. (1) holds by Lemma A.5.2.3(2), combined with Lemma A.5.2.3(1) and Lemma A.5.2.5.

(2) and (3) : Since, by Lemma A.5.2.12(3), $\bigcup_{1\leq i\leq h_{F,S}} g_i B_{F,S}^1$, for $\{g_1 = 1, \ldots, g_{h_{F,S}}\}$ a choice of representatives in $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \prod_{v\notin S} \mathcal{O}_{F_v})^1$ of the $h_{F,S}$ ideal classes in $\mathrm{Pic}(\mathcal{O}_{F,S})$, is a fundamental domain for the continuous left-translation action of $\mathcal{O}_{F,S}^{\times}$ on $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \prod_{v\notin S} \mathcal{O}_{F_v})^1$, and for every integer $k \geq 0$ we have that $U_{F,k}$ is contained in it, then (2) and (3) both hold by (1) and Proposition A.5.2.6. □

We conclude with the following:

Lemma A.5.2.17. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. We let $\{B_k\}_{k\geq 0}$ be a choice of a sequence of compact open subgroups of $\widehat{\mathcal{O}}_F^{\times}$ such that $B_{k+1} \subset B_k$ for every integer $k \geq 0$ such that $\{B_k\}_{k\geq 0}$ is a unit-neighborhood base in $\widehat{\mathcal{O}}_F^{\times}$, we let $\{U_{F,k}\}_{k\geq 0}$ be the sequence of open subsets of $\mathrm{GL}_1(\mathbf{A}_{F,S_\infty})^1 \times \widehat{\mathcal{O}}_F^{\times}$ defined in* (A.5.2.23).

We let $\tau : \mathbf{R}_{>0} \to \mathrm{GL}_1(\mathbf{A}_F)$ be a choice of continuous homomorphic section to the idelic norm.

(1) *The sequence:*

$$\{U_k := \tau((1-1/k, 1+1/k)) \cdot U_{F,k}\}_{k\geq 0} \tag{A.5.2.24}$$

is a symmetric open D'-sequence for $\mathrm{GL}_1(\mathbf{A}_F)$. For $\overline{U}_k$ the image of U_k in C_F for each integer $k \geq 0$, $\{\overline{U}_k\}_{k\geq 0}$ is a symmetric open D'-sequence for C_F.

(2) *If τ factors through $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S}) \subset \mathrm{GL}_1(\mathbf{A}_F)$, then the sequence $\{U_k\}_{k\geq 0}$ is a symmetric open D'-sequence for $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$. For $\overline{U}_k$ the image of U_k in $C_{F,S}$ for each integer $k \geq 0$, $\{\overline{U}_k\}_{k\geq 0}$ is a symmetric open D'-sequence for $C_{F,S}$.*

Proof. (1) : By Lemma A.5.2.5, the sequence in (A.5.2.24) is a symmetric D'-sequence for $\mathrm{GL}_1(\mathbf{A}_F) \simeq \mathbf{R}_{>0} \times \mathrm{GL}_1(\mathbf{A}_F)^1$, the isomorphism of Hausdorff locally compact abelian groups being induced by the choice

of τ, and since $\{(1-1/k, 1+1/k)\}_{k\geq 0}$ and $U_{F,k}$ each satisfy all the assertions as sequences within $\mathbf{R}_{>0}$, respectively $\mathrm{GL}_1(\mathbf{A}_F)^1$, the latter by Lemma A.5.2.16(1).

The assertions on $\{\overline{U}_k\}_{k\geq 0}$ and $\{\overline{U}_k, \overline{U}_{2k}\}_{k\geq 0}$ then follow from Proposition A.5.2.6, since $\bigcup_{1\leq i\leq h_F} g_i B_F^1$ is, by Lemma A.5.2.12(1), for $\{g_1 = 1, \ldots, g_{h_F}\}_{1\leq i\leq h_F} \subset \mathrm{GL}_1(\mathbf{A}_F)^1$ a choice of representatives for the h_F ideal classes in $\mathrm{Pic}(\mathcal{O}_F)$ a fundamental domain for the continuous left-translation action of $F^\times$ on $\mathrm{GL}_1(\mathbf{A}_F)^1$. □

We introduce the following terminology, for later use:

Definition A.5.2.18. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

We refer to the open symmetric D'-sequences:

(1) $\{U_k\}_{k\geq 0}$ from (A.5.2.24) for $\mathrm{GL}_1(\mathbf{A}_F)$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$.
(2) $\{\overline{U}_k\}_{k\geq 0}$ from Lemma A.5.2.17, respectively Lemma A.5.2.15, for C_F, respectively $C_{F,S}$.
(3) $\{U_{F,k}\}_{k\geq 0}$ from (A.5.2.23) for $\mathrm{GL}_1(\mathbf{A}_F)^1$, respectively $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$.
(4) $\{\overline{U}_{F,k}\}_{k\geq 0}$ from Lemma A.5.2.16(2), for C_F^1, $\{\overline{U}_{F,S,k}\}_{k\geq 0}$ from Lemma A.5.2.16(3), for $C_{F,S}^1$.
(5) $\{U_k\}_{k\geq 0}$ from (A.5.2.22) for $\mathrm{GL}_1(\mathbf{A}_{F,S})$.
(6) $\{\overline{U}_k\}_{k\geq 0}$ from Lemma A.5.2.15, for $J_{F,S}$.
(7) $\{U_{S,k}\}_{k\geq 0}$ from (A.5.2.21), for $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$.
(8) $\{\overline{U}_{S,k}\}_{k\geq 0}$ from Lemma A.5.2.14(2), for $J_{F,S}^1$.

as the *standard D'-sequence* for $G = \mathrm{GL}_1(\mathbf{A}_F)$,$\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$, C_F, $C_{F,S}$, $\mathrm{GL}_1(\mathbf{A}_F)^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$, C_F^1, $C_{F,S}^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, $J_{F,S}$, $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, $J_{F,S}^1$, respectively.

We will be using the notation $\{U_k\}_{k\geq 0}$ to denote the standard symmetric D'-sequence for $\mathrm{GL}_1(\mathbf{A}_F)$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$, $\mathrm{GL}_1(\mathbf{A}_F)^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$, and $\{\overline{U}_k\}_{k\geq 0}$ to denote the standard symmetric D'-sequence for their respective class groups.

Remark A.5.2.19. Let F be a number field, S a finite set of places of F containing all the Archimedean places.

We observe that for the standard D'-sequences for $\mathrm{GL}_1(\mathbf{A}_F)^1$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})^1$, C_F^1, $C_{F,S}^1$, the structural constant C from (A.5.2.1) can be chosen[13] to be $A^{r_1+r_2-1}$ for any real number $A > 2$.

For $\mathrm{GL}_1(\mathbf{A}_{F,S})^1$ and $J_{F,S}^1$, it can be chosen to be $A^{|S|-1}$ for any real number $A > 2$.

For $\mathrm{GL}_1(\mathbf{A}_F)$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$ and their class groups, it is enough to multiply the above constants by A.

A.5.3. Hardy-Littlewood Maximal Theorem on Hausdorff locally compact abelian groups. We now know that the Hausdorff locally compact abelian groups $\mathrm{GL}_1(\mathbf{A}_F)$, $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{F,\Sigma_F - S})$, $\mathrm{GL}_1(\mathbf{A}_{F,S})$, their closed subgroups of norm-1 elements, their class groups C_F, $C_{F,S}$, $J_{F,S}$, and their subgroups of norm-1 elements, admit symmetric open D'-sequences.

We will now show how to construct an analog, for Hausdorff locally compact abelian groups that admit Borel D-sequences, and, at times, D'-sequences as the need arises, of the Hardy-Littlewood maximal function, and we will be discussing analogs of the Hardy-Littlewood Maximal Theorem and Lebesgue's Differentiation Theorem.

We begin with the following result from [**HR-2**, Ch. XI, §44]:

Theorem A.5.3.1. *Let G be a Hausdorff locally compact abelian group, μ_G a Haar measure on G.*

Let $\{U_k\}_{k\geq 0}$ be a D-sequence for G. We let $\mathcal{F} \subset \mathcal{P}(G)$ be the collection of all subsets of G of the form xU_k, for $x \in G$ and $k \geq 0$ an integer. Let $\mathcal{F}' \subset \mathcal{F}$ be a subset.

Let $B \subset G$ be a μ_G-measurable subset such that:

[13] Should briefly explain calculation to get these sufficient constants.

(1) $\mu_G(B \cdot U_0)$ *is finite.*[14]
(2) *For each* $x \in B$, *there exists an integer* $k \geq 0$ *such that* $xU_k \in \mathcal{F}'$.
Then there exists a finite subset $\{x_i U_{k_i}\}_{1 \leq i \leq N} \subset \mathcal{F}'$ *such that* $x_i \in B$ *for all* $1 \leq i \leq N$, *the subsets* $x_i U_{k_i}$ *are pairwise disjoint, and:*

$$\sum_{1 \leq i \leq N} \mu_G(U_{k_i}) > \frac{\mu_G(B)}{C} \tag{A.5.3.1}$$

where $C > 0$ *is the structural constant from* (A.5.2.1) *associated to the D-sequence* $\{U_k\}_{k \geq 0}$.

REMARK A.5.3.2. The Theorem is [**HR-2**, Ch. XI, Thm. 44.11, p. 638], and it is a generalization of the Vitali Covering Lemma (see [**StSha-III**, Ch. 3, §1, Lemma.1.2, p. 102]).

We note that in [**HR-2**, Ch. XI, Thm.44.11, p. 638], the subset $B \subset G$, denoted by E, is not assumed to be measurable. The notation $\lambda(E)$ and $\lambda(E \cdot U_1)$ is to mean $\lambda^*(E)$, $\lambda^*(E \cdot U_1)$, in the sense of [**Bou-I**, Ch. IV, §1, No. 4, Def. 4, INT IV.10].

We will now discuss some aspects of the proofs of the main results from [**HR-2**, Ch. XI, §44.50], resting on [**HR-2**, Ch. XI, Thm. 44.13].

We will be assuming the Hausdorff locally compact abelian group G under consideration is second countable, which, in light of the following Lemma, leads to promoting identities and properties valid locally almost everywhere (i.e almost everywhere within every compact subset), into assertions valid almost everywhere.

We recall that, for a Hausdorff locally compact[15] topological space X, and a measure μ on X, a subset $A \subset X$ is locally μ-negligible if and only if for every $x \in X$ there exists a neighborhood U of x such that $A \cap U$ is μ-negligible (see [**Bou-I**, Ch. IV, §5, No. 2, Def. 4, INT IV.61]).

LEMMA A.5.3.3. *Let* G *be a Hausdorff locally compact abelian group. Assume* G *is second countable. Let* μ_G *be a choice of Haar measure on* G.
Let $A \subset G$ *be a subset. If* A *is locally* μ_G*-negligible, then* A *is* μ_G *measurable and* μ_G*-negligible.*

PROOF. Let $\{K_m\}_{m \geq 0}$ be a countable exhaustion of G by compact subsets. Then $A \cap K_m$ is μ_G-negligible and $A = \bigcup_{m \geq 0}(A \cap K_m)$, which gives $\mu_G(A) = 0$. In particular, A is μ_G-measurable. □

A.5.3.1. *Hardy-Littlewood Maximal Theorems on Hausdorff locally compact abelian groups.* The following is a special case of [**HR-2**, Ch.XI,§44,Thm.44.13,p.640]:

THEOREM A.5.3.4 (Measure-theoretic Maximal Theorem - I). *Let* G *be a Hausdorff locally compact abelian group,* μ_G *a choice of Haar measure on* G. *Assume* G *is second countable.*
Let $\{U_k\}_{k \geq 0}$ *be a Borel D-sequence in* G. *Let* ν *be a positive Radon measure on* G.
We define, for each $x \in G$:[16]

$$(M\nu)(x) := \limsup_{k \geq 0} \frac{\nu(xU_k)}{\mu_G(U_k)}. \tag{A.5.3.2}$$

Let B *be a* μ_G*-measurable subset of* G *such that* $\mu_G(B \cdot U_0)$ *is finite.*
(1) *For any real number* $\alpha > 0$, *we have:*

$$\mu_G(B \cap \{x \in G : (M\nu)(x) > \alpha\}) \leq \frac{C}{\alpha} \nu((B \cap \{x \in G : (M\nu)(x) > \alpha\}) \cdot U_0) \tag{A.5.3.3}$$

and equality holds if and only if $B \cap \{x \in G : (M\nu)(x) > \alpha\} = \varnothing$.

[14] Should address why $B \cdot U_0$ is automatically measurable; maybe for this we need sufficiently small or somehow nice U_0?
[15] no second-countable hypothesis?
[16] Must address $M\nu$ being measurable. warn reader that $\nu(xU_k)$ may be infinite.

(2) *If $\overline{U}_0$ is compact, then $M\nu$ is finite μ_G-almost everywhere.*

PROOF. (1) : As in the proof of [**HR-2**, Ch.XI,§44,Thm.44.13,p.640], we first assume $B \cap \{x \in G : (M\nu)(x) > \alpha\}$, and the Theorem is obtained by defining $\mathcal{F}$ to be the collection of the subsets of G of the form xU_k for $x \in G$ and each integer $k \geq 0$, and $\mathcal{F}' \subset \mathcal{F}$ the sub-collection of those subsets xU_k with $x \in B \cap \{y \in G : (M\nu)(y) > \alpha\}$ satisfying $\nu(xU_k) > \alpha\mu_G(U_k)$. Indeed, then applying Theorem A.5.3.1 to the choice $\{U_k\}_{k\geq 0}$ of D-sequence for G, and to the above choices of $\mathcal{F}$ and $\mathcal{F}'$ gives (1).

The strict inequality in (A.5.3.1) from Theorem A.5.3.1 ensures[17] that if $B \cap \{x \in G : (M\nu)(x) > \alpha\}$ is non-empty then the inequality in (A.5.3.3) holds and is strict. Equality holds if $B \cap \{x \in G : (M\nu)(x) > \alpha\}$ is empty.

(2) : We define $B_\infty \subset G$ to be the subset $\{x \in G : (M\nu)(x) = \infty\}$. By Lemma A.5.3.3 it is enough to show that for every compact subset $K \subset G$, we have $\mu_G(B_\infty \cap K) = 0$.

By assumption, $\overline{U}_0$ is compact, and hence so is $K \cdot \overline{U}_0$. Since both μ_G and ν are Radon measures, both $\mu_G(K \cdot \overline{U}_0)$ and $\nu(K \cdot \overline{U}_0)$ are finite.

For every positive real number $\alpha > 0$ we have

(A.5.3.4)

$$\mu_G(K \cap B_\infty) \leq \mu_G(K \cap \{x \in G : (M\nu)(x) > \alpha\}) \leq \frac{C}{\alpha}\,\nu((K \cap \{x \in G : (M\nu)(x) > \alpha\}) \cdot U_0) \leq \frac{C}{\alpha}\,\nu(K \cdot \overline{U}_0)$$

(middle inequality via (1) applied to $B = K$) and since $\alpha > 0$ is arbitrary, we may let α tend to $+\infty$ in (A.5.3.4) and deduce $\mu_G(K \cap B_\infty) = 0$. □

In the presence of a D'-sequence in G, a more precise result is available. The following is a special case of [**HR-2**, Ch. XI, Thm.44.14, p. 641]:

THEOREM A.5.3.5 (Measure-theoretic Maximal Theorem - II). *Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G. Assume G is second countable.*

Let $\{U_k\}_{k\geq 0}$ be a D'-sequence in G. Let ν be a positive Radon measure on G.

We define, for each $x \in G$:[18]

$$(\overline{M}\nu)(x) := \limsup_{k\geq 0} \frac{\nu(xU_k)}{\mu_G(U_k)}. \tag{A.5.3.5}$$

Let $K \subset G$ be a compact subset. For any real number $\alpha > 0$, we have:

$$\mu_G(K \cap \{x \in G : (\overline{M}\nu)(x) > \alpha\}) \leq \frac{C}{\alpha}\,\nu(K). \tag{A.5.3.6}$$

PROOF. Let $N \geq 0$ be a nonnegative integer, and consider, as in the proof of [**HR-2**, Ch. XI, Thm.44.14, p. 641], the D-sequence in G defined by $\{U_k\}_{k\geq N}$. Since $\{U_k\}_{k\geq 0}$ is a D'-sequence, for any compact unit-neighborhood V there exists a large enough integer $k_0 \geq 0$ such that U_{k_0}, and hence U_k for all $k \geq k_0$, is contained in V. This implies that the closure of each U_k for every $k \geq k_0$ is compact.

We fix $N \geq k_0$. Let $K \subset G$ be a compact subset. Then $K \cdot \overline{U}_k$ is compact for every $k \geq N$, and hence $\mu_G(K \cdot U_N)$ is finite. By Theorem A.5.3.4, we have, for every real number $\alpha > 0$:

$$\mu_G(K \cap \{x \in G : (\overline{M}\nu)(x) > \alpha\}) \leq \frac{C}{\alpha}\,\nu(K \cdot U_N). \tag{A.5.3.7}$$

For any open subset V containing K, there exists a unit-neighborhood U in G such that $K \cdot U \subset V$. Since $\{U_k\}_{k\geq 0}$ is a D'-sequence, there exists a large enough integer $k_1 \geq k_0$ such that $U_{k_1} \subset U$, and hence $K \cdot U_{k_1} \subset V$.

---

[17] Why? Must say more.

[18] This seems the same as $(M\nu)(x)$ above, so why now denoted differently?

Since ν is regular by [Fol, Thm. 7.8], (in fact, [Fol, Prop. 7.5] suffices), and hence

$$\nu(K) \leq \liminf_{N\to+\infty} \nu(K \cdot U_N) \leq \inf\{\nu(V) : K \subset V, V \text{ open}\} = \nu(K),$$

this gives $\lim_{N\to+\infty} \nu(K \cdot U_N) = \nu(K)$. □

We are ready to state and prove the following version of the Hardy-Littlewood Maximal Theorem:

THEOREM A.5.3.6 (Hardy-Littlewood Local Maximal Theorem). *Let G be a Hausdorff locally compact abelian group, and μ_G a choice of Haar measure. Assume G is second countable.*

Let $\{U_k\}_{k\geq 0}$ be a D-sequence such that $\overline{U}_0$ is compact. Let $f \in L^1_{\mathrm{loc}}(G, \mu_G)$, and define, for every $x \in G$:

$$(Mf)(x) := \limsup_{k\geq 0} \left\{ \frac{1}{\mu_G(U_k)} \int_{xU_k} |f(y)|\, \mathrm{d}\mu_G(y) \right\}. \tag{A.5.3.8}$$

Then Mf is μ_G-measurable, and if $f \log^+|f| \in L^1_{\mathrm{loc}}(G, \mu_G)$ then $Mf \in L^1_{\mathrm{loc}}(G, \mu_G)$, so Mf is finite μ_G-almost everywhere.

If $\{U_k\}_{k\geq 0}$ is a D'-sequence then

(1) *for every μ_G-measurable subset $B \subset G$ such that $B \cdot U_0$ is μ_G-measurable[19] with finite μ_G-measure, any real number $\alpha > 0$, and any real number $\delta \in (0, 1)$, we have*

$$\mu_G(B \cap \{x \in G : (Mf)(x) > \alpha\}) \leq \frac{C}{(1-\delta)\alpha} \int_{(B\cdot U_0)\cap\{x\in G:(Mf)(x)>\delta\alpha\}} |f(x)|\, \mathrm{d}\mu_G(x). \tag{A.5.3.9}$$

(2) *For every real number $p > 1$, we have:*

$$\int_B |(Mf)(x)|^p\, \mathrm{d}\mu_G(x) \leq \frac{C\, p^{p+1}}{(p-1)^p} \int_{B\cdot U_0} |f(x)|^p\, \mathrm{d}\mu_G(x). \tag{A.5.3.10}$$

PROOF. The proof is analogous to the proof of Theorem A.5.1.1. We will be following the proof of [HR-2, Ch. XI, Thm.44.50, p. 669].

We note that both Mf and, if $\{U_k\}_{k\geq 0}$ is a D'-sequence,[20] $\overline{M}f$, are μ_G-measurable by [Ru-2, Ch. 1, Thm. 1.14, p. 14].

We note that by [Ru-2, Ch. 1, Corollaries,(b), p. 15], for $f \in L^1_{\mathrm{loc}}(G, \mu_G)$, then $\log^+|f|$ and hence $f \log^+|f|$ is μ_G-measurable.

We now prove that Mf is finite μ_G-almost everywhere.

If $f \log^+|f| \in L^1_{\mathrm{loc}}(G, \mu_G)$ then for any compact subset $K \subset G$ we have, by [HR-2, Ch. XI, Thm.44.50, p. 669] (a consequence of Fubini's Theorem and [Fol, Thm.7.27]):

$$\int_K |(Mf)(x)|\, \mathrm{d}\mu_G(x) \leq \frac{1}{2}\mu_G(K) + \frac{C}{2} \int_{K\cdot U_0} |f(x)| \log^+|f|\, \mathrm{d}\mu_G(x) \tag{A.5.3.11}$$

which proves that for every compact subset $K \subset G$, $(Mf) \cdot 1_K$ is absolutely μ_G-integrable. Since G is second countable, this implies that $Mf \in L^1_{\mathrm{loc}}(G, \mu_G)$ holds. In particular, Mf is finite locally μ_G-almost everywhere, and hence (by Lemma A.5.3.3) μ_G-almost everywhere.

(1) : We fix $\alpha > 0$, and $\delta \in (0, 1)$. Let $F_\delta := |f| \cdot 1_{A_{f,\delta,\alpha}}$ where $1_{A_{f,\delta,\alpha}}$ is the indicator function of the set:

$$A_{f,\delta,\alpha} := \{x \in G : |f(x)| > \delta\alpha\}. \tag{A.5.3.12}$$

Since $f \in L^1_{\mathrm{loc}}(G, \mu_G)$, the set $A_{f,\delta,\alpha}$ is μ_G-measurable. The function F_δ is locally μ_G-integrable, and it satisfies:

$$Mf \leq MF_\delta + \delta\alpha \tag{A.5.3.13}$$

[19] Is this measurability not automatically satisfied?

[20] We really sequence D'-sequence property to ensure such measurability? Why?

and hence:

$$\{x \in G : (Mf)(x) > \alpha\} \subset \{x \in G : (MF_\delta)(x) > (1-\delta)\alpha\}. \tag{A.5.3.14}$$

We define $\nu := F_\delta\, \mu_G$. Since F_δ is locally μ_G-integrable, ν is a positive Radon measure on G.

We first apply Theorem A.5.3.4. We obtain, from (A.5.3.14) and (A.5.3.3):

$$\begin{aligned} \mu_G(B \cap \{x \in G : (Mf)(x) > \alpha\}) &\leq \mu_G(B \cap \{x \in G : (MF_\delta)(x) > (1-\delta)\alpha\}) \\ &\leq \frac{C}{(1-\delta)\alpha} \int_{(B \cap \{x \in G : (MF_\delta)(x) > (1-\delta)\alpha\}) \cdot U_0} F_\delta(x)\, \mathrm{d}\mu_G(x) \\ &\leq \frac{C}{(1-\delta)\alpha} \int_{B \cdot U_0} F_\delta(x)\, \mathrm{d}\mu_G(x). \end{aligned} \tag{A.5.3.15}$$

This proves (A.5.3.9). (2) : To prove that (A.5.3.10) holds, the argument is identical to that provided in the liminf analogue [**HR-2**, Ch. XI, Thm. 44.50(iii), p. 670], an application of Fubini's Theorem and [**Fol**, Thm. 7.27] by choosing $\delta = 1 - 1/p$. □

The following consequence of Theorem A.5.3.6 is a global version:

THEOREM A.5.3.7 (Hardy-Littlewood Global Maximal Theorem). *Let G be a Hausdorff locally compact abelian group. Assume G is second countable.*

Let $\{U_k\}_{k\geq 0}$ be a D'-sequence such that $\overline{U}_0$ is compact. Let $f \in L^1_{\mathrm{loc}}(G, \mu_G)$, and define for every $x \in G$:

$$(Mf)(x) := \limsup_{k\geq 0} \left\{ \frac{1}{\mu_G(U_k)} \int_{xU_k} |f(y)|\, \mathrm{d}\mu_G(y) \right\}. \tag{A.5.3.16}$$

Then:

(1) *Assume $f \in L^1(G, \mu_G)$. For every real number $\alpha > 0$ we have:*

$$\mu_G(\{x \in G : (Mf)(x) > \alpha\}) \leq \frac{2C}{\alpha} \|f\|_{L^1(G,\mu_G)}. \tag{A.5.3.17}$$

(2) *For every real number $p > 1$, if $f \in L^p(G, \mu_G)$ then $Mf \in L^p(G, \mu_G)$ also holds and we have:*

$$\int_G |(Mf)(x)|^p\, \mathrm{d}\mu_G(x) \leq \frac{Cp^{p+1}}{(p-1)^p} \|f\|^p_{L^p(G,\mu_G)}. \tag{A.5.3.18}$$

PROOF. By Theorem A.5.3.6 with $\delta = 1/2$, (1) and (2) both follow from the Theorem and an application of Dominated Convergence, noting that we may choose B to range over an exhaustion of G by a sequence of compact subsets. □

A.5.4. Lebesgue Differentiation on Hausdorff locally compact abelian groups. In this subsection, we establish an analog of Lebesgue's Differentiation Theorem:

THEOREM A.5.4.1 (Lebesgue's Differentiation Theorem - Hausdorff Locally Compact Abelian Groups). *Let G be a Hausdorff locally compact abelian group. Let μ_G be a Haar measure on G. Assume G is second countable. Let $\{U_k\}_{k\geq 0}$ be a D'-sequence for G. Let $f \in L^1_{\mathrm{loc}}(G, \mu_G)$. Then:*

(1) *For each integer $k \geq 0$, the integral*

$$A_{U_k}(f)(x) := \frac{1}{\mu_G(U_k)} \int_{xU_k} f(y)\, \mathrm{d}\mu_G(y) \tag{A.5.4.1}$$

is a uniformly continuous function of $x \in G$.

(2) *The sequence $\{A_{U_k}(f)(x)\}_{k\geq 0}$ converges for μ_G-almost all $x \in G$.*

(3) *Let $\mathrm{Reg}_{\{U_k\}_{k\geq 0}, f} \subset G$ be the subset of those points $x \in G$ at which $\{A_{U_k}(f)(x)\}_{k\geq 0}$ converges.*

Then the complement of $\mathrm{Reg}_{\{U_k\}_{k\geq 0},f}$ *is* μ_G*-negligible, and the identity*

$$f(x) = \lim_{k\to+\infty} \frac{1}{\mu_G(U_k)} \int_{xU_k} f(y)\,\mathrm{d}\mu_G(y) \tag{A.5.4.2}$$

holds at μ_G*-almost every* $x \in \mathrm{Reg}_{\{U_k\}_{k\geq 0},f}$ *and, in particular,* μ_G*-almost everywhere.*

Terminology A.5.4.1 (Smoothening). We will be referring to the function defined as the limit of the sequence in (A.5.4.1), where the defining limit exists pointwise μ_G-almost everywhere by Theorem A.5.4.1(2), as the *smoothening* of f.

Remark A.5.4.2. The set $\mathcal{L}_f(\{U_k\}_{k\geq 0}) := \mathrm{Reg}_{\{U_k\}_{k\geq 0},f} \cap D_f$, abbreviated $\mathcal{L}_f$, is a generalization of the Lebesgue set of a locally integrable complex-valued function on a finite-dimensional real Euclidean space. Each of the sets $\mathrm{Reg}_{\{U_k\}_{k\geq 0},f}$ and D_f is allowed to be strictly larger.[21]

Proof. (1) : Let $\varepsilon > 0$ be an arbitrary real number.

We analyze the following condition on $x, y \in G$:

$$\left| \int_{x_1U_k} f(y)\,\mathrm{d}\mu_G(y) - \int_{x_2U_k} f(y)\,\mathrm{d}\mu_G(y) \right| < \varepsilon. \tag{A.5.4.3}$$

We have, by invariance of the Haar measure μ_G:

$$\int_{x_iU_k} f(y)\,\mathrm{d}\mu_G(y) = \int_{U_k} f(x_iy)\,\mathrm{d}\mu_G(y) \tag{A.5.4.4}$$

for $i = 1, 2$.

Since μ_G is a Radon measure on G, the functional $\mu_G : L^1_{\mathrm{loc}}(G,\mu_G) \to \mathbf{C}$ is continuous with respect to the Hausdorff locally convex topology on $L^1_{\mathrm{loc}}(G,\mu_G)$ defined by the countable family of seminorms $\{\|\cdot\|_{L^1(K_m,\mu_G)}\}$ for $\{K_m\}_{m\geq 0}$ an exhaustion by compact subsets of G (with each K_m having non-empty interior).

To see this, it is enough to show that the functional $\mu_G : L^1(K_m,\mu_G) \to \mathbf{C}$ defined by $\mu_G(\phi) := \int_{K_m} \phi(x)\,\mathrm{d}\mu_G(x)$ is norm-continuous, which is indeed the case since we have:

$$|\mu_G(\phi_1 - \phi_2)| \leq \|\phi_1 - \phi_2\|_{L^1(K_m,\mu_G)}. \tag{A.5.4.5}$$

We note that $C_c(G) \subset L^1_{\mathrm{loc}}(G,\mu_G)$ is dense with respect to the resulting Hausdorff locally convex topology, since the image into each $L^1(K_m,\mu_G)$ is norm-dense and the topology on $L^1_{\mathrm{loc}}(G,\mu_G)$ is metrizable.

We now define $\rho_G : G \times L^1_{\mathrm{loc}}(G,\mu_G) \to L^1_{\mathrm{loc}}(G,\mu_G)$ the map $\rho_G(g,\phi)(x) := \phi(g^{-1}x)$ for μ_G-almost all $x \in G$. Equivalently, we may define ρ_G on $C_c(G)$ as the left-translation action by G on compactly supported continuous complex-valued functions on G, and extend the left-translation action by continuity to $L^1_{\mathrm{loc}}(G,\mu_G)$. Such action is indeed continuous, since, for each $g \in G$, we have:

$$\|\rho_G(g,\cdot)\|_{L^1(K_m,\mu_G)} \leq \|\cdot\|_{L^1(K_0,\mu_G)} + \ldots + \|\cdot\|_{L^1(K_{m_N},\mu_G)} \tag{A.5.4.6}$$

for finitely many integers $0, \ldots, m_N$ determined by K_{m_N} enclosing $g'K_m$ in its interior for all g' near g.

It follows that ρ_G is uniformly continuous (see [**Bou-GT**, Ch. II, §2, No. 1, Def. 1, p. 174]). The adjoint action ρ'_G on the strong dual $L^1_{\mathrm{loc}}(G,\mu_G)'_\beta$ is therefore also uniformly continuous, in light of (A.5.4.5).

(2) : We will be following the proof of Theorem A.5.1.3. It is enough to treat the case when f is real-valued.

Let $V \subset G$ be an arbitrary open neighborhood of $x \in G$ with compact closure. Since G is Hausdorff, locally compact and second-countable, by [**Fol**, Prop. 7.5] (a regularity property of μ in the σ-finite case, which applies since $\overline{V}$ is compact), a compact subset $K \subset V$ containing x and such that $\mu_G(V - K)$ is as small as we please.

Clearly there exists an open unit-neighborhood $U \subset G$ such that $U \cdot K \subset V$, and since $\{U_k\}_{k\geq 0}$ is a D'-sequence, there exists a large enough integer $k_0 \geq 0$ such that $U_k \subset U$ for every integer $k \geq k_0$ and each

[21] Should make clearer to the reader why this Remark is being recorded here.

U_k has compact closure. It follows that we have:

$$A_{U_k}(f)(x) = A_{U_k}(f \cdot 1_V)(x) \tag{A.5.4.7}$$

for every $k \geq k_0$.

We define the "local oscillation" of f at x:

$$\omega_f(x) := \omega_{\{U_k\}_{k\geq 0},f}(x) := \left| \limsup_{k\to+\infty} A_{U_k}(f)(x) - \liminf_{k\to+\infty} A_{U_k}(f)(x) \right|. \tag{A.5.4.8}$$

We note that we are assuming f is real-valued, and hence the right side in (A.5.4.8) makes sense. In light of (A.5.4.7), we have:

$$\omega_f(x) = \omega_{f\cdot 1_V}(x). \tag{A.5.4.9}$$

Since $f \cdot 1_V \in L^1(G, \mu_G)$, all $\mu_G(E_j)$ are finite and for any $\varepsilon > 0$ there exists $\varphi = \sum_{1\leq j\leq N} c_j 1_{E_j}$ for μ_G-measurable subsets E_j and $c_j \in \mathbf{R}$ such that $\|f \cdot 1_V - \varphi\|_{L^1(G,\mu_G)} < \varepsilon$. We define $F := f \cdot 1_V - \varphi$, and note that we have[22]

$$\omega_{f\cdot 1_V}(x) \leq \omega_F(x) \tag{A.5.4.10}$$

for almost all x so long as we show that we have $\omega_\varphi(x) = 0$ for almost all $x \in G$. We will be granting this for the moment.

We have:

$$\omega_{f\cdot 1_V}(x) \leq 2(MF)(x). \tag{A.5.4.11}$$

Now observe that by Theorem A.5.3.7(1) we have

$$\mu_G(\{y \in G : (MF)(y) > \alpha\}) \leq \frac{2C}{\alpha}\|F\|_{L^1(G,\mu_G)} < \frac{2C\varepsilon}{\alpha} \tag{A.5.4.12}$$

and from (A.5.4.11) we deduce that we have:

$$\mu_G(\{x \in G : \omega_{f\cdot 1_V}(x) > \alpha\}) \leq \frac{4C}{\alpha}\|F\|_{L^1(G,\mu_G)} < \frac{4C\varepsilon}{\alpha} \tag{A.5.4.13}$$

which implies (since $\varepsilon > 0$ is arbitrary) that $\mu_G(\{x \in G : \omega_{f\cdot 1_V}(x) > \alpha\}) = 0$. Since this holds for every $\alpha > 0$, we deduce that $\omega_{f\cdot 1_V}(x) = 0$ for μ_G-almost every $x \in G$.

This implies that the sequence $A_{U_k}(f)(x)$ converges, for μ_G-almost every $x \in V$, for V any open subset in G with $\overline{V}$ compact, and hence locally μ_G-almost everywhere. Thus, it converges μ_G-almost everywhere.

We may therefore assume $f = \varphi$; i.e., f is an integrable simple function whose coefficients are real. We want to show $\omega_f(x) = 0$ for all x. By subadditivity and the identity $\omega_{cf} = |c|\omega_f$, it is enough to treat the case $f = 1_E$ for $E \subset G$ a μ_G-measurable subset of finite μ_G-measure. This amounts to showing that the sequence of nonnegative real numbers:

$$\frac{\mu_G(xU_k \cap E)}{\mu_G(U_k)}$$

converges for all $x \in G$.

By outer regularity of the Radon measure μ_G (see [**Fol**, Prop. 7.5]), we may replace[23] E by an open subset $U \subset G$ with finite μ_G-measure.

Assume $x \in U$. There exists an open unit-neighborhood $V \subset G$ such that $xV \subset U$, and since $\{U_k\}_{k\geq 0}$ is a D'-sequence there exists a enough integer $k_0 \geq 0$ such that $U_k \subset V$ for all $k \geq k_0$ and U_k has compact closure. It follows that for all integers $k \geq k_0$, we have $xU_k \subset U$, and hence $A_{U_k}(f)(x) = 1$ for all $k \geq k_0$.

---

[22] Be more careful: *first* should choose V, then $x \in V$, then $K \subset V$ containing x, etc.

[23] Should say a bit more, not hard by comparing limsup and liminf

Assume now $x \notin \overline{U}$. Then there exists an open neighborhood U' of x in G entirely contained in the open complement $G - \overline{U}$, and upon repeating the above argument we deduce that there exists a large enough integer $k_0 \geq 0$ such that for all integers $k \geq k_0$, we have $xU_k \subset U'$, and xU_k has compact closure. It follows that we have $A_{U_k}(f)(x) = 0$ for all $k \geq k_0$.

We now assume x is in the boundary of U in G. By outer regularity of μ_G (since $\mu(U) < \infty$), there exists an open subset $U' \subset G$ containing U such that $\mu_G(U' - U)$ is as small as we please, which shows that we have $\mu_G(\partial U) = 0$. It follows that indeed the convergence assertion holds for μ_G-almost all x for such f.

(3) : To prove that (A.5.4.2) holds for μ_G-almost all $x \in G$, it is enough to assume[24] that f is compactly supported, hence in $L^p(G, \mu_G)$ for every $p \geq 1$. We write:

$$\frac{1}{\mu_G(U_k)} \int_{xU_k} f(y)\, \mathrm{d}\mu_G(y) = \frac{1}{\mu_G(U_k)} \int_G f(y) 1_{U_k}(yx^{-1})\, \mathrm{d}\mu_G(y)$$

and upon defining $u_k(y) := \frac{1}{\mu_G(U_k)} 1_{U_k}(y^{-1})$, we have $A_{U_k}(f) = f * u_k$, for each integer $k \geq 0$.

For any real number $q > 1$, $\|u_k\|_{L^q(G,\mu_G)} = 1$. By [**Fol**, Thm. 2.26] there exists a sequence $\{\varphi_m\}_{m \geq 0}$ of simple functions with complex coefficients such that $\|f - \varphi_m\|_{L^p(G,\mu_G)} \to 0$ as $m \to +\infty$. We have, for μ_G-almost all $x \in G$ and for $p > 1$ with $1/p + 1/q = 1$,[25] by [**HR-1**, Ch.V,§20,Thm.20.16,p.295] and Hölder's Inequality:

$$|(f * u_k)(x) - f(x) - (\varphi_m * u_k)(x) + \varphi_m(x)| \leq \|f - \varphi_m\|_{L^p(G,\mu_G)} \|u_k\|_{L^q(G,\mu_G)} + |\varphi_m(x) - f(x)|$$

and since $\varphi_m \to f$ in $L^p(G, \mu_G)$, there exists a subsequence converging μ_G-almost everywhere. Thus, the result for each φ_m yields the same for f.

We may therefore assume f is a simple function with complex coefficients, and by linearity reduce to the case $f = 1_E$, which was checked throughout the proof of (2). □

We discuss the following consequence of Theorem A.5.4.1:

COROLLARY A.5.4.3. *Let G be a Hausdorff locally compact abelian group. Let μ_G be a choice of Haar measure on G. Assume G is second countable. Let $\{U_k\}_{k \geq 0}$ be a D'-sequence for G. Let $f \in L^1_{\mathrm{loc}}(G, \mu_G)$. Let $\varphi \in C_c(G)$. Then the limit of the sequence:*

$$A_{U_k}(\varphi \cdot f)(x) := \frac{1}{\mu_G(U_k)} \int_{xU_k} (\varphi \cdot f)(y)\, \mathrm{d}\mu_G(y) \tag{A.5.4.14}$$

exists at all the points $x \in G$ at which the limit of the sequence $\{A_{U_k}(f)(x)\}_{k \geq 0}$ exists. Its value is

$$\lim_{k \to +\infty} A_{U_k}(\varphi \cdot f)(x) = \varphi(x) \cdot \lim_{k \to +\infty} A_{U_k}(f)(x). \tag{A.5.4.15}$$

PROOF. Since $\{U_k\}_{k \geq 0}$ is a D'-sequence, there exists an integer $k_0 \geq 0$, such that for every integer $k \geq k_0$, U_k has compact closure.

We consider the sequence:

(A.5.4.16)

$$|A_{U_k}(\varphi \cdot f)(x) - \varphi(x) \cdot A_{U_k}(f)(x)| \leq \frac{1}{\mu_G(U_k)} \int_{xU_k} |(\varphi(y) - \varphi(x)) f(y)|\, \mathrm{d}\mu_G(y) \leq \sup_{y \in xU_k} |\varphi(y) - \varphi(x)| \cdot \|f\|_{L^1(xU_k, \mu_G)}$$

for all $k \geq k_0$.

Since $xU_k \subset xU_{k_0}$ for every integer $k \geq k_0$, we deduce the inequality:

$$|A_{U_k}(\varphi \cdot f)(x) - \varphi(x) \cdot A_{U_k}(f)(x)| \leq \sup_{y \in xU_k} |\varphi(y) - \varphi(x)| \cdot \|f\|_{L^1(xU_{k_0}, \mu_G)} \tag{A.5.4.17}$$

for all $k \geq k_0$.

[24] Should briefly explain, using $\mu_G(\partial U)$ for open U with $\overline{U}$ compact

[25] Why not just use $p = q = 2$?

Since φ is continuous, for every real number $\varepsilon > 0$ there exists an open neighborhood V of $x \in G$ such that $|\varphi(y) - \varphi(x)| < \varepsilon$ for all $y \in V$. There exists an open unit-neighborhood $V' \subset G$ such that $xV' \subset V$, and there exists an integer $k_1 \geq k_0$ such that $U_k \subset V'$ for all integers $k \geq k_1$.

In particular, $\sup_{y \in xU_k} |\varphi(y) - \varphi(x)| < \varepsilon$ for all $k \geq k_1$. In light of (A.5.4.17), the sequence

$$\{A_{U_k}(\varphi \cdot f)(x)\}_{k \geq 0}$$

converges if and only if $\{A_{U_k}(f)(x)\}_{k \geq 0}$ does. Its limit at such x is therefore $\varphi(x) \cdot \lim_{k \to +\infty} A_{U_k}(f)(x)$, by letting k tend to $+\infty$ within the left side of (A.5.4.17). □

APPENDIX B

Preparations from Analytic Number Theory

In this Appendix we collect some results from analytic number theory and cover them in the generality of all Hecke L-functions of an arbitrary number field.

We review classical facts on the growth of "finite" Hecke L-functions (i.e., without Γ-factors) over vertical strips, classical zero-free regions[1] for Hecke L-functions, and convergence of partial Euler products over the 1-line (i.e. the vertical line $\{\mathrm{Re}(s) = 1\}$). We also discuss Weil's own idelic formulation of the Explicit Formulas, incorporating Connes' observations on the local Fourier transform in relation to Weil's principal values.

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[1] perhaps omit this mention if in the end the discussion of zero-free regions is removed if not needed

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B.1. Convexity bound on $L_f(s,\chi)$

In this section we record classical facts about growth bounds for Hecke L-functions and their derivatives.

B.1.1. Growth order of $L_f(s,\chi)$ over the critical strip. Recall that $L_f(s,\chi)$ is defined as

$$L_f(s,\chi) := \prod_{\mathfrak{p}\subset \mathcal{O}_F} (1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}$$

for $\mathrm{Re}(s)>1$, where the product is indexed by the prime ideals of $\mathcal{O}_F$ (noting that $\chi(\mathfrak{p}):=0$ if χ is ramified at $v_{\mathfrak{p}}$). and is absolutely convergent for $\mathrm{Re}(s)>1$, uniformly on $\mathrm{Re}(s)\geq 1+\varepsilon$ for each $\varepsilon>0$.

The function $L_f(s,\chi)$ has a Dirichlet series expansion

$$L_f(s,\chi) = \sum_{0\neq I\subset \mathcal{O}_F} \frac{\chi(I)}{N(I)^s}$$

which is absolutely convergent for $\mathrm{Re}(s)>1$, uniformly on $\mathrm{Re}(s)\geq 1+\varepsilon$ for each $\varepsilon>0$, where $\chi(I):=0$ if I is not coprime to the conductor of χ.

B.1.1.1. *Phragmen-Lindelöf and convexity bound on $L_f(\sigma+it,\chi)$.* In this subsection we record the arguments to establish the classical polynomial growth bound on $L_f(1/2+it,\chi)$ for arbitrary continuous

characters $\chi \in \widehat{C_F}$ (i.e. $L(1/2+it,\chi) = O_\varepsilon(|t|^{1/4+\varepsilon})$ for every $\varepsilon > 0$), the bound

$$L_f(1+it,\chi) = O_\varepsilon(|t|^\varepsilon)$$

for every $\varepsilon > 0$, and deduce the classical convexity bound $L_f(s,\chi)$ for $s \in \mathbf{C}_{[1/2,1]}$ from the Phragmen-Lindelöf Theorem.

The results in this subsection can be sharpened and have been so in the existing literature (see, for instance, [**MVe**, Thm. 5.1] for $\pi = 1$). We will be requiring them only in the form and strength discussed below, which is classical and obtained by following the methods in [**T2**, Ch. V, §5.1].

We recall from [**T1**, Ch. 9, §9.41] that for $f(s)$ a meromorphic function on a half-plane $\mathbf{C}_{>a}$ whose poles, if any, are all contained within a horizontal vertically bounded strip within the complex s-plane, if f is of finite order within $\mathbf{C}_{>a}$ (see [**T1**, Ch. 9, §9.4]) then we may define the function $\mu(\sigma)$ for $\sigma > a$ as the infimum of the real numbers $\xi \in \mathbf{R}_{\geq 0}$ such that

$$f(\sigma+it) = O_\sigma(|t|^\xi)$$

as $|t| \to \infty$. The function μ is continuous and convex downwards as a consequence of the Phragmen-Lindelöf Theorem, as we'll soon review. Its value is zero if f is bounded on $\mathrm{Re}(s) = \sigma$.

In particular, if f is an absolutely convergent Dirichlet series over $\mathbf{C}_{>b}$ with $b > a$ then $\mu(\sigma) = 0$ for every $\sigma > b$, and hence $\mu(b+\varepsilon) = 0$ for every $\varepsilon > 0$ since any Dirichlet series which is absolutely convergent is also bounded over closed half-planes of absolute convergence. It follows that we have $\mu(b) = 0$ by continuity. We refer the reader to [**T1**, Ch. 9, §9.41], where the continuity and downward convexity properties of $\mu(\sigma)$ are deduced from the Phragmen-Lindelöf Theorem. Since in [**T1**, Ch. 9, §9.41] such properties are only proved in the case f is holomorphic over $\mathbf{C}_{>0}$, we include a discussion on the function μ and its properties below.

LEMMA B.1.1.1. *Let $a, b \in \mathbf{R}$ be real numbers, $a < b$, $f : \mathbf{C}_{>a} \to \mathbf{C}$ a meromorphic function. Assume:*

(1) *The poles of f are simple and finitely many.*
(2) *There exists a real number $\xi > 0$ such that*

$$f(\sigma+it) \ll |t|^\xi \tag{B.1.1.1}$$

as $|t| \to +\infty$, uniformly with respect to $\sigma > a$.

Then for every $\sigma_1 < \sigma_2 \in (a, +\infty)$ the function

$$\mu(\sigma) := \inf\{k \in \mathbf{R}_{\geq 0} : f(\sigma+it) \ll |t|^k \text{ as } |t| \to \infty\} < \infty \tag{B.1.1.2}$$

satisfies

$$\mu(\sigma) \leq t\mu(\sigma_2) + (1-t)\mu(\sigma_1) \tag{B.1.1.3}$$

for every $t \in [0,1]$ and $\sigma := t\sigma_2 + (1-t)\sigma_1 \in [\sigma_1, \sigma_2]$. Moreover, μ is continuous as a function $(a, +\infty) \to \mathbf{R}_{\geq 0}$ and for all $\varepsilon > 0$ we have $f(\sigma+it) = O_{\varepsilon,\sigma_1,\sigma_2}(|t|^{\mu(\sigma)+\varepsilon})$ as $|t| \to \infty$ uniformly in $\sigma \in [\sigma_1, \sigma_2]$.

PROOF. Let $s_1, \ldots, s_N$ be the poles of f. We consider the function

$$g(s) := \prod_{1 \leq k \leq N} (s - s_k) f(s) \tag{B.1.1.4}$$

that is a holomorphic function over $\mathbf{C}_{>a}$ by assumption (1). We call μ_g the μ-function of g, which is defined (and finite) by assumption (2).

We have $\mu_g(\sigma) = \mu(\sigma) + N$ for all $\sigma > a$. Indeed, we have $\mu_g(\sigma) \leq \mu(\sigma) + N$ by definition, and since $\prod_{1 \leq k \leq N}(s - s_k)$ is nowhere vanishing for all ordinates of large enough absolute value, and bounded above uniformly with respect to all such ordinates by a nonzero positive real number, the reverse inequality also holds.

Let $\varepsilon > 0$, and choose arbitrary $\sigma_1, \sigma_2 > a$ with $\sigma_1 < \sigma_2$. Then as $|t| \to \infty$,

$$|g(\sigma_1 + it)| \ll |t|^{N+\mu(\sigma_1)+\varepsilon}, \quad |g(\sigma_2 + it)| \ll |t|^{N+\mu(\sigma_2)+\varepsilon}.$$

By [**T1**, Ch. V, §5.65, pp. 180-1] (which applies since (B.1.1.1) implies that for all $\varepsilon > 0$, $f(\sigma + it) = O(e^{\varepsilon|t|})$ as $|t| \to \infty$ uniformly $\sigma > a$), we deduce

$$|g(\sigma + it)| \ll |t|^{k_\varepsilon(\sigma)}$$

uniformly for all $\sigma \in [\sigma_1, \sigma_2]$ as $|t| \to \infty$, where

$$k_\varepsilon(\sigma) := \frac{(\sigma_2 - \sigma)(\mu_g(\sigma_1) + \varepsilon) + (\sigma - \sigma_1)(\mu_g(\sigma_2) + \varepsilon)}{\sigma_2 - \sigma_1} = \frac{(\sigma_2 - \sigma)(\mu(\sigma_1) + \varepsilon) + (\sigma - \sigma_1)(\mu(\sigma_2) + \varepsilon)}{\sigma_2 - \sigma_1} + N + \varepsilon$$

from which we deduce $\mu_g(\sigma) \leq k_\varepsilon(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_2]$. Since $\varepsilon > 0$ is arbitrary, we may let $\varepsilon \to 0^+$ to obtain:

$$\mu(\sigma) + N = \mu_g(\sigma) \leq \lim_{\varepsilon \to 0^+} k_\varepsilon(\sigma) = \frac{(\sigma_2 - \sigma)\mu(\sigma_1) + (\sigma - \sigma_1)\mu(\sigma_2)}{\sigma_2 - \sigma_1} + N \tag{B.1.1.5}$$

which gives

$$\mu(\sigma) \leq \frac{(\sigma_2 - \sigma)\mu(\sigma_1) + (\sigma - \sigma_1)\mu(\sigma_2)}{\sigma_2 - \sigma_1}. \tag{B.1.1.6}$$

Upon calling $t := (\sigma - \sigma_1)/(\sigma_2 - \sigma_1)$ in (B.1.1.6), we see that (B.1.1.3) holds. Continuity of μ follows from (B.1.1.3) by [**T1**, Ch. V, §5.31]. □

The following Lemma is [**IK**, Ch. 5, §5.1, Lemma 5.2, p. 97].

LEMMA B.1.1.2. *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. The finite Hecke L-function $L_f(s, \chi)$ is $O(|t|^{M_\chi})$ for some $M_\chi > 0$ as $|t| \to \infty$ uniformly in $\sigma \in [a, b]$ for any fixed $a < b$.*

In light of Lemma B.1.1.2, the function μ is defined for every Hecke L-function. Moreover, since each $L_f(s, \chi)$ is defined by an absolutely convergent Dirichlet series over $\mathbf{C}_{>1}$ uniformly over $\mathrm{Re}(s) \geq 1 + \varepsilon$, it satisfies $\mu(\sigma) = 0$ for every $\sigma > 1$ and hence $\mu(1) = 0$ by continuity (see Lemma B.1.1.1 with $a < 1$).

Recall that the function $\gamma(\chi, s)$ defined in (3.2.1.3) is a product of Γ-functions of the form

$$L(|\cdot|_v^s) := \pi^{-s/2}\Gamma(s/2), \; L(\mathrm{sgn}|\cdot|_v^s) = \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right) \tag{B.1.1.7}$$

for v real and $\chi_v = 1$, respectively $\chi_v = \mathrm{sgn}$, and

$$L(\chi_v|\cdot|_v^s) := (2\pi)^{-\left(s+\frac{|n|}{2}\right)}\Gamma\left(s + \frac{|n|}{2}\right) \tag{B.1.1.8}$$

for v complex and $\chi_v|\cdot|_v^s = \chi_{n,s}$, the continuous quasi-character character of $\mathbf{C}^\times$ defined as $(r, \theta) \mapsto r^s e^{in\theta}$. On $\mathbf{C}_{>0}$ the function $\gamma(\chi, s)$ is holomorphic and nowhere vanishing.

In light of Proposition 3.2.1.3 and (3.2.1.8)-(3.2.1.10), the finite Hecke L-function $L_f(s, \chi)$ extends to a meromorphic function on the complex s-plane and satisfies the functional equation

$$L_f(s, \chi) = W(s, \chi)L_f(1 - s, \chi^{-1}). \tag{B.1.1.9}$$

where for $a, b \in \mathbf{R}$ with $a < b$ fixed arbitrary real numbers we have

$$|W(s, \chi)| \ll_{a,b} |t|^{[F:\mathbf{Q}]\left(\frac{1}{2}-\sigma\right)} \tag{B.1.1.10}$$

uniformly as $s = \sigma + it$ satisfies $|t| \to +\infty$ with $\sigma \in [a, b]$.

Up to a constant multiplier $W(\chi)$ of absolute value 1 (see [**Neu**, Ch. VII, Thm. 8.5]), the function $W(s, \chi)$ is equal to

$$|d_F|^{\frac{1}{2}-s} \frac{N(\mathfrak{f}_{\overline{\chi}})^{\frac{1-s}{2}}}{N(\mathfrak{f}_\chi)^{\frac{s}{2}}} \frac{\gamma(\overline{\chi}, 1 - s)}{\gamma(\chi, s)} \tag{B.1.1.11}$$

that is entire and with no zeros (see [**Neu**, Ch. VII, Prop. 4.4]). The estimate in (B.1.1.10) is obtained from the product structure of $W(s,\chi)$.

PROPOSITION B.1.1.3 (Convexity Bound). *Let F be a number field, χ a continuous character on C_F. For every $\varepsilon > 0$, upon writing $\sigma := \mathrm{Re}(s)$, we have uniformly in $\sigma \in [0,1]$:*[2]

$$L_f(s,\chi) \ll |t|^{[F:\mathbf{Q}]\frac{1-\sigma}{2}+\varepsilon} \tag{B.1.1.12}$$

as $|t| \to +\infty$, where the implied constant depends on χ and on F but not on ε nor σ.

PROOF. For $\sigma := \mathrm{Re}(s)$, we have, for $\sigma < 0$, $L_f(1-s,\chi^{-1})$ is given by an absolutely and locally uniformly convergent Dirichlet series, by the very design of the finite Hecke L-function from (3.2.1.7). In particular, $L_f(1-s,\chi^{-1})$ is bounded for $\sigma \leq -\delta$ for any real number $\delta > 0$. By the functional equation (see (B.1.1.9)) and Stirling's Approximation, giving (B.1.1.10), we have

$$|L_f(s,\chi)| \sim |W(s,\chi)| \ll |t|^{[F:\mathbf{Q}]\left(\frac{1}{2}-\sigma\right)}$$

as $|t| \to +\infty$ while s remains within a fixed and horizontally bounded vertical strip.

It follows that we have

$$\mu(\sigma) \leq [F:\mathbf{Q}]\left(\frac{1}{2}-\sigma\right) \tag{B.1.1.13}$$

for $\sigma < 0$.

For any real number $\delta > 0$, $L_f(s,\chi)$ is bounded for $\sigma \geq 1+\delta$, and hence $\mu(\sigma) = 0$ for $\sigma > 1$. By downward convexity of μ (see Lemma B.1.1.1), we deduce

$$\mu(\sigma) \leq (1-\sigma)\mu(0) + \sigma\mu(1) = [F:\mathbf{Q}]\left(\frac{1-\sigma}{2}\right). \tag{B.1.1.14}$$

We deduce via the uniform growth bound at the end of Lemma B.1.1.1 that for every $\varepsilon > 0$ we have that (B.1.1.12) holds, with an implied constant that depends only on χ and F. □

REMARK B.1.1.4 (Polynomial growth of the holomorphic derivatives). We note that from the functional equation

$$L_f(s,\chi) = W(s,\chi)L_f(1-s,\chi^{-1})$$

from (B.1.1.9), the holomorphic derivative $\partial_s^k L_f(s,\chi)$ of order $k \geq 0$ also grows polynomially on all vertical lines in $\mathbf{C}$.[3]

Indeed, we have:

$$\partial_s^k L_f(s,\chi) = \sum_{0\leq m\leq k}\binom{k}{m}(\partial_s^m W(s,\chi))(\partial_s^{k-m}L_f(1-s,\chi^{-1})) \tag{B.1.1.15}$$

as meromorphic functions on $\mathbf{C}$. For $s \in \mathbf{C}_{>1}$, $\partial_s^k L_f(s,\chi)$ is an absolutely convergent Dirichlet series, and as such it is bounded on vertical lines within the half-plane $\mathbf{C}_{>1}$. For $s \in \mathbf{C}_{<0}$, we use (B.1.1.15) and boundedness on vertical lines of each $\partial_s^{k-m}L_f(1-s,\chi^{-1})$ to reduce to showing that $\partial_s^m W(s,\chi)$ grows polynomially over the complex s-plane[4] for all integers $m \geq 0$. This is indeed the case by Proposition 3.3.1.5 applied to $S = S_\infty$.

B.2. Asymptotic Cauchy Integral Formula

In this section we record arguments enabling the computation of the values of certain line integrals by contour integration, when the meromorphic functions involved are products of a slowly increasing meromorphic

[2] Need to treat $\sigma \in [-1,2]$ for use in Theorem B.6.1.7.

[3] Need to address uniform control with $\sigma \in [a,b]$ if needed elsewhere.

[4] Is this really supposed to be about vertical strips of bounded width? With uniformity in σ?

function and the Mellin transform of a Schwartz function supported over $\mathbf{R}_{>0}$ and which decays faster than any positive power of $e^{-|\log(t)|}$ as $|\log(t)| \to +\infty$.

B.2.1. Complements on the Mellin Transform. Recall the definition of the Hardy space $H^p(\mathbf{C}_{<a})$ of the half-plane $\mathbf{C}_{<a} = \{s \in \mathbf{C} : \mathrm{Re}(s) < a\}$, for each real number $p > 0$: the space of holomorphic functions $\Phi(s)$ on $\mathbf{C}_{<a}$ such that

$$\|\Phi\|_{H^p(\mathbf{C}_{<a})} := \sup_{\sigma<a} \|\Phi(\sigma + iy)\|_{L^p(\mathbf{R},\mathrm{d}y)} < \infty.$$

The assignment $\Phi \mapsto \|\Phi\|_{H^p(\mathbf{C}_{<a})}$ makes $H^p(\mathbf{C}_{<a})$ a complex Banach space for $p \geq 1$ (as can be deduced[5] from [Nik, Thm. 2.2.2(2)]), and in the case $p = 2$ to a complex Hilbert space.

From [Dur, Cor. to Thm. 11.1], we note that for all $p > 0$, the boundary function

$$f_\Phi(y) := \lim_{\sigma \to a^-} \Phi(\sigma + iy)$$

for each $\Phi \in H^p(\mathbf{C}_{<a})$ exists for almost all $y \in \mathbf{R}$, and it is an element of $L^p(\mathbf{R}, \mathrm{d}y)$.

By [Dur, Cor. to Thm. 11.2, p. 190] we further have that the assignment $\Phi \mapsto f_\Phi$ is an isometric embedding of complex Banach spaces for $p \geq 1$:

$$H^p(\mathbf{C}_{<a}) \to L^p(\mathbf{R}, \mathrm{d}y)$$

since it is norm-preserving in this p-range. It is not surjective, and its image can be described (see[6] [Nik, Thm. 2.2.2(2)]).

It is a consequence of the defining conditions on the holomorphic functions on $\mathbf{C}_{<a}$ in the class $H^p(\mathbf{C}_{<a})$, for $p \geq 1$, that the following formulas hold: for every $a' \leq a$ and every $z \in \mathbf{C}$ with $\mathrm{Re}(z) \neq a'$, we have

$$\frac{1}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} \frac{\Phi(s)}{s-z}\, \mathrm{d}s = \Phi(z)$$

if $\mathrm{Re}(z) < a'$ and:

$$\frac{1}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} \frac{\Phi(s)}{s-z}\, \mathrm{d}s = 0$$

if $\mathrm{Re}(z) > a'$. This is [Dur, Ch.11,Thm.11.8].

In this section we provide formulas similar to those above, when the holomorphic function $\Phi(s)$ is not in the class $H^p(\mathbf{C}_{<a})$, but rather it satisfies certain decay properties towards infinity uniformly only on vertical strips[7] and not necessarily over the half-plane $\mathbf{C}_{<a}$. Such analogs will involve integrating over the vertical lines bounding a fixed vertical strips.

We will also be needing an asymptotic version of Cauchy's Residue Theorem, valid under the same assumptions on $\Phi(s)$ as above. The holomorphic functions $\Phi(s)$ we will be considering are Mellin transforms of Schwartz functions on the real line whose support is contained within $\mathbf{R}_{>0}$, which satisfy decay conditions towards both ∞ and 0.

B.2.1.1. *Schwartz functions with exponential and semi-exponential decay.* We recall that a function $\phi : \mathbf{R} \to \mathbf{C}$ is *Schwartz* if and only if $\phi \in C^\infty(\mathbf{R})$ and

$$\|\phi\|_{m,N} := \sup_{x \in \mathbf{R}} (1 + |x|^N)\, |\partial_x^m \phi(x)| < \infty$$

for all integers $N, m \geq 0$.

The space of all Schwartz functions on $\mathbf{R}$ is denoted by $\mathcal{S}(\mathbf{R})$, topologized by the family of seminorm $\{\|\cdot\|_{m,N}\}_{m,N\geq 0}$. The locally convex space $\mathcal{S}(\mathbf{R}_{>0})$ of Schwartz-Bruhat functions on the real abelian Lie group

[5] How? Give clearer reference. Maybe discuss passing from $\mathbf{C}_{<a}$ to $\mathbf{D}$ with $\sigma = y$ going over to concentric circles.
[6] Need to explain how reference applies, including passing from $\mathbf{C}_{<a}$ to $\mathbf{D}$
[7] Do you mean on $\{\sigma \leq a - \varepsilon\}$? Clarify if issue is being bounded away from $\sigma = a$.

$\mathbf{R}_{>0}$ is isomorphic to $\mathcal{S}(\mathbf{R})$ under the exponential map $\exp : \mathbf{R} \to \mathbf{R}_{>0}$, with the topology generated by the family of seminorms resulting from the exponential change of variables. Namely, for $\phi \in C^\infty(\mathbf{R}_{>0})$ the rapid decay condition is defined by

$$\|\phi\|_{m,N} := \sup_{t \in \mathbf{R}_{>0}} (1 + |\log(t)|^N) \, |(t\partial_t)^m \phi(t)| < \infty$$

for all $m, N \geq 0$.

We now define a subspace of $\mathcal{S}(\mathbf{R}_{>0})$ comprised of those functions that decay very rapidly towards both ∞ and 0.

Definition B.2.1.1 (Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ with (semi-)exponential decay). Let $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function. Say ϕ has:

(1) *forward semi-exponential decay* if ϕ and its derivatives decay faster than any positive power of t^{-1} as $t \to +\infty$. We denote by $\mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$ the space of ϕ's with foward semi-exponential decay.

Likewise say ϕ has *backward semi-exponential decay* if ϕ and its derivatives decay faster than any positive power of t as $t \to 0^+$. We denote by $\check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$ the space of ϕ's with backward semi-exponential decay.

(2) *exponential decay* if ϕ and its derivatives decay faster than any positive power $e^{-|\log(t)|}$, as $|\log(t)| \to +\infty$. We denote by $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ the $\mathbf{C}$-linear space of Schwartz-Bruhat functions of exponential decay.

Remark B.2.1.2. The role of applying ∂_t in the preceding definition can be replaced by applying $t\partial_t$; the latter has the merit of being invariant and negating under inversion on t. It is elementary to check that membership in $\mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$ is equivalent to the requirement $t^a\phi \in \mathcal{S}(\mathbf{R}_{>0})$ for all $a \geq 0$, and via the exponential isomorphism $\mathbf{R} \simeq \mathbf{R}_{>0}$ this corresponds to functions $f \in \mathcal{S}(\mathbf{R})$ such that $f^{(j)} \in O(e^{-x})$ as $x \to +\infty$ for all $j \geq 0$.

We will often abbreviate "forward exponential decay" into "exponential decay", when no confusion is expected to arise.[8]

Remark B.2.1.3. The notion of "polynomial growth" over $\mathbf{R}_{>0}$ is defined via the usual concept for functions on the real line upon performing the exponential change of variables $t = e^x$: a function $f : \mathbf{R}_{>0} \to \mathbf{C}$ has *at most polynomial growth over* $\mathbf{R}_{>0}$ if for every integer $N \geq 0$ there exists a real number $C_N > 0$ such that $|f(t)| \leq C_N |\log(t)|^N$.

Lemma B.2.1.4. *Let $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function. We define $\check{\phi}$ the function on $\mathbf{R}_{>0}$ assigned by:*

$$\check{\phi}(t) := \phi(1/t) \tag{B.2.1.1}$$

for $t \in \mathbf{R}_{>0}$ (so $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R})$ if and only if $\check{\phi} \in \mathcal{S}_{\mathrm{E}}(\mathbf{R})$),

(1) *If $\phi \in \check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$ then the extension of ϕ to $\mathbf{R}$ by zero for $t \in (-\infty, 0]$ is Schwartz over $\mathbf{R}$.*
(2) *The involution $\phi \mapsto \check{\phi}$ is a linear topological automorphism of $\mathcal{S}(\mathbf{R}_{>0})$ respecting the linear subspace $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, and identifying $\mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$ with $\check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$.*
(3) *The extension by zero of any function in $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ to the real line is Schwartz.*
(4) *The operator $t\partial_t$ on $\mathcal{S}(\mathbf{R}_{>0})$ preserves $\mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$, $\check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$, and $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$.*

Proof. (1) : We need to check smoothness at $t = 0$ of $\partial_t^N \phi$ for every integer $N \geq 0$. We know that $\partial_t^N \phi$ exists and is smooth over $\mathbf{R}^\times$. It is therefore enough to check that it is continuous at $t = 0$: that says $\partial_t^N \phi \to 0$ as $t \to 0^+$ for each N. For $t = e^x$ we have $\partial_t = e^{-x}\partial_x$, so we are given that $f(x) := \phi(e^x) \in \mathcal{S}(\mathbf{R})$ and its derivatives all have decay more rapid than $e^{-M|x|}$ as $x \to -\infty$ for each $M > 0$ and we want to show

[8] Can such abuse of terminology be avoided, as it will cause confusion, and then remove this sentence?

$(e^{-x}\partial_x)^N f \to 0$ as $x \to -\infty$. Via the product rule, $(e^{-x}\partial_x)^N = e^{-Nx}(\sum_{j=1}^N a_{j,N}\partial_x^i j)$ for some constants $a_{j,N}$, the desired decay as $x \to -\infty$ clearly holds.

(2) and (3) follow from the definitions and from (1).

(4) : If $\phi \in \mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$ then for any real number $A > 0$, we have $t^A\phi \in \mathcal{S}(\mathbf{R}_{>0})$. It follows that $(t\partial_t)(t^A\phi) \in \mathcal{S}(\mathbf{R}_{>0})$. But $(t\partial_t)(t^A\phi) = At^A\phi + t^A(t\partial_t)\phi \in \mathcal{S}(\mathbf{R}_{>0})$, which gives $t^A(t\partial_t)\phi \in \mathcal{S}(\mathbf{R}_{>0})$. It follows that $(t\partial_t)\phi$ has semi-exponential decay. The same assertion holds for backward semi-exponential decay (by passing to $1/t$) and exponential decay in place of semi-exponential decay. □

Let $\phi : \mathbf{R}_{>0} \to \mathbf{C}$ be a measurable function, and $s \in \mathbf{C}$. We define the Mellin transform $M(\phi)$ by

$$M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t)t^s \frac{\mathrm{d}t}{t} \tag{B.2.1.2}$$

for $s \in \mathbf{C}$ when the integrand is absolutely integrable over $\mathbf{R}_{>0}$.

We will be discussing the regularity properties of $M(\phi)(s)$ when ϕ is a Schwartz function on the real line with semi-exponential decay, respectively exponential decay.

PROPOSITION B.2.1.5. *Let ϕ be a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ with forward semi-exponential decay. Let a, b be real numbers with $0 < a < b$.*

The Mellin transform $M(\phi)$ defined by

$$\Phi(s) := \int_{\mathbf{R}_{>0}} \phi(t)t^s \frac{\mathrm{d}t}{t} \tag{B.2.1.3}$$

is absolutely convergent for all $s \in \mathbf{C}_{>0}$ and:

(1) *Φ is holomorphic over the half-plane $\mathbf{C}_{>0}$.*

(2) *For every integer $N \geq 1$, there exists a real number $C_{N,a,b}$ depending only on N, a, b, ϕ (but not y) such that for every $a \leq \sigma \leq b$ and every $y \neq 0$, we have:*

$$|\Phi(\sigma + iy)| \leq C_{N,a,b}|y|^{-N}. \tag{B.2.1.4}$$

In particular, as a function of $y \in \mathbf{R}$ and uniformly in $\sigma \in [a, b]$ the value $\Phi(\sigma + iy)$ decays faster than any positive power of $|y|^{-1}$ as $|y| \to +\infty$.

(3) *For every real number $p \geq 1$, for every integer $N > 1$, there exists a real number $B_{N,a,b} > 0$ depending also on ϕ such that for every $a \leq \sigma \leq b$, we have the inequality:*

$$\|\Phi(\sigma + iy)\|_{L^p(\mathbf{R},\mathrm{d}y)} \leq \left\{ B_{N,a,b}^p + C_{N,a,b}^p \int_{|y|\geq 1/2} \frac{1}{|y|^{pN}}\,\mathrm{d}y \right\}^{1/p}. \tag{B.2.1.5}$$

To prove this, we need the following:

LEMMA B.2.1.6. *Let $\phi \in C^\infty(\mathbf{R}_{>0})$ be a smooth function. For every integer $N \geq 1$, we have:*

$$((t\partial_t)^N\phi)(e^x) = \partial_x^N(\phi(e^x)) \tag{B.2.1.6}$$

as smooth functions of $x \in \mathbf{R}$.

PROOF. The identity in (B.2.1.6) for $N = 1$ is the Chain Rule, and for $N > 1$ it holds by induction. □

PROOF OF PROPOSITION B.2.1.5. The absolute integrability for $M(\phi)(s)$ with all $s \in \mathbf{C}_{>0}$ is clear near $t = 0$ since $\mathrm{Re}(s) > 0$, and near $t = \infty$ it follows from the forward semi-exponential decay hypothesis on ϕ.

(1) : For holomorphicity on $\mathbf{C}_{>0}$ we want to use differentiation under the integral sign and so must analyze integrability near 0 and near ∞ after applying $\mathrm{d}/\mathrm{d}s$ to the integrand. For $\int_0^1$ the s-differentiated integrand is absolutely integrable since $\phi(t)$ decays faster than $|\log(t)|^{-m}$ for all $m \geq 0$ as $t \to 0^+$. For $\int_1^\infty$, the forward semi-exponential decay hypothesis on ϕ does the job (even allowing all $s \in \mathbf{C}$, not just $s \in \mathbf{C}_{>0}$).

(2) : We have, by (B.2.1.6) from Lemma B.2.1.6, for every integer $N \geq 1$:

$$((t\partial_t)^N \phi)(e^x) = \partial_x^N(\phi(e^x))$$

as smooth functions of $x \in \mathbf{R}$.

For every integer $N \geq 1$, every $\sigma > 0$, and every $y \neq 0$, we have:

$$\Phi(\sigma + iy) = \int_{\mathbf{R}} \phi(e^x) e^{(\sigma+iy)x}\, \mathrm{d}x = \frac{(-1)^N}{(\sigma+iy)^N} \int_{-\infty}^{+\infty} \partial_x^N(\phi(e^x)) e^{(\sigma+iy)x}\, \mathrm{d}x. \tag{B.2.1.7}$$

To see this, we observe that (B.2.1.7) holds for $N = 1$ by integration by parts, noting that the integral converges since $\partial_x(\phi(e^x)) = (t\partial_t\phi)(e^x)$ is Schwartz on the real line and decays faster than any positive power of e^{-x} as $x \to +\infty$ by the forward semi-exponential decay hypothesis. Then one inducts on $N \geq 1$.

For all $0 < a \leq \sigma \leq b$ we deduce

$$|\Phi(\sigma + iy)| \leq \frac{1}{(\sigma^2 + y^2)^{N/2}} \int_{-\infty}^{+\infty} |\partial_x^N(\phi(e^x))| e^{\sigma x}\, \mathrm{d}x \tag{B.2.1.8}$$

and upon defining

$$C_{N,a,b} := \int_0^{+\infty} |\partial_x^N(\phi(e^x))| e^{bx}\, \mathrm{d}x + \int_{-\infty}^{0} |\partial_x^N(\phi(e^x))| e^{ax}\, \mathrm{d}x \tag{B.2.1.9}$$

(finite since $\partial_x^N(\phi(e^x))$ decays faster than all e^{-mx} as $x \to \infty$ (with $m \geq 0$) and is bounded as $x \to -\infty$) that for all $y \neq 0$ and all $0 < a \leq \sigma \leq b$ we have

$$|\Phi(\sigma + iy)| \leq C_{N,a,b} |y|^{-N}. \tag{B.2.1.10}$$

(3) : We have, for every $y \in \mathbf{R}$ (allowing $y = 0$) and every $a \leq \sigma \leq b$:

$$|\Phi(\sigma + iy)| \leq \|\phi(e^x) e^{bx}\|_{L^1([0,+\infty),\mathrm{d}x)} + \|\phi(e^x) e^{ax}\|_{L^1((-\infty,0],\mathrm{d}x)} \tag{B.2.1.11}$$

and hence, upon calling $B_{N,a,b}$ the right side of (B.2.1.11), we deduce from (B.2.1.11) and (B.2.1.10) the inequality

$$|\Phi(\sigma + iy)| \leq B_{N,a,b} \cdot 1_{|y|<1/2}(y) + \frac{C_{N,a,b}}{|y|^N} \cdot 1_{|y|\geq 1/2}(y) \tag{B.2.1.12}$$

for all $a \leq \sigma \leq b$. We deduce from (B.2.1.12) that for every $p \geq 1$, upon choosing $N > 1$:

$$\|\Phi(\sigma + iy)\|_{L^p(\mathbf{R},\mathrm{d}y)} \leq \left\{ B_{N,a,b}^p + C_{N,a,b}^p \int_{|y|\geq 1/2} \frac{1}{|y|^{pN}}\, \mathrm{d}y \right\}^{1/p} \tag{B.2.1.13}$$

for all $a \leq \sigma \leq b$. □

B.2.1.2. *Asymptotic Expansions and Mellin Inversion.* The following result encodes meromorphic continuation of $M(\phi)$ to a wider strip in terms of asymptotic expansion conditions on ϕ. [9]

THEOREM B.2.1.7 (Asymptotic Expansions and Mellin Inversion - I). *Pick $a \in \mathbf{R}_{>0}$ and $b \in \mathbf{R}_{>0} \cup \{+\infty\}$ with $a < b$. Pick $\phi \in C^\infty(\mathbf{R}_{>0})$ such that $t^\sigma \cdot \phi \in \mathcal{S}(\mathbf{R}_{>0})$ for every $\sigma \in (a,b)$ (so the integral $M(\phi)(s)$ is absolutely convergent and holomorphic in s on $\mathbf{C}_{(a,b)}$).*

Let $\Sigma \subset \mathbf{C}$ be a finite set of points with an associated finite multiplicity $m_\alpha \geq 1$ for each $\alpha \in \Sigma$. Let $A < 0$ be a real number such that $-\Sigma \subset \mathbf{C}_{(A,a]}$.

(1) *Assume:*

(a) *$M(\phi)(s)$ from (B.2.1.2) admits a meromorphic continuation f over $\mathbf{C}_{(A,b)}$ with a pole at $s = -\alpha$ of order m_α for each $\alpha \in \Sigma$ and no other poles,*

[9] Originally there were references given to [HL, §2], but this seems useless since the referenced items include no proofs!

(b) *$\varepsilon > 0$ is a real number such that $A + \varepsilon < 0$ and f has no pole over $\mathbf{C}_{(A,A+\varepsilon]}$,*

(c) *there exists $\beta > 1$ such that $|s|^\beta |f(s)|$ is bounded over $\mathbf{C}_{[A+\varepsilon/2,B]}$ for some real number $B \in (a,b)$.*

Then for some $c_{\alpha,k} \in \mathbf{C}$ for $1 \le k \le m_\alpha$ with $c_{\alpha,m_\alpha} \neq 0$ for all α, as $t \to 0^+$ we have

$$\phi(t) - \sum_{\alpha\in\Sigma} \sum_{1\le k\le m_\alpha} c_{\alpha,k} t^\alpha (\log(t))^{k-1} = O_\varepsilon(t^{-A-\varepsilon}). \tag{B.2.1.14}$$

(2) *Assume for each $\alpha \in \Sigma$ there exist $c_{\alpha,k} \in \mathbf{C}$ for $1 \le k \le m_\alpha$ with $c_{\alpha,m_\alpha} \neq 0$ for all α such that as $t \to 0^+$,*

$$\phi(t) - \sum_{\alpha\in\Sigma} \sum_{1\le k\le m_\alpha} c_{\alpha,k} t^\alpha (\log(t))^{k-1} = O(t^{-A}). \tag{B.2.1.15}$$

Then $M(\phi)(s)$ admits a meromorphic function over $\mathbf{C}_{(A,b)}$ with a pole at $s = -\alpha$ of order m_α.

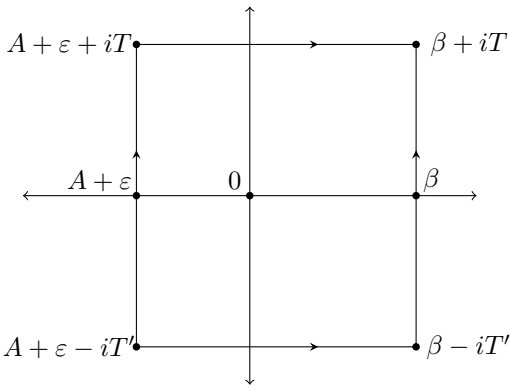


FIGURE 1. A contour with upward vertical sides in $\sigma = A + \varepsilon < 0$ and $\sigma = \beta > 1$

PROOF. (1) : Consider integration along the contour in Figure 1. Applying the Residue Theorem, we can use (a) and (c) (with $\beta > 0$) to see that the contributions along the top and bottom sides decay to 0 as $T, T' \to \infty$, yielding

$$\frac{1}{2\pi i}\int_{B-i\infty}^{B+i\infty} f(s)t^{-s}\,\mathrm{d}s = \sum_{\alpha\in\Sigma} \mathrm{Res}_{s=-\alpha}(f(s)t^{-s}) + \frac{1}{2\pi i}\int_{A+\varepsilon-i\infty}^{A+\varepsilon+i\infty} f(s)t^{-s}\,\mathrm{d}s \tag{B.2.1.16}$$

with the vertical integrals on both sides absolutely convergent since $\beta > 1$ in (c). The left integral as a function of t is $\phi(t)$ by Fourier inversion (using the change of variables $t = e^x$), and the right integral is $O_\varepsilon(t^{-A-\varepsilon})$ as $t \to 0^+$.

Letting $a_k(f, -\alpha)$ be the coefficient of $(s+\alpha)^{-k}$ in the Laurent series expansion of $f(s)$ at $s = -\alpha$ (so this is nonzero when $k = m_\alpha$), we compute

$$\mathrm{Res}_{s=-\alpha}(f(s)t^{-s}) = \sum_{1\le k\le m_\alpha} a_k(f,-\alpha)\frac{(-1)^{k-1}}{(k-1)!} t^\alpha (\log(t))^{k-1}.$$

Defining

$$c_{\alpha,k} := a_k(f,-\alpha)\frac{(-1)^{k-1}}{(k-1)!},$$

we obtain (B.2.1.14).

(2) : We write

$$\psi(t) := \phi(t) - 1_{(0,1]}(t) \sum_{\alpha\in\Sigma} \sum_{1\le k\le m_\alpha} c_{\alpha,k} t^\alpha (\log(t))^{k-1} \tag{B.2.1.17}$$

for $t > 0$ and note that ψ is smooth away from $t = 1$. If (B.2.1.15) holds then the integral

$$\int_0^\infty t^s \psi(t)\,\frac{\mathrm{d}t}{t} \tag{B.2.1.18}$$

converges absolutely and uniformly on the compact subsets of the vertical strip $\mathbf{C}_{(A,b)}$ (so it is holomorphic on that strip).

For $z \in \mathbf{C}_{>0}$ and any integer $k \geq 0$, it is elementary to verify that

$$\int_0^1 t^z (\log t)^k\,\frac{\mathrm{d}t}{t} = \int_{-\infty}^0 e^{zx} x^k\,\mathrm{d}x = \frac{(-1)^k\,k!}{z^{k+1}}$$

(the final equality using induction on k via either z-differentiation under the integral sign or integration by parts). Thus, for $\mathrm{Re}(s) > -\mathrm{Re}(\Sigma)$, so $\mathrm{Re}(s+\alpha) > 0$ for all $\alpha \in \Sigma$, the value of the integral in (B.2.1.18) is

$$g(s) := M(\phi)(s) - \sum_{\alpha\in\Sigma}\sum_{1\leq k\leq m_\alpha} \frac{c_{\alpha,k}(-1)^{k-1}\,(k-1)!}{(s+\alpha)^k}$$

but $g(s) = M(\psi)(s)$ is holomorphic over $\mathbf{C}_{(A,b)}$, so we get a meromorphic continuation of $M(\phi)$ over that same strip with poles exactly at $-\Sigma$ with pole-order m_α at $-\alpha$ for each $\alpha \in \Sigma$. □

For any integer $m \geq 0$, define

$$C_0^m(\mathbf{R}_{>0}) := \{f \in C^m(\mathbf{R}_{>0}) : \partial_x^k(f(e^x)) \in C_0(\mathbf{R}) \text{ for all } 0 \leq k \leq m\} \tag{B.2.1.19}$$

a Banach space equipped with the norm $f \mapsto \|f\|_m := \sum_{0\leq k\leq m} \|(t\partial_t)^k f\|_u$ (a sum of sup-norms over $\mathbf{R}_{>0}$). These arise in the following result that is a partial converse to Theorem B.2.1.7.

Proposition B.2.1.8 (Asymptotic Expansions and Mellin Inversion - II). *Pick $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}_{>0} \cup \{\infty\}$ with $a < b$. Let $N > 2$ be a real number. Let $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ be a holomorphic function such that $|s|^N |f(s)|$ is bounded on each closed strip of bounded width contained inside $\mathbf{C}_{(a,b)}$.*

Then for any integer $0 \leq m < N-1$ there exists $\phi \in C^m(\mathbf{R}_{>0})$ such that $t^\sigma \cdot \phi \in C_0^m(\mathbf{R}_{>0}) \cap L^2(\mathbf{R}_{>0}, \mathrm{d}t/t)$ for all $\sigma \in (a,b)$ and the integral defining $M(\phi)(s)$ is absolutely convergent over $\mathbf{C}_{(a,b)}$ and equal to $f(s)$.

Proof. Since $N > 1$, we deduce that $f|_{\sigma+i\mathbf{R}} \in L^2(\mathbf{R}, \mathrm{d}x) \cap L^1(\mathbf{R}, \mathrm{d}x)$ for every $\sigma \in (a,b)$. It follows that we have $|s|^k |f(s)|^2$ is integrable on every vertical line of $\mathbf{C}_{(a,b)}$ for $0 \leq k \leq N$.

For every $\sigma \in (a,b)$, on $\mathbf{R}_{>0}$ the Fourier transform $\phi_\sigma(x) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty} f(\sigma+i\tau)x^{-i\tau}\,\mathrm{d}\tau$ of $f|_{\sigma+i\mathbf{R}}$ (which indeed is given by the integral $\frac{1}{2\pi i}\int_{-\infty}^{+\infty} f(\sigma+i\tau)x^{-i\tau}\,\mathrm{d}\tau$, by absolute integrability of f on $\sigma + i\mathbf{R}$) therefore satisfies $\phi_\sigma \in C_0^m(\mathbf{R}_{>0})$ for all integers $0 \leq m < N-1$ via x-differentiation under the integral sign and the Riemann-Lebesgue Lemma. Each ϕ_σ is itself square-integrable by Plancherel's Theorem (see [**Fol**, Thm. 8.29]). By using the Cauchy integral formula for $f(s)x^{-s}$ on rectangles with vertical edges along $\mathrm{Re}(s) = \sigma, \sigma' \in (a,b)$ and heights $T, -T'$ with $T, T' \to \infty$, the decay $f(s) = O(|s|^{-N})$ ensures $x^{-\sigma}\phi_\sigma(x) = x^{-\sigma'}\phi_{\sigma'}(x)$ for all $x > 0$; call this $\phi(x)$.

We claim ϕ_σ is absolutely $(\mathrm{d}t/t)$-integrable over $\mathbf{R}_{>0}$ for all $\sigma \in (a,b)$. Granting this for now, for all $\sigma \in (a,b)$ the inverse Fourier transform of ϕ_σ is given by $\int_{\mathbf{R}_{>0}} \phi_\sigma(t)t^{ix}\,\frac{\mathrm{d}t}{t}$ as a function of $x \in \mathbf{R}$, and this function belongs to $C_0(\mathbf{R})$. Thus, we have

$$M(\phi)(\sigma+ix) := \int_{\mathbf{R}_{>0}} \phi(t)t^{\sigma+ix}\,\frac{\mathrm{d}t}{t} = \int_{\mathbf{R}_{>0}} \phi_\sigma(t)t^{ix}\,\frac{\mathrm{d}t}{t} = f(\sigma+ix)$$

for all $x \in \mathbf{R}$, where the integrand defining $M(\phi)$ is absolutely integrable and the final equality initially holds initially only for $\mathrm{d}x$-almost all x but then holds for all x since both sides are continuous in x. This yields that over $\mathbf{C}_{(a,b)}$ the Mellin transform $M(\phi)$ is an absolutely convergent integral that coincides with the holomorphic f.

It remains to show $\int_{\mathbf{R}_{>0}} |\phi_\sigma(t)|\,\mathrm{d}t/t < \infty$ for all $\sigma \in (a,b)$. Since f is holomorphic over $\mathbf{C}_{(a,b)}$, Lemma D.3.1.1 applies f and each of the finitely many integers $0 \leq m \leq N$ to give that each holomorphic derivative $\partial_s^k f$ on $\mathbf{C}_{(a,b)}$ satisfies

$$|s^m \partial_s^k f(s)| \leq A_{k,J} \tag{B.2.1.20}$$

on $\mathbf{C}_J$ for each closed bounded subinterval $J \subset (a,b)$, for some constant $A_{k,J}$ (which also depends on f). In particular, since we can use $m = 2$ (as $N \geq 2$), we see that each $\partial_s^k f$ is absolutely integrable on each vertical line in $\mathbf{C}_{(a,b)}$.

We claim

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} (\partial_s^2 f)(\sigma + i\tau) t^{-i\tau}\,\mathrm{d}\tau = (\log(t))^2 \phi_\sigma(t) \tag{B.2.1.21}$$

as functions of $t \in \mathbf{R}_{>0}$. To see this, we proceed as follows. As holomorphic functions of $s \in \mathbf{C}_{(a,b)}$ we have

$$\partial_s(f(s)t^{-s}) = (\partial_s f)t^{-s} - (\log(t))f(s)t^{-s}$$

with $s^2\partial_s f$ and $s^2 f(s)$ bounded on each $\mathbf{C}_J$, and t^{-s} is clearly bounded on each $\mathbf{C}_J$. Integrating over $[\sigma - iT, \sigma + iT]$ for any $T > 0$ then gives

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} (\partial_s f)(s)t^{-s}\,\mathrm{d}s = \log(t)\,\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(s)t^{-s}\,\mathrm{d}s + \frac{1}{2\pi i}\left\{(\partial_s f)(\sigma+iT)t^{-\sigma-iT} - (\partial_s f)(\sigma-iT)t^{-\sigma+iT}\right\}, \tag{B.2.1.22}$$

and then letting T tends to $+\infty$ yields

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} (\partial_s f)(s)t^{-s}\,\mathrm{d}s = \log(t)\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(s)t^{-s}\,\mathrm{d}s. \tag{B.2.1.23}$$

As a consequence of Lemma D.3.1.1 and as observed above, we may replace f with $\partial_s f$ and so obtain:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} (\partial_s^2 f)(s)t^{-s}\,\mathrm{d}s = \log(t)^2\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(s)t^{-s}\,\mathrm{d}s = \log(t)^2 t^{-\sigma}\,\phi_\sigma(t). \tag{B.2.1.24}$$

Multiplying both sides by t^σ then gives (B.2.1.21).

From (B.2.1.21), upon breaking up the integration over $\mathbf{R}$ into integration over $[-1,1]$ and its complement, multiplying and dividing the integrand by $(\sigma + i\tau)^2$ in the latter integral (over that complement) yields

$$|(\log(t))^2\phi_\sigma(t)| \leq \frac{1}{2\pi}\int_{-1}^{1} |\partial_s^2 f(\sigma+i\tau)|\,\mathrm{d}\tau + \frac{A_{2,\{\sigma\}}}{2\pi}\int_{|\tau|>1} \frac{1}{\sigma^2+\tau^2}\,\mathrm{d}\tau \tag{B.2.1.25}$$

for all $t \in \mathbf{R}_{>0}$. Thanks to this constant upper bound (depending on σ), the desired absolute $(\mathrm{d}t/t)$-integrability of ϕ_σ over $\mathbf{R}_{>0}$ reduces to that of $1/\log(t)^2$ is over $[2,+\infty)$ and $(0,1/2]$, and those are readily verified. □

Proposition B.2.1.9. *Let ϕ be a Schwartz-Bruhat function on $\mathbf{R}_{>0}$ with exponential decay.*

Let Φ denote the Mellin transform $M(\phi)$, i.e. the complex valued function of $s \in \mathbf{C}$:

$$\Phi(s) := \int_{\mathbf{R}_{>0}} \phi(t)t^s\,\frac{\mathrm{d}t}{t}. \tag{B.2.1.26}$$

Let a, b be arbitrary real numbers with $a < b$. Then:

(1) *Φ is entire.*

(2) *For every integer $N \geq 1$, there exists a real number $C_{N,a,b}$ depending only on N, a, b, and ϕ such that for every $a \leq \sigma \leq b$ and every $y \neq 0$, we have:*

$$|\Phi(\sigma+iy)| \leq C_{N,a,b}|y|^{-N}. \tag{B.2.1.27}$$

In particular, as a function of $y \in \mathbf{R}$, for each $\sigma \in \mathbf{R}$, $\Phi(\sigma+iy)$ decays faster than any positive power of $|y|^{-1}$ as $|y| \to +\infty$.

(3) *For every real number $p \geq 1$ and any integer $N > 1$, we have the inequality:*

$$\|\Phi(\sigma+iy)\|_{L^p(\mathbf{R},\mathrm{d}y)} \leq \left\{ \|\phi(t)e^{(\max\{|a|,|b|\}+N)|\log(t)|}\|^p_{L^1(\mathbf{R}_{>0},\frac{\mathrm{d}t}{t})} + C^p_{N,a,b}\int_{|y|\geq 1/2}\frac{1}{|y|^{pN}}\,\mathrm{d}y \right\}^{1/p}. \tag{B.2.1.28}$$

(4) *If ϕ is supported over $\mathbf{R}_{>1}$, then $C_{N,a,b}$ from* (B.2.1.26) *depends only on N, b, and ϕ but not on a, and $\Phi \in H^p(\mathbf{C}_{<b})$ for every real number $p \geq 1$.*

Proof. (1) : We write $t = e^x$ and obtain

$$\Phi(s) = \int_{-\infty}^{+\infty} \phi(e^x)e^{xs}\,\mathrm{d}x. \tag{B.2.1.29}$$

We have for every real number $A > 0$ and $|\sigma| \leq A$ that

$$\int_{-\infty}^{+\infty} |\phi(e^x)||e^{\sigma x}|\,\mathrm{d}x \leq \int_{-\infty}^{+\infty} |\phi(e^x)|e^{|Ax|}\,\mathrm{d}x \tag{B.2.1.30}$$

since $e^{\sigma x} \leq e^{|Ax|}$ for all $x \in \mathbf{R}$. The integral on the right side in (B.2.1.30) converges since $\phi(e^x) \in \mathcal{S}(\mathbf{R})$ decays faster than $e^{-N|x|}$ for any $N > A$ as $|x| \to +\infty$. It follows that the integral defining $\Phi(s)$ in (B.2.1.29) converges absolutely and uniformly over the vertical strip $\mathbf{C}_{(-A,A)}$. Since this holds for every $A > 0$, Φ is entire.

(2) : We have, as in the proof of Proposition B.2.1.5(2) (see (B.2.1.7)), for every integer $N \geq 1$ and every $y \neq 0$:

$$\Phi(\sigma+iy) = \frac{(-1)^N}{(\sigma+iy)^N}\int_{-\infty}^{+\infty} \partial_x^N(\phi(e^x))e^{(\sigma+iy)x}\,\mathrm{d}x. \tag{B.2.1.31}$$

Since $|e^{(\sigma+iy)x}| = e^{\sigma x} \leq e^{\max\{|a|,|b|\}\cdot|x|}$ for all $x \in \mathbf{R}$ and all $a \leq \sigma \leq b$, we deduce the inequality, for $y \neq 0$ and all $a \leq \sigma \leq b$:

$$|\Phi(\sigma+iy)| \leq \frac{1}{(\sigma^2+y^2)^{N/2}}\int_{-\infty}^{+\infty} |\partial_x^N(\phi(e^x))|e^{\max\{|a|,|b|\}\cdot|x|}\,\mathrm{d}x \tag{B.2.1.32}$$

$$\leq \frac{1}{|y|^N}\int_{-\infty}^{+\infty} |\partial_x^N(\phi(e^x))|e^{\max\{|a|,|b|\}\cdot|x|}\,\mathrm{d}x. \tag{B.2.1.33}$$

Upon defining

$$C_{N,a,b} := \int_{-\infty}^{+\infty} |\partial_x^N(\phi(e^x))|e^{(\max\{|a|,|b|\})|x|}\,\mathrm{d}x \tag{B.2.1.34}$$

which indeed converges by Lemma B.2.1.4(4), we deduce that we have, for all $y \neq 0$ and all $a \leq \sigma \leq b$:

$$|\Phi(\sigma+iy)| \leq C_{N,a,b}|y|^{-N}. \tag{B.2.1.35}$$

(3) : We have, for every $y \in \mathbf{R}$:

$$|\Phi(\sigma+iy)| \leq \|\phi(e^x)e^{\max\{|a|,|b|\}\cdot|x|}\|_{L^1(\mathbf{R},\mathrm{d}x)} \tag{B.2.1.36}$$

and hence from (B.2.1.36) and (B.2.1.34):

$$|\Phi(\sigma+iy)| \leq \|\phi(e^x)e^{\max\{|a|,|b|\}\cdot|x|}\|_{L^1(\mathbf{R},\mathrm{d}x)}\cdot 1_{|y|<1/2}(y) + \frac{C_{N,a,b}}{|y|^N}\cdot 1_{|y|\geq 1/2}(y). \tag{B.2.1.37}$$

We deduce from (B.2.1.37) that for every $p \geq 1$, upon choosing $N > 1$:

$$\|\Phi(\sigma+iy)\|_{L^p(\mathbf{R},\mathrm{d}y)} \leq \left\{ \|\phi(e^x)e^{\max\{|a|,|b|\}\cdot|x|}\|^p_{L^1(\mathbf{R},\mathrm{d}x)} + C^p_{N,a,b}\int_{|y|\geq 1/2}\frac{1}{|y|^{pN}}\,\mathrm{d}y \right\}^{1/p}. \tag{B.2.1.38}$$

This proves that (B.2.1.28) holds.

(4) : If ϕ is supported over $\mathbf{R}_{>1}$, then the majorant $|\partial_x^N(\phi(e^x))|e^{\max\{|a|,|b|\}\cdot|x|}$ for $|\partial_x^N(\phi(e^x))e^{sx}|$ can be replaced with the sharper majorant $|\partial_x^N(\phi(e^x))|e^{|b|\cdot|x|}$, which does not depend on a.

We deduce that we have $\Phi \in H^p(\mathbf{C}_{<b})$ for every real number $p \geq 1$. □

COROLLARY B.2.1.10. *Let $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function with exponential decay. Then:*

(1) *For every integer $n \geq 0$, the functions $M(\phi)(n-s)$ and $M(s+n)$ are entire and satisfy the inequality from* (B.2.1.28) *on each vertical strip $\mathbf{C}_{[a,b]}$. In particular, they are bounded on vertical strips of bounded width.*

(2) *Let $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$, for $a < b$ fixed real numbers, be a meromorphic function with finitely many poles contained in $\mathbf{C}_{(a,b)}$. Assume that there exist real numbers $A, B > 0$ such that for $a < \sigma < b$ we have*

$$|f(\sigma + iy)| \leq A|y|^B \tag{B.2.1.39}$$

for all $|y| \gg 0$ (largeness uniform in such σ). Then for every integer $n \geq 1$, the meromorphic functions $f(s)M(\phi)(s+n)$ and $f(s)M(\phi)(-s+n)$ satisfy the inequality from (B.2.1.26) *for all large enough $|y| > 0$.*

PROOF. (1) : We have $M(\phi)(-s) = M(\check{\phi})(s)$, and since ϕ is Schwartz with exponential decay, so is $\check{\phi}$. Proposition B.2.1.9(1) and (2) therefore apply and gives (1) for $n = 0$.

We have $M(\phi)(s+n) = M(t^n\phi)(s)$, and since ϕ is Schwartz with exponential decay, so is $t^n\phi$ for every integer $n \in \mathbf{Z}$. We may therefore assume $n = 0$ and the foregoing applies.

(2) : By (1) and (B.2.1.26), the inequality in (B.2.1.27) holds for $f(s)\Phi(s)$ so long as (B.2.1.39) does, and hence both hold for all large enough $|y| \gg 0$. □

B.2.2. Asymptotic Cauchy Integral Formula. We begin by studying the Cauchy Integral Formula applied to the special case of holomorphic functions Φ over an open bounded vertical strip such that $\Phi(s)$ decays rapidly as $|s| \to \infty$ uniformly on the vertical strip.

B.2.2.1. *Cauchy Integrals.* We introduce a class of holomorphic functions that comprises the Mellin transforms of all Schwartz functions with exponential decay.

DEFINITION B.2.2.1. Let $a < b$ be real numbers. Let $\Phi : \mathbf{C}_{(a,b)} \to \mathbf{C}$ be a holomorphic function. We say that Φ is *uniformly Schwartz over the vertical strip* $\mathbf{C}_{(a,b)}$ if and only if for every $\varepsilon > 0$ such that $a + \varepsilon < b - \varepsilon$, and for every integer $N \geq 1$, there exists a real number $C_{N,\varepsilon} > 0$ such that for all sufficiently large $|y|$ uniformly over all $a + \varepsilon \leq \sigma \leq b - \varepsilon$ we have

$$|\Phi(\sigma + iy)| \leq \frac{C_{N,\varepsilon}}{|y|^N} \tag{B.2.2.1}$$

(so in particular Φ is bounded on each $\mathbf{C}_{(a+\varepsilon,b-\varepsilon)}$).

If Φ is entire, we say it is *Schwartz on vertical strips* if and only if for every bounded open interval $(a, b) \subset \mathbf{R}$ it is uniformly Schwartz over the vertical strip $\mathbf{C}_{(a,b)}$. If $\Phi : \mathbf{C}_{<b} \to \mathbf{C}$ is holomorphic, we say it is *Schwartz on the half-plane* $\mathbf{C}_{<b}$ if and only if for every $\varepsilon > 0$ and for every integer $N \geq 1$, $\Phi(\sigma + iy)$ satisfies condition (B.2.2.1) for all large enough $|y|$ uniformly over all $\sigma \leq b - \varepsilon$.

The "justification" of this terminology is that by Lemma D.3.1.1 each of the derivatives $\partial_y^j(\Phi(\sigma + iy)$ satisfy the same uniform decay estimates as $|y| \to \infty$ when σ varies in a compact interval. In other words, the Schwartz decay properties along each vertical line have uniform constants and uniform largeness conditions on $|y|$ as σ varies in a compact interval.

EXAMPLE B.2.2.2. If $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ is a Schwartz-Bruhat with exponential decay, then $\Phi := M(\phi)$ is Schwartz on vertical strips by Proposition B.2.1.9(2). Indeed, for any non-empty vertical strip of the form $\mathbf{C}_{(a,b)}$, we may any real number $\varepsilon > 0$ and let $C_{N,\varepsilon} := C_{N,a-\varepsilon,b+\varepsilon}$. By Proposition B.2.1.9(2), the condition in (B.2.2.1) is satisfied by the restriction of Φ to $\mathbf{C}_{(a,b)}$.

By Proposition B.2.1.5(1,2), Schwartz-Bruhat functions ϕ of exponential or semi-exponential decay over $\mathbf{R}_{>0}$ provide examples of holomorphic functions $\Phi(s) := M(\phi)(s)$ on either $\mathbf{C}$, $\mathbf{C}_{>0}$ or $\mathbf{C}_{<0}$ that are uniformly Schwartz over the open vertical strips of bounded width contained within such domains in $\mathbf{C}$.

We record here the following general estimate for the Mellin transform of smooth functions on $\mathbf{R}_{>0}$ that are Schwartz-Bruhat, but neither exponentially nor semi-exponentially decaying. It is a uniform version of the Riemann-Lebesgue Lemma, and provides examples of holomorphic functions over a vertical strip $\mathbf{C}_{(a,b)}$ of bounded width that are uniformly Schwartz on all open vertical sub-strips.

PROPOSITION B.2.2.3 (Uniform Riemann-Lebesgue - I). *Let $a, b \in \mathbf{R}$ be two real numbers, with $a < b$. Let $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ be a Schwartz-Bruhat function.*

Assume that we have, for any $\sigma \in [a, b]$, $t^\sigma \phi \in \mathcal{S}(\mathbf{R}_{>0})$.

Then:

(1) *The Mellin transform $M(\phi)(s)$ is a continuous function over $\mathbf{C}_{[a,b]}$ whose restriction to every vertical line in $\mathbf{C}_{[a,b]}$ is Schwartz, and it is holomorphic over $\mathbf{C}_{(a,b)}$.*

(2) *For any integer $N \geq 1$, there exists a real number $C_{N,a,b} > 0$ such that for any $a \leq \sigma \leq b$, we have:*

$$|M(\phi)(s)| \leq \frac{C_{N,a,b}}{|y|^N} \tag{B.2.2.2}$$

as $|y| \to +\infty$, uniformly in such σ. In particular, $M(\phi)$ is uniformly Schwartz over all open vertical strips contained within $\mathbf{C}_{[a,b]}$.

PROOF. We first claim $(t\partial_t)^N \phi$ satisfies the same assumptions as ϕ for every integer $N \geq 1$. We have, for every $\sigma \in [a, b]$:

$$(t\partial_t)(t^\sigma \phi) = \sigma \cdot t^\sigma \phi + t^\sigma (t\partial_t)\phi$$

and since $(t\partial_t)(t^\sigma\phi)$ is Schwartz-Bruhat, as is $t^\sigma\phi$ (and hence $\sigma \cdot t^\sigma \phi$) by the assumptions on ϕ, we deduce that $t^\sigma(t\partial_t)\phi$ is Schwartz-Bruhat over $\mathbf{R}_{>0}$ for all $\sigma \in [a, b]$.

We have, as in the proof of Proposition B.2.1.5(2) (see (B.2.1.7)), for every integer $N \geq 1$, every $y \neq 0$, and every $\sigma \in [a, b]$ that

$$\Phi(\sigma + iy) = \frac{(-1)^N}{(\sigma + iy)^N} \int_{-\infty}^{+\infty} \partial_x^N(\phi(e^x)) e^{(\sigma+iy)x} \, \mathrm{d}x \tag{B.2.2.3}$$

where the integral on the right side of (B.2.2.3) converges by Lemma B.2.1.6 and the foregoing since $t^\sigma (t\partial_t)^N \phi \in \mathcal{S}(\mathbf{R}_{>0})$ for every integer $N \geq 1$.

This proves that $\Phi(s)$ is holomorphic over $\mathbf{C}_{(a,b)}$, Schwartz on every vertical line of $\mathbf{C}_{[a,b]}$ since $t^\sigma\phi$ is Schwartz-Bruhat for every $\sigma \in [a, b]$ and the restriction of the Mellin transform to each such vertical line $\mathrm{Re}(s) = \sigma$ is the (inverse) Fourier transform of $t^\sigma\phi$ on $\mathbf{R}_{>0}$.

The absolute value of the integral in (B.2.2.3) is bounded above by $C_{N,a,b} := \|\partial_x^N(\phi(e^x)) \max(e^{ax}, e^{bx})\|_{L^1(\mathbf{R}, \mathrm{d}x)}$, and hence the estimate in (B.2.2.2) holds by (B.2.2.3). □

The following is an asymptotic version of Cauchy's Integral Formula.

THEOREM B.2.2.4 (Asymptotic Cauchy Integral Formula). *Let a, b be real numbers, $a < b$, and $\Phi : \mathbf{C}_{(a,b)} \to \mathbf{C}$ holomorphic. Assume that Φ is uniformly Schwartz on the vertical strip $\mathbf{C}_{(a,b)}$. Let $\varepsilon > 0$ be a real number such that $a + \varepsilon < b - \varepsilon$.*

Let $N \geq 0$ be an integer.

(1) *For every $z \in \mathbf{C}$,*

$$-\frac{N!}{2\pi i} \int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty} \frac{\Phi(s)}{(s-z)^{N+1}} \, \mathrm{d}s + \frac{N!}{2\pi i} \int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty} \frac{\Phi(s)}{(s-z)^{N+1}} \, \mathrm{d}s = \Phi^{(N)}(z) \tag{B.2.2.4}$$

if $z \in \mathbf{C}_{(a+\varepsilon,b-\varepsilon)}$ *and*

$$-\frac{N!}{2\pi i}\int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s+\frac{N!}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s=0 \tag{B.2.2.5}$$

if $z \notin \mathbf{C}_{[a+\varepsilon,b-\varepsilon]}$*, with all integrals absolutely convergent.*

(2) *If* $\Phi : \mathbf{C}_{<b} \to \mathbf{C}$ *is holomorphic and Schwartz on the half-plane* $\mathbf{C}_{<b}$*, then for every* $\varepsilon > 0$ *we have:*

$$\frac{N!}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s=\Phi^{(N)}(z) \tag{B.2.2.6}$$

if $\mathrm{Re}(z) < b-\varepsilon$ *and*

$$\frac{N!}{2\pi i}\int_{b-\varepsilon-i\infty}^{b-\varepsilon+i\infty}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s=0 \tag{B.2.2.7}$$

if $\mathrm{Re}(z) > b-\varepsilon$*, with all integrals absolutely convergent.*

Proof. (1) : We first claim

$$\lim_{|T|\to+\infty}\int_{a+\varepsilon}^{b-\varepsilon}|\Phi(\sigma+iT)|\,\mathrm{d}\sigma=0. \tag{B.2.2.8}$$

For each integer $M \geq 1$ and for all large enough $|T| \gg 0$, there exists a constant $C_{M,\varepsilon} > 0$ such that $|\Phi(\sigma+iT)| \leq C_{M,\varepsilon}|T|^{-M}$ for all $a+\varepsilon \leq \sigma \leq b-\varepsilon$. It follows that we have:

$$\int_{a+\varepsilon}^{b-\varepsilon}|\Phi(\sigma+iT)|\,\mathrm{d}\sigma \leq C_{M,\varepsilon}\int_{a+\varepsilon}^{b-\varepsilon}\frac{1}{|T|^M}\,\mathrm{d}\sigma \leq C_{M,\varepsilon}(b-a)|T|^{-M}. \tag{B.2.2.9}$$

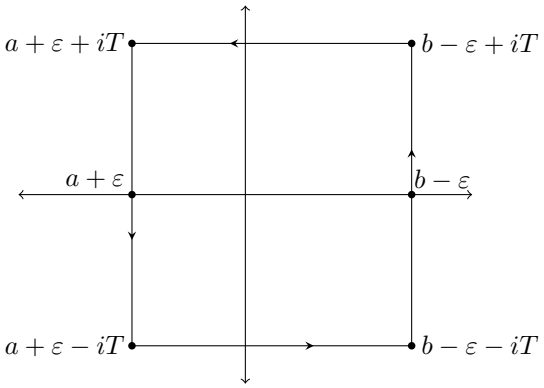


Figure 2. A contour Γ_T with counterclockwise orientation (it may or may not enclose 0)

We call Γ_T the contour as in Figure 2. Thus, if $\mathrm{Re}(z) \in (a+\varepsilon, b-\varepsilon)$ and $|T| > \mathrm{Im}(z)$ then

$$\Phi^{(N)}(z)=\frac{N!}{2\pi i}\int_{\Gamma_T}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s, \tag{B.2.2.10}$$

and if $\mathrm{Re}(z) \notin [a+\varepsilon, b-\varepsilon]$ then

$$\frac{N!}{2\pi i}\int_{\Gamma_T}\frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s=0. \tag{B.2.2.11}$$

Upon letting $T \to +\infty$, we obtain assertion (1) upon noting the absolute convergence of the integrals in (1) follows from the bound $|\Phi(\sigma+iy)| \leq C_{1,\varepsilon}|y|^{-1}$ for $\sigma = a+\varepsilon, b-\varepsilon$ and sufficiently large $|y|$.

(2) : The absolute convergence of the integrals is seen as in the proof of (1). Since $a+\varepsilon < \mathrm{Re}(z)$, the function $(s-z)^{-1}$ is smooth and square-integrable over $a+\varepsilon - i\mathbf{R}$, and by the Cauchy-Schwartz inequality we obtain:[10]

$$\left|\int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty} \frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s\right| \leq \|\Phi(a+\varepsilon+it)\|_{L^2(\mathbf{R},\mathrm{d}t)} \left\{\int_{-\infty}^{\infty} \frac{1}{|a+\varepsilon+it-z|^{2(N+1)}}\,\mathrm{d}t\right\}^{1/2} \leq \|\Phi\|_{H^2(\mathbf{C}_{<b-\varepsilon})} \left\{\int_{-\infty}^{\infty} \frac{1}{|a+\varepsilon+it-z|^{2(N+1)}}\,\mathrm{d}t\right\}^{1/2}. \tag{B.2.2.12}$$

By the argument as in (1), we may move the line of integration $(a+\varepsilon-i\infty, a+\varepsilon+i\infty)$ as leftwards as we please leaving the left side of (B.2.2.12) unchanged. Thus, from (B.2.2.12) and elementary estimates (after multiplying t by $a+\varepsilon$), we have

$$\lim_{a\to-\infty} \int_{a+\varepsilon-i\infty}^{a+\varepsilon+i\infty} \frac{\Phi(s)}{(s-z)^{N+1}}\,\mathrm{d}s = 0, \tag{B.2.2.13}$$

so (B.2.2.4) and (B.2.2.5) hold by arguing as in (1). □

Remark B.2.2.5. For Φ entire and Schwartz on vertical strips, we may define

$$\phi(t) := \int_{-i\infty}^{+i\infty} \Phi(s) t^{-s}\,\mathrm{d}s.$$

on $\mathbf{R}_{>0}$. The integral is absolutely convergent in light of (B.2.1.27), and (by the Dominated Convergence Theorem) ϕ is continuous as a function of $t \in \mathbf{R}_{>0}$.

The function $f(u) = \int_{-\infty}^{+\infty} \Phi(ix) e^{-ixu}\,\mathrm{d}x$ is Schwartz since Φ is Schwartz on vertical lines, and it is in particular smooth. It follows that $\phi(t) = f(\log(t))$ is smooth over $\mathbf{R}_{>0}$ and we may extend it by zero over $\mathbf{R}$ preserving smoothness since the derivative of ϕ on $\mathbf{R}_{>0}$ of any order decays to 0 as $t \to 0^+$.

In some cases, Φ is not only the Mellin transform of a smooth function ϕ on $\mathbf{R}$ supported over $\mathbf{R}_{\geq 0}$ and such that $\phi(e^x)$ is Schwartz, but it is the Mellin transform of a Schwartz function on $\mathbf{R}_{>0}$ with exponential decay. The entire functions Φ such that (B.2.1.27) holds with a constant $C_{N,a,b}$ which does not depend on a, b, but only on $N \geq 1$, are the Mellin transform of a smooth compactly supported function over $\mathbf{R}$ by [**Horm**, Ch.VII,Thm.7.3.1] (the Paley-Wiener-Schwartz Theorem).[11]

In the proof of the Asymptotic Cauchy Integral Formula, we can choose the top and bottom edges of the rectangles wherever we wish sufficiently high up and low down, so it isn't necessary to require the uniformly Schwartz decay property across the entirety of the strips $a+\varepsilon \leq \sigma \leq b-\varepsilon$ for all sufficiently small $\varepsilon > 0$. We only need the uniform vertical Schwartz decay "near" the vertical edges $\sigma = a, b$ and a sufficient supply of horizontal segments arbitrarily high up and low down on which there is such decay:

Corollary B.2.2.6. *For real numbers $a < b$, let $\Phi : \mathbf{C}_{(a,b)} \to \mathbf{C}$ be a holomorphic function that is uniformly Schwartz on $\mathbf{C}_{(a,a')}$ and on $\mathbf{C}_{(b',b)}$ for some $a < a' < b' < b$. Pick $a'' \in (a, a')$ and $b'' \in (b', b)$.*

Assume there is a sequence of real numbers $\{T_k\}_{k\in\mathbf{Z}}$ with $T_k < T_{k+1}$ and $|T_k| \to +\infty$ as $|k| \to +\infty$ such that for each integer $M \geq 1$ we have

$$|\Phi(\sigma + iT_k)| = O_{M,\varepsilon}(|T_k|^{-M}) \tag{B.2.2.14}$$

uniformly in $\sigma \in [a'', b'']$ as $|k| \to +\infty$.

The assertions in Theorem B.2.2.4(1) *hold for Φ.*

[10] Why is the H^2-norm of Φ at the end finite? Maybe use Cauchy Integral Formula to replace that with L^2-norm of $\Phi(b-\varepsilon+it)$ that is independent of a?

[11] This latter theorem seems to involve a different type of upper bound, so must either rewrite for accuracy or remove it; if not needed, I recommend to remove.

PROOF. The uniformly Schwartz hypothesis on Φ does not address behavior in the strip $\{a' \leq \sigma \leq b'\}$, but the proof of Theorem B.2.2.4(1) doesn't need precise control in that entire region as one goes far up and far down: we can carry out the method of the proof by choosing the rectangular contours to have top and bottom sides at the heights $t = T_k$ for sufficiently positive and sufficiently negative k since (B.2.2.14) provides the required decay of integrals along the top and bottom of the rectangle. □

B.3. Asymptotic Residue Theorem

In this section we discuss a consequence of Cauchy's Residue Theorem under a Schwartz condition uniform on upper and lower halves of a vertical strip, combined with a condition on the distribution of the poles of the meromorphic function across the vertical strip.

B.3.1. Slow growth near the poles. We discuss a growth condition for meromorphic functions on vertical strips, near their poles, to identify a class of such functions that, when multiplied by an entire function that is Schwartz on vertical strips, decays on bounded horizontal segments whose ordinate tends to $\pm\infty$.

B.3.1.1. *Polar locations and vertical growth.* We introduce a condition on the location of the poles of a meromorphic function f in a vertical strip that will combine correctly with the Schwartz condition uniformly on vertical strips introduced in Definition B.2.2.1 for a holomorphic function Φ, leading to an asymptotic version of Cauchy's Residue Theorem for $\Phi(s)f(s)$.

DEFINITION B.3.1.1. Let $a < b$ be real numbers. Let $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ be a meromorphic function. We say that f has *slow growth near its poles* if and only if:

(1) there exists a real number $\alpha > 0$ such that for each integer k there exists a real number T_k, satisfying the following properties:
   (a) $|T_{\pm(k+1)}| \geq |T_{\pm k}|$.
   (b) $\lim_{|k|\to+\infty} |T_k| = +\infty$.
   (c) $T_k = O(|k|^\alpha)$ as $|k| \to +\infty$.
   and f is holomorphic over a small enough horizontal strip containing the horizontal line $\mathbf{R} + iT_k$ in its interior,

(2) there exists a real number $\beta > 0$ and a real number $A_{a,b,\beta} > 0$, neither depending on σ, such that for every integer $k \in \mathbf{Z}$ and $a < \sigma < b$,

$$|f(\sigma + iT_k)| \leq A_{a,b}|k|^\beta, \tag{B.3.1.1}$$

(3) there exist real numbers $\varepsilon_1, \varepsilon_2 > 0$ such that $0 < \varepsilon_1 < \varepsilon_2 < (b-a)/2$ and such that $f(a + \varepsilon_1 + it)$ and $f(b - \varepsilon_2 + it)$ are continuous functions of $t \in \mathbf{R}$ growing at most polynomially as $|t| \to +\infty$. In particular, no pole of f is contained within the vertical lines $a + \varepsilon_1 + i\mathbf{R}$ and $b - \varepsilon_2 + i\mathbf{R}$.

We refer to the sequence $\{T_k\}_{k\in\mathbf{Z}}$ as a sequence that *witnesses the slow growth of f near its poles.*

We now show that if $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ is meromorphic and with slow growth near its poles, and if $\Phi : \mathbf{C}_{(a-\varepsilon,b+\varepsilon)} \to \mathbf{C}$ is holomorphic and uniformly Schwartz on the vertical strip $\mathbf{C}_{(a-\varepsilon,b+\varepsilon)}$ for some $\varepsilon > 0$ then line integrals of $\Phi(s)f(s)$ over horizontal segments contained within $\mathbf{C}_{(a,b)}$ vanish asymptotically along the ordinates $\{T_k\}_{k\in\mathbf{Z}}$ as $|k| \to \infty$.

LEMMA B.3.1.2. *Let $a < b$ be real numbers, $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ a meromorphic function with slow growth near its poles within $\mathbf{C}_{(a,b)}$. Let $\varepsilon > 0$ be a real number and $\Phi : \mathbf{C}_{(a-\varepsilon,b+\varepsilon)} \to \mathbf{C}$ a holomorphic function, uniformly Schwartz over the vertical strip $\mathbf{C}_{(a-\varepsilon,b+\varepsilon)}$. Then for $\{T_k\}_{k\in\mathbf{Z},|k|\geq 2} \subset \mathbf{R}$ witnessing the slow growth condition*

on f near its poles, we have:

$$\lim_{|k|\to+\infty}\int_{a-iT_k}^{b-iT_k}|f(s)\Phi(s)|\,|\mathrm{d}s| = 0 = \lim_{|k|\to+\infty}\int_{a+iT_k}^{b+iT_k}|f(s)\Phi(s)|\,|\mathrm{d}s|. \tag{B.3.1.2}$$

PROOF. Let $N \geq 1$ be a fixed integer. Then there exists a real number $C_{N,\varepsilon} > 0$, such that for all large enough $|k|$, we have $|\Phi(\sigma + iT_k)| \leq C_{N,\varepsilon}|T_k|^{-N}$, for all $\sigma \in [a,b]$. We may choose N such that $N\gamma > \delta$, and we have that there exists a real number $A > 0$ such that:

$$|f(\sigma+iT_k)\Phi(\sigma+iT_k)| \leq A_{a,b}\cdot C_{N,\varepsilon}|k|^{\delta}T_k^{-N} \leq A\cdot A_{a,b}\cdot C_{N,\varepsilon}|k|^{\delta}|k|^{-N\gamma}$$

for all large enough $|k|$.

We therefore have:

$$\lim_{|k|\to+\infty}\int_{a\pm iT_k}^{b\pm iT_k}|f(s)\Phi(s)|\,|\mathrm{d}s| \leq \lim_{|k|\to+\infty} A\cdot A_{a,b}\cdot C_{N,\varepsilon}(b-a)|k|^{\delta-N\gamma} = 0. \tag{B.3.1.3}$$

□

B.3.2. The Asymptotic Residue Theorem. We now discuss a consequence of Cauchy's Residue Theore, for those meromorphic functions of the form $f(s)\Phi(s)$ with f meromorphic and having slow growth around its poles over a vertical strip $\mathbf{C}_{(a,b)}$ (see Definition B.3.1.1) and Φ holomorphic over a larger open vertical strip containing $\mathbf{C}_{[a,b]}$ and uniformly Schwartz on that open strip.

B.3.2.1. *Residues over asymptotic contours.* Consider a vertical strip $\mathbf{C}_{(a,b)}$ with real $a < b$, a meromorphic function $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ with slow growth near its poles, and a holomorphic function Φ on $\mathbf{C}_{(a-\varepsilon,b+\varepsilon)}$ that is uniformly Schwartz.

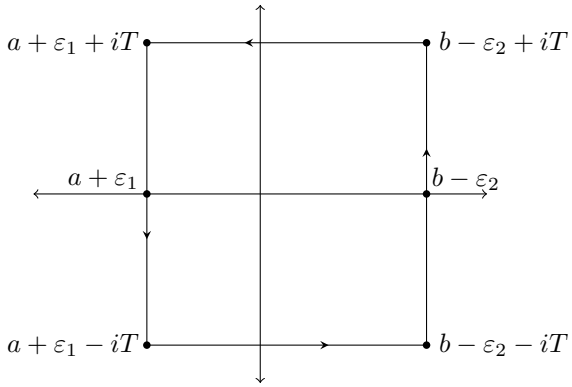


FIGURE 3. A contour $\Gamma_{T,\varepsilon_1,\varepsilon_2}$ with counterclockwise orientation (it may or may not enclose 0)

We shall analyze the product function $\Phi(s)f(s)$ over growing contours $\Gamma_{T,\varepsilon_1,\varepsilon_2}$ of the form as in Figure 3 for all real numbers $T > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $a + \varepsilon_1 < b - \varepsilon_2$. We denote by $U_{T,\varepsilon_1,\varepsilon_2}$ the open subset of $\mathbf{C}_{(a,b)}$ enclosed by $\Gamma_{T,\varepsilon_1,\varepsilon_2}$ and having that rectangle as its boundary.

THEOREM B.3.2.1 (Asymptotic Residue Theorem). *Let $a < b$ be real numbers, $f : \mathbf{C}_{(a,b)} \to \mathbf{C}$ a meromorphic function with slow growth near its poles within $\mathbf{C}_{(a,b)}$. Let $a' < b'$ be real numbers such that $[a,b] \subset (a',b')$, and $\Phi : \mathbf{C}_{(a',b')} \to \mathbf{C}$ a holomorphic function that is uniformly Schwartz.*

Let $\varepsilon_1, \varepsilon_2 > 0$ be real numbers such that $a + \varepsilon_1 < b - \varepsilon_2$, and such that f has no poles on the vertical lines $\sigma = a + \varepsilon_1, b - \varepsilon_2$. Then for $\{T_k\}_{k\geq 0}$ the sequence of real numbers witnessing the slow growth of f near its poles, we have:

(1) *the following limit exists:*

$$\lim_{|k|\to+\infty} \sum_{\rho\in U_{T_k,\varepsilon_1,\varepsilon_2}} \Phi(\rho) \operatorname*{Res}_{s=\rho} f(s) \tag{B.3.2.1}$$

(*where the sum in* (B.3.2.1) *is indexed by the poles of* f *contained in* $U_{T_k,\varepsilon_1,\varepsilon_2}$),

(2)

$$-\frac{1}{2\pi i}\int_{a+\varepsilon_1-i\infty}^{a+\varepsilon_1+i\infty} \Phi(s)f(s)\,\mathrm{d}s + \frac{1}{2\pi i}\int_{b-\varepsilon_2-i\infty}^{b-\varepsilon_2+i\infty} \Phi(s)f(s)\,\mathrm{d}s = \lim_{|k|\to+\infty} \sum_{\rho\in U_{T_k,\varepsilon_1,\varepsilon_2}} \Phi(\rho) \operatorname*{Res}_{s=\rho} f(s) \tag{B.3.2.2}$$

with absolutely convergent integrals on the left side.

Proof of Theorem B.3.2.1. We will apply the Residue Theorem to the rectangles $\Gamma_{T_k,\varepsilon_1,\varepsilon_2}$ and let $|k|$ tend to $+\infty$, using Lemma B.3.1.2 to get the asymptotic vanishing of the integrals over the horizontal segments of $\Gamma_{T_k,\varepsilon_1,\varepsilon_2}$.

The Residue Theorem applied to the meromorphic function $\Phi(s)f(s)$ whose poles are not on the boundary $\Gamma_{T_k,\varepsilon_1,\varepsilon_2}$ yields

$$\frac{1}{2\pi i}\int_{\Gamma_{T_k,\varepsilon_1,\varepsilon_2}} \Phi(s)f(s)\,\mathrm{d}s = \sum_{\rho\in U_{T_k,\varepsilon_1,\varepsilon_2}} \Phi(\rho)\operatorname{Res}_{s=\rho} f(s). \tag{B.3.2.3}$$

The left side of (B.3.2.3) is

$$-\frac{1}{2\pi i}\int_{a+\varepsilon_1-iT_k}^{a+\varepsilon_1+iT_k} + \frac{1}{2\pi i}\int_{b-\varepsilon_2-iT_k}^{b-\varepsilon_2+iT_k} - \frac{1}{2\pi i}\int_{a+\varepsilon_1-iT_k}^{b-\varepsilon_2-iT_k} + \frac{1}{2\pi i}\int_{a+\varepsilon_1+iT_k}^{b-\varepsilon_2+iT_k} (\Phi(s)f(s)\,\mathrm{d}s), \tag{B.3.2.4}$$

The uniformly Schwartz condition on Φ and slow growth condition on f (which includes at most polynomial growth along vertical lines) ensure that $\Phi(s)f(s)$ is absolutely integrable on the two vertical lines $\mathrm{Re}(s) = a+\varepsilon_1, b-\varepsilon_2$, and if we let $|k|\to+\infty$ then by Lemma B.3.1.2 the third and fourth integrals in (B.3.2.4) tend to 0 as $|k|\to\infty$, so as $|k|\to\infty$ the left side of (B.3.2.3) has a limit that is equal to

$$-\frac{1}{2\pi i}\int_{a+\varepsilon_1-i\infty}^{a+\varepsilon_1+i\infty} \Phi(s)f(s)\,\mathrm{d}s + \frac{1}{2\pi i}\int_{b-\varepsilon_2-i\infty}^{b-\varepsilon_2+i\infty} \Phi(s)f(s)\,\mathrm{d}s. \tag{B.3.2.5}$$

We deduce that the limit in (B.3.2.1) converges, and that (B.3.2.2) holds (with absolutely convergent integrals). □

B.4. Explicit Formulas for Hecke L-functions

In this section we review for ease of access the explicit formulas for Hecke L-functions proved by Weil in [**Weil-b**]. Barner's generalization in [**Bar**, §2] has the advantage of identifying a class of functions on which the functionals arising in the explicit formulas and corresponding to the Archimedean places attain a simpler form.

The main references for this section are Lang's book [**Lang-b**, Ch. XVII], where the explicit formulas are reproved under conditions that mildly generalize Barner's in [**Bar**, §2].

Occasional reference is also made to Lang's book [**Lang-a**, Ch. X], an earlier edition to [**Lang-b**] (where the arguments in Weil's original article [**Weil-b**] are presented and, specifically, the continuity properties of the functionals arising in the explicit formulas for Hecke L-functions are proved).

B.4.1. Complements on Poisson Summation and Cauchy Integrals. In this subsection we will be proving identities between values of Dirac combs on piece-wise smooth functions on the positive real half-line with isolated jump singularities, by means of Poisson Summation and contour integration of their Mellin transform.

B.4.1.1. *Slow decay in the presence of isolated jump singularities.* In this subsection we recall basic properties of the Mellin transform of smooth functions supported over the positive reals and decaying rapidly at zero and infinity, and we recall the Poisson Summation Formula over $\mathbf{R}_{>0}$ and its quotients by the discrete subgroups $\theta^{\mathbf{Z}}$, for $\theta \in \mathbf{R}_{>0} - \{1\}$.

Let $\theta \in \mathbf{R}_{>0} - \{1\}$. A continuous character $\chi : \theta^{\mathbf{Z}} \to \mathrm{U}(1)$ is of the form $\chi(\theta^m) = \theta^{imx}$ for some $x \in \mathbf{R}$. The imaginary number ix is uniquely determined modulo the discrete group $2\pi i\mathbf{Z}/\log(\theta)$. We will refer to a choice of ix as $\log \chi(\theta)$, and all constructions and assertions made throughout this section involving $\log \chi(\theta)$ will, a priori, depend on such choice.

For a function $\phi : \mathbf{R} \to \mathbf{C}$, we say ϕ is *piece-wise smooth* if and only if there is a discrete subset $\Sigma \subset \mathbf{R}$ such that ϕ is smooth on $\mathbf{R} - \Sigma$ and all $\phi^{(j)}$'s have both one-sided limits at each $x \in \Sigma$. We refer the reader to [**Fol-2**, Ch. 2, §2.2, p. 32] for more on piece-wise smooth functions on the real line. A function $\phi : \mathbf{R}_{>0} \to \mathbf{C}$ is called *piece-wise smooth* if and only if $\phi(e^x)$ is piece-wise smooth as a function of $x \in \mathbf{R}$.

We begin with the following:

LEMMA B.4.1.1 (Slow decay in the presence of isolated jump singularities). *Let $\phi : \mathbf{R}_{>0} \to \mathbf{C}$ be a function. Assume:*

(1) *ϕ is smooth away from finitely many jump singularities; i.e., there exist finitely many points $\{t_1, \ldots, t_N\} \subset \mathbf{R}_{>0}$ such that for each integer $k \geq 0$, the derivative $(t\partial_t)^k\phi$ exists and is continuous away from $\{t_1, \ldots, t_N\}$ and its one-sided limits as $t \to t_m^{\pm}$ exist for each $1 \leq m \leq N$,*
(2) *$|\phi(t)|$ and $|(t\partial_t)\phi(t)|$ are bounded and decay faster than any positive power of $e^{-|\log(t)|}$ as $|\log(t)| \to +\infty$; i.e., for every integer $N \geq 0$, there exist real numbers $C_N > 0$ and $T > 0$ such that for all $t \in \mathbf{R}_{>0}$ with $|\log(t)| > T$, we have:*

$$|\phi(t)|, |(t\partial_t)\phi(t)| < C_N e^{-N|\log(t)|}. \tag{B.4.1.1}$$

Then the Mellin transform

$$\Phi(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \frac{\mathrm{d}t}{t}, \ s \in \mathbf{C} \tag{B.4.1.2}$$

is entire, and we have

$$|\Phi(s)| \ll |s|^{-1} \tag{B.4.1.3}$$

uniformly over the vertical strips of bounded width in $\mathbf{C}$ as $|s| \to +\infty$ over each fixed such strip.

PROOF. Let $\{t_1, \ldots, t_N\} \subset \mathbf{R}_{>0}$ be the finitely many singularities of ϕ, and call $A > 0$ a real number such that $\{t_1, \ldots, t_N\} \subset (1/A, A)$. Then $\int_{1/A}^{A} \phi(t) t^s \frac{\mathrm{d}t}{t}$ is an entire function of $s \in \mathbf{C}$ since (i) ϕ is bounded over $[1/A, A]$ and (ii) by (2), $\int_0^{1/A} + \int_A^{+\infty} \left(\phi(t) t^s \frac{\mathrm{d}t}{t}\right)$ (limit of $\int_{1/n}^{1/A} + \int_A^n$ as $n \to \infty$) is an entire function of $s \in \mathbf{C}$ since both integrals converge absolutely and uniformly on all vertical strips of bounded width in $\mathbf{C}$. This proves Φ is entire.

It remains to prove (B.4.1.2). Let $\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ be a smooth partition of unity on $\mathbf{R}_{>0}$ with φ_m compactly supported near t_m for $1 \leq m \leq N$ and equal to 1 near t_m (so φ_0 vanishes near each t_m), we can write

$$\phi = \varphi_0\phi + \varphi_1\phi + \cdots + \varphi_N\phi.$$

We can apply the *proof* of Proposition B.2.1.9(2) (with the integer N there, not to be confused with N above, equal to 1) to the function $\varphi_0\phi$ because the given exponential decay of ϕ and ϕ' near $0, \infty$ is all that proof requires to attain the conclusion there with the integer N there equal to 1. Thus, we can pass to $\phi - \varphi_0\phi$ and then separately treat each $\varphi_m\phi$ with $1 \leq m \leq N$; i.e., it is enough to treat the case ϕ is compactly

supported over $(1/A, A)$ for some $A > 1$ and is smooth except for one jump singularity at $t = 1$. Define $a := \phi(1^+) - \phi(1^-)$.

We have

$$a - \int_{1/A}^{1} \phi'(t)t^{s+1}\,\frac{\mathrm{d}t}{t} - \int_{1}^{A} \phi'(t)t^{s+1}\,\frac{\mathrm{d}t}{t} = s\int_{1/A}^{1} \phi(t)t^{s}\,\frac{\mathrm{d}t}{t} + s\int_{1}^{A} \phi(t)t^{s}\,\frac{\mathrm{d}t}{t}, \tag{B.4.1.4}$$

which gives

$$\Phi(s) = \frac{a}{s} - \frac{1}{s}\int_{0}^{+\infty} \phi'(t)t^{s+1}\,\frac{\mathrm{d}t}{t}. \tag{B.4.1.5}$$

We fix a bounded interval $I \subset \mathbf{R}$, and since ϕ' is also compactly supported over $(1/A, A)$ with at most one jump discontinuity there exists a real number $C_I > 0$ such that $\int_0^{+\infty} |\phi'(t)|t^{\sigma}\,\mathrm{d}t \leq C_I$ for all $\sigma \in I$.

We deduce that (B.4.1.3) holds for ϕ with implied constant $C_I + |a|$. □

Proposition B.4.1.2 (Poisson Summation Formula - jump singularities). *Let $\theta \in \mathbf{R}_{>0} - \{1\}$ be a real number, $\phi : \mathbf{R}_{>0} \to \mathbf{C}$ a function as in Lemma* B.4.1.1. *Let*

$$\Phi(s) := \int_{\mathbf{R}_{>0}} \phi(t)t^{s}\,\frac{\mathrm{d}t}{t},\ s \in \mathbf{C} \tag{B.4.1.6}$$

be the Mellin transform of ϕ (an entire function by Lemma B.4.1.1*) and χ a continuous character of the discrete subgroup $\theta^{\mathbf{Z}} \subset \mathbf{R}_{>0}$.*

The following identity holds:

$$\log(\theta)\sum_{m\in\mathbf{Z}} \chi(\theta^m)\frac{\lim_{t\to(\theta^m)^+}\phi(t) + \lim_{t\to(\theta^m)^-}\phi(t)}{2} = \sum_{m\in\mathbf{Z}} \Phi\left(\frac{\log\chi(\theta) + 2\pi i m}{\log(\theta)},\right) \tag{B.4.1.7}$$

where the series in the right side is convergent and independent of the choice of $\log\chi(\theta)$.

We refer the reader to [**Pinsky**, Ch. 4, §4.2, Exercise 4.2.9, p. 226] for a generalization in the case $\chi = 1$. Before proving Proposition B.4.1.2, we make some remarks.

Remark B.4.1.3. There are at most finitely many integers $m \in \mathbf{Z}$ such that $\theta^m \in \{t_1, \ldots, t_N\}$, and so, away from finitely many integers $m \in \mathbf{Z}$ we have

$$\frac{\lim_{t\to(\theta^m)^+}\phi(t) + \lim_{t\to(\theta^m)^-}\phi(t)}{2} = \phi(\theta^m).$$

Remark B.4.1.4. The series in the right side of (B.4.1.7) is generally not absolutely convergent under the stated assumptions on ϕ,[12] though it is so if ϕ is smooth (as opposed to only piece-wise smooth). An example of ϕ that satisfies the assumptions but is not smooth is

$$\phi(t) := 1_{[1,2]}(t). \tag{B.4.1.8}$$

We have:

$$\Phi(s) = \int_{1}^{2} t^{s}\,\frac{\mathrm{d}t}{t} = \frac{2^s - 1}{s} \tag{B.4.1.9}$$

which is entire, with value $\log(2)$ at $s = 0$.

Upon choosing, for instance, $\chi = 1$ and $\theta = 2$, we obtain $\Phi\left(\frac{2\pi i m}{\log(2)}\right) = 0$ for all $m \in \mathbf{Z} - \{0\}$ whereas $\Phi(0) = \log(2)$. The left side of (B.4.1.7) is

$$\log(2)\sum_{m\in\mathbf{Z}} \frac{1_{[1,2]}((2^m)^+) + 1_{[1,2]}((2^m)^-)}{2} = \log(2)\frac{1_{[1,2]}(1)}{2} + \log(2)\frac{1_{[1,2]}(2)}{2} = \log(2).$$

[12] Can you give an example in which absolute convergence of the sum provably does not hold?

PROOF OF PROPOSITION B.4.1.2. For $\psi(u) = \phi(u^{\log\theta})$ with $u > 0$, the analogous decay hypotheses hold (as $u \mapsto u^{\log\theta}$) is an automorphism of $\mathbf{R}_{>0}$) and its Mellin transform is

$$\Psi(s) = \frac{1}{\log\theta}\,\Phi\left(\frac{s}{\log\theta}\right).$$

One thereby checks that (B.4.1.7) for ϕ, Φ, θ is the same as (B.4.1.7) for ψ, Ψ, e. Thus, now we may and do assume $\theta = e$, which simplifies the notation and algebraic work considerably.

For ϕ_0 satisfying the hypotheses (with $\theta = e$), define the $\mathbf{Z}$-periodic function

$$\alpha(x) := \sum_{m\in\mathbf{Z}} \phi_0(e^{x+m}) \tag{B.4.1.10}$$

of $x \in \mathbf{R}$, where the series converges absolutely and locally uniformly since $\phi_0(e^x)$ (as a function of $x \in \mathbf{R}$) decays faster than any positive power of $e^{-|x|}$ as $|x| \to +\infty$. Since $(\mathrm{d}/\mathrm{d}x)(\phi_0(e^x))$ also has such exponential decay, clearly $\alpha(x)$ is C^1 away from the t_m's, at which α and α' have at most jump discontinuities.

Since α is a $\mathbf{Z}$-periodic piecewise-C^1 function on $\mathbf{R}$, by [Fol-2, §2.2, Thm. 2.1, p. 35] (where "piecewise smooth" *means* "piecewise C^1" by definition), its Fourier series converges pointwise:

$$\lim_{N\to+\infty} \sum_{m\in\mathbf{Z}:|m|\le N} a_m e^{2\pi i m x} \tag{B.4.1.11}$$

exists, where the Fourier coefficients are given by

$$a_m := \int_0^1 \alpha(x)\, e^{-2\pi i m x}\,\mathrm{d}x. \tag{B.4.1.12}$$

and the pointwise limit of (B.4.1.11) is:

$$\frac{\alpha(x^+) + \alpha(x^-)}{2} \tag{B.4.1.13}$$

where for each sign $\pm$ we define the notation $f(x^\pm) := \lim_{u\to x^\pm} f(u)$ for a function f defined on a punctured neighborhood of x. In particular, the limit in (B.4.1.11) is $\alpha(x)$ at each point $x \in \mathbf{R}$ at which α is continuous.

We have:

$$a_m = \int_0^1 \alpha(x)\, e^{-2\pi i m x}\,\mathrm{d}x = \int_0^1 \left\{\sum_{k\in\mathbf{Z}} \phi_0(e^{x+k})\right\} e^{-2\pi i m x}\,\mathrm{d}x = \sum_{k\in\mathbf{Z}} \int_0^1 \phi_0(e^{x+k})\, e^{-2\pi i m x}\,\mathrm{d}x.$$

The exchange in the order of summation and integration is valid due to $\phi_0(e^x)$ being in $L^1(\mathbf{R})$.

We further have:

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} \int_0^1 \phi_0(e^{x+k})\, e^{-2\pi i m x}\,\mathrm{d}x &= \sum_{k\in\mathbf{Z}} \int_k^{k+1} \phi_0(e^x)\, e^{-2\pi i m(x-k)}\,\mathrm{d}x \\
&= \sum_{k\in\mathbf{Z}} \int_k^{k+1} \phi_0(e^x)\, e^{-2\pi i m x}\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \phi_0(e^x)\, e^{-2\pi i m x}\,\mathrm{d}x \\
&= \mathcal{F}(\phi_0(e^x))(m)
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbf{R}$. We deduce the identity of sums of pointwise convergent series:

$$\sum_{m\in\mathbf{Z}:|m|\le N_0} \frac{\phi_0(e^{x^+ + m}) + \phi_0(e^{x^- + m})}{2} + \sum_{m\in\mathbf{Z}:|m|>N_0} \phi_0(e^{x+m}) = \sum_{m\in\mathbf{Z}} \mathcal{F}(\phi_0(e^x))(m) e^{2\pi i m x} \tag{B.4.1.14}$$

where $N_0 > 0$ is a large enough integer locally dependent on x such that for every $m \in \mathbf{Z}$ with $|m| > N_0$ we have $x + m \notin \{e^{t_1}, \dots, e^{t_N}\}$.

Any continuous character of $e^{\mathbf{Z}}$ lifts to a continuous character of $\mathbf{R}_{>0}$. The dual group $\widehat{e^{\mathbf{Z}}}$ is the quotient $\mathbf{R}/2\pi\mathbf{Z}$, and it is described as the group of characters of the form $\chi(e) = e^{-iu}$ for $u \in \mathbf{R}$. For χ one such character, u is uniquely determined modulo $2\pi\mathbf{Z}$.

For $u_0 \in \mathbf{R}$ the unique real number contained in the interval $[0, 2\pi)$ such that $\chi(e) = e^{-iu_0}$, we see that $\widetilde{\chi}(x) := x^{-iu_0}$ is a lift of χ to a continuous character of $\mathbf{R}_{>0}$. As such, $\widetilde{\chi}$ is smooth and multiplication by $\widetilde{\chi}$ preserves the hypotheses on ϕ.

Now set $\phi_0 := \widetilde{\chi}\phi$ and apply (B.4.1.14) to this choice of ϕ_0 to obtain, for every $x \in \mathbf{R}_{>0}$:

$$\sum_{m\in\mathbf{Z}:|m|\le N_0} \chi(e^m)\frac{\phi(e^{x^+ + m}) + \phi(e^{x^- + m})}{2} + \sum_{m\in\mathbf{Z}:|m|>N_0} \chi(e^m)\phi(e^{x+m}) = \sum_{m\in\mathbf{Z}} \mathcal{F}(\phi_0(e^x))(m)e^{2\pi imx} \tag{B.4.1.15}$$

and we compute:

$$\mathcal{F}(\phi_0(e^x))(y) = \int_{\mathbf{R}} \widetilde{\chi}(e^x)\phi(e^x)e^{-2\pi ixy}\,\mathrm{d}x = \int_{\mathbf{R}} e^{-2\pi ix\frac{u_0}{2\pi}}\phi(e^x)e^{-2\pi ixy}\,\mathrm{d}x = \Phi(-iu_0 - 2\pi iy). \tag{B.4.1.16}$$

Upon combining (B.4.1.10) and (B.4.1.16), we deduce that we have

$$\sum_{m\in\mathbf{Z}:|m|>N_0} \chi(e^m)\frac{\phi((e^m)^+) + \phi((e^m)^-)}{2} + \sum_{m\in\mathbf{Z}:|m|>N_0} \chi(e^m)\phi(e^m) = \sum_{m\in\mathbf{Z}} \Phi(-iu_0 - 2\pi im) \tag{B.4.1.17}$$

with $\phi(t^\pm) := \lim_{u\to t^\pm}\phi(u)$ for every $t \in \mathbf{R}_{>0}$.

We note that the series on the right side of (B.4.1.17) is convergent and invariant under exchanging positive and negative integers, and invariant under the left-translation action on Φ by $2\pi i\mathbf{Z}$. In particular, it is independent of the choice of $\log\chi(e) = -iu_0$, since such choice is unique modulo translation of $i\mathbf{R}$ by $2\pi i\mathbf{Z}$. □

Remark B.4.1.5. For the identity in (B.4.1.7) with $\chi = 1$, we further refer the reader to [**Horm**, Ch. VII, Thm. 7.2.1] in its form of [**Horm**, Ch. VII, p. 178, (7.2.1)′] applied to $\phi(e^x)$, $a := 2\pi/\log(\theta)$ and $n = 1$ (noting that our convention in defining the Fourier transform on the real line follows [**Fol**, Ch. 8, p. 249], resting upon the choice of $\exp(-2\pi ix)$ as additive character of $\mathbf{R}$, whereas in [**Horm**, Ch. VII, Def. 7.1.1, (7.1.3)] the renormalizing factor of 2π is missing from the chosen additive character $\exp(ix)$).

B.4.1.2. *Cauchy Integrals of $\Phi(s)L_v'(\chi,s)/L_v(\chi,s)$ and weighted Dirac combs.* We give preparations on using the Asymptotic Residue Theorem to compute line integrals of $\Phi(s)L_v'(\chi,s)/L_v(\chi,s)$ with v a finite place of a number field F. In what follows we write $L_f(s,\chi)$ for the finite Hecke L-function of a continuous (unitary) character $\chi \in \widehat{C_F}$, and Φ for the entire Mellin transform of a choice of Schwartz function on $\mathbf{R}_{>0}$ with exponential decay (Definition B.2.1.1(2)).

For F a number field, $\chi \in \widehat{C_F}$ a continuous character, and $\sigma \in \mathbf{R}$ a real number, the formal sum of Dirac measures

$$D_{\chi,\mathfrak{p}} := \sum_{m>0} \chi(\mathfrak{p}^m)N(\mathfrak{p})^{m\sigma}\delta_{N(\mathfrak{p})^m} \tag{B.4.1.18}$$

will be referred to as the *local Dirac comb* at the maximal ideal $\mathfrak{p} \subset \mathcal{O}_F$, assuming χ is unramified at $\mathfrak{p}$.

The formal sum $D_{\chi,\mathfrak{p}}$ consists of terms in the topological dual space $\mathcal{S}(\mathbf{R})'$ (called "tempered distributions") of the locally convex Schwartz space $\mathcal{S}(\mathbf{R})$, and it makes sense as an element of $\mathcal{S}(\mathbf{R})'$ too: if $f \in \mathcal{S}(\mathbf{R})$ satisfies

$|f(x)| \leq A|x|^{-(\sigma+2)}$ for $|x| \geq 1$ then

$$\sum_{m>0} N(\mathfrak{p})^{m\sigma}|f(N(\mathfrak{p})^m)| \leq A\sum_{m\geq 1} N(\mathfrak{p})^{-2} = \frac{A}{N(\mathfrak{p})^2-1},$$

so for any $\varepsilon > 0$ if we require $|f(x)| \leq B/(1+|x|^{\sigma+2})$ for all $x \in \mathbf{R}$ and sufficiently small B (depending on ε and $N(\mathfrak{p})$) then $|D_{\chi,\mathfrak{p}}(f)| < \varepsilon$. The sum $D_{\chi,\mathfrak{p}}$ therefore converges for the weak topology $\sigma(\mathcal{S}(\mathbf{R})', \mathcal{S}(\mathbf{R}))$ on $\mathcal{S}(\mathbf{R})'$ since it converges pointwise.

REMARK B.4.1.6. Although not needed for our purposes, the sum $D_{\chi,\mathfrak{p}}$ even converges for the strong-dual topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbf{R})$ in the sense of topological vector spaces (not to be confused with metric-boundedness; see §C.2.1–§C.2.2 for a discussion of topologies on topological duals of Hausdorff locally convex topological vector spaces).

To prove the convergence in the strong-dual topology $\mathcal{S}(\mathbf{R})'_\beta$, we first note that $\mathcal{S}(\mathbf{R})$ is complete and barreled (as is any Fréchet space [**Scha**, Ch. II, 7.1]) and its closed bounded subsets (in the sense of topological vector spaces) are compact by Lemma D.1.3.7(3). For any barreled Hausdorff locally convex topological vector space E, a sequence $\{\varphi_n\}$ in E' that is pointwise convergent (i.e., $\{\varphi_n(x)\}$ is convergent in $\mathbf{C}$ for all $x \in E$) is pointwise bounded, which is to say it is bounded for the weak topology $\sigma(E', E)$. Thus, by [**Scha**, Ch. IV, 4.6] the pointwise limit $\varphi : E \to \mathbf{C}$ belongs to E' and $\varphi_n \to \varphi$ uniformly on all precompact subsets of E. Since bounded subsets have bounded closure [**Scha**, Ch. I, 5.1], if all closed bounded subsets of E are compact (such as for $E = \mathcal{S}(\mathbf{R})$) then $\varphi_n \to \varphi$ uniformly on bounded subsets; i.e., $\varphi_n \to \varphi$ in $E'_{\beta(E',E)}$.

In this subsection we provide a method to compute the value of $D_{\chi,\mathfrak{p}}$ at certain Schwartz functions on the real line by means of a version of the Poisson Summation Formula from Proposition B.4.1.2 combined with the Asymptotic Residue Theorem (i.e., Theorem B.3.2.1).

LEMMA B.4.1.7 (Poisson Summation and Cauchy Integrals). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character, $\mathfrak{p} \subset \mathcal{O}_F$ a prime ideal such that χ is unramified at $\mathfrak{p}$. Let $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ be a Schwartz function on $\mathbf{R}_{>0}$ with exponential decay (Definition B.2.1.1(2)).*

Let

$$\Phi(s) := \int_{\mathbf{R}_{>0}} \phi(t)t^s \frac{\mathrm{d}t}{t} \tag{B.4.1.19}$$

be the entire and uniformly Schwartz Mellin transform of ϕ.

Then, for every real number $a > 0$, the following identities with absolutely convergent sums hold:

(B.4.1.20)

$$\log N(\mathfrak{p}) \sum_{m\in\mathbf{Z}} \chi(\mathfrak{p}^m)\phi(N(\mathfrak{p})^m) = \sum_{m\in\mathbf{Z}} \Phi\left(\frac{\log\chi(\mathfrak{p}) + 2\pi i m}{\log N(\mathfrak{p})}\right) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Phi(s)\frac{\log N(\mathfrak{p})}{1-\chi(\mathfrak{p})^{-1}N(\mathfrak{p})^s}\,\mathrm{d}s$$

for any choice of $\log\chi(\mathfrak{p})$.

We provide a proof of Lemma B.4.1.7 below.

REMARK B.4.1.8. The identity is obtained upon combining [**Den**, (2.13)] with [**Den**, (2.18)] and [**Den**, (2.19)], noting that Φ fulfils the requirements of [**Den**, Thm.1.7] (which are only needed for $\widetilde{a} = a$, in the notation of loc.cit.). Indeed, since ϕ is Schwartz and with exponential decay, then $\Phi(\sigma + ix)$ is Schwartz as a function of x for every σ, and in fact uniformly for σ varying over bounded intervals, by Lemma B.2.1.9(2). This implies that Φ satisfies conditions [**Den**, Thm.1.7,(1),(2)]. We need only show that condition [**Den**, Thm.1.7,(3)] is also satisfied. This is a consequence of Lemma B.2.1.9(3).

The identity in (B.4.1.20) therefore holds by [**Den**, (2.18)]. Note that the restriction on a to satisfy $a \leq 2$ in [**Den**, Thm.1.17] (with $w = 0$ there) is unnecessary for the arguments proving [**Den**, (2.18)] to hold, and it

is only used to obtain the identity in [Den, Thm.1.7,(4)] by accessing the estimates from [Lang-b, Ch. XVII, Prop. 2.7].

LEMMA B.4.1.9. *The following identity holds for $z \in \mathbf{C} - \mathbf{Z}$:*

$$(1 - e^{2\pi i z})^{-1} = \frac{1}{2} - \frac{1}{2\pi i z} - \frac{1}{2\pi i} \sum_{m \in \mathbf{Z} - \{0\}} \left(\frac{1}{z - m} + \frac{1}{m} \right) \tag{B.4.1.21}$$

where the series in the rightmost side of (B.4.1.21) *is absolutely and locally uniformly convergent over $\mathbf{C} - \mathbf{Z}$. Moreover, for every real number $a \neq 0$, we have on the horizontal line $\{\mathrm{Im}(z) = a\}$:*

$$\left| \sum_{m \in \mathbf{Z} - \{0\}} \left(\frac{1}{z - m} + \frac{1}{m} \right) \right| \leq C_a \tag{B.4.1.22}$$

and

$$\sum_{m \in \mathbf{Z} - \{0\}} \left| \frac{1}{z - m} + \frac{1}{m} \right| \leq C'_a \, |z| \tag{B.4.1.23}$$

where $C_a, C'_a > 0$ are constants depending only on a.

PROOF. Let $A > 0$ be a real number, and $z \in \mathbf{C}$ be such that $|z| \leq A$. For any integer m such that $|m| \geq 2A$, we have

$$\left| \frac{1}{z - m} + \frac{1}{m} \right| = \frac{|z|}{|m(z - m)|} \leq \frac{A}{|m| \cdot ||z| - |m||} \leq \frac{2A}{|m|^2}$$

and so for any $A > 0$, the series in the rightmost side of (B.4.1.21) is the sum of a series that converges absolutely and uniformly over the closed ball of radius A and centered at the origin, and finitely many meromorphic functions of the form $\frac{1}{z-m} + \frac{1}{m}$. This shows that the series converges absolutely and uniformly on the compact subsets of $\mathbf{C} - \mathbf{Z}$.

From the classical identity

$$\psi(1 - z) = -\gamma + \sum_{m \geq 1} \left(\frac{1}{z - m} + \frac{1}{m} \right) \tag{B.4.1.24}$$

for $\psi(z) := \Gamma'(z)/\Gamma(z)$ over $\mathbf{C} - \mathbf{N}_{>0}$, we likewise have (evaluating with $-z$)

$$\psi(z + 1) = -\gamma - \sum_{m \leq -1} \left(\frac{1}{m} + \frac{1}{z - m} \right) \tag{B.4.1.25}$$

over $\mathbf{C} - \mathbf{N}_{<0}$. Moreover, from $\Gamma(z + 1) = z\Gamma(z)$ we obtain

$$\psi(z + 1) = \psi(z) + 1/z \tag{B.4.1.26}$$

over $\mathbf{C} - \mathbf{N}_{\leq 0}$. Taking the log-derivative of Euler's reflection formula for the Γ-function (i.e. $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$, valid over $\mathbf{C} - \mathbf{Z}$) yields

$$\psi(1 - z) - \psi(z) - \frac{1}{z} = \pi \cot(\pi z) - \frac{1}{z} \tag{B.4.1.27}$$

over $\mathbf{C} - \mathbf{Z}$.

By (B.4.1.24), (B.4.1.25), and (B.4.1.26), the left side of (B.4.1.27) agrees with $\sum_{m \in \mathbf{Z} - \{0\}} \left(\frac{1}{z-m} + \frac{1}{m} \right)$. Since we have

$$\pi \cot(\pi z) = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \frac{2\pi i}{2} \frac{e^{2\pi i z} - 1 + 2}{e^{2\pi i z} - 1} = \pi i + 2\pi i \frac{1}{e^{2\pi i z} - 1}, \tag{B.4.1.28}$$

the identity in (B.4.1.21) holds over $\mathbf{C}-\mathbf{Z}$. From (B.4.1.27) and (B.4.1.28) we deduce for $z=\sigma+ia$ with $\sigma\in\mathbf{R}$ and real $a\neq 0$ that

$$\left|\sum_{m\in\mathbf{Z}-\{0\}}\left(\frac{1}{z-m}+\frac{1}{m}\right)\right| = \left|\pi i+\frac{2\pi i}{e^{2\pi i\sigma}e^{-2\pi a}-1}-\frac{1}{\sigma+ia}\right|\leq C_a \tag{B.4.1.29}$$

which is (B.4.1.22).

To obtain (B.4.1.23) we proceed as follows. Over $\mathbf{C}-(\mathbf{Z}-\{0\})$ we have

$$\sum_{m\in\mathbf{Z}-\{0\}}\left|\frac{1}{z-m}+\frac{1}{m}\right| = |z|\sum_{m\in\mathbf{Z}-\{0\}}\left|\frac{1}{(z-m)m}\right|\leq |z|(2\zeta(2))^{1/2}\left(\sum_{m\in\mathbf{Z}-\{0\}}\frac{1}{|z-m|^2}\right)^{1/2} \tag{B.4.1.30}$$

by the Cauchy-Schwartz inequality. The series $\sum_{m\in\mathbf{Z}-\{0\}}\frac{1}{|z-m|^2}$ in (B.4.1.30) is absolutely and locally uniformly convergent over $\mathbf{C}-(\mathbf{Z}-\{0\})$, and its sum is continuous and horizontally 1-periodic. In particular, it is bounded on all horizontal lines $\{\mathrm{Im}(z)=a\}$ for all real numbers $a\neq 0$. We call its sum A_a, and define $C'_a:=(2\zeta(2))^{1/2}A_a^{1/2}$ to see that (B.4.1.23) holds. □

Proof of Lemma B.4.1.7. We proceed as in [Den, p. 8] and observe that upon setting

$$z:=\frac{1}{2\pi i}(s\log N(\mathfrak{p})-\log\chi(\mathfrak{p})) \tag{B.4.1.31}$$

in (B.4.1.21), since $\mathrm{d}/\mathrm{d}s=((\log N(\mathfrak{p}))/(2\pi i))\,\mathrm{d}/\mathrm{d}z$ and

$$L_{\mathfrak{p}}(s,\chi)=\frac{1}{1-\chi(\mathfrak{p})(N\mathfrak{p})^{-s}}=\frac{1}{1-e^{-2\pi iz}}$$

has z-derivative $2\pi i/(1-e^{2\pi iz})$, we obtain

$$\frac{L'_{\mathfrak{p}}(s,\chi)}{L_{\mathfrak{p}}(s,\chi)}=\frac{1}{2}\log N(\mathfrak{p})-\frac{1}{s-\frac{\log\chi(\mathfrak{p})}{\log N(\mathfrak{p})}}-\sum_{m\in\mathbf{Z}-\{0\}}\left(\frac{1}{s-\frac{\log\chi(\mathfrak{p})+2\pi im}{\log N(\mathfrak{p})}}+\frac{\log N(\mathfrak{p})}{2\pi im}\right). \tag{B.4.1.32}$$

The change of variables in (B.4.1.31) is affine-linear and hence by Lemma B.4.1.9 the series on the right side of (B.4.1.32) is absolutely and locally uniformly convergent on compact subsets away from the image of the integers under the inverse of (B.4.1.31), which is the discrete set $\log\chi(\mathfrak{p})+(2\pi i/N(\mathfrak{p}))\mathbf{Z}$.

For any $\sigma\neq 0$, the estimate in (B.4.1.23) from Lemma B.4.1.9 applied to

$$z=\left(\frac{\log N(\mathfrak{p})}{2\pi}t-\frac{\log\chi(\mathfrak{p})}{2\pi i}\right)+\frac{\log N\mathfrak{p}}{2\pi}\,\sigma i$$

(decomposition into real and imaginary parts) translates into the estimate over $\sigma+i\mathbf{R}$:

$$\frac{1}{\left|\sigma+it-\frac{\log\chi(\mathfrak{p})}{\log N(\mathfrak{p})}\right|}+\sum_{m\in\mathbf{Z}-\{0\}}\left|\frac{1}{\sigma+it-\frac{\log\chi(\mathfrak{p})+2\pi im}{\log N(\mathfrak{p})}}+\frac{\log N(\mathfrak{p})}{2\pi im}\right|\leq C_{\mathfrak{p},\sigma}\,|t|. \tag{B.4.1.33}$$

As in [Den, (2.13)], we obtain:

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\frac{L'_{\mathfrak{p}}(s,\chi)}{L_{\mathfrak{p}}(s,\chi)}\,\mathrm{d}s &= \frac{1}{4\pi i}\log N(\mathfrak{p})\int_{a-i\infty}^{a+i\infty}\Phi(s)\,\mathrm{d}s-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Phi(s)}{s-\frac{\log\chi(\mathfrak{p})}{\log N(\mathfrak{p})}}\,\mathrm{d}s\\ &\quad-\sum_{m\in\mathbf{Z}-\{0\}}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\left(\frac{1}{s-\frac{\log\chi(\mathfrak{p})+2\pi im}{\log N(\mathfrak{p})}}+\frac{\log N(\mathfrak{p})}{2\pi im}\right)\mathrm{d}s\end{aligned}$$

where the exchange of the order of summation and integration is valid by Dominated Convergence using $C_{\mathfrak{p},\sigma}\,|\Phi(\sigma+it)||t|$ as a majorant, as provided by (B.4.1.33) which is integrable over $\sigma+i\mathbf{R}$ for each $\sigma\neq 0$ since Φ is Schwartz on vertical lines by Lemma B.2.1.9.

We have $\int_{a+i\mathbf{R}}\Phi(s)\,\mathrm{d}s=\int_{a+i\mathbf{R}}(s\Phi(s)/s)\,\mathrm{d}s=0$ by Theorem B.2.2.4(2) with $N=0$, so

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\frac{L'_{\mathfrak{p}}(s,\chi)}{L_{\mathfrak{p}}(s,\chi)}\,\mathrm{d}s=-\frac{1}{2\pi i}\sum_{m\in\mathbf{Z}}\int_{a-i\infty}^{a+i\infty}\frac{\Phi(s)}{s-\frac{\log\chi(\mathfrak{p})+2\pi im}{\log N(\mathfrak{p})}}\,\mathrm{d}s \tag{B.4.1.34}$$

which by (B.2.2.6) from Theorem B.2.2.4 (with $N=0$ and pure imaginary $z=(\log\chi(\mathfrak{p})+2\pi im)/\log N\mathfrak{p}$) gives

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\frac{L'_{\mathfrak{p}}(\chi,s)}{L_{\mathfrak{p}}(\chi,s)}\,\mathrm{d}s=-\sum_{m\in\mathbf{Z}}\Phi\left(\frac{\log\chi(\mathfrak{p})+2\pi im}{\log N(\mathfrak{p})}\right). \tag{B.4.1.35}$$

By Proposition B.4.1.2 with $\theta=N(\mathfrak{p})$, and noting that $L'_{\mathfrak{p}}(s,\chi)/L_{\mathfrak{p}}(s,\chi)=\log N(\mathfrak{p})/(1-\chi(\mathfrak{p})^{-1}N(\mathfrak{p})^{s})$, we see that (B.4.1.20) holds. □

B.4.2. Explicit formulas from the Asymptotic Residue Theorem. We let $\phi\in\mathcal{S}(\mathbf{R}_{>0})$ be a Schwartz function with exponential decay, and $\Phi(s):=M(\phi)(s)$ its entire Mellin transform.

B.4.2.1. *Logarithmic derivative of completed $L(s,\chi)$ and the Asymptotic Residue Formula.* By [**Lang-b**, Ch. XVII, Prop. 2.3, Cor. 2.5, Prop. 2.7], the function

$$f(s)=\frac{L'(s,\chi)}{L(s,\chi)} \tag{B.4.2.1}$$

has slow growth near its poles (in the sense of Definition B.3.1.1)) over the vertical strip $\mathbf{C}_{(1-a,a)}$ for any $1<a\leq 2$. Moreover, Φ is uniformly Schwartz over any vertical strip. Upon replacing a with $a-\varepsilon$ for small enough $\varepsilon>0$ so that $a>1$ and f has no poles on $\sigma=a,1-a$, we may apply the Asymptotic Residue Theorem (namely Theorem B.3.2.1) to obtain the identity with convergence on both sides

$$\delta_\chi(\Phi(0)+\Phi(1))-\sum_{\rho:L(\rho,\chi)=0}\Phi(\rho)=\lim_{T\to+\infty}\sum_{\rho:|\mathrm{Im}(\rho)|<T}\Phi(\rho)n_\rho$$

where the right side has terms indexed by the poles of $L'(s,\chi)/L(s,\chi)$ (counted *without* multiplicity) contained within $\mathbf{C}_{(1-a,a)}$ and $n_\rho:=\mathrm{ord}_\rho(L(s,\chi))$ the residue at ρ, and the left side having terms indexed by the zeros of $L(s,\chi)$ (counted *with* multiplicity) and being shorthand notation for the limit:

$$\lim_{T\to+\infty}\sum_{\rho:L(\rho,\chi)=0,|\mathrm{Im}(\rho)|<T}\Phi(\rho).$$

The Asymptotic Residue Formula says

$$-\frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Phi(s)\frac{L'(s,\chi)}{L(s,\chi)}\,\mathrm{d}s+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\frac{L'(s,\chi)}{L(s,\chi)}\,\mathrm{d}s=-\delta_\chi(\Phi(0)+\Phi(1))+\sum_{\rho:L(\rho,\chi)=0}\Phi(\rho) \tag{B.4.2.2}$$

Upon using the functional equation from Proposition 3.2.1.3(3) (and expressing[13] $\epsilon(s,\chi)$ there as a product involving the factor $A_\chi^{\alpha s+\beta}$), we have

$$\frac{L'(s,\chi)}{L(s,\chi)}=\frac{L'(\overline{\chi},1-s)}{L(\overline{\chi},1-s)}+\log(A_\chi) \tag{B.4.2.3}$$

where

$$A_\chi:=|N(\mathfrak{d}_{F/\mathbf{Q}})|\cdot N(\mathfrak{f}_\chi).$$

[13]Should a precise reference be given here for the expression for $\epsilon(s,\chi)$ being invoked?

Thus, the following identity holds:

(B.4.2.4)
$$\frac{\log(A_\chi)}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Phi(s)\,\mathrm{d}s+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(1-s)\frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}\,\mathrm{d}s+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Phi(s)\frac{L'(s,\chi)}{L(s,\chi)}\,\mathrm{d}s$$
$$=-\delta_\chi(\Phi(0)+\Phi(1))+\sum_{\rho:L(\rho,\chi)=0}\Phi(\rho).$$

Since $a>1$, absolute and uniform convergence of the Euler product for $L_f(s,\chi)$ in right half-planes $\mathrm{Re}(s)\geq 1+\varepsilon$ implies the absolute and uniform convergence of the series:

(B.4.2.5)
$$\frac{L'(s,\chi)}{L(s,\chi)}=\sum_{v\in\Sigma_F}\frac{L_v'(s,\chi)}{L_v(s,\chi)}$$

over the vertical line $\mathrm{Re}(s)=a$, where $L_v(\chi,s)$ are the local factors of $L(s,\chi)$ defined in (3.2.1.1) and (3.2.1.3). The same holds for

(B.4.2.6)
$$\frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}=\sum_{v\in\Sigma_F}\frac{L_v'(s,\overline{\chi})}{L_v(s,\overline{\chi})}.$$

We further observe that

(B.4.2.7)
$$\frac{L_v'(s,\chi)}{L_v(s,\chi)}=-\log N(\mathfrak{p}_v)\sum_{m>0}\frac{\chi(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ms}},\quad \frac{L_v'(s,\overline{\chi})}{L_v(s,\overline{\chi})}=-\log N(\mathfrak{p}_v)\sum_{m>0}\frac{\overline{\chi}(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ms}}$$

for every finite place v of F, where the series converge absolutely and uniformly over the vertical line $\mathrm{Re}(s)=a$, since $a>1$. This is the starting point of the proof of the Explicit Formulas of Weil (see [**Weil-b**], [**Lang-b**, Ch. XVII, Thm. 3.1], and [**Bar**, Thm. 1]).

B.4.2.2. *Non-Archimedean local factors.* We analyze, for each finite place v of F, the integral:

(B.4.2.8)
$$\log N(\mathfrak{p}_v)\int_{a-i\infty}^{a+i\infty}\Phi(s)\sum_{m>0}\frac{\chi(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ms}}\,\mathrm{d}s.$$

By Dominated Convergence, we are allowed to interchange the order of summation and integration to get

(B.4.2.9)
$$\log N(\mathfrak{p}_v)\sum_{m>0}\chi(\mathfrak{p}_v)^m\int_{a-i\infty}^{a+i\infty}\frac{\Phi(s)}{N(\mathfrak{p}_v)^{ms}}\,\mathrm{d}s=\log N(\mathfrak{p}_v)\sum_{m>0})\frac{i\chi(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ma}}\int_{-\infty}^{+\infty}\frac{\Phi(a+it)}{N(\mathfrak{p}_v)^{imt}}\,\mathrm{d}t.$$

By viewing $\Phi(a+it)\in\mathcal{S}(\mathbf{R})$ and $N(\mathfrak{p}_v)^{imt}$ as tempered distributions for $m>0$, Mellin/Fourier inversion (for $\phi\in\mathcal{S}(\mathbf{R}_{>0})$) gives the identity:

(B.4.2.10)
$$\log N(\mathfrak{p}_v)\frac{\chi(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ma}}\int_{-\infty}^{+\infty}\frac{\Phi(a+it)}{N(\mathfrak{p}_v)^{imt}}\,\mathrm{d}t=\log N(\mathfrak{p}_v)\chi(\mathfrak{p}_v)^m\,\delta_{N(\mathfrak{p}_v)^m}(\phi)$$

where $\delta_{N(\mathfrak{p}_v)^m}$ is Dirac's δ-distribution. Similarly, we obtain (using that $\mathrm{d}s=i\,\mathrm{d}t$ and $1-s=(1-a)-it$)

(B.4.2.11)
$$\log N(\mathfrak{p}_v)\int_{a-i\infty}^{a+i\infty}\Phi(1-s)\frac{\overline{\chi}(\mathfrak{p}_v)^m}{N(\mathfrak{p}_v)^{ms}}\,\mathrm{d}s=-i\log N(\mathfrak{p}_v)\frac{\overline{\chi}(\mathfrak{p}_v^m)}{N(\mathfrak{p}_v)^m}\,\delta_{N(\mathfrak{p}_v)^{-m}}(\phi)$$

B.4.2.3. *Archimedean Local Factors.* We now fix v an Archimedean place.

We have:

(B.4.2.12)
$$\frac{L_v'(s,\chi)}{L_v(s,\chi)}=-\alpha_v+\frac{1}{a_v}\frac{\Gamma'\left(\frac{s-s_v}{a_v}\right)}{\Gamma\left(\frac{s-s_v}{a_v}\right)}$$

where $\alpha_v = -\frac{1}{2}\log(\pi)$ if v is real, $\alpha_v = -\log(2\pi)$ if v is complex, and $a_v = [\mathbf{C} : F_v]$. The complex number s_v is purely imaginary and completely determined by the restriction of the continuous character $\chi \in \widehat{C_F}$ to $F_v^\times$.

We analyze the line integral from (B.4.2.4):

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left\{\Phi(s)\frac{L_v'(s,\chi)}{L_v(s,\chi)} + \Phi(1-s)\frac{L_v'(s,\overline{\chi})}{L_v(s,\overline{\chi})}\right\}\,\mathrm{d}s. \tag{B.4.2.13}$$

PROPOSITION B.4.2.1. *Let F be a number field, $\chi \in \widehat{F_v^\times}$ a continuous character of $F_v^\times$ for an Archimedean place v of F. For any real number $a > 1$, the line integral:*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left\{\Phi(s)\frac{L_v'(s,\chi)}{L_v(s,\chi)} + \Phi(1-s)\frac{L_v'(s,\overline{\chi})}{L_v(s,\overline{\chi})}\right\}\,\mathrm{d}s \tag{B.4.2.14}$$

equals:

$$\int_0^{+\infty}\left\{\frac{N_v\phi(1)}{x} - (\phi(e^x)e^{s_v x} + \phi(e^{-x})e^{-s_v x})\frac{e^{\left(\frac{2}{N_v}-\frac{1}{2}\right)x}}{1-e^{-\frac{2}{N_v}x}}\right\}\,\mathrm{d}x \tag{B.4.2.15}$$

where $N_v = 1$ if v is real, $N_v = 2$ if v is complex, and $s_v \in i\mathbf{R}$ satisfies $\chi_v(t) = t^{s_v}$ for $t \in \mathbf{R}_{>0} \subset F_v^\times$.

PROOF. [14] Upon writing $s_v = -i\varphi$, the proof in [**Bar**, p.150] applies, upon observing that by Stirling's Formula for $\Gamma(z)$ we may and do apply the Asymptotic Residue Theorem (see Theorem B.3.2.1) to replace the vertical line of integration $\mathrm{Re}(s) = a$ with the vertical line $\mathrm{Re}(s) = 1/2$, without picking up any residue.

The line integral in (B.4.2.14) therefore equals:

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\Phi(s)\left\{\frac{\Gamma'}{\Gamma}\left(\frac{N_v(s+i\varphi)}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{N_v(1-s-i\varphi)}{2}\right)\right\}\,\mathrm{d}s. \tag{B.4.2.16}$$

A change of variables in (B.4.2.16) combined with [**Bar**, Proposition, p. 146], applied to ϕ Schwartz with exponential decay, gives (B.4.2.15). We note that in this special case, [**Bar**, Proposition, p. 146] is classical and due to Euler. The equality between (B.4.2.15) and (B.4.2.16) is a consequence of Mellin/Fourier inversion (which is especially tractable since $\phi \in \mathcal{S}(\mathbf{R}_{>0})$ with exponential decay). □

B.4.2.4. *Explicit Formulas for Hecke L-functions.* We can now collect the results obtained in the previous subsections into the following result:[15]

THEOREM B.4.2.2 (Explicit Formulas for Hecke L-functions). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character, $\phi \in \mathcal{S}(\mathbf{R})$ a Schwartz function with exponential decay, $\Phi(s) := M(\phi)(s)$ its Mellin transform. The following identity holds:*

(B.4.2.17)

$$\begin{aligned}-\Phi(-1/2) - \Phi(1/2) + \sum_{\rho: L(\rho,\chi)=0}\Phi(\rho - 1/2) &= \phi(1)\log(A_\chi)\\ &- \sum_{v\in\Sigma_{F,f}}\sum_{m>0}\frac{\log N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)^{m/2}}\left\{\chi(\mathfrak{p}_v)^m\,\phi(N(\mathfrak{p}_v)^m) + \overline{\chi}(\mathfrak{p}_v)^m\,\phi(N(\mathfrak{p}_v)^{-m})\right\}\\ &+ \sum_{v\in S_\infty}\int_0^{+\infty}\left\{\frac{N_v\phi(1)}{x} - (\phi(e^x)e^{s_v x} + \phi(e^{-x})e^{-s_v x})\frac{e^{\left(\frac{2}{N_v}-\frac{1}{2}\right)x}}{1-e^{-\frac{2}{N_v}x}}\right\}\,\mathrm{d}x.\end{aligned}$$

[14] Brian still needs to check proof of this by reading [**Bar**, pp. 150-1], make sure formulas are correct.

[15] Brian should double-check this.

PROOF. The identity in (B.4.2.17) is obtained by combining (B.4.2.2) with (B.4.2.5), (B.4.2.6), (B.4.2.7), (B.4.2.10) and (B.4.2.11), together with Proposition B.4.2.1. □

We note that the main difficulty in the proof of the main result in [**Weil-b**], explained in [**Lang-a**, Ch. X], and in [**Bar**], further explained in [**Lang-b**, Ch. XVII] stems from dropping any smoothness assumptions on the test function ϕ. As discussed in this section, under the additional assumption that ϕ be Schwartz and with exponential decay the Explicit Formulas are ultimately a consequence of the Residue Theorem in its asymptotic version of Theorem B.3.2.1, and Mellin/Fourier inversion.

B.4.3. Fourier transform of $\log|\cdot|_v$. Let F be a number field, v a fixed place of F.

In this subsection we illustrate results of Connes from [**Co**, Appendix II, pp. 72-76], identifying the Fourier transform of the function $J_v := \log|\cdot|_v$, as a tempered distribution on F_v; i.e., an element of $\mathcal{S}(F_v)'$ (we further refer the reader to §2.1 for more on Schwartz-Bruhat functions on a Hausdorff locally compact abelian group; see also [**Bru**, §9]), the topological dual of the Hausdorff locally convex space $\mathcal{S}(F_v)$ of Schwartz-Bruhat functions on the Hausdorff locally compact additive abelian group $(F_v, +)$ of the local field F_v.

B.4.3.1. *Calculation of the additive local Fourier transform of* $\log|\cdot|_v$. We recall that we denote by:

$$\rho_v : F_v^\times \times \mathcal{S}(F_v) \to \mathcal{S}(F_v) \tag{B.4.3.1}$$

the continuous left-translation action of $F_v^\times$ on $\mathcal{S}(F_v)$ (see also, in the case v is non-Archimedean, Lemma 6.1.2.2 for a study of the invariants and co-invariants under the induced action of $\mathcal{O}_{F_v}^\times \subset F_v^\times$, and to Theorem 3.4.3.3 for a classification of the homogeneous elements of $\mathcal{S}(F_v)'_\beta$ and Proposition 5.5.3.1 for a study of $\mathcal{S}(F_v)'_\beta$ in terms of its homogeneous elements). Explicitly, $\rho_v(g,f)(x) = f(g^{-1}x)$.

Let F be a number field, v a place of F. We denote by $\alpha_{v,0}$ the standard additive character of the local field F_v, as defined in Example A.3.1.8 and Example A.3.1.9. With respect to this choice of character, the additive Tamagawa measure ν_v on F_v is self-dual, i.e. it is its own Plancherel measure under the isomorphism of Hausdorff locally compact abelian groups $F_v \simeq \widehat{F_v}$ induced by the choice of $\alpha_{v,0}$.

PROPOSITION B.4.3.1 (Fourier transform of $\log|\cdot|_v$ - non-Archimedean places). *Let F be a number field, v a non-Archimedean place. Let $|\cdot|_v$ be the normalized absolute value on F and F_v the completion of F with respect to $|\cdot|_v$.*

The Fourier transform $\widehat{J_v} \in \mathcal{S}(F_v)'$, defined relative to the standard additive character $\alpha_{v,0}$ of F_v, of the function

$$J_v(x) := \frac{\log|x|_v}{\log(q_v)} \tag{B.4.3.2}$$

of $x \in F_v$ (where q_v is the size of the residue field at v) satisfies the following conditions:

(1) *As tempered distributions on F_v, we have:*

$$\rho'_v(g, \widehat{J_v}) = \widehat{J_v} - \frac{\log|g|_v}{\log(q_v)}\,\delta_0 \tag{B.4.3.3}$$

where ρ'_v is the adjoint action on $\mathcal{S}(F_v)'$. Explicitly, for any $f \in \mathcal{S}(F_v)$, we have

$$\langle \widehat{J_v}, \rho_v(g,f)\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = \langle \widehat{J_v}, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} - \frac{\log|g|_v}{\log(q_v)}\, f(0) \tag{B.4.3.4}$$

for every $g \in F_v^\times$.

(2) *For every $f \in C_c^\infty(F_v^\times) \subset \mathcal{S}(F_v)$, we have*

$$\langle \widehat{J_v}, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = -\frac{1}{\mu_v(\mathcal{O}_{F_v}^\times)} \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) \tag{B.4.3.5}$$

where μ_v is the local multiplicative Tamagawa measure on $F_v^\times$ from Definition A.3.2.2.
The function $J_v \in \mathcal{S}(F_v)$ is unique for the properties of vanishing at 1 and satisfying (B.4.3.5).

The meaning of "adjoint action" ρ_v' in (B.4.3.3) is the effect of the linear dual of the action of g.

PROOF. The identity, for all $g \in F_v^\times$:

$$\int_{F_v} \rho_v(g,f)(x)\overline{\psi}_v(xy)\,\mathrm{d}\nu_v(x) = |g|_v \rho_v(g^{-1},\widehat{f})(y) \tag{B.4.3.6}$$

where ν_v is the local additive Tamagawa measure on F_v from Definition A.3.1.2, and ψ_v is the normalized additive character chosen as in [**CF**, Ch. XV, §2.5, p. 319] implies

$$\langle \widehat{J}_v, \rho_v(g,f)\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = \langle J_v, \widehat{\rho_v(g,f)}\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'_\beta} = \langle J_v, |g|_v\rho_v(g^{-1},\widehat{f})\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'}. \tag{B.4.3.7}$$

We compute

$$\langle J_v, |g|_v\rho_v(g^{-1},\widehat{f})\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = |g|_v\int_{F_v}\widehat{f}(gx)\frac{\log|x|_v}{\log(q_v)}\,\mathrm{d}\nu_v(x) = \int_{F_v}\widehat{f}(x)\frac{\log|x|_v}{\log(q_v)}\,\mathrm{d}\nu_v(x) - \frac{\log|g|_v}{\log(q_v)}f(0) \tag{B.4.3.8}$$

for all $g \in F_v^\times$, so the left side of (B.4.3.7) equals

$$\langle \widehat{J}_v, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} - \frac{\log|g|_v}{\log(q_v)}\langle \delta_0, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'}.$$

This says that (B.4.3.3) holds.

If $f \in C_c^\infty(F_v^\times) \subset \mathcal{S}(F_v)$ then $f(0) = 0$, so (B.4.3.3) gives that $\widehat{J}_v$ is $F_v^\times$-invariant on $C_c^\infty(F_v^\times)$. By [**?**, Rem. p. 27], for any locally profinite group G the space of left G-invariant linear functionals on $C_c^\infty(G)$ is 1-dimensional. Thus, for $G = F_v^\times$ that space is generated by

$$L_0 : f \mapsto \langle L_0, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} := \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g).$$

There therefore exists a unique complex number $A \in \mathbf{C}$, $A \neq 0$, such that $\widehat{J}_v = AL_0$ on the subspace $\ker(\delta_0) \subset C_c^\infty(F_v^\times)$ of $\mathcal{S}(F_v)$. This determines $\widehat{J}_v$ up to a nonzero constant and adding a scalar multiple of δ_0.

We have, upon choosing $f = 1_{\mathcal{O}_{F_v}^\times}$:

$$\langle L_0, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) = \mu_v(\mathcal{O}_{F_v}^\times). \tag{B.4.3.9}$$

We write $1_{\mathcal{O}_{F_v}^\times} = 1_{\mathcal{O}_{F_v}} - \rho_v(\pi_v, 1_{\mathcal{O}_{F_v}})$ and from (B.4.3.4) we obtain

$$\langle \widehat{J}_v, 1_{\mathcal{O}_{F_v}^\times}\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = \langle \widehat{J}_v, 1_{\mathcal{O}_{F_v}} - \rho_v(\pi_v, 1_{\mathcal{O}_{F_v}})\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = \langle \widehat{J}_v, 1_{\mathcal{O}_{F_v}}\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} - \langle \widehat{J}_v, 1_{\mathcal{O}_{F_v}}\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} - 1 \tag{B.4.3.10}$$

which reads:

$$\langle \widehat{J}_v, 1_{\mathcal{O}_{F_v}^\times}\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = -1. \tag{B.4.3.11}$$

Upon combining (B.4.3.10) with (B.4.3.9), we deduce that indeed $A = -1/\mu_v(\mathcal{O}_{F_v}^\times)$, and so (B.4.3.5) holds. The condition (2) on $\widehat{J}_v$ thereby determines $\widehat{J}_v$ up to adding an element of $\mathbf{C}\delta_0$ and so determines J_v as a distribution up to adding an element of $\mathbf{C}\widehat{\delta}_0 = \mathbf{C}$. Thus, requiring $J_v(1) = 0$ gives uniqueness. □

We now proceed to compute the Fourier transform of the tempered distribution $\log|\cdot|_v$ on F_v for v an Archimedean place of the number field F. Throughout the Proposition below, γ denotes Euler's constant determined by the equation

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -\gamma - 2\log(2).$$

PROPOSITION B.4.3.2 (Fourier transform of $\log|\cdot|_v$ - Archimedean places). *Let F be a number field, v an Archimedean place. Let $|\cdot|_v$ be the normalized absolute value on F and F_v the completion of F with respect to $|\cdot|_v$.*

The Fourier transform $\widehat{J}_v \in \mathcal{S}(F_v)'$, defined relative to the standard additive character $\alpha_{v,0}$ of F_v, of the function

$$J_v(x) := \log|x|_v \tag{B.4.3.12}$$

of $x \in F_v$, is computed according to the following formulas:

(1) *If v is real, then we have:*

$$\langle \widehat{J}_v, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = -\lim_{\varepsilon\to 0^+}\left(\int_{|x|\geq\varepsilon} f(x)\,\frac{\mathrm{d}x}{|x|} + 2f(0)\log(\varepsilon)\right) - 2(\log(2\pi)+\gamma)f(0) \tag{B.4.3.13}$$

for every $f \in \mathcal{S}(\mathbf{R})$.

(2) *If v is complex, then we have:*

$$\langle \widehat{J}_v, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'} = -\lim_{\varepsilon\to 0^+}\left(\int_{|z|^2\geq\varepsilon} f(x)\,\frac{|\mathrm{d}z\wedge\mathrm{d}\overline{z}|}{|z|^2} + 2\pi f(0)\log(\varepsilon)\right) - 4\pi(\log(2\pi)+\gamma)f(0) \tag{B.4.3.14}$$

for every $f \in \mathcal{S}(\mathbf{C})$.

For the above calculations, we refer the reader to the proof of [Co, Appendix II, Lemma 2, p. 72],[16] and specifically to [Co, (32), (33), (34), p .73], and [Co, (41), (42), p. 75]. Note that the local multiplicative Tamagawa measures μ_v we are using in (B.4.3.2) are not the canonical measures on the $\mathbf{R}_{>0}$-modulated groups $\mathbf{R}^\times$ and $\mathbf{C}^\times$: these are obtained by dividing by 2 for v real, and by 2π for v complex, thus explaining the additional factors of 2, respectively 2π (i.e. the volumes of the respective "local unit groups" $\{|x|=1\}\subset F_v^\times$) in the above formulas compared with the formulas in [Co, Appendix II, pp. 72-75].[17]

B.4.4. Idelic Explicit Formulas. Let F be a number field. We conclude this section by rephrasing the Explicit Formulas as an identity between continuous linear functionals on $\mathcal{D}(C_F) = C_c^\infty(C_F)$.

B.4.4.1. *Weil's Principal Values from the Explicit Formulas via additive local Fourier transform.*

DEFINITION B.4.4.1 (Connes-Weil Principal Values). Let F be a number field. For every place v of F, we define, relative to the standard additive character $\alpha_{v,0}$ of F_v:[18]

(1) The continuous[19] functional on $\mathcal{S}(F_v)$ assigned by:[20]

$$\mathrm{C}_{\alpha_{v,0}}\int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) := \begin{cases} -\langle \widehat{J}_v, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'}, & v\mid\infty,\\ -\mu_v(\mathcal{O}_{F_v}^\times)\,\langle \widehat{J}_v, f\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'}, & v\nmid\infty \end{cases} \tag{B.4.4.1}$$

for all $f\in\mathcal{S}(F_v)$, where the notation $\langle \widehat{J}_v,\cdot\rangle_{\mathcal{S}(F_v),\mathcal{S}(F_v)'}$ means $\int_{F_v} J_v\widehat{f}\,\mathrm{d}\nu_v$ by definition, where ν_v is the self-dual measure on F_v with respect to $\alpha_{v,0}$ and the Fourier transform $\widehat{f}$ also rests on ν_v.[21] We refer to $\mathrm{C}_{\alpha_{v,0}}\int_{F_v^\times}(\cdot)\,\mathrm{d}\mu_v$ as *Connes' Principal Value* relative to $\alpha_{v,0}$, for $\int_{F_v^\times}(\cdot)\,\mathrm{d}\mu_v$.

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[16] Brian needs to recheck the proof, both for factors of 2 and 2π in the integrals and for the coefficient of $f(0)$ in both parts

[17] For part (1), should we tell the reader that Weil *defines* the right side to be the meaning of $\int_{F_v^\times} f(g)|g|_v^{1/2}/|1-g|_v\,\mathrm{d}\mu_v(g)$?

[18] It would be much more convenient to give the definition directly here for C_α for all non-trivial $\alpha\in\widehat{F}_v$, not just for $\alpha=\alpha_{v,0}$. We should also then comment that the definition below using α is unaffected by pre-composing α with multiplication by $\{|x|=1\}$!

[19] Continuity of right side in f must be proved as a separate lemma before this definition, treating $v|\infty$ and $v\nmid\infty$ separately

[20] In a clear way, should emphasize that the notation μ_v on the left side is *just notation*, and that the choice of α is baked into the right side.

[21] This should go similarly for all $\alpha\in\widehat{F}_v-\{1\}$ and the associated self-dual measure ν.

(2) The continuous[22] functional on $\mathcal{D}(F_v^\times) = C_c^\infty(F_v^\times)$:

$$\int_{F_v^\times,\alpha_{v,0}}' \frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_v(x) := \mathrm{C}_{\alpha_{v,0}} \int_{F_v^\times} f(1+x)|1+x|_v^{-1/2}\,\mathrm{d}\mu_v(g) \tag{B.4.4.2}$$

for all $f \in \mathcal{D}(F_v^\times)$. We refer to (B.4.4.2) as the *Connes-Weil Principal Value* relative to $\alpha_{v,0}$, for $\int_{F_v^\times}(\cdot)\frac{|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_v(x)$. This notation is "justified" by (B.4.4.5) below.

Remark B.4.4.2 (On normalizations for the Haar measure on $F_v^\times$). Let F be a number field. We note that the local multiplicative Tamagawa measure μ_v chosen on $F_v^\times$ for v a non-Archimedean place of F is not the canonical measure defined in Definition D.2.1.1 on any modulated group. The absolute value $|\cdot|_v$ promotes $F_v^\times$ to a $q_v^{\mathbf{Z}}$-modulated group, and the canonical measure μ_{can} is related to μ_v by the following formula:

$$\mu_{\mathrm{can}} = \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\mu_v. \tag{B.4.4.3}$$

Indeed, this choice of Haar measure gives the subset:

$$A_T := \{x \in F_v^\times : |x|_v \in [1,T)\} \subset F_v^\times$$

for $T > 1$ any real number, volume:

$$\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\mu_v(A_T) = \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \sum_{0\le m<\log_{q_v}(T)} \mu_v(\pi_v^{-m}\mathcal{O}_{F_v}^\times) = \log(T) + O(1)$$

which is the defining requirement of the canonical measure on the modulated group $F_v^\times$ (see (D.2.1.4) from Definition D.2.1.1).

When v is Archimedean, the local multiplicative Tamagawa measure μ_v is not the canonical measure either.[23] The canonical measure μ_{can} on $F_v^\times$ for v Archimedean is related to μ_v by the formula:

$$\mu_{\mathrm{can}} = \frac{1}{M_v}\mu_v \tag{B.4.4.4}$$

where M_v is the volume of the local unit group, just as for the case v is non-Archimedean: $M_v = 2$ if v is real, and $M_v = 2\pi$ if v is complex.

The following is the main calculation of this section:

Lemma B.4.4.3 (Weil's Principal Values at the non-Archimedean places). *Let F be a number field, v a non-Archimedean place of F. Let $\mathfrak{p}_v \subset \mathcal{O}_F$ be the unique prime ideal corresponding to the finite place v.*

For any $f \in C_c^\infty(F_v^\times)$ such that $f(1) = 0$, we have

$$\int_{F_v^\times,\alpha_{v,0}}' \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = \int_{F_v^\times} \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{B.4.4.5}$$

and for $T \in C_c^\infty(F_v^\times)' = \mathcal{D}(F_v^\times)'$ unique such that $\widehat{T}(1_{\mathcal{O}_{F_v}^\times}) = 0$ and its restriction to $\ker(\delta_1) \subset C_c^\infty(F_v^\times)$ is the functional

$$\frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\frac{\nu_v}{|1-x|_v},$$

we have

$$\int_{F_v^\times,\alpha_{v,0}}' \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = -\mu_v(\mathcal{O}_{F_v}^\times)\,\langle T, f(x)|x|_v^{-1/2}\rangle_{\mathcal{D}(F_v^\times),\mathcal{D}(F_v^\times)'} \tag{B.4.4.6}$$

for all $f \in \mathcal{D}(F_v^\times) = C_c^\infty(F_v^\times)$ with $f(1) = 0$.

[22] Continuity of right side in f must be proved as a separate lemma before this definition, treating $v|\infty$ and $v\nmid\infty$ separately

[23] Should we refer in some way to the cautionary comment at the end of §B.4.3?

PROOF. By Lemma A.3.2.3, we have:

$$\mu_v = \frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\frac{\nu_v}{|\cdot|_v}$$

where $\mathfrak{p}_v$ is the unique prime ideal of $\mathcal{O}_F$ corresponding to the finite place v.

Assume first $f(1)=0$. Then, upon defining

$$u(x) := \frac{f(1+x)}{|1+x|_v^{1/2}} \tag{B.4.4.7}$$

(which vanishes near $x=-1$ since f vanishes near 0), we have $u(0)=0$ and $u \in C_c^\infty(F_v^\times)$. Thus, by (B.4.3.5) from Proposition B.4.3.1 and (B.4.4.1) we have that (B.4.4.2) is equal to

$$\begin{aligned}
\mathrm{C}_{\alpha_{v,0}} \int_{F_v^\times} u(x)\,\mathrm{d}\mu_v(x) &= \int_{F_v^\times} \frac{f(1+x)}{|1+x|_v^{1/2}}\,\mathrm{d}\mu_v(x) \\
&= \frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\int_{F_v} \frac{f(1+x)}{|1+x|_v^{1/2}}\,\frac{\mathrm{d}\nu_v(x)}{|x|_v} \\
&= \frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\int_{F_v} \frac{f(1+y-1)}{|1+y-1|_v^{1/2}}\,\frac{\mathrm{d}\nu_v(y-1)}{|y-1|_v} \\
&= \frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\int_{F_v} \frac{f(y)|y|_v}{|y-1|_v\,|y|_v^{1/2}}\,\frac{\mathrm{d}\nu_v(y)}{|y|_v} \\
&= \int_{F_v^\times} \frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_v(x),
\end{aligned} \tag{B.4.4.8}$$

We deduce that (B.4.4.5) holds on the hyperplane $\ker(\delta_1) \subset C_c^\infty(F_v^\times)$.

We have:

$$\begin{aligned}
\langle \widehat{J}_v, f(1+x)|1+x|_v^{-1/2}\rangle_{\mathcal{D}(F_v^\times),\mathcal{D}(F_v^\times)'} &= \langle J_v, \int_{F_v} f(1+y)|1+y|_v^{-1/2}\overline{\alpha_{v,0}(xy)}\,\mathrm{d}\nu_v(y)\rangle_{\mathcal{D}(F_v^\times),\mathcal{D}(F_v^\times)'} \\
&= \langle J_v, \alpha_{v,0}(x)\int_{F_v} f(z)|z|_v^{-1/2}\overline{\alpha_{v,0}(xz)}\,\mathrm{d}\nu_v(z)\rangle_{\mathcal{D}(F_v^\times),\mathcal{D}(F_v^\times)'} \\
&= \langle \widehat{\alpha_{v,0}\cdot J_v}, f(x)|x|_v^{-1/2}\rangle_{\mathcal{D}(F_v^\times),\mathcal{D}(F_v^\times)'}
\end{aligned} \tag{B.4.4.9}$$

which, combined with (B.4.4.5), shows that $T := \widehat{\alpha_{v,0}\cdot J_v}$ is the unique element in $\mathcal{D}(F_v^\times)'$ that restricts on $\ker(\delta_1) \subset \mathcal{D}(F_v^\times)$ to the functional:

$$\frac{N(\mathfrak{p}_v)}{N(\mathfrak{p}_v)-1}\frac{\nu_v}{|1-x|_v}$$

and satisfies $\widehat{T}(1_{\mathcal{O}_{F_v}^\times}) = 0$:

$$\langle \widehat{T}, 1_{\mathcal{O}_{F_v}^\times}\rangle = \langle T, \widehat{1_{\mathcal{O}_{F_v}^\times}}\rangle = \langle \widehat{\alpha_{v,0}J_v}, \widehat{1_{\mathcal{O}_{F_v}^\times}}\rangle = \langle \alpha_{v,0}J_v, 1_{\mathcal{O}_{F_v}^\times}\rangle = \int_{\mathcal{O}_{F_v}^\times} \alpha_{v,0}J_v\,\mathrm{d}\mu_v = 0$$

because $J_v|_{\mathcal{O}_{F_v}^\times} = 0$. □

B.4.4.2. *Change of local additive character.* Upon replacing $\alpha_{v,0}$ with a different additive character of the local field F_v, the following Lemma explains how the Connes-Weil Principal Value changes. We refer the reader to the calculation in [**Co**, Lemma 3,p.76].

LEMMA B.4.4.4 (Change of additive character Lemma). *Let F be a number field, v a fixed finite place of F. We write q_v for the size of the residue field at v. Let $\alpha_{v,0}$ be the standard additive character of the local*

field F_v and let $\alpha \in \widehat{F_v}$ be an arbitrary non-trivial additive character, so there exists a unique $x_v \in F_v^\times$ such that $\alpha(x) = \alpha_{v,0}(x_v x)$.

We have:

(1) *for any $f \in \mathcal{S}(F_v)$:*

$$\mathrm{C}_\alpha \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) = f(0)\,\mu_v(\mathcal{O}_{F_v}^\times)\frac{\log|x_v|_v}{\log(q_v)} + \mathrm{C}_{\alpha_{v,0}} \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) \tag{B.4.4.10}$$

(2) *for any $f \in \mathcal{D}(F_v^\times)$:*

$$\int'_{F_v^\times,\alpha} \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = f(1)\,\mu_v(\mathcal{O}_{F_v}^\times)\frac{\log|x_v|_v}{\log(q_v)} + \int'_{F_v^\times,\alpha_{v,0}} \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{B.4.4.11}$$

Proof. (1) : The Lemma follows from (B.4.3.4) from Proposition B.4.3.1, as we now explain. The measure $|x_v|_v^{1/2}\nu_v$ is self-dual relative to the choice of character α (by Plancherel's Theorem), using [**CF**, Ch. XV, Ch. 4, §4.1, Thm. 4.1.2]. Thus,

(B.4.4.12)

$$\begin{aligned}
\mathrm{C}_\alpha \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) &= \mu_v(\mathcal{O}_{F_v}^\times) \int_{F_v} \left\{ \frac{-\log|y|_v}{\log(q_v)} \int_{F_v} f(u)\overline{\alpha}(uy)\,(|x_v|_v^{1/2}\,\mathrm{d}\nu_v(u)) \right\} (|x_v|_v^{1/2}\mathrm{d}\nu_v(y)) \\
&= \mu_v(\mathcal{O}_{F_v}^\times) \int_{F_v} \left\{ \frac{-\log|y|_v}{\log(q_v)} \int_{F_v} f(u)\overline{\alpha_{v,0}}(x_v uy)\,(|x_v|_v^{1/2}\,\mathrm{d}\nu_v(u)) \right\} (|x_v|_v^{1/2}\mathrm{d}\nu_v(y)) \\
&= \mu_v(\mathcal{O}_{F_v}^\times) \int_{F_v} \left\{ \frac{-\log|y|_v}{\log(q_v)} |x_v|_v \int_{F_v} f(u)\overline{\alpha_{v,0}}(x_v uy)\,\mathrm{d}\nu_v(u) \right\} \mathrm{d}\nu_v(y) \\
&= -\mu_v(\mathcal{O}_{F_v}^\times)\,\langle J_v(y), |x_v|_v \widehat{f}(x_v y)\rangle_{\mathcal{S}(F_v)',\mathcal{S}(F_v)}.
\end{aligned}$$

Now use (B.4.3.4) with $g = x_v^{-1}$ and (B.4.4.1) to see this equals recalling that ρ_v denotes the left-translation action of $F_v^\times$ on F_v, which by (B.4.3.4) equals:

$$\mathrm{C}_{\alpha_{v,0}} \int_{F_v^\times} f(g)\,\mathrm{d}\mu_v(g) + f(0)\,\mu_v(\mathcal{O}_{F_v}^\times)\frac{\log|x_v|_v}{\log(q_v)}. \tag{B.4.4.13}$$

(2) : We have:

$$\int'_{F_v^\times,\alpha} \frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_v(x) := \mathrm{C}_\alpha \int_{F_v^\times} f(1+x)|1+x|_v^{-1/2}\,\mathrm{d}\mu_v(g)$$

and so (B.4.4.11) follows from (B.4.4.10). □

B.4.4.3. *Calculation of Weil's Principal Values.*

Lemma B.4.4.5 (Weil's Principal Values at the non-Archimedean places). *Let F be a number field, v a non-Archimedean place of F, q_v the size of the residue field at v, and p the residue characteristic of F_v.*

For any $f \in \mathcal{D}(F_v^\times) = C_c^\infty(F_v^\times)$ we have

$$\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \int'_{F_v^\times,\alpha_{v,0}} \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = \mathrm{PF}_0 \int_{F_v^\times} \frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_{\mathrm{can},v}(x) - \log|\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}|_v f(1), \tag{B.4.4.14}$$

where the principal value $\mathrm{PF}_0 \int_{F_v^\times} \frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_{\mathrm{can},v}(x)$ (for the canonical measure $\mu_{\mathrm{can},v}$ on the modulated group $(F_v^\times, |\cdot|_v)$) is defined as in [**Weil-e**, §15] *to be*

$$\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \int_{F_v^\times - \mathcal{O}_{F_v}^\times} \frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) + \frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \int_{\mathcal{O}_{F_v}^\times} \frac{(f(g)-f(1))|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g).$$

REMARK B.4.4.6. Note that $f(g)-f(1)$ vanishes in any small enough open neighborhood of $1\in F_v^\times$. The integrand in the second integral used for defining Weil's principal value therefore makes sense. For $f=1_{\mathcal{O}_{F_v}^\times}$, this lemma is also addressed in [**Co**, Lemma 2, p. 72] with different normalizations for the Haar measures.

PROOF. Assume first that f vanishes at $g=1$. Then (B.4.4.14) holds by (B.4.4.5) and (B.4.4.11). Since (B.4.4.14) is stable under replacement of f with $f-1_{\mathcal{O}_{F_v}^\times}\cdot f(1)$, to treat the general case it remains to check both sides for $f=1_{\mathcal{O}_{F_v}^\times}$.

Let $d=\mathrm{ord}_v(\mathfrak{d}_{F_v/\mathbf{Q}_p})\geq 0$. By design of $\mathfrak{d}_{F_v/\mathbf{Q}_p}$, the additive character $\alpha(x)=\alpha_{v,0}(\pi_v^{-d}x)$ (for a uniformizer π_v at v) is trivial on $\mathcal{O}_{F_v}$ and non-trivial on $\mathfrak{m}_v^{-1}$, and the self-dual measure relative to it is $q_v^{d/2}\nu_v=N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{1/2}\nu_v$. By Lemma B.4.4.4(2), we have

$$\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times,\alpha}\frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g)=f(1)\log|\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}|_v+\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times,\alpha_{v,0}}\frac{f(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g). \tag{B.4.4.15}$$

Since $\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\mu_v=\mu_{\mathrm{can},v}$ as per Remark B.4.4.2, the calculation of $\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times,\alpha_{v,0}}\frac{f(x)|x|_v^{1/2}}{|1-x|_v}\,\mathrm{d}\mu_v(x)$ for $f=1_{\mathcal{O}_{F_v}^\times}$ can be done directly from the definitions, using that $\alpha_{v,0}$ has self-dual measure $\nu_v=N(\mathfrak{d}_{F_v/\mathbf{Q}_p})^{-1/2}\mu$ for the Haar measure μ on F_v giving $\mathcal{O}_{F_v}$ volume 1. The key calculation[24] is

$$\int_{F_v}\frac{1_{\mathcal{O}_{F_v}^\times}(1+y)}{\sqrt{|1+y|_v}}\alpha_{v,0}(xy)\,\mathrm{d}\mu(y)=\alpha_{v,0}(-x)\left(1_{\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}}(x)-\frac{1}{q_v}1_{\mathfrak{m}_v^{-1}\mathfrak{d}_{F_v/\mathbf{Q}_p}}(x)\right)$$

and shows that (B.4.4.14) holds in this case too. □

B.4.4.4. *Reformulation of Weil's Explicit Formulas.* We can now rewrite Weil's Explicit Formulas as an identity of distributions on the idele class group; i.e., as elements of $\mathcal{D}(C_F)'$.

THEOREM B.4.4.7 (Idelic Explicit Formulas). *Let F be a number field. For every $\varphi\in\mathcal{D}(C_F)$, the sums on each side of the identity:*

(B.4.4.16)

$$\begin{aligned}\sum_{c\in\Omega(C_F):L(c)=0}\frac{1}{\overline{\mu}_1(C_F^1)}\int_{C_F}\varphi(g)c(g)|g|_F^{-1/2}\,\mathrm{d}\overline{\mu}(g)&=\frac{1}{\overline{\mu}_1(C_F^1)}\int_{C_F}\varphi(g)|g|_F^{-1/2}\,\mathrm{d}\overline{\mu}(g)+\frac{1}{\overline{\mu}_1(C_F^1)}\int_{C_F}\varphi(g)|g|_F^{1/2}\,\mathrm{d}\overline{\mu}(g)\\&-\sum_{v\in S_{F,\infty}}\frac{1}{M_v}\int'_{F_v^\times,\alpha_{v,0}}\frac{\varphi(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g)\\&-\sum_{v\in S_{F,f}}\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)}\int'_{F_v^\times,\alpha_{v,0}}\frac{\varphi(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g)\end{aligned}$$

(where $M_v=2$ if v is real, and $M_v=2\pi$ if v is complex) absolutely converge for every $\varphi\in\mathcal{D}(C_F)$ and converge in $\mathcal{D}(C_F)'_\beta$ defining the same continuous functional on $\mathcal{D}(C_F)$.

The topology on $\mathcal{D}(C_F)$ is the countable strict locally convex inductive limit $\varinjlim\mathcal{D}(C_F/K)$ where K varies through a decreasing countable cofinal system of compact subgroups of C_F for which C_F/K is a Lie group with finitely generated component group (see [**Bru**, pp. 44-45]). For the proof of Theorem B.4.4.7 we shall use the following (see Definition D.2.1.1 and the subsequent discussion after that for the concept and main results on modulated Hausdorff locally compact abelian groups):

[24] the right side should be double-checked

LEMMA B.4.4.8. *Let G be an $\mathbf{R}_{>0}$-modulated group abelian Lie group (possibly disconnected). Let μ_G be a choice of Haar measure on G. Let $\mathcal{Z} \subset \Omega(G)$ be a subset whose intersection with every connected component of $\Omega(G)$ is discrete and contained within a single vertical strip of bounded width of the connected component.*[25]

For any $\varphi \in \mathcal{D}(G)$, the sum

$$\sum_{c \in \mathcal{Z}} \int_G \varphi(g) c(g) \, \mathrm{d}\mu_G(g) \tag{B.4.4.17}$$

is absolutely convergent.

PROOF. We choose the Haar measure μ_1 on G^1 compatible with the given μ_G on G and the measure $\mathrm{d}t/t$ on $\mathbf{R}_{>0}$ via the exact sequence of topological groups $1 \to G^1 \to G \stackrel{|\cdot|_G}{\to} \mathbf{R}_{>0} \to 1$. For a choice of continuous homomorphic section τ to $|\cdot|_G$, (B.4.4.17) can be rewritten termwise as

$$\sum_{c \in \mathcal{Z}} \int_{\mathbf{R}_{>0}} t^s c(\tau(t)) \int_{G^1} \varphi(g\tau(t)) \chi_1(g) \, \mathrm{d}\mu_1(g) \, \frac{\mathrm{d}t}{t} \tag{B.4.4.18}$$

where $\chi_1 := c|_{G^1}$.

Since G^1 is a (possibly disconnected) compact abelian Lie group, up to splitting the sum into finitely many sub-sums and adjusting φ accordingly, we may focus on integrating over $(G^1)^0 = \mathbf{T}^N$ for an integer $N \geq 0$. We therefore have for all $z = (z_k)_{1 \leq k \leq N} \in \mathbf{T}^N$ that $\chi_1(z) = \prod_{1 \leq k \leq N} z_k^{a_k}$ for some $(a_k)_{1 \leq k \leq N} \in \mathbf{Z}^N$.

As a function of $(s, (a_k)_{1 \leq k \leq N}) \in \mathbf{C} \times \mathbf{Z}^N$, the term for c in (B.4.4.18) is, up to a positive multiplicative constant (depending on $\mu_1((G^1)^0)$),

$$\Phi(s, (a_k)_{1 \leq k \leq N}) = \int_{\mathbf{R}_{>0}} t^s c(\tau(t)) \int_{\mathbf{T}^N} \varphi(z\tau(t)) \prod_{1 \leq k \leq N} z_k^{a_k} \, \mathrm{d}\mu_N(z) \, \frac{\mathrm{d}t}{t}$$

where μ_N is the volume-1 Haar measure on $\mathbf{T}^N$. This decays rapidly on the vertical strips $\mathbf{C}_I$ of bounded width of the Riemann surface $\mathbf{C} \times \mathbf{Z}^N$: for every real number $T > 0$ and integer $M \geq 0$, there exists a real number $C_{T,M} > 0$ independent of c such that:

$$\sup_{(s,(a_k)_{1 \leq k \leq N}) \in \mathbf{C}_I \times \mathbf{Z} : |\mathrm{Im}(s)| > T, \|(a_k)_{1 \leq k \leq N}\|_\infty > T} |\Phi(s, (a_k))| < C_{T,M} \, T^{-3M}.$$

This rapid decay condition is a consequence of the Uniform Riemann-Lebesgue Lemma (Theorem D.3.4.2 with $D = (t\partial_t)^M (\sum_{1 \leq k \leq N} (z_k \partial_{z_k})^2)^M$), and thereby implies the absolute convergence of the sum in (B.4.4.17).[26] □

LEMMA B.4.4.9 (Convergence of sums of Mellin transforms). *Let F be a number field, $\varphi \in \mathcal{D}(C_F)$.*

Then for any subset $\mathcal{Z}$ of the set:

$$\{c \in \Omega(C_F) : L(c) = 0\} \subset \Omega(C_F)$$

where $L(c)$ is the Hecke L-function of F, and for any real number $a \in \mathbf{R}$, the sum:

$$\sum_{c \in \mathcal{Z}} \int_{C_F} \varphi(g) c(g) |g|_F^a \, \mathrm{d}\mu(g) \tag{B.4.4.19}$$

converges absolutely.

PROOF. Multiplication by $|\cdot|_F^a$ is a linear topological automorphism on $\mathcal{D}(C_F)$, and so we may and do assume $a = 0$.

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[25] Need some hypothesis on $\mathcal{Z}$ preventing "too many" $c \in \mathcal{Z}$ near a given large height.

[26] Must explain clearly how we are bringing in that $\mathcal{Z}$ doesn't accumulate too much at a given height.

We let $k \subset C_F^1$ be a compact subgroup with open compactly generated orthogonal such that φ is k-invariant. The contribution from c that are not k-invariant vanishes since

$$\int_{C_F} \varphi(g)c(g)\,\mathrm{d}\mu(g) = \int_{C_F/k} \varphi(\overline{g}) \left(\int_k c(k)\,\mathrm{d}k \right) \mathrm{d}\overline{\mu}(g),$$

so we may replace C_F with $G := C_F/k$, an abelian Lie group, and μ with the unique Haar measure μ_G on C_F/k prescribed by μ and the volume-1 Haar measure on k as per Theorem D.1.2.1, in (B.4.4.19). The result then follows from Lemma B.4.4.8. □

Proof of Theorem B.4.4.7. By Proposition B.4.3.2 and Lemma B.4.4.5, for each place v of F and $\varphi_v \in \mathcal{D}(F_v^\times)$ we have

$$\int'_{F_v^\times, \alpha_{v,0}} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = \mathrm{PF}_0 \int_{F_v^\times} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) \tag{B.4.4.20}$$

for Archimedean places v (where the principal value on the right side is defined to be the right sides of (B.4.3.13) and (B.4.3.14), depending on whether v is real or complex) and

$$\frac{\log(q_v)}{\mu_v(\mathcal{O}_{F_v}^\times)} \int'_{F_v^\times, \alpha_{v,0}} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = \mathrm{PF}_0 \int_{F_v^\times} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_{\mathrm{can},v}(g) - \log|\mathfrak{d}_{F_v/\mathbf{Q}_p}^{-1}|_v \varphi_v(1) \tag{B.4.4.21}$$

for non-Archimedean v.

For each place v of F, define the test function $\varphi_v \in \mathcal{D}(F_v^\times)$ to be the composition of $\varphi \in \mathcal{D}(C_F)$ with the closed homomorphism $F_v^\times \subset C_F$. The right side of (B.4.4.16) agrees with the sum on the right side of the identity[27]

$$\Delta(\varphi) = \varphi(1)\log|\mathfrak{d}_{F/\mathbf{Q}}^{-1}|_F + D(\varphi) - \sum_{v \in \Sigma_F} D_v(\varphi_v)$$

displayed at the end of [**Weil-e**, §16], where

$$D_v(\varphi_v) := \mathrm{PF}_0 \int_{F_v^\times} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_{\mathrm{can},v}(g)$$

(with $\mu_{\mathrm{can},v}$ the canonical measure on the modulated group $F_v^\times$ as in Definition D.2.1.1: $\mathrm{d}t/t$ for real v, $|\mathrm{d}z \wedge \mathrm{d}\overline{z}|/(2\pi z\overline{z})$ for complex v, and $(\log(q_v)/\mu_v(\mathcal{O}_{F_v}^\times))\mu_v$ for any Haar measure μ_v on $F_v^\times$ for finite v),

$$D(\varphi) := \int_{C_F} \varphi(g)|g|_F^{-1/2}\,\mathrm{d}\overline{\mu}(g) + \int_{C_F} \varphi(g)|g|_F^{1/2}\,\mathrm{d}\overline{\mu}(g),$$

and $\Delta(\varphi)$ defined to be the sum on the left side of (B.4.4.16) (absolutely convergent by Lemma B.4.4.9).

We note that for each $\varphi \in \mathcal{D}(C_F)$, the sum in the right side of (B.4.4.16) has all but finitely many terms vanishing: to see this, for any non-Archimedean place v of F we compute

$$\mathrm{PF}_0 \int_{F_v^\times} \frac{\varphi_v(g)|g|_v^{1/2}}{|1-g|_v}\,\mathrm{d}\mu_v(g) = \mathrm{PF}_0 \sum_{m \in \mathbf{Z}} \int_{\mathcal{O}_{F_v}^\times} \frac{\varphi_v(g\pi_v^m)|g\pi_v^m|_v^{1/2}}{|1-g\pi_v^m|_v}\,\mathrm{d}\mu_{v,0}(g) \tag{B.4.4.22}$$

as is shown by deducing the case $\varphi_v(1) = 0$ from Theorem D.1.2.1 and then verifying the case $\varphi_v = 1_{\mathcal{O}_{F_v}^\times}$ (for which case both sides vanish). For all v non-Archimedean with large enough residue characteristic (depending on the support of $\varphi \in \mathcal{D}(C_F)$), $\varphi_v(\pi_v^m g) = 0$ for all $g \in \mathcal{O}_{F_v}^\times$ and all $m \neq 0$, and so the right side of (B.4.4.22) vanishes.

To verify that both sides of (B.4.4.16) as linear functionals on $\mathcal{D}(C_F)$ are continuous and as an infinite sum of functionals is convergent in $\mathcal{D}(C_F)'_\beta$, we can use the argument from Remark B.4.1.6 with $\mathcal{S}(\mathbf{R})$ replaced by $\mathcal{D}(C_F)$ provided that each term in the sums on both sides is continuous on $\mathcal{D}(C_F)$ and that $\mathcal{D}(C_F)$ is

[27] Must explain to the reader how to obtain this identity from Theorem B.4.2.2.

complete and barreled with its closed bounded subsets all compact. The continuity of the local terms on the right side is immediate from combining the continuity of each restriction map $\mathcal{D}(C_F) \to \mathcal{D}(F_v)$ (which follows from $F_v^\times \to C_F$ being a closed embedding and hence proper [**Bru**, p. 58]) and continuity in Definition B.4.4.1; the continuity of the global terms on both sides is seen by passage to the locally convex inductive limit from the straightforward case of φ with a fixed compact support.

Since $\mathcal{D}(C_F)$ is a strict countable locally convex inductive limit of Fréchet spaces [**Bru**, p. 46], completeness and barreledness of $\mathcal{D}(C_F)$ are inherited from the Fréchet space via passage to the inductive limit [**Scha**, Ch. II, 6.6, Cor. 1 to 7.2]. The compactness of closed bounded subsets in $\mathcal{D}(C_F)$ is part of [**Bru**, §4, Cor. 2 to Thm. 2]. □

B.5. Recurrence and equidistribution

In this section we record a proof of a uniform version of the Kronecker-Weyl Theorem, and prove, as an application required in Chapter **??** of this work, the strong recurrence on vertical strips of the reciprocal Euler factors of Hecke L-functions. We will be discussing the concept of recurrence for holomorphic functions on vertical strips, and basic permanence properties as the need arises.

B.5.1. Preparations. This section is devoted to reviewing the small amount of ergodic theory required, providing some complements with a view towards the intended applications. Most of the results herein are well-known or direct consequences of well-known mathematics, and can be found in the referred literature. We include the arguments here for completeness, ease of access, and to discuss minor variations needed for applicability.

We will not be requiring the general Ergodic Theorem of Birkhoff-Khinchin (see, for instance, [**CFSS**, Ch.1,§2,Thm.1]), nor theory and lexicon from ergodic theory. We will be very briefly pointing out to the reader how the results discussed below fit within terminology and principles of ergodic theory. Each will be an incarnation of the classical Kronecker-Weyl Theorem, expressing the unique ergodicity of certain linear flows on tori.

Main references for the results and respective preparations throughout this section are [**HK**, Part 1,Ch.4,§4.1,4.2] (focusing on the one and two-dimensional cases only), and the unpublished [**Hu1**, §4.3] and [**Hu2**, Appendix A], for proofs of the finite-dimensional continuous and discrete versions, respectively, of the classical Kronecker-Weyl Theorem.

We denote by

$$\mathbf{T} := \mathbf{R}/\mathbf{Z}$$

the additive compact Hausdorff topological group of the unit circle. For any integer $N \geq 1$, we write $[N]$ for the ordered subset

$$[N] := \{1, \ldots, N\} \subset \mathbf{Z}.$$

Definition B.5.1.1. We define

(1) The compact Hausdorff topological product group

$$\mathbf{T}^N := \prod_{1 \leq k \leq N} \mathbf{T}. \tag{B.5.1.1}$$

(2) For a subset $U \subset \mathbf{R}^N$ and a continuous 1-parameter subgroup $\gamma : \mathbf{R} \to \mathbf{R}^N$, we write

$$\gamma(t) \in U \bmod 1 \tag{B.5.1.2}$$

to mean that the image of $\gamma(t)$ in $\mathbf{T}^N$ under the quotient map θ is contained in the image of U, i.e. $\overline{\gamma}(t) \in \theta(U)$. Equivalently, $\gamma(t) \in U + \mathbf{Z}^N$. We write, for any interval $I \subset \mathbf{R}$

$$I_U := \{t \in I : \gamma(t) \in U \bmod 1\} \subset I. \tag{B.5.1.3}$$

We identify $\mathbf{T}$ with the unit circle within $\mathbf{C}^\times$ via the quotient topological group homomorphism

$$\exp(2\pi it) : \mathbf{R} \to \mathbf{T}$$

whose kernel is the discrete subgroup $\mathbf{Z} \subset \mathbf{R}$.

For any integer $N \geq 1$ and any vector $(\tau_1, \ldots, \tau_N)$ in $\mathbf{R}^N$, we may consider the smooth map:

$$\overline{\gamma}_N : \mathbf{R} \to \mathbf{T}^N$$

defined as the composition of $\gamma_N(t) := t \cdot (\tau_1, \ldots, \tau_N) \in \mathbf{R}^N$ followed by the projection $\theta_N : \mathbf{R}^N \to \mathbf{T}^N$.

When $\tau_1, \ldots, \tau_N$ are $\mathbf{Q}$-linearly independent, $\overline{\gamma}_N$ is injective with dense image, as we will be recalling in Theorem B.5.2.1. In this case, $\overline{\gamma}_N$ is a one-parameter subgroup of the torus $\mathbf{T}^N$, and it induces a continuous flow on $\mathbf{T}^N$ by left-translation with dense image.

B.5.2. Kronecker-Weyl. The following Theorem is the multi-dimensional version of [HK, Part 1, Thm.4.1.15] and its method of proof is analogous to the argument [HK, Part 1, §4.1.6]. We will only be requiring the case when $(\tau_1, \ldots, \tau_N) \in \mathbf{R}^N$ has $\mathbf{Q}$-independent coordinates. A full proof in the general case when the vector $(\tau_1, \ldots, \tau_N) \in \mathbf{R}^N$ in the statement has not necessarily $\mathbf{Q}$-independent coordinates can be found in the appendix of [Hu1, §4.3, Proof of Lemma 4.3], and we will be following it closely to establish the case of continuous functions.

B.5.2.1. *The classical Kronecker-Weyl Theorem.* For any integer $N \geq 1$, we say a function $u : \mathbf{T}^N \to \mathbf{C}$ is *Riemann-integrable* if and only if the composition $u \circ \theta : [0,1)^N \to \mathbf{C}$ is a function whose real and imaginary parts are both Riemann-integrable on the half-open cube $[0,1)^N$ in the standard sense of Riemann-integrability on convex subsets within Euclidean spaces.

We denote by μ_N the volume-1 Haar measure on $\mathbf{T}^N$, and recall that μ_N agrees under θ_N with the measure induced by restriction to $[0,1)^N$ of the Lebesgue measure λ_N on $\mathbf{R}^N$, i.e.

$$\mu_N = (\theta_N)_*(\lambda_N|_{[0,1)^N})$$

since $[0,1)^N$ is a fundamental domain for the continuous action by left-translations of the closed discrete subgroup $\mathbf{Z}^N \subset \mathbf{R}^N$.

THEOREM B.5.2.1 (Kronecker-Weyl). *Let $\tau := (\tau_1, \ldots, \tau_N) \in \mathbf{R}^N$ be a vector. Let $0 \leq d \leq N$ be the dimension of the $\mathbf{Q}$-linear subspace generated within $\mathbf{R}$ by the coordinates of τ.*

Then

(1) *τ generates a continuous 1-parameter group $\overline{\gamma}_\tau : \mathbf{R} \to \mathbf{T}^N$ whose image has closure a d-dimensional closed torus G within $\mathbf{T}^N$.*
(2) *For every Riemann integrable function $u : G \to \mathbf{C}$, the following identity holds*

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} u(\overline{\gamma}(t))\, \mathrm{d}t = \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} u(\overline{\gamma}(t))\, \mathrm{d}t = \int_G u(\xi)\, \mathrm{d}\mu_G(\xi) \tag{B.5.2.1}$$

where μ_G is the volume-1 Haar measure on G.

We will use the following:

LEMMA B.5.2.2. *Let X be a compact Hausdorff space, μ a regular Borel measure on X, and $U \subset X$ an open subset such that $\mu(\partial U) = 0$. For every $\varepsilon > 0$, there exist continuous functions $u_1, u_2 : X \to [0,1]$ such that:*

$$u_1(x) \leq 1_U(x) \leq u_2(x) \tag{B.5.2.2}$$

for all $x \in X$, and:

$$\int_X (u_2 - u_1)(x)\, \mathrm{d}\mu(x) < \varepsilon. \tag{B.5.2.3}$$

PROOF. We fix $\varepsilon > 0$. Since μ is a finite regular Borel measure, by [Fol, Prop. 7.7(a)] there exists a closed subset $K_1 \subset U$ such that $\mu(U - K_1) < \varepsilon/2$. By openness of U, $K_2 := X - U$ is closed. By Urysohn's Lemma ([Fol, Thm. 4.32], one of whose equivalent forms is that for any $K \subset U \subset X$ with K closed and U open, there exists a continuous compactly-supported function $f : X \to [0,1]$ such that $1_K(x) \leq f(x) \leq 1_U(x)$ for all $x \in X$), there exists a continuous function $u_1 : X \to [0,1]$ such that $K_1 \subset u_1^{-1}(1)$ and $K_2 \subset u_1^{-1}(0)$. In particular, we have $u_1(x) \leq 1_U(x)$ for all $x \in X$.

We now let $\overline{U} \subset X$ be the closure of U. Since μ is regular, there exists an open subset $U' \subset X$ containing $\overline{U}$ and such that $\mu(U' - \overline{U}) < \varepsilon/2$. Calling $H_1 := \overline{U}$ and $H_2 := X - U'$, two disjoint closed subsets of X, by Urysohn's Lemma again there exists a continuous function $u_2 : X \to [0,1]$ such that $H_1 \subset u_2^{-1}(1)$ and $H_2 \subset u_2^{-1}(0)$. In particular, we have

$$1_{K_1}(x) \leq u_1(x) \leq 1_U(x) \leq 1_{\overline{U}}(x) \leq u_2(x) \leq 1_{U'}(x)$$

for all $x \in X$. We now observe that we have

$$\int_X (u_2 - u_1)(x)\, \mathrm{d}\mu(x) \leq \mu(U') - \mu(K_1) = \mu(U' - \overline{U}) + \mu(\partial U) + \mu(U) - \mu(K_1) < \frac{\varepsilon}{2} + \mu(\partial U) + \frac{\varepsilon}{2} = \varepsilon$$

since we assumed $\mu(\partial U) = 0$. This completes the proof of the Lemma. □

REMARK B.5.2.3. (1) Let $U \subset [0,1)^N$ be a subset such that ∂U (in $[0,1)^N$) is a Lebesgue null-set. Then, upon denoting by $\widehat{U}$ the image of U under θ_N in $\mathbf{T}^N$, we have $\mu_N(\partial \widehat{U}) = 0$. To see this, we proceed as follows. Since $\theta_N : \mathbf{R}^N \to \mathbf{T}^N$ is a covering map, it is open. When restricted to $[0,1]^N$, it is also closed since $\mathbf{T}^N$ is compact and $[0,1]^N$ is Hausdorff. Upon calling $\overline{U}$ the closure of U in $[0,1]^N$, the Lebesgue measure of the boundary of U in $[0,1]^N$ is again zero, since $(\overline{U} - U) \cap [0,1)^N$ has measure zero by assumption, and $[0,1]^N - [0,1)^N$ is a null-set. Since we have $\theta_N(\overline{U} - U) = \partial \widehat{U}$, it follows that $\partial \widehat{U}$ is a μ_N-null set.

(2) For any Jordan-measurable subset $\Delta \subset [0,1)^N$, we deduce that the image $\widehat{\Delta}$ in $\mathbf{T}^N$ has boundary of μ_N-measure zero, and so its interior qualifies for the U in Lemma B.5.2.2 and has the same μ_N-measure as $\widehat{\Delta}$.

The proof recorded below follows closely the argument provided in [Hu1, §4.3,Proof of Lemma 4.3].

PROOF OF THEOREM B.5.2.1. (1) : The case $N = 1$ is easy, so assume $N > 1$. We start off by discussing the special case in which the coordinates of τ are $\mathbf{Q}$-linearly independent.

In this case, $\overline{\gamma}$ is injective. Indeed, if $\gamma(t) = \gamma(t')$ for distinct $t, t' \in \mathbf{R}$ then

$$(t - t') \cdot \tau \in \mathbf{Z}^N$$

gives a $\mathbf{Q}$-linear dependence relation among the N coordinates of τ since $N > 1$, a contradiction. Thus, $\overline{\gamma}$ is injective.

We now call G the closure of the image of $\overline{\gamma}$ in $\mathbf{T}^N$. This is a closed connected subgroup of $\mathbf{T}^N$ and as such it is contained in the kernel a non-trivial character $\chi : \mathbf{T}^N \to \mathbf{T}$ if the dimension of G is less than N. But $\chi(z_1, \ldots, z_N) = \prod z_j^{a_j}$ for some nonzero $a \in \mathbf{Z}^N$ (viewing $\mathbf{T}$ multiplicatively), so this would give a $\mathbf{Q}$-linear dependence relation on the coordinates of τ, contradicting the assumption. We deduce that we have $G = \mathbf{T}^N$.

In the general case, let G be the closure of the image of $\overline{\gamma}$. This is a closed connected subgroup of $\mathbf{T}^N$, and as such it is itself a torus. We call r its dimension. Then $G^\perp \subset \widehat{\mathbf{T}^N}$ is a finitely generated free subgroup of rank $N - r$. Let $\chi \in G^\perp$.

Then χ is of the form $\chi(z_1 \cdots z_N) = \prod_{1\le k\le N} z_k^{a_k}$ for a unique vector $(a_k)_{1\le k\le N} \in \mathbf{Z}^N$, and the defining condition of $G^\perp$ is $1 = \chi(e^{2\pi i\tau_1 t}\cdots e^{2\pi i\tau_N t}) = e^{2\pi i t(\sum \tau_k a_k)}$ for all $t \in \mathbf{R}$, which reads $\sum_{1\le k\le N} \tau_k a_k = 0$.

It follows that $G^\perp$ is contained within a $\mathbf{Q}$-linear subspace of $\mathbf{Q}^N$ of codimension d, and it equals the intersection of such a subspace with $\mathbf{Z}^N$. The rank of $G^\perp$ is therefore $N-d$, and $\widehat{\mathbf{T}^N}/G^\perp \simeq \widehat{G}$ by [DE, Prop. 3.6.1(c)] (see also Lemma D.1.4.6(3)), from which it follows that G is a d-dimensional closed subtorus of $\mathbf{T}^N$.

(2) : We first assume that the coordinates of τ are $\mathbf{Q}$-linearly independent, so $\overline{\gamma(\mathbf{R})}$ has closure $\mathbf{T}^N$ by (1). Assume that (B.5.2.1) holds for all continuous functions $u \in C(\mathbf{T}^N)$. Let $\Delta \subset [0,1)^N$ be any Jordan-measurable open subset (so $\mu(\partial\Delta) = 0$) and denote by $\widehat{\Delta}$ its image in $\mathbf{T}^N$.

Since Δ is Jordan-measurable, its boundary is a null-set for the Lebesgue measure on $\mathbf{R}^N$, and hence $\mu_N(\partial\widehat{\Delta}) = 0$. Since integration on $\mathbf{T}^N$ with respect to μ_N is insensitive to μ_N-null-sets, by openness of θ (a continuous quotient homomorphism of topological groups) we may replace Δ by its interior, to the effect of replacing $\widehat{\Delta}$ by its interior.

By Lemma B.5.2.2, for every $\varepsilon > 0$ there exist continuous functions $u_1, u_2 \in C(\mathbf{T}^N)$ such that u_1, u_2 are real-valued, and

$$u_1(\xi) \le 1_{\widehat{\Delta}}(\xi) \le u_2(\xi) \tag{B.5.2.4}$$

for all $\xi \in \mathbf{T}^N$, $1_{\widehat{\Delta}}$ being the indicator function of $\widehat{\Delta} \subset \mathbf{T}^N$, and such that

$$\int_{\mathbf{T}^N} (u_2 - u_1)(\xi)\,\mathrm{d}\mu_N(\xi) < \varepsilon. \tag{B.5.2.5}$$

By (B.5.2.1) applied to the continuous function $u = u_2 - u_1$, we also obtain

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} (u_2 - u_1)(\overline{\gamma}(t))\,\mathrm{d}t < \varepsilon \tag{B.5.2.6}$$

and therefore

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} u_2(\overline{\gamma}(t))\,\mathrm{d}t - \varepsilon < \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} 1_{\widehat{\Delta}}(\overline{\gamma}(t))\,\mathrm{d}t < \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} u_1(\overline{\gamma}(t))\,\mathrm{d}t + \varepsilon, \tag{B.5.2.7}$$

and likewise

$$\int_{\mathbf{T}^N} u_2(\xi)\,\mathrm{d}\mu(\xi) - \varepsilon < \int_{\mathbf{T}^N} 1_{\widehat{\Delta}}(\xi)\,\mathrm{d}\mu_N(\xi) < \int_{\mathbf{T}^N} u_1(\xi)\,\mathrm{d}\mu_N(\xi) + \varepsilon \tag{B.5.2.8}$$

Upon letting ε tend to zero, we deduce from the assumed continuous case (applied to u_1 and u_2) that (B.5.2.1) holds for the indicator function of $\widehat{\Delta}$, for any Jordan-measurable subset $\Delta \subset [0,1)^N$.

This case then propagates (B.5.2.1) to arbitrary Riemann sums, since each Riemann sum is an integral on $\mathbf{T}^N$ with respect to μ_N, of a finite $\mathbf{C}$-linear combination of indicator functions of images in $\mathbf{T}^N$ of Jordan-measurable subsets in $[0,1)^N$.

We next deduce that (B.5.2.1) holds for all Riemann-integrable $\mathbf{R}$-valued functions u on $\mathbf{T}^N$. For $\varepsilon > 0$, we can pick a Riemann step function $\widetilde{u}$ on $\mathbf{T}^N$ so that $|\widetilde{u} - u| \le \varepsilon$ pointwise on $\mathbf{T}^N$. Thus,

$$\left|\frac{1}{T}\int_0^T \widetilde{u}(\overline{\gamma}(t))\,\mathrm{d}t - \frac{1}{T}\int_0^T u(\overline{\gamma}(t))\,\mathrm{d}t\right| \le \frac{1}{T}\int_0^T |\widetilde{u} - u| \circ \overline{\gamma}(t)\,\mathrm{d}t \le \varepsilon,$$

and similarly with $(1/(2T))\int_{-T}^{T}$ and $\int_{\mathbf{T}^N} \mathrm{d}\mu_N$. By the settled Riemann step-function case for $\widetilde{u}$, if T is sufficiently large then

$$\left|\int_{\mathbf{T}^N} \widetilde{u}\,\mathrm{d}\mu_N - \frac{1}{T}\int_0^T \widetilde{u}(\overline{\gamma}(t))\,\mathrm{d}t\right| \le \varepsilon, \quad \left|\int_{\mathbf{T}^N} \widetilde{u}\,\mathrm{d}\mu_N - \frac{1}{2T}\int_{-T}^{T} \widetilde{u}(\overline{\gamma}(t))\,\mathrm{d}t\right| \le \varepsilon.$$

Thus, we get analogous estimates for 3ε with u in place of $\widetilde{u}$, so the desired conclusion holds for u. By $\mathbf{R}$-linearity, this conclusion holds for general $\mathbf{C}$-valued Riemann-integrable functions on $\mathbf{T}^N$ by analyzing real and imaginary parts separately.

We are thereby reduced to the case of continuous u in (B.5.2.1) when the coordinates of τ are $\mathbf{Q}$-linearly independent. For a $\mathbf{C}$-sub-algebra $A \subset C(\mathbf{T}^N)$ dense with respect to the uniform-norm topology on $C(\mathbf{T}^N)$, with A to be chosen, we now assume that (B.5.2.1) holds for all the elements of A and deduce it for all continuous functions u on $\mathbf{T}^N$. Pick $\varepsilon > 0$ and $v \in A$ such that $|v - u| \leq \varepsilon$ pointwise on $\mathbf{T}^N$. We then have $|\int_{\mathbf{T}^N} v\,\mathrm{d}\mu_N - \int_{\mathbf{T}^N} u\,\mathrm{d}\mu_N| \leq \varepsilon$ and for all sufficiently large T (depending on ε and v) we have

$$\left|\frac{1}{T}\int_0^T v(\overline{\gamma}(t))\,\mathrm{d}t - \int_{\mathbf{T}^N} v\,\mathrm{d}\mu_N\right| \leq \varepsilon, \ \ \left|\frac{1}{2T}\int_{-T}^T v(\overline{\gamma}(t))\,\mathrm{d}t - \int_{\mathbf{T}^N} v\,\mathrm{d}\mu_N\right| \leq \varepsilon,$$

so each of $(1/T)\int_0^T v(\overline{\gamma}(t))\,\mathrm{d}t$ and $(1/(2T))\int_{-T}^T v(\overline{\gamma}(t))\,\mathrm{d}t$ are within 2ε of $\int_{\mathbf{T}^N} u\,\mathrm{d}\mu_N$. But also

$$\left|\frac{1}{T}\int_0^T v(\overline{\gamma}(t))\,\mathrm{d}t - \frac{1}{T}\int_0^T u(\overline{\gamma}(t))\,\mathrm{d}t\right| \leq \frac{1}{T}\int_0^T |v-u|(\overline{\gamma}(t))\,\mathrm{d}t \leq \varepsilon$$

and likewise for $(1/(2T))\int_{-T}^T$, so both $(1/T)\int_0^T u(\overline{\gamma}(t))\,\mathrm{d}t$ and $(1/(2T))\int_{-T}^T u(\overline{\gamma}(t))\,\mathrm{d}t$ are within 3ε of $\int_{\mathbf{T}^N} u\,\mathrm{d}\mu_N$ for all sufficiently large T. This yields the desired result for u once we have such an algebra A.

We now choose A. Upon denoting by $\widehat{\mathbf{T}^N}$ the group of continuous unitary characters of $\mathbf{T}^N$, we consider the subset

$$A := \{f \in C(\mathbf{T}^N) : f(\xi) = \sum_{0 \leq k \leq N_f} c_k \cdot \chi_k(\xi),\ N_f \in \mathbf{N}, c_k \in \mathbf{C}, \chi_k \in \widehat{\mathbf{T}^N} \text{ for all } k\} \subset C(\mathbf{T}^N) \tag{B.5.2.9}$$

of all finite sums of continuous unitary characters of $\mathbf{T}^N$ with complex coefficients. A is a nonzero $\mathbf{C}$-sub-algebra whose elements separate the points of $\mathbf{T}^N$. Indeed, A is the $\mathbf{C}$-sub-algebra generated by the continuous unitary characters of $\widehat{\mathbf{T}^N}$, and to check the separation condition it is enough to check that given two distinct points $\xi, \xi' \in \mathbf{T}^N$, there exists a continuous unitary character $\chi \in \widehat{\mathbf{T}^N}$ such that $\chi(\xi) \neq \chi(\xi')$. Since $\xi \neq \xi'$, there exists $k \in \mathbf{Z}$ with $1 \leq k \leq N$, such that $\xi_k \neq \xi'_k$. We let $a \in \mathbf{Z}^N$ be such that $a_j = 0$ for all $j \neq k$, and $a_k = 1$. The corresponding character χ under the group-isomorphism $\widehat{\mathbf{T}^N} \simeq \mathbf{Z}^N$ separates ξ and ξ'.

By the Stone-Weierstrass Theorem, $A \subset C(\mathbf{T}^N)$ is dense with respect to the uniform-norm topology on $C(\mathbf{T}^N)$. By the foregoing, we are reduced to proving that (B.5.2.1) holds for u the arbitrary element in A. Before treating such u, we return to the general case (allowing $\mathbf{Q}$-linear dependence among the coordinates of τ). Since G is a closed connected subgroup of $\mathbf{T}^N$, it is itself isomorphic, as a Lie group, to a torus. Any choice of isomorphism $G \simeq \mathbf{T}^d$ of Lie groups pushes forward μ_G to the standard volume-1 Haar measure on $\mathbf{T}^d$.

We may replace μ_N with the finite regular Borel measure μ on $\mathbf{T}^N$ defined on the Borel sets A of $\mathbf{T}^N$ as $\mu(A) := \mu_G(G \cap A)$, and apply Lemma B.5.2.2 with this choice of μ, to propagate the assertion in (B.5.2.1) from the case of continuous complex-valued functions on G, to the case of Riemann integrable functions on G. The rest of the reductions in the foregoing apply upon replacing $\mathbf{T}^N$ with G. Now it remains to treat the $\mathbf{Q}$-linearly independent case (since using a uniformization of G yields τ' that that satisfies $\mathbf{Q}$-linear independence for its d coordinates).

Any $u \in A$ is of the form

$$u(\xi) = \sum_{a \in \mathbf{Z}^N} c_a \cdot \xi_1^{a_1} \cdot \ldots \cdot \xi_N^{a_N} \tag{B.5.2.10}$$

with $c_a \neq 0$ for at most finitely many $a \in \mathbf{Z}^N$, and where $\xi \in \mathbf{T}^N$ is written in coordinates $\xi = (\xi_1, \ldots, \xi_N)$ since each $\chi \in \widehat{\mathbf{T}^N} \simeq \mathbf{Z}^N$ is of the form

$$\chi(\xi) = \prod_{k=1}^{N} \xi_k^{a_k}$$

for a unique $a = (a_1, \ldots, a_N) \in \mathbf{Z}^N$. We write χ_a for the continuous character of $\mathbf{T}^N$ corresponding to $a \in \mathbf{Z}^N$.

We have

(B.5.2.11)
$$\widehat{u}(\chi_b) = \int_{\mathbf{T}^N} u(\xi) \cdot \overline{\chi_b(\xi)}\, \mathrm{d}\mu(\xi) = \int_{[0,1)^N} \sum_{a \in \mathbf{Z}^N} c_a \cdot e^{2\pi i a_1 x_1} \cdot \ldots \cdot e^{2\pi i a_N x_N} \cdot \overline{e^{2\pi i b_1 x_1} \cdot \ldots \cdot e^{2\pi i b_N x_N}}\, \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

By orthogonality of the exponentials $\{e^{2\pi i t N}, N \in \mathbf{Z}\} \subset L^2([0,1))$, we obtain

$$\widehat{u}(\chi_b) = c_b. \tag{B.5.2.12}$$

We have

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} u(e^{2\pi i t \tau_1}, \ldots, e^{2\pi i t \tau_N})\, \mathrm{d}t = \lim_{T \to +\infty} \frac{1}{2T} \sum_{b \in \mathbf{Z}^N} c_b \int_{-T}^{T} e^{2\pi i (b_1 \tau_1 + \ldots + b_N \tau_N) t}\, \mathrm{d}t \tag{B.5.2.13}$$

where we are allowed to interchange the summation and the integral since $c_b = 0$ for all but finitely many b.

Suppose $b_1\tau_1 + \ldots + b_N\tau_N \neq 0$ for one of the finitely many b such that $c_b \neq 0$. Then we have:

(B.5.2.14)
$$\left| \frac{1}{2T} \int_{-T}^{T} e^{2\pi i (b_1 \tau_1 + \ldots + b_N \tau_N) t} \mathrm{d}t \right| \leq \frac{1}{2T\pi (b_1\tau_1 + \ldots + b_N\tau_N)} |e^{2\pi i (b_1 \tau_1 + \ldots + b_N \tau_N) T} - e^{-2\pi i (b_1 \tau_1 + \ldots + b_N \tau_N) T}| = o_b(1)$$

as $T \to +\infty$.

The right side of (B.5.2.13) therefore equals $\sum_{b : b_1\tau_1 + \ldots + b_N\tau_N = 0} c_b$. The left side of (B.5.2.13) is also written as

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} u(e^{2\pi i t \tau_1}, \ldots, e^{2\pi i t \tau_N})\, \mathrm{d}t = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} u(\overline{\gamma}(t))\, \mathrm{d}t.$$

But $\sum_{b : b_1\tau_1 + \ldots + b_N\tau_N = 0} c_b = \int_{G^\perp} \widehat{u}(\chi)\chi(1)\, \mathrm{d}\mu_{G^\perp}(\chi)$ with $\mu_{G^\perp}$ the counting measure on $G^\perp$ dual to the volume-1 measure on $\mathbf{T}^N/G$. This integral over $G^\perp$ is equal to

$$\int_G u(\xi)\, \mathrm{d}\mu_G(\xi)$$

by the Poisson summation formula (see Theorem D.1.2.7(3)). The case of $(1/T)\int_0^T$ goes similarly. □

Corollary B.5.2.4 (Uniform distribution). *Pick $\tau = (\tau_1, \ldots, \tau_N) \subset \mathbf{R}^N$. Let $\gamma_\tau : \mathbf{R} \to \mathbf{T}^N$ be the continuous 1-parameter group generated by τ. Let G be the closure of the image of γ_τ in $\mathbf{T}^N$, and μ_G the volume-1 Haar measure on G.*

For every open subset $U \subset [0,1)^N$, upon denoting by $\widehat{U}$ the image of U in $\mathbf{T}^N$ under the quotient map θ, if $\mu_G(\partial(\widehat{U} \cap G)) = 0$ then we have:

$$\lim_{T \to +\infty} \frac{\lambda([-T,T]_U)}{2T} = \lim_{T \to +\infty} \frac{\lambda([0,T]_U)}{T} = \mu_G(\widehat{U} \cap G). \tag{B.5.2.15}$$

Proof. The function $u := 1_{\widehat{U} \cap G}$ is Riemann-integrable on G since $\mu_G(\partial(\widehat{U} \cap G)) = 0$. The identities in (B.5.2.15) are then a special case of (B.5.2.1) from Theorem B.5.2.1. □

Remark B.5.2.5. There is a discrete-time version of both Theorem B.5.2.1 and Corollary B.5.2.4. Both are proved by closely adapting the arguments provided above. We further refer the reader to [**Hu2**, Lemma A.14, p. 93].

B.5.3. Strong recurrence of non-Archimedean Euler factors. We begin by recalling basic concepts on self-approximation properties of holomorphic and meromorphic functions on an open non-empty vertical strip in $\mathbf{C}$.

B.5.3.1. *Strong recurrence.* Let $I \subset \mathbf{R}$ be a non-empty interval. Recall that we write $\mathbf{C}_I$ to denote the vertical strip

$$\mathbf{C}_I := \{s \in \mathbf{C} : \mathrm{Re}(s) \in I\}. \tag{B.5.3.1}$$

In particular, if $I = (a, b)$ is a bounded open interval, we have

$$\mathbf{C}_I = \{s \in \mathbf{C} : a < \mathrm{Re}(s) < b\}$$

and if $I = [a, b)$, (a, ∞), respectively $[a, \infty)$, we have

$$\mathbf{C}_I = \{s \in \mathbf{C} : a \leq \mathrm{Re}(s) < b\},\ \mathbf{C}_I = \{s \in \mathbf{C} : \mathrm{Re}(s) > a\},\ \mathbf{C}_I = \{s \in \mathbf{C} : \mathrm{Re}(s) \geq a\}.$$

Definition B.5.3.1. (1) For a subset $X \subset \mathbf{R}$ we define the *upper density* of X to be

$$\limsup_{T \to +\infty} \frac{\lambda([-T, T] \cap X)}{2T}. \tag{B.5.3.2}$$

(2) Let $I \subset \mathbf{R}$ be an open non-empty interval. We say that a holomorphic function $f : \mathbf{C}_I \to \mathbf{C}$ is *strongly recurrent* on a compact subset $K \subset \mathbf{C}_{>R}$ if and only if for every $\varepsilon > 0$, the set $X_K(\varepsilon)$ of all $t \in \mathbf{R}$ such that

$$\sup_{s \in K} |f(s + it) - f(s)| < \varepsilon$$

has positive upper density in $\mathbf{R}$.

We recall that the notion of (strong) "recurrence" for a holomorphic function on a vertical strip $\mathbf{C}_I$, for $I \subset \mathbf{R}$ a non-empty open interval, was studied in B. Bagchi's PhD thesis [**Ba1**, Ch. 1,2]. Strong recurrence is a self-approximation property of holomorphic functions on vertical strips. We refer the reader to [**Ba1**, Ch. 1, §1.4] (respectively [**Ba1**, Ch. 2, §2.3]).

B.5.3.2. *Strong recurrence of reciprocal partial Euler products on open strips containing $i\mathbf{R}$.* The main result of this section is the following:

Proposition B.5.3.2. *Let F be a number field, S a finite set of places of F containing all the Archimedean places. We write $S_f := S \cap \Sigma_{F,f}$. For each $v \in S_f$, we denote by π_v a choice of uniformizer at v. Let $\chi \in \widehat{C_F}$ be a continuous character unramified at the finite places of F contained in S.*

There exists a real number $N_S > 0$ and a sequence $\{T_k\}_{k \in \mathbf{Z}} \subset \mathbf{R}$, such that:

(1) *$T_k \geq 0$ if $k \geq 0$, $T_k \leq 0$ if $k \leq 0$, and $T_k \leq T_{k+1}|$ for every $k \in \mathbf{Z}$.*
(2) *$\lim_{|k| \to +\infty} |T_k| = +\infty$.*
(3) *For all $k \in \mathbf{Z}$, we have:*

$$\prod_{v \in S_f} |1 - \chi(\pi_v) q_v^{-iT_k}|^{-1} \leq N_S. \tag{B.5.3.3}$$

In this subsection we provide a direct argument for Proposition B.5.3.2, resting on the Kronecker-Weyl Theorem and a uniform version of a classical Theorem of Carlson An alternative argument has been provided in Proposition 7.1.1.1 using the results of [**Ba1**, Ch. 2].

We prove the following stronger assertion:

Proposition B.5.3.3 (Strong recurrence of reciprocal partial Euler products). *Let F be a number field, S a finite set of finite places of F. Let $\chi \in \widehat{C_F}$ be a continuous character unramified at all places in S.*

Let $I \subset \mathbf{R}$ be an open non-empty bounded interval containing $0 \in \mathbf{R}$. Let $K \subset \mathbf{C}_I$ be an arbitrary compact subset with non-empty interior. Then for any real number $\varepsilon > 0$, the subsets:

$$\left\{ t \in [0, +\infty) : \sup_{s \in K} \left| \prod_{v \in S} L_v(\chi_v | \cdot |_v^{s+it})^{-1} - \prod_{v \in S} L_v(\chi_v | \cdot |_v^{s})^{-1} \right| < \varepsilon \right\} \subset \mathbf{R} \tag{B.5.3.4}$$

and

$$\left\{ t \in (-\infty, 0] : \sup_{s \in K} \left| \prod_{v \in S} L_v(\chi_v | \cdot |_v^{s+it})^{-1} - \prod_{v \in S} L_v(\chi_v | \cdot |_v^{s})^{-1} \right| < \varepsilon \right\} \subset \mathbf{R} \tag{B.5.3.5}$$

both have positive upper-density.

B.5.3.3. *Preparations on recurrence.* We begin with a few permanence properties of strong recurrence for holomorphic functions on non-empty open vertical strips of bounded width.[28]

LEMMA B.5.3.4. *Let $I \subset \mathbf{R}$ be an open non-empty bounded interval, and $f : \mathbf{C}_I \to \mathbf{C}$ a holomorphic function.*

(1) *If f is strongly recurrent on the compact subset $K \subset \mathbf{C}_I$ then it is strongly recurrent on any compact subset $K' \subset K$.*
(2) *If f is strongly recurrent on any closed disc contained within $\mathbf{C}_I$ then it is strongly recurrent on any compact subset contained within $\mathbf{C}_I$.*

PROOF. We prove (1). If, for every $\varepsilon > 0$, the set $X_K(\varepsilon) \subset \mathbf{R}$ has positive upper-density, then so does $X_{K'}(\varepsilon)$, since $X_K(\varepsilon) \subset X_{K'}(\varepsilon)$. This proves (1).

Let $K \subset \mathbf{C}_I$ be an arbitrary compact subset. There exists $c_0 > 0$ such that

$$r_0 := \inf\{\sigma \in I : \sigma = \mathrm{Re}(s), s \in K\} > \inf I + c_0.$$

We call $r_1 := \sup\{\sigma \in I : \sigma = \mathrm{Re}(s), s \in K\}$, and $r := r_1 - r_0$. Then K is contained within a closed disc D of radius r centered at some $s \in \mathbf{C}_I$. By assumption, f is strongly recurrent on D, and hence, by (2), so it is on K. □

Let $f : \mathbf{C}_I \to \mathbf{C}$ be a holomorphic function, with $I \subset \mathbf{R}$ an open non-empty bounded interval.

Sub-harmonicity of $|f|^2$ as a real valued function and on any domain of holomorphy of f, ensures that the sup-norm condition in the definition of strong recurrence can be verified through the following Lemma.

We denote by μ_A the Radon measure associated to the area form $\mathrm{d}s \wedge \mathrm{d}\overline{s}$ on the complex s-plane.

LEMMA B.5.3.5. *Let $I \subset \mathbf{R}$ be a non-empty open bounded interval, and $f : \mathbf{C}_I \to \mathbf{C}$ a holomorphic function. Let $K \subset \mathbf{C}_I$ be a non-empty compact subset, and $U \subset \mathbf{C}_I$ an open subset, such that:*

(1) *The closure of U is compact and contained within $\mathbf{C}_I$.*
(2) *$K \subset U$ and the distance δ between K and the complement of U is positive.*

Fix $T > 0$. Upon defining the subsets

$$A_{\overline{U},r}(X, T) := \left\{ t \in [-T, T] : \int_{\overline{U}} |f(s + it) - f(s)|^2 \, \mathrm{d}\mu_A(s) < X \right\} \subset \mathbf{R} \tag{B.5.3.6}$$

$$A_K(X, T) := \left\{ t \in [-T, T] : \sup_{s \in K} |f(s + it) - f(s)| < X \right\} \subset \mathbf{R} \tag{B.5.3.7}$$

for $X > 0$, and the function $c(X) := \pi\delta^2 X^2$, for $X > 0$, we have

$$A_{\overline{U},2}(c(X), T) \subset A_K(X, T) \tag{B.5.3.8}$$

[28] Brian needs to check the next two lemmas and how they are used in the subsequent proof of the preceding Proposition.

for all $T > 0$.

PROOF. We first show that an open subset $U \subset \mathbf{C}_I$ satisfying the assumptions (1),(2) exists, to conclude that the statement is non-vacuous. Let $a > 0$ be the distance between K and the complement of $\mathbf{C}_I$, and cover K with finitely many discs of radius $a/2$. The union of their interiors is an example of such U. It is totally bounded by compactness of K. Its boundary is contained in a finite union of circles, and hence it is contained within $\mathbf{C}_I$. The closure of U is therefore also contained within $\mathbf{C}_I$.

For any $s \in K$, we therefore have $\overline{D}_\delta(s) \subset U$.

We fix $s \in K$ arbitrary. Since, for every $t \in \mathbf{R}$, $|f(z+it) - f(z)|^2$ is a sub-harmonic function of $z \in U$, it follows that

$$|f(s+it) - f(s)|^2 \leq \frac{1}{\pi\delta^2} \int_{\overline{D}_\delta(s)} |f(z+it) - f(z)|^2 \, \mathrm{d}\mu_A(z) \leq \frac{1}{\pi\delta^2} \int_{\overline{U}} |f(z+it) - f(z)|^2 \, \mathrm{d}\mu_A(z) \tag{B.5.3.9}$$

where $\mathrm{d}\mu_A(z) = |\mathrm{d}x \, \mathrm{d}y|$, for $z = x + iy$, is the normalized area measure on the complex z-plane. If $t \in \mathbf{R}$ satisfies the condition

$$\int_{\overline{U}} |f(s+it) - f(s)|^2 \, \mathrm{d}\mu_A(s) < \pi\delta^2 X^2 = c(X) \tag{B.5.3.10}$$

then by (B.5.3.9) we have, for all $s \in K$

$$|f(s+it) - f(s)| < X. \tag{B.5.3.11}$$

Since (B.5.3.10) implies (B.5.3.11) we therefore have $A_{\overline{U},2}(c(X), T) \subset A_K(X, T)$. □

B.5.3.4. *Strong recurrence of reciprocal finite Euler products.* We can now prove Proposition B.5.3.3.

PROOF OF PROPOSITION B.5.3.3. We denote by λ the Lebesgue measure on the real line, normalized so as to give the interval $[0,1]$ volume 1. We define:

$$u\left(s, (t_v)_{v \in S}\right) := \prod_{v \in S} L_v(\chi_v| \cdot |_v^{s + it_v \frac{2\pi}{\log(q_v)}})^{-1} - \prod_{v \in S} L_v(\chi_v| \cdot |_v^s)^{-1} \tag{B.5.3.12}$$

for $q_v := N(\mathfrak{p}_v)$, with $\mathfrak{p}_v \subset \mathcal{O}_F$ the prime ideal corresponding to the finite place v of F.

Since χ is unramified at $v \in S$, we have $L_v(\chi_v| \cdot |_v^s)^{-1} = 1 - \chi_v(\pi_v) q_v^{-s}|$. Thus, the function $u : \mathbf{C} \times \mathbf{T}^S \to \mathbf{C}$ is continuous and its holomorphic derivatives of arbitrary order with respect to s, formed pointwise over $\mathbf{C} \times \mathbf{T}^S$, are also continuous over $\mathbf{C} \times \mathbf{T}^S$.

Let $K \subset \mathbf{C}_I$ be a compact subset with non-empty interior. We define, for every real number $\varepsilon > 0$:

$$X(\varepsilon) := \left\{ (t_v)_{v \in S} : \int_K |u(s, (t_v)_{v \in S})|^2 \, \mathrm{d}\mu_A(s) < \varepsilon \right\} \subset \mathbf{T}^S. \tag{B.5.3.13}$$

Since $u(s, \cdot)$ is a continuous function on $\mathbf{T}^S$ and K is compact, $\int_K |u(s, (t_v)_{v \in S}|^2 \, \mathrm{d}\mu_A(s)$ is also continuous on $\mathbf{T}^S$ and has a zero at $0 \in \mathbf{T}^S$ (i.e. at $(t_v)_{v \in S} = 0$). It follows that $X(\varepsilon)$ is open and non-empty.

We define $\tau = (\tau_v)_{v \in S}$ to be the vector in $\mathbf{R}^S$ whose coordinates are $\tau_v := \frac{\log(q_v)}{2\pi}$.

We denote by $\gamma_\tau : \mathbf{R} \to \mathbf{R}^S$ the continuous 1-parameter subgroup generated by τ, and by $\overline{\gamma}_\tau : \mathbf{R} \to \mathbf{T}^S$ the continuous 1-parameter group obtained by composition with the quotient homomorphism $\theta_S : \mathbf{R}^S \to \mathbf{T}^S$. Note that

$$u(s, \overline{\gamma}_\tau(t)) = \prod_{v \in S} L_v(\chi_v| \cdot |_v^{s+it})^{-1} - \prod_{v \in S} L_v(\chi_v| \cdot |_v^s)^{-1}.$$

We denote by G the closure of the image of γ_τ in $\mathbf{T}^S$, a closed sub-torus. We call μ_G the volume-1 Haar measure on G.

Assume for the moment that we have $\mu_G(\partial(X(\varepsilon) \cap G)) = 0$. Then by Corollary B.5.2.4 we have

$$\lim_{T \to +\infty} \frac{\lambda(\{t \in [-T, T] : \overline{\gamma}_\tau(t) \in X(\varepsilon)\})}{2T} = \lim_{T \to +\infty} \frac{\lambda(\{t \in [0, T] : \overline{\gamma}_\tau(t) \in X(\varepsilon)\})}{T} = \mu_G(X(\varepsilon) \cap G) \tag{B.5.3.14}$$

and since $X(\varepsilon) \cap G$ is open in G and non-empty (it contains the neutral element), its μ_G-measure is positive.

Since we have:

$$\{t \in I_T : \overline{\gamma}_\tau(t) \in X(\varepsilon)\} = \left\{ t \in I_T : \int_K |\prod_{v \in S} L_v(\chi_v|\cdot|_v^{s+it})^{-1} - \prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1}|^2 \, \mathrm{d}\mu_A(s) < \varepsilon \right\} \tag{B.5.3.15}$$

for I_T either of the two intervals $[-T, T]$ and $[0, T]$, we deduce from (B.5.3.14), from positivity of $\mu_G(X(\varepsilon) \cap G)$, and from Lemma B.5.3.5, that $\prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1}$ is indeed strongly recurrent over $\mathbf{C}_I$ (with a different $\varepsilon > 0$ dependent on ε and the distance between K and the boundary of the strip) for $K \subset \mathbf{C}_I$ any compact subset with non-empty interior, and hence in particular for K satisfying the assumptions of the Proposition. By combining (B.5.3.15) with (B.5.3.14) all assertions of the Proposition follow.

We now prove that we have $\mu_G(\partial(X(\varepsilon) \cap G)) = 0$, so that the Kronecker-Weyl Theorem (see Theorem B.5.2.1), through its Corollary B.5.2.4, applies. Let d the dimension of the $\mathbf{Q}$-linear subspace of $\mathbf{R}$ generated by $(\tau_v)_{v \in S}$. The locus $\{(t_v)_{v \in S} \in G : \int_K |u(s, (t_v)_{v \in S})|^2 \, \mathrm{d}\mu_A(s) = \varepsilon\}$ contains $\partial(X(\varepsilon) \cap G)$, and it has μ_G-measure zero. To see this, we proceed as follows.

The subset:

$$U := \left\{ (t_v)_{v \in S} \in G : \int_K |u(s, (t_v)_{v \in S})|^2 \, \mathrm{d}\mu_A(s) = \varepsilon \right\}$$

which reads

$$\left\{ (t_v)_{v \in S} \in G : \int_K ((\mathrm{Re}(u)(s, (t_v)_{v \in S}))^2 + (\mathrm{Im}(u)(s, (t_v)_{v \in S}))^2) \, \mathrm{d}\mu_A(s) = \varepsilon \right\}$$

as a subset of G pulls back to an analytic subset of $\mathbf{T}^d$ (the zero set of a nonzero real-analytic function) under any isomorphism $\mathbf{T}^d \simeq G$ of Hausdorff locally compact abelian groups, upon observing that any isomorphism of Hausdorff locally compact abelian groups $\mathbf{T}^d \simeq G$ is also a real-analytic diffeomorphism.

This is so, because the continuous function of $(t_v)_{v \in S}$:

$$(t_v)_{v \in S} \mapsto \int_K |u(s, (t_v)_{v \in S})|^2 \, \mathrm{d}\mu_A(s)$$

is itself analytic, and U is one of its level sets. Note that since $\varepsilon > 0$, U does not contain $(t_v)_{v \in S} = 0$, and hence this function is analytic and nonzero, thus exhibiting U as an analytic subset of G.

Since θ_S is itself analytic (i.e. its coordinate functions are complex-valued real-analytic functions) then the pre-image of U in $\mathbf{R}^S$ under θ_S is a real-analytic subset of $\mathbf{R}^S$, and its intersection with $[0, 1)^S$ therefore has Lebesgue measure zero by [**Mit**, Prop. 1]. Its image under the smooth composition of θ_S followed by the real-analytic diffeomorphism $\mathbf{T}^d \to G$ therefore has μ_G-measure zero. □

Proposition B.5.3.2 follows from Proposition B.5.3.3 by choosing a small disc whose interior is non-empty and intersects $i\mathbf{R}$ in a non-empty open vertical interval containing none of the zeros of $\prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1}$, nor in its interior nor on its boundary, and choosing it as K in Proposition B.5.3.3. Over K, the absolute value of $\prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1}$ is therefore strictly positive, and since $|\prod_{v \in S} L_v(\chi_v|\cdot|_v^s)|$ is continuous over K, it attains a maximum $M > 0$, and hence $|\prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1}|$ is bounded below by $1/M$. By choosing $\varepsilon > 0$ in Proposition B.5.3.3 tiny compared to $1/M$, the Proposition implies that there exists a subset of $t \in \mathbf{R}$ of the real line of positive upper density such that $\prod_{v \in S} L_v(\chi_v|\cdot|_v^{s+it})$ is, for all such t's, bounded above in absolute value by a positive real number independent of $s \in K$ (so including $s = 0 \in K$).

The positivity of the upper density implies the existence of a sequence $\{T_k\}$ satisfying the requirements of Proposition B.5.3.2.

B.6. Classical zero-free regions for Hecke L-functions

In this section, we recall well known results on the existence of classical zero-free regions for general Hecke L-functions of arbitrary number fields, and discuss estimates on the polynomial growth of the finite L-function, its holomorphic derivatives of any order on vertical strips of bounded width, and of its reciprocal on the 1-line.

B.6.1. Zero-free regions for Hecke L-functions and estimates on $L'_f(s,\chi)/L_f(s,\chi)$. We record here analogs of the main results on zero-free regions for the Riemann ζ-function, and estimates for $\zeta'(s)/\zeta(s)$ on the 1-line, adapted to the case of finite Hecke L-functions.

B.6.1.1. *Classical zero-free region for $L_f(s,\chi)$.* We recall the statement of the following well-known result, whose proof goes back to the argument of Hadamard and de la Vallée Poussin:

PROPOSITION B.6.1.1 (Classical zero-free region for $L_f(s,\chi)$). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. There exists an absolute constant $A > 0$, and a real number $A_{F,\chi} > 0$ depending only on F and χ, such that $L_f(s,\chi)$ has no complex zeros in the region:*

$$\sigma \geq 1 - \frac{A}{[F:\mathbf{Q}]^4 \log(A_{F,\chi}(|t|+3))} \tag{B.6.1.1}$$

and if $L_f(\sigma,\chi) = 0$ for some $\sigma \in [1/2,1]$ then χ is quadratic and σ is unique, simple, and satisfies $\sigma < 1$.

The real number $A_{F,\chi}$ can in fact be chosen to be $|N(\mathfrak{d}_{F/\mathbf{Q}})||N_{F/\mathbf{Q}}(\mathfrak{f}_\chi)|$ where $\mathfrak{f}_\chi$ is the conductor of χ [**IK**, Ch. 5, §5.1, (5.4)]. Proposition B.6.1.1 can be deduced from [**IK**, §5.4, Thm. 5.10, p. 106] upon setting $d = [F:\mathbf{Q}]$, $A_{F,\chi} := \mathfrak{q}(f,0)$ in the notation of loc.cit. to denote the analytic conductor of the L-function (see [**IK**, Ch. 5, (5.7), p. 95]), and noting that the Rankin-Selberg convolutions $L_f(s,\chi\otimes\chi)$ and $L_f(s,\chi\otimes\overline{\chi})$ both exist with the former entire when $\chi \neq \overline{\chi}$ (equivalently, when $\chi^2 \neq 1$) since $L_f(s,\chi^2)$ is entire when χ is not quadratic, and the latter is $\zeta_F(s)$ (so it has a simple pole at $s=1$).

B.6.1.2. *Estimates for $L'_f(s,\chi)/L_f(s,\chi)$.* Let F be a number field and $\chi \in \widehat{C_F}$ a continuous character. On $\mathbf{C}_{>1}$, the Dirichlet series expansion of the associated "finite" Hecke L-function is

$$L_f(s,\chi) = \sum_{0\neq I\subset \mathcal{O}_F} \frac{\chi(I)}{N(I)^s} \tag{B.6.1.2}$$

(sum over nonzero ideals) and we define the summatory function

$$S_{F,\chi}(x) := \sum_{I\subset\mathcal{O}_F : N(I)\leq x} \chi(I). \tag{B.6.1.3}$$

We note that we may define, for every real $x \geq 0$ the coefficients

$$a_x := \sum_{I\subset\mathcal{O}_F : N(I)=x} \chi(I) \tag{B.6.1.4}$$

as a sum over nonzero ideals (so $a_x := 0$ if the set indexing the sum is empty). For instance, if x is not of the form $N(I)$ for a nonzero ideal $I \subset \mathcal{O}_F$ (such as whenever x is not an integer, or when $x = 0$) then $a_x = 0$.

We recall from [**CF**, Ch. VIII, §1, p. 205] that a continuous character $\chi \in \widehat{C_F}$ of the idele class group (equivalently a continuous character χ of the idele group $\mathrm{GL}_1(\mathbf{A}_F)$ such that $\chi(F^\times) = 1$) is unramified outside a finite set S of finite places of F and so uniquely determines a character χ of the group $\mathrm{Div}(\mathcal{O}_{F,S})$ of fractional ideals via $\chi(I) := \chi(\mathfrak{a}_I)$ where $\mathfrak{a}_I \in \mathrm{GL}_1(\mathbf{A}_F)$ is the idele defined to have v-indexed coordinate 1 if $v \in S \cup \Sigma_{F,\infty}$ and equal to $\pi_v^{\mathrm{ord}_v(I)}$ for π_v a choice of local uniformizer in F_v for finite $v \notin S$. This function χ of $\mathrm{Div}(\mathcal{O}_{F,S})$ satisfies $\chi(IJ) = \chi(I)\chi(J)$, so we can rewrite the Dirichlet series in (B.6.1.2) in the more

familiar form

$$L_f(s,\chi)=\sum_{m\geq 1}\frac{a_m}{m^s}. \tag{B.6.1.5}$$

We have $a_{mn}=a_m a_n$ for coprime integers $m,n\geq 1$ because any ideal $I\subset \mathcal{O}_F$ satisfying $N(I)=mn$ can be uniquely written as $I=JK$ for ideals $J,K\subset\mathcal{O}_F$ with $N(J)=m$ and $N(K)=n$.

The mth Dirichlet series coefficient a_m of $L_f(\chi,s)$ satisfies

$$|a_m|\leq \#\{I\subset\mathcal{O}_F : N(I)=m\}\leq \tau(m)^{[F:\mathbf{Q}]}$$

where $\tau(m):=\#\{d|m\}$, so $a_m=O_{F,\varepsilon}(m^\varepsilon)$ for all $\varepsilon>0$. By similar reasoning, for every $\varepsilon>0$ we have

$$S_{F,\chi}(x)\ll_\varepsilon x^{1+\varepsilon}. \tag{B.6.1.6}$$

We record here the following consequence of the analytic behavior of $L_f(s,\chi)$:

LEMMA B.6.1.2. *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character.*

(1) *If $\chi|_{C_F^1}$ is trivial then for the unique real number $t_0\in\mathbf{R}$ such that $\chi=|\cdot|_F^{-it_0}$ we have*

$$\frac{S_{F,\chi|\cdot|_F^{it_0}}(x)}{x}=N(\mathfrak{d}_{F/\mathbf{Q}})^{-1/2}\overline{\mu}_1(C_F^1)+o(1) \tag{B.6.1.7}$$

(2) *If $\chi|_{C_F^1}$ is non-trivial then*

$$\frac{S_{F,\chi}(x)}{x}=o(1). \tag{B.6.1.8}$$

PROOF. (1) : The Hecke L-function $L_f(s,\chi|\cdot|_F^{it_0})=\zeta_F(s)$ has a simple pole at $s=1$ whose residue is $N(\mathfrak{d}_{F/\mathbf{Q}})^{-1/2}\overline{\mu}_1(C_F^1)$. Upon applying [**MVa**, Cor. 8.8] with $c=N(\mathfrak{d}_{F/\mathbf{Q}})^{-1/2}\overline{\mu}_1(C_F^1)$, we see that (B.6.1.7) holds.

(2) : The Hecke L-function $L_f(s,\chi)$ is holomorphic over $\mathbf{C}_{>0}$. Upon applying [**MVa**, Cor. 8.8] with $c=0$, we see that (B.6.1.8) holds. □

The following result is well-known to experts (see, for instance, [**SW**, §4, (4.1)]) and it is recorded in [**T2**, Ch. III, (3.11.7)] for $F=\mathbf{Q}$ and $\chi=1$. We provide the argument here below, following [**Ingh3**, Thm. 20, p. 60-62].

THEOREM B.6.1.3. *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let*

$$\eta:\mathbf{R}_{\geq 0}\to\mathbf{R}_{>0} \tag{B.6.1.9}$$

be a differentiable decreasing function satisfying the following conditions:

(1) $\eta(\mathbf{R}_{\geq 0})\subset(0,1/2]$.
(2) $\lim_{t\to+\infty}\eta'(t)=0$.
(3) $1/\eta(t)=O(\log(t))$.

Let $\alpha=\alpha_F$ be a real number satisfying $0<\alpha<1$ such that $\zeta_F(s)$ is non-vanishing on $\{0<|s-1|<\alpha\}$. Assume that $L_f(s,\chi)$ has no zeros in the region

$$\sigma>1-\eta(|t|).$$

Then

$$\frac{L_f'(s,\chi)}{L_f(s,\chi)}=O_{\alpha,\chi}((\log(t))^2) \tag{B.6.1.10}$$

uniformly in the region

$$\sigma\geq 1-\alpha\eta(|t|) \tag{B.6.1.11}$$

as $|t| \to +\infty$.

PROOF. The function $L_f'(s,\chi)/L_f(s,\chi)$ is holomorphic over $\mathbf{C}_{>1}$ and there it is an absolutely convergent Dirichlet series

$$-\sum_{I\subset\mathcal{O}_F}\frac{\Lambda_{F,\chi}(I)}{N(I)^s} \tag{B.6.1.12}$$

where

$$\Lambda_{F,\chi}(I) := \chi(I)\,\Lambda_F(I)$$

with the von Mangoldt function Λ_F for F defined by

$$\Lambda_F(I) := \begin{cases}\log(N\mathfrak{p}), & I=\mathfrak{p}^k \text{ a prime power with } k\geq 1,\\ 0, & \text{otherwise (including } I=(1)).\end{cases} \tag{B.6.1.13}$$

We define, for every integer $m \geq 1$:

$$A_m := \sum_{I\subset\mathcal{O}_F : N(I)=m}\chi(I)\Lambda_F(I), \tag{B.6.1.14}$$

as a sum over the ideals of $\mathcal{O}_F$ whose norm is m (so $A_1 = 0$ since $\Lambda_F((1)) = 0$).

LEMMA B.6.1.4. *For* A_m *as in* (B.6.1.14), $A_m = O_F(\log m)$ *as* $m \to \infty$, *where the* O*-constant depends on* $[F:\mathbf{Q}]$ *but not* χ.

PROOF. From the definition, $|A_m| \leq \sum_{\mathfrak{p},k:N(\mathfrak{p}^k)=m}\log(N\mathfrak{p})$ where the sum is over prime ideals $\mathfrak{p}$ and integers $k \geq 1$ (recall $A_1 = 0$). If m is not a prime power > 1 then $A_m = 0$, so we can assume $m = p^r$ with a rational prime p and exponent $r \geq 1$. Then the equality $N(\mathfrak{p}^k) = m = p^r$ forces $\mathfrak{p}|p$, allowing at most $[F:\mathbf{Q}]$ possibilities for $\mathfrak{p}$ (given $m = p^r$), and for each such $\mathfrak{p}$ there is at most one possibility for k. Thus,

$$|A_m| \leq [F:\mathbf{Q}]\max_{\mathfrak{p}|p}\log(N\mathfrak{p}) \leq [F:\mathbf{Q}]^2\log(p) \leq [F:\mathbf{Q}]^2\log(m).$$

□

We will now be using Proposition B.1.1.3 and the method of proof of [**Ingh3**, Thm.20] to establish the Theorem. Without loss of generality, we may assume $t > 0$ (e.g., pass to $\overline{L_f(s,\chi)} = L_f(\overline{s},\overline{\chi})$). Assume first that $\sigma > 1 + \alpha\eta(t)$. Then we have, from the absolute and locally uniform convergence of the Dirichlet series in (B.6.1.12):

$$\left|\frac{L_f'(\chi,s)}{L_f(s,\chi)}\right| \leq \sum_{I\subset\mathcal{O}_F}\frac{\Lambda_F(I)}{N(I)^{1+\alpha\eta(t)}} = -\frac{\zeta_F'(1+\alpha\eta(t))}{\zeta_F(1+\alpha\eta(t))} \leq \frac{A}{\alpha\eta(t)} \tag{B.6.1.15}$$

for some $A = A_F > 0$ since $-\zeta_F'(s)/\zeta_F(s) = 1/(s-1) + O(1)$ as $s \to 1$. By condition (3) on η, $A/\alpha\eta(t) = O(\log(t))$.

Since $L_f(s,\chi)$ is holomorphic and nonzero over the domain

$$\Omega := \{s = \sigma + it \in \mathbf{C} : \sigma > 1 - \eta(t), t > 0\}$$

and Ω is simply connected, the Dirichlet series

$$Z_{F,\chi}(s) := \sum_{\mathfrak{p},m}\frac{\chi(\mathfrak{p})^m}{mN(\mathfrak{p})^{ms}} \tag{B.6.1.16}$$

on $\mathbf{C}_{>1}$ (where the sum is indexed by the prime ideals of $\mathcal{O}_F$ and by the positive integers $m \geq 1$) defines a branch of $\log L_f(s,\chi)$ that extends holomorphically over Ω.

For $T > 1$, we define

$$s_0 := 1 + \alpha\eta(T) + iT \tag{B.6.1.17}$$

and define the two circle

$$\gamma_1(T) := \{s \in \mathbf{C} : |s - s_0| = 2\alpha\eta(T)\}, \quad \gamma_2(T) := \left\{s \in \mathbf{C} : |s - s_0| = \frac{1+3\alpha}{2}\eta(T)\right\}. \tag{B.6.1.18}$$

The two circles $\gamma_1(T), \gamma_2(T)$, whose radii we call $r_1(T) < r_2(T) < 1$, are the same as in the proof of [**Ingh3**, Thm. 20, p. 62] (see [**Ingh3**, Thm. 20, Fig. 4, p. 61]), they share the same center s_0 defined in (B.6.1.17), and $\gamma_1(T)$ is enclosed within $\gamma_2(T)$.

There exists $T_1 > 1$ such that for all $T > T_1$, we have $\gamma_1(T), \gamma_2(T) \subset \Omega$: the argument to see this is provided in the proof of [**Ingh3**, Thm. 20, p. 62], and it rests on the fact that η is decreasing, and on assumptions (1) and (2).

It follows that for all $T > T_1$, then $Z_{F,\chi}$ is holomorphic over the open disc $D_2(T)$ whose boundary is $\gamma_2(T)$. By Proposition B.1.1.3, applied to $\sigma = 1/2$ (in the notation of the Proposition), we have that there exists $T_2 > T_1$ such that

$$\mathrm{Re}(Z_{F,\chi}(\sigma + it)) = \log|L_f(\sigma + it, \chi)| \leq \log(A_2 t^{1+[F:\mathbf{Q}]/4}) \leq B_F \log(T) \tag{B.6.1.19}$$

over $D_2(T)$ for all $T > T_2$, since $T - 1 < t < T + 1$ for each $s = \sigma + it \in D_2(T)$ (as $r_2(T) < 1$ and $\mathrm{Im}(s_0) = T$).

We further note that since $\#\{\mathfrak{p}|p\} \leq [F : \mathbf{Q}]$ for all rational primes p,

$$\begin{aligned} |\mathrm{Re}(Z_{F,\chi}(s_0))| \leq |Z_{F,\chi}(s_0)| \leq \sum_{\mathfrak{p},m} \frac{1}{mN(\mathfrak{p})^{1+\alpha\eta(T)}} &< [F : \mathbf{Q}] \sum_{m \geq 2} \frac{1}{m^{1+\alpha\eta(T)}} = [F : \mathbf{Q}](\zeta_{\mathbf{Q}}(1 + \alpha\eta(T)) - 1) \\ &< \frac{2[F : \mathbf{Q}]}{\alpha\eta(T)} \\ &\leq C_F \log(T) \end{aligned}$$

for all sufficiently large T.

By [**Ingh3**, Thm. E, p. 50] applied to $f = Z_{F,\chi}$, $z_0 = s_0$, $r = r_1(T)$, $R = r_2(T)$, $U = C_F \log(T)$, we obtain

$$|Z'_{F,\chi}(s)| < \frac{2r_2(T)}{(r_2(T) - r_1(T))^2}\{C_F \log(T) - \mathrm{Re}(Z_{F,\chi}(s_0))\} < \frac{4(1+2\alpha)}{(1-\alpha^2)\eta(T)}\left(C_F \log(T) + \frac{1}{\alpha\eta(T)}\right) \tag{B.6.1.20}$$

over the closure of the smaller disc $\overline{D}_1(T)$. The inequality in (B.6.1.20) holds in particular on the horizontal radial segment from the center s_0 to $1 \pm \alpha\eta(T) + iT$ on $\gamma_1(T)$. Equivalently, the inequality holds at all $s = \sigma + iT$ with $1 - \alpha\eta(T) \leq \sigma \leq 1 + \alpha\eta(T)$.

By assumption (3) on η and (B.6.1.20), we obtain $|Z'_{F,\chi}(s)| = O((\log(T))^2)$, and since $Z_{F,\chi}(s) = \log L_f(s, \chi)$ over $\overline{D}_2(T) \subset \Omega$, (B.6.1.10) follows on $\{1 - \alpha\eta(t) \leq \sigma \leq 1 + \alpha\eta(t) : t \geq 1\}$. We already treated $\sigma > 1 + \alpha\eta(t)$ for $t \geq 1$, so we are done. □

Corollary B.6.1.5 (Logarithmic derivative on the 1-line). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Then*

$$\frac{L'_f(1 + it, \chi)}{L_f(1 + it, \chi)} = O((\log(t))^2) \tag{B.6.1.21}$$

as $|t| \to +\infty$, where the implied constant depends only on F and χ.

Proof. Choose A small enough so that

$$\eta(t) := \frac{A}{[F : \mathbf{Q}]^4 \log(A_{F,\chi}(|t| + 3))}$$

satisfies $\eta(\mathbf{R}_{\geq 0}) \subset (0, 1/2]$. By Proposition B.6.1.1 and Theorem B.6.1.3, $L'_f(s, \chi)/L_f(s, \chi) = O((\log(t))^2)$ uniformly in the region $\sigma \geq 1 - \alpha\eta(|t|)$ for some $\alpha \in (0, 1)$, the implied constant depending on α. In particular, (B.6.1.21) holds. □

B.6.1.3. *Polynomial growth of the reciprocal finite Hecke L-function on the* 1*-line.* We record here below a polynomial growth estimate for the reciprocal of the finite Hecke L-function over the 1-line. We will not be needing sharper sub-polynomial estimates.

THEOREM B.6.1.6 (Polynomial growth of reciprocals on the 1-line). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. There exists an integer $N = N_\chi > 0$ such that we have:*

$$\frac{1}{L_f(1+it,\chi)} = O_N(|t|^N) \tag{B.6.1.22}$$

as $|t| \to +\infty$.

PROOF. By the inequality $3 + 4\cos(\theta) + \cos(2\theta) = 2(\cos\theta + 1)^2 \geq 0$ for all $\theta \in \mathbf{R}$ and by absolute convergence of the Euler product of $L_f(s,\chi)$ for $s \in \mathbf{C}_{>1}$, we have, following the proof of [**CF**, Ch. VIII, Thm. 2, p. 212]:

$$|\zeta_F(\sigma)^3 L(\sigma+it,\chi)^4 L(\sigma+2it,\chi^2)| \geq 1 \tag{B.6.1.23}$$

for $\sigma > 1$ and all $t \in \mathbf{R}$. By the Convexity Bound from Proposition B.1.1.3, for every $\varepsilon > 0$, there exists a real number $A_1 > 0$ depending on $\varepsilon > 0$ and χ but not on $s \in \mathbf{C}$, and a real number $t_0(\varepsilon) > 0$ such that

$$|L_f(s,\chi)| \leq A_1 |t|^{[F:\mathbf{Q}]\frac{1-\sigma}{2}+\varepsilon} \tag{B.6.1.24}$$

for all $\sigma \in [0,1]$ and $t \in \mathbf{R}$ such that $|t| > t_0(\varepsilon)$.

By Corollary B.6.1.5 we obtain that there exists a real number $A_2 > 0$ and a real number $t_1 > 0$ such that for all real numbers $t \in \mathbf{R}$ such that $|t| > t_1$, we have:

$$\left|\frac{L_f'(1+it,\chi)}{L_f(1+it,\chi)}\right| \leq A_2 |\log(t)|^2. \tag{B.6.1.25}$$

Since $L_f'(s,\chi)/L_f(s,\chi)$ is bounded on vertical lines within $\mathbf{C}_{>1}$ (it is an absolutely convergent Dirichlet series on $\mathbf{C}_{>1}$), by (B.6.1.10) applied over $\mathbf{C}_{\geq 1}$ we can use the Phragmen-Lindelöf Theorem [**Lang-c**, Ch. XII, §6, Cor. 6.3] to deduce that there exists a real number $A_3 > 0$ and a real number $t_2 > 0$ such that for all real numbers $|t| > t_2$ we have:

$$\left|\frac{L_f'(s,\chi)}{L_f(s,\chi)}\right| \leq A_3 |t| \tag{B.6.1.26}$$

uniformly over the vertical strip $\mathbf{C}_{[1,\sigma_0]}$ for any fixed $\sigma_0 > 1$, with A_3 and t_2 depending on χ and the choice of σ_0.

By (B.6.1.24) and (B.6.1.26), there exists an integer $N' > 0$, a real number $A_4 > 0$, a real number $t_3 > 0$, such that for every real number $|t| > t_2$ we have uniformly over the vertical strip $\mathbf{C}_{[1,\sigma_0]}$:

$$\left|L_f'(\chi,s)\right| \leq A_4 |t|^{N'}. \tag{B.6.1.27}$$

By (B.6.1.23) and (B.6.1.24) (applied to χ^2), we obtain that there exists an integer $M > 0$ and a real number $B > 0$ such that for every $|t| > T > \max\{t_0(\varepsilon), t_1, t_2, t_3\}$ we have:

$$(\sigma-1)^3 \leq ((\sigma-1)\zeta_F(\sigma))^3 |L_f(\sigma+it,\chi)|^4 |L_f(\sigma+it,\chi^2)| \leq B|t|^{[F:\mathbf{Q}]\frac{1-\sigma}{2}+\varepsilon} |L_f(\sigma+it,\chi)|^4 \tag{B.6.1.28}$$

for $\sigma \in (1,\sigma_0]$ and all $|t| > T$. We therefore have the inequality:

$$|L_f(\sigma+it,\chi)| \geq \frac{(\sigma-1)^{\frac{3}{4}}}{B|t|^{[F:\mathbf{Q}]\frac{1-\sigma}{8}+\frac{\varepsilon}{4}}} \tag{B.6.1.29}$$

for $\sigma \in (1,\sigma_0]$ and all $|t| > T$, which remains valid for $\sigma = 1$ too.

We fix $\sigma_1 \in (1, \sigma_0]$ and obtain:

$$|L_f(1+it,\chi) - L_f(\sigma_1+it,\chi)| \leq \int_1^{\sigma_1} |L_f'(x+it,\chi)|\,\mathrm{d}x \leq (\sigma_1-1)A_4|t|^{N'} \tag{B.6.1.30}$$

for all $|t| > T$. We obtain from (B.6.1.29) and (B.6.1.30) that

$$|L_f(1+it,\chi)| \geq \frac{(\sigma_1-1)^{\frac{3}{4}}}{B|t|^{[F:\mathbf{Q}]\frac{1-\sigma_1}{8}+\frac{\varepsilon}{4}}} - (\sigma_1-1)A_4|t|^{N'} \geq \frac{(\sigma_1-1)^{\frac{3}{4}}}{B|t|^{\frac{\varepsilon}{4}}} - (\sigma_1-1)A_4|t|^{N'} \tag{B.6.1.31}$$

for all $|t| > T$.

We now solve for σ_1 in the equation

$$\frac{(\sigma_1-1)^{\frac{3}{4}}}{B|t|^{\frac{\varepsilon}{4}}} = 2(\sigma_1-1)A_4|t|^{N'} \tag{B.6.1.32}$$

to get

$$\sigma_1(t) = \frac{1}{16\cdot B^4 A_4^4 |t|^{\varepsilon+4N'}} + 1, \tag{B.6.1.33}$$

and upon enlarging T if necessary to ensure $\sigma_1(t) \in (1, \sigma_0]$ whenever $|t| > T$, we obtain:

$$|L_f(1+it,\chi)| \geq \frac{1}{16\cdot B^4 A_4^3 |t|^{\varepsilon+3N'}} \tag{B.6.1.34}$$

for all $|t| > T$. This proves that (B.6.1.22) holds, with an implied constant that depends on ε and N'. □

The next result is an analog for the reciprocal $1/L_f(s,\chi)$ of the finite Hecke L-function, of [**Lang-b**, Ch. XVII, §2, Prop. 2.7] on the existence of horizontal segments on which the logarithmic derivative of the completed Hecke L-function has no singularities and grows at most polynomially with respect to the absolute ordinate. In the special $F = \mathbf{Q}$ and $\chi = 1$ it recovers [**T2**, Thm. 9.7].

We recall the following definition from (6.3.1.6) of Terminology 6.3.1.1: for F a number field, $\chi \in \widehat{C_F}$ a continuous character, and S a finite set of places of F containing all the Archimedean places and with S_f its subset of finite places, we define:

$$L_{\Sigma_F - S}(\chi, s) := \prod_{v \in S} L_v(\chi_v|\cdot|_v^s)^{-1} L(s,\chi)$$

a meromorphic function on $\mathbf{C}$ which agrees with the product of the finite Hecke L-function $L_f(s,\chi)$ and the finite product of *entire* functions $\prod_{v\in S_f} L_v(\chi_v|\cdot|_v^s)^{-1}$, where $L_v(\chi_v|\cdot|_v^s)$ is the local Euler factor at $v \in S_f$ of the Hecke L-function $L(s,\chi)$. The follow

THEOREM B.6.1.7 (Slow growth near the poles of $1/L_{\Sigma_F-S}(\chi,s)$ - I). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S be a finite set of places of F containing all the Archimedean places.*

There exists a real number $A > 0$ such that for all sufficiently large T (depending on χ and S), the interval $(T, T+1)$ contains a real number t such that

$$|L_{\Sigma_F-S}(\chi, \sigma+it)| > |t|^{-A} \tag{B.6.1.35}$$

for all $-1 \leq \sigma \leq 2$.

For any real number $B > 1$ and all sufficiently large T (independently of B), the inequality

$$|L_{\Sigma_F-S}(\chi, \sigma+it)| > T^{-AB} \tag{B.6.1.36}$$

for all $-1 \leq \sigma \leq 2$ holds for all $t \in [T, T+1]$ away from a subset of $[T, T+1]$ of measure $1/B$.

PROOF. Pick $\sigma_0 \geq 2$ and let $r = 4\sqrt{1+(\sigma_0+1)^2} > 4(\sigma_0+1)$. We fix an integer $N > \frac{[F:\mathbf{Q}]}{2}$ and choose $A_1 > 0$ with the following property: there exists $T_0 > 1$ such that:

$$|L_{\Sigma_F - S}(\chi, s)| \leq A_1 |\mathrm{Im}(s)|^N \tag{B.6.1.37}$$

for all $s \in \mathbf{C}_{[\sigma_0 - r, \sigma_0 + r]}$ with $|\mathrm{Im}(s)| > T_0$. That such $A_1 > 0$ exists is a consequence of boundedness of the entire function $\prod_{v \in S_f} L_v(\chi_v | \cdot |_v^s)^{-1}$, uniformly on the vertical strips of bounded width in $\mathbf{C}$, combined with Proposition B.1.1.3 which provides a real number $A_{0,N} > 0$ such that:

$$|L_f(s, \chi)| \leq A_{0,N} |\mathrm{Im}(s)|^N \tag{B.6.1.38}$$

for all $s \in \mathbf{C}_{[\sigma_0 - r, \sigma_0 + r]}$ with $|\mathrm{Im}(s)| > T_0$.

We fix $T \in \mathbf{R}$ such that $|T| > T_0$. By (B.6.1.37) and [**T2**, Ch. III, §3, Lemma α, p. 56] applied to $f(s) = L_{\Sigma_F - S}(\chi, s)$ with $s_0 = \sigma_0 + iT$, $M = A_2 \log(T)$, $r = 4((\sigma_0+1)^2+1)^{1/2}$, with

$$A_2 := \sup_{T \geq T_0 + r} \left(N + \frac{1}{\log T} \log(A_1 \zeta_F(\sigma_0)) \right)$$

(noting that $|1/L_{\Sigma_F - S}(\chi, \sigma_0 + iT)| \leq \zeta_F(\sigma_0)$), we obtain

$$\left| \frac{L'_{\Sigma_F - S}(\chi, s)}{L_{\Sigma_F - S}(\chi, s)} - \sum_{\rho : |\rho - \sigma_0 - iT| \leq r/2} \frac{1}{s - \rho} \right| \leq \frac{A A_2}{r} \log(T) \tag{B.6.1.39}$$

where $A > 0$ is provided by the cited Lemma in [**T2**] and the sum is indexed by the zeros of $L_{\Sigma_F - S}(\chi, s)$ (counted with multiplicity) within the disc $|s - s_0| \leq r/2$. The application of [**T2**] uses that the disc $|s - s_0| \leq r$ is contained inside $\mathbf{C}_{\geq \sigma_0 - r}$.

Note that the box $[-1, \sigma_0] + i(T + [-1, 1])$ is entirely contained within the disc $|s - s_0| \leq r/4$. We now choose $T = t$ with $|t| > T_0 + r$, and keep it fixed, and deduce

$$\left| \frac{L'_{\Sigma_F - S}(\chi, \sigma + it)}{L_{\Sigma_F - S}(\chi, \sigma + it)} - \sum_{\rho : |\rho - \sigma_0 - it| \leq r/2} \frac{1}{\sigma + it - \rho} \right| \leq \frac{A A_2}{r} \log(|t|) \tag{B.6.1.40}$$

for all $\sigma \in [-1, \sigma_0]$. We further note that if $\rho = \beta + i\gamma$ is a zero of $L_{\Sigma_F - S}(\chi, s)$ satisfying $|\gamma - t| \leq 1$ then since $0 < \sigma_0 - \beta \leq \sigma_0$ we see that ρ is contained within the disc $|s - \sigma_0 - it| \leq r/2$.

The zeros $\rho = \beta + i\gamma$ of $L_{\Sigma_F - S}(\chi, s)$ contained in the disc $|s - \sigma_0 - it| \leq r/2$ are either zeros of $\prod_{v \in S_f} L_v(\chi_v | \cdot |_v^s)^{-1}$ or of $L_f(s, \chi)$. In the former case, there is at most $\sum_{v \in S_f} \frac{r \log(q_v)}{2\pi} = O_{\sigma_0}(1)$ for $q_v := N(\mathfrak{p}_v)$ for each $v \in S_f$ and $\mathfrak{p}_v \subset \mathcal{O}_F$ the unique prime ideal corresponding to the finite place v. In the latter case, there is at most $O(\log(|t|))$ such zeros of $L_f(s, \chi)$, by [**Lang-b**, Ch. XVII, §2, Prop. 2.4].

We denote by $[r]$ the highest nonnegative integer $x \leq r$. For each zero ρ of $L_{\Sigma_F - S}(\chi, s)$ satisfying $|\gamma - t| > 1$ we have

$$\left| \frac{1}{s - \rho} \right| = \frac{1}{(|\sigma - \beta|^2 + |t - \gamma|^2)^{1/2}} \leq 1 \tag{B.6.1.41}$$

and so, up to replacing the implied constant AA_2/r by a larger one, we obtain

$$\frac{L'_{\Sigma_F - S}(\chi, \sigma + it)}{L_{\Sigma_F - S}(\chi, \sigma + it)} = \sum_{\rho : |\gamma - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log(|t|)) \tag{B.6.1.42}$$

for $|t| > T_0$ and for all $\sigma \in [-1, \sigma_0]$. Since $\sigma_0 \geq 2$, we obtain that (B.6.1.42) holds for all $|t| > T_0$ and all $\sigma \in [-1, 2]$.

We now integrate (B.6.1.42) across the horizontal segment from $2+it$ to $\sigma+it$ for $\sigma\in[-1,2]$ assuming t is not the ordinate of a zero of $L_{\Sigma_F-S}(\chi,s)$. We obtain

$$\log(L_{\Sigma_F-S}(\chi,\sigma+it))-\log(2+it)=\sum_{\rho:|\gamma-t|\leq 1}(\log(\sigma+it-\rho)-\log(2+it-\rho))+O(\log|t|) \tag{B.6.1.43}$$

for all $|t|$ large enough and uniformly with respect to $\sigma\in[-1,2]$ (in the finite sum over zeros, we use the standard logarithm $\log(re^{i\theta}):=\log(r)+i\theta$ for $|\theta|<\pi$ on $\mathbf{C}-\mathbf{R}_{\leq 0}\ni s-\rho$).

Since $\log(2+it)=O(\log|t|)$ and $\log(2+it-\rho)=O(1)$ for $|\gamma-t|\leq 1$ with $O(\log|t|)$ such ρ's, we obtain

$$\log(L_{\Sigma_F-S}(\chi,s))=\left(\sum_{\rho:|\gamma-t|\leq 1}\log(s-\rho)\right)+O(\log|t|) \tag{B.6.1.44}$$

for $s=\sigma+it$ with $|t|$ large enough that are not ordinates of a zero of $L_{\Sigma_F-S}(\chi,s)$ (only finitely many omissions for $t\in[a,b]$) and uniformly with respect to $\sigma\in[-1,2]$.

Upon taking real parts of both sides of (B.6.1.44), we proceed as in [**T2**, Ch. IX, §9.7, p. 219] and deduce that both (B.6.1.35) and (B.6.1.36) hold as asserted, for some real number $A>0$. □

We deduce the following:

Corollary B.6.1.8 (Slow growth near the poles of $1/L_{\Sigma_F-S}(s,\chi)$ - II). *For each integer $m\geq 0$, there exist $A,\beta>0$ and a sequence of real numbers $\{T_k\}_{k\in\mathbf{Z}}$ such that $T_k\leq T_{k+1}$, $\lim_{|k|\to+\infty}|T_k|=+\infty$, $T_k=O(|k|)$ such that*

$$\sup_{s\in[-1,2]+iT_k}|\partial_s^m(1/L_{\Sigma_F-S}(s,\chi))|\leq A|T_k|^{\beta} \tag{B.6.1.45}$$

for all $k\in\mathbf{Z}$.

Proof. This is a consequence of Theorem B.6.1.7 (which handles $t\in[-T-1,-T]$ by passing to $\overline{\chi}$) using[29] boundedness on vertical strips (including the uniformly Schwartz property) for all derivatives of the completed L-functions of χ and exponential control on Γ-factors to apply the Phragmen-Lindelöf Theorem and the convexity bound on the derivatives $L_f^{(k)}(s,\chi)$. □

B.7. Reciprocals and Partial Euler products on the 1-line

Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. In this section we will be establishing a precise estimate to the reciprocal partial Euler products for the finite Hecke L-function $L_f(s,\chi)$, on the vertical line $\mathrm{Re}(s)=1$.

B.7.1. The Dirichlet polynomials of $L_f'(s,\chi)/L_f(s,\chi)$ on the 1-line. Leading to the analysis of partial Euler products and their reciprocals, for $L_f(s,\chi)$, we need to study the Dirichlet polynomials of the Dirichlet series that sums to $L_f'(s,\chi)/L_f(s,\chi)$.

B.7.1.1. *Logarithmic derivatives of finite Hecke L-functions.* We consider the following Dirichlet series:

$$\sum_{0\neq I\subset\mathcal{O}_F}\frac{\Lambda_{F,\chi}(I)}{N(I)^s} \tag{B.7.1.1}$$

which for $\mathrm{Re}(s)>1$ converges absolutely to $-L_f'(\chi,s)/L_f(s,\chi)$. Recall that $\Lambda_{F,\chi}(I)=\chi(I)\Lambda_F(I)$, where Λ_F is the von Mangoldt function for F as defined in (B.6.1.13) (and $\chi(I)=0$ if I is not coprime to the conductor of χ). In particular, $\Lambda_{F,\chi}$ is nonzero only on prime powers $\mathfrak{p}^k$ with $k\geq 1$.

[29] This should be spelled out in some more precision/detail.

B.7.1.2. *A version of Perron's Formula from Titchmarsh.* We recall the following result from [T2, Ch. III]:

LEMMA B.7.1.1. *Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers such that:*

$$f(s) := \sum_{n\geq 1} \frac{a_n}{n^s} \tag{B.7.1.2}$$

is absolutely convergent over $\mathbf{C}_{>1}$. Assume:

(1) *$a_n = O(\psi(n))$ as $n \to +\infty$, where $\psi : \mathbf{N} \to \mathbf{R}_{>0}$ is a non-decreasing function,*

(2) *there exists $\alpha > 0$ such that*

$$\sum_{n\geq 1} \frac{|a_n|}{n^\sigma} = O_\alpha\left(\frac{1}{(\sigma-1)^\alpha}\right) \tag{B.7.1.3}$$

as $\sigma \to 1^+$, the implied constant depending on α and the O-constant in (1).

Let $c > 0$ be a fixed real number and consider $s = \sigma + iT$ with $\sigma > 1 - c$. If $x \in \mathbf{R}_{>0}$ is not an integer and N is the integer nearest to x, we have for $T \geq 1$:

(B.7.1.4)

$$\sum_{1\leq n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,\mathrm{d}w + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) + O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right) + O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right)$$

where the implied constants are independent of x, σ and T and only depend on α and the O-constant in (1).

If $x \in \mathbf{R}_{>0}$ is an integer, we have:

(B.7.1.5)

$$\sum_{1\leq n\leq x-1} \frac{a_n}{n^s} + \frac{a_x}{2x^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,\mathrm{d}w + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) + O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right) + O\left(\frac{c\psi(x)x^{-\sigma}}{T}\right)$$

where the implied constants are independent of x, σ and T and only depend on α and the O-constant in (1).

Lemma B.7.1.1 is [T2, Ch. III, Lemma 3.12]. It is a version of Perron's formula whose classical form is [T1, Ch. IX, §9.42]. It rests on the calculus of residues, and specifically on the calculation of $(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(x/m)^w\,\mathrm{d}w/w$ (as $\lim_{t\to+\infty}\int_{c-it}^{c+it}$) being 1 if $m < x$, 0 if $m > x$, and $1/2$ if $m = x$.

PROOF OF LEMMA B.7.1.1. We know $f(s) = \sum_{m\geq 1} a_m m^{-s}$ has abscissa of absolute convergence $\sigma_a \leq 1$. Let σ and c be real numbers, $c > 0$, such that $\sigma + c > \sigma_a$, for $\sigma := \mathrm{Re}(s)$. Let $T, x \geq 1$ be real numbers. If x is not an integer then we have

$$\sum_{1\leq m<x} \frac{a_m}{m^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,\mathrm{d}w + O\left(\frac{x^c}{T}\sum_{m\geq 1}\frac{|a_m|}{m^{\sigma+c}|\log(x/m)|}\right) \tag{B.7.1.6}$$

where the O-constant is $1/\pi$.

The formula in (B.7.1.6) follows from [QQ, Ch. 4, Thm. 4.2.3], applied as follows. We choose the coefficients a_m in loc. cit. to be $a_m m^{-s}$, in which case the abscissa of absolute convergence σ_a in loc. cit. is $\sigma_a - \sigma$. The positive real number ρ in loc.cit. is c, and the s-variable of integration is w in (B.7.1.6). With these notational changes, (B.7.1.6) is [QQ, Ch. 4, (4.2.5)] up to recognizing the O-constant as $1/\pi$, which emerges from the method of proof.

If $m < x/2$ or $m > 2x$, we have $|\log(x/m)| > A := \log(2) > 0$, so the parts of the sum $\sum_{m\geq 1}\frac{|a_m|}{m^{\sigma+c}|\log(x/m)|}$ indexed by these integers are $O(\sum_{m\geq 1}|a_m|m^{-\sigma-c})$, the implied constant being $1/\log(2)$. By (B.7.1.3), this is $O(1/(\sigma+c-1)^\alpha)$, the implied constant depending on α and the O-constant in (1): this gives the first O-term in (B.7.1.4).

In the range $N < m \leq 2x$, the relevant estimate is contained at the top of [T2, Ch. III, p. 62], and the implied constant can be taken to be an absolute constant times the implied constant in $a_n = O(\psi(n))$

(ultimately because $\log(1+t) \geq t/2$ for $0 < t \leq 1$). This gives the second O-term in (B.7.1.4). For the same reason, the implied constant in the estimate

$$\frac{|a_N|}{N^{\sigma+c}|\log(x/N)|} = O\left(\frac{\psi(N)x^{1-\sigma-c}}{|x-N|}\right)$$

depends only on the one in assumption (1) on a_N.

The same reasoning applies to the range $x/2 \leq m < N$, by writing $m = N - r$ where $1 \leq r \leq N - x/2$. We therefore have

$$\log\left(\frac{x}{m}\right) \geq \log\left(\frac{N-1/2}{N-r}\right) = \log\left(1 + \frac{r-1/2}{N-r}\right) \geq \frac{1}{2}\left(\frac{r-1/2}{N-r}\right) \geq \frac{1}{2}\,\frac{r}{N} \geq \frac{1}{3}\,\frac{r}{x}.$$

The part of the sum $\sum_{m\geq 1} \frac{|a_m|}{m^{\sigma+c}|\log(x/m)|}$ indexed by the integers m in that range is therefore

$$O\left(\frac{\psi(x)}{x^{-1+\sigma+c}} \sum_{1\leq r\leq N-x/2} \frac{1}{r}\right)$$

where the implied constant is absolute, leading to another contribution to the error term $O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right)$ in (B.7.1.4).

For x not an integer, we deduce that all of the implied constants in (B.7.1.4) are as desired. Now assuming x is an integer, as asserted in [**T2**, Ch. III, p. 62] the same method applies to the sum $\sum_{1\leq n<x-1/2} a_n n^{-s}$, where $N = x - 1$, since the line integral estimate

$$\frac{a_x}{2\pi i x^s}\int_{c-iT}^{c+iT} \frac{\mathrm{d}w}{w} = \frac{a_x}{2\pi i x^s}\log\left(\frac{c+iT}{c-iT}\right) = \frac{a_x}{2\pi i x^s}\{\pi i + O(1/T)\} = \frac{a_x}{2x^s} + O\left(\frac{\psi(x)x^{-\sigma}}{T}\right) \tag{B.7.1.7}$$

has implied constant in the second-to-last expression equal to c/π and so in the final error term the implied constant is the one in hypothesis (1) times c/π. Combining (B.7.1.4) with (B.7.1.7) yields (B.7.1.5) by observing that

$$O\left(\frac{\psi(N)(x-1/2)^{1-\sigma}}{T|x-1/2-N|}\right) = O\left(\frac{\psi(x-1)(x-1/2)^{1-\sigma}}{T|x-1/2-x+1|}\right) = O\left(\frac{\psi(x)x^{1-\sigma}}{T}\right) = O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right)$$

by the assumption on ψ being non-decreasing, with all the implied constants depending only on α and the one in (1). □

B.7.1.3. *Convergence and oscillation of the Dirichlet polynomials of $L'_f(s,\chi)/L_f(s,\chi)$.* By closely following the method in [**T2**, Ch.3,§3.14], we now prove the following:

THEOREM B.7.1.2 (Convergence and oscillation on the 1-line). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Fix a compact subset $K \subset 1 + i\mathbf{R}$.*

(1) *Assume $\chi|_{C_F^1}$ is trivial, so $\chi = |\cdot|_F^{-it_0}$ for some $t_0 \in \mathbf{R}$. Assume $1 + it_0 \notin K$, pick $b > 0$, and let $J_b \subset \mathbf{R}$ be the union of $\mathbf{Z}$ and the set of real numbers x whose distance $d(x,\mathbf{Z})$ from $\mathbf{Z}$ is at least b.*

For $s \in 1 + i(\mathbf{R} - \{t_0\})$ and $x > 1$ we have

$$\sum_{0\neq I\subset \mathcal{O}_F : N(I)\leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^s} = -\frac{\zeta'_F(s-it_0)}{\zeta_F(s-it_0)} + \frac{x^{1+it_0-s}}{1+it_0-s} + f_x(s) \tag{B.7.1.8}$$

where $f_x(s)$ is an analytic function of $s \in 1 + i\mathbf{R}$ such that $\lim_{x\to+\infty, x\in J_b} f_x(s) = 0$ uniformly in such $s \in K$. Moreover, for $s = 1 + it_0$:

$$\sum_{0\neq I\subset \mathcal{O}_F : N(I)\leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^{1+it_0}} = \log(x) + O(1). \tag{B.7.1.9}$$

(2) *Assume* $\chi|_{C_F^1}$ *is non-trivial. For* $s \in 1 + i\mathbf{R}$ *we have that for* $x > 1$,

$$\sum_{0 \neq I \subset \mathcal{O}_F : N(I) \leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^s} = -\frac{L_f'(s,\chi)}{L_f(s,\chi)} + f_x(s) \tag{B.7.1.10}$$

where $f_x(s)$ *is an analytic function of* $s \in 1 + i\mathbf{R}$ *such that* $\lim_{x \to +\infty, x \in J_b} f_x(s) = 0$ *uniformly in such* $s \in K$.

PROOF. (1) : We take $a_n := \sum_{I \subset \mathcal{O}_F : N(I) = n} \Lambda_{F,\chi}(I)$ in Lemma B.7.1.1, with $\alpha = 1$ in (B.7.1.3). We therefore obtain by Lemma B.6.1.4 for $s \in 1 + i(\mathbf{R} - \{t_0\})$ and $T \geq 1$ that

(B.7.1.11)

$$\sum_{0 \neq I \subset \mathcal{O}_F : N(I) \leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^s} = -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w} \, \mathrm{d}w + O\left(\frac{x^c}{Tc}\right) + O_F\left(\frac{\log(x)^2}{T}\right) + \begin{cases} O_F\left(\frac{\log x}{T \, d(x, \mathbf{Z})}\right), & x \notin \mathbf{Z} \\ O_F\left(\frac{c \log x}{Tx}\right), & x \in \mathbf{Z} \end{cases}$$

because we can choose $\psi(x) \ll_F \log(x)$: the implied constants in (B.7.1.11) only depend on α and χ by Lemma B.7.1.1.

We now perform a contour-shift to $\int_{-\delta - iT}^{-\delta + iT}$ (with $\delta > 0$ to be chosen) via the Residue Theorem for the integrand $-(\zeta_F'(s - it_0 + w)/\zeta_F(s - it_0 + w))(x^w/w)$ using the counterclockwise rectangle in Figure 4. We'll pick δ to ensure the rectangle is contained inside a classical zero-free region for $w \mapsto \zeta_F(s - it_0 + w)$. Estimates for the integrals over the top, bottom, and left sides of the rectangle will then yield what we need.

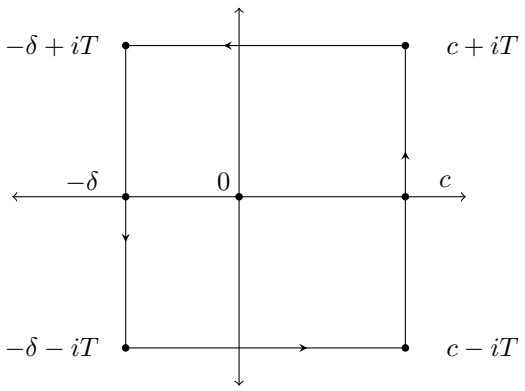


FIGURE 4. A rectangle oriented counterclockwise for shifting a vertical contour of integration

We have

$$\begin{aligned} -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w} \, \mathrm{d}w \; &- \frac{1}{2\pi i} \int_{-\delta - iT}^{c-iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w} \, \mathrm{d}w \\ &+ \frac{1}{2\pi i} \int_{-\delta - iT}^{-\delta + iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w} \, \mathrm{d}w \\ &+ \frac{1}{2\pi i} \int_{-\delta + iT}^{c+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w} \, \mathrm{d}w \\ &= -\frac{\zeta_F'(s - it_0)}{\zeta_F(s - it_0)} + \frac{x^{1 + it_0 - s}}{1 + it_0 - s} \end{aligned}$$

where the final expression is the sum of the residues at the two poles of the integrand: the pole at $w = 0$ with residue $-\zeta_F'(s - it_0)/\zeta_F(s - it_0)$ and the pole at $w = 1 + it_0 - s$ (nonzero since $s \neq 1 + it_0$) with residue $x^{1 + it_0 - s}/(1 + it_0 - s)$.

Choose $C > 0$ small enough so that

$$\eta(t) := \frac{C}{\log(A_{F,\chi}(|t|+3))}$$

satisfies $\eta(\mathbf{R}_{\geq 0}) \subset (0, 1/2]$ holds, and choose $\alpha \in (0,1)$. Then Proposition B.6.1.1 and Theorem B.6.1.3 give that $\zeta_F(s - it_0)$ has no zeros in the region $\sigma > 1 - \eta(|t-t_0|)$, and uniformly in the region $\sigma \geq 1 - \alpha\eta(|t-t_0|)$ with $|t-t_0| \geq 1$ the logarithmic derivative of ζ_F satisfies

$$\frac{\zeta_F'(\sigma + it - it_0)}{\zeta_F(\sigma + it - it_0)} \ll_{\alpha,F} (\log|t-t_0|)^2.$$

We now take $T \geq 2 + |t_0| + |t|$ and choose $\delta := \alpha\eta(|T-t_0|)$, so δ has magnitude on the order of $1/\log|T-t_0|$. The contour in Figure 4 is contained in the classical zero-free region for $\zeta_F(s - it_0)$.

Let $A_F = A_{F,\alpha}$ be an upper bound on $|\zeta_F'/\zeta_F|$ over the compact overlap of the classical zero-free region $\sigma \geq 1 - \alpha\eta(|t|)$ with the vertical strip $|t| \leq 1$. Since $T \geq 2 + |t_0| + |t|$, we have

$$\begin{aligned}\left|\int_{-\delta-iT}^{-\delta+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w}\,\mathrm{d}w\right| &\leq \int_{-1}^{1} A_F \frac{x^{-\delta}\,\mathrm{d}u}{\sqrt{\delta^2+u^2}} + \left(\int_{-T-t_0}^{-1} + \int_{1}^{T-t_0}\right) \frac{x^{-\delta}(\log(u+t))^2}{\sqrt{\delta^2+u^2}}\,\mathrm{d}u \\ &\leq (A_F + (\log(T + |t_0| + |t|))^2) x^{-\delta} \int_{-T-t_0}^{T-t_0} \frac{\mathrm{d}u}{\sqrt{\delta^2+u^2}}.\end{aligned}$$

where the final integral can be rewritten as $\int_{-(T+t_0)/\delta}^{(T-t_0)/\delta} \mathrm{d}v/\sqrt{1+v^2}$.

Since $\int \mathrm{d}v/\sqrt{1+v^2} = \log(v + \sqrt{1+v^2}) = \log|v| + O(1)$ (implicit constant at most $\log(1+\sqrt{2})$), we get the upper bound

$$(A_F + (\log(T + |t| + |t_0|))^2) x^{-\delta} \cdot 2(\log(T + |t_0|) + B) = O_{F,\alpha}((\log(T + |t| + |t_0|))^3 x^{-\delta})$$

(using that $T + |t_0| \geq 2 + 2|t_0| \geq 2$). To summarize, we have an estimate for integrating along the left side of the contour in Figure 4:

$$\int_{-\delta-iT}^{-\delta+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w}\,\mathrm{d}w \ll_{\alpha,F} x^{-\delta}(\log(T + |t| + |t_0|))^3 \tag{B.7.1.12}$$

for $x \geq 1$.

Along the top and bottom sides of the rectangle we have the estimates

$$\int_{-\delta+iT}^{c+iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w}\,\mathrm{d}w \ll_{\alpha} \frac{(\log|T + t - t_0|)^2}{T} \int_{-\delta}^{c} x^u\,\mathrm{d}u = O_{\alpha,F}\left(\frac{x^c\,(\log|T+t-t_0|)^2}{T\log(x)}\right) \tag{B.7.1.13}$$

for $x > 1$, and:

$$\int_{-\delta-iT}^{c-iT} \frac{\zeta_F'(s - it_0 + w)}{\zeta_F(s - it_0 + w)} \frac{x^w}{w}\,\mathrm{d}w \ll_{\alpha} \frac{(\log|T + t - t_0|)^2}{T} \int_{-\delta}^{c} x^u\,\mathrm{d}u = O_{\alpha,F}\left(\frac{x^c\,(\log|T+t-t_0|)^2}{\log(x)T}\right). \tag{B.7.1.14}$$

We define

$$f_x(s) := \sum_{0 \neq I \subset \mathcal{O}_F : N(I) \leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^s} + \frac{\zeta_F'(s - it_0)}{\zeta_F(s - it_0)} - \frac{x^{1+it_0-s}}{1 + it_0 - s} \tag{B.7.1.15}$$

for $s \in 1 + i(\mathbf{R} - \{t_0\})$. As a function of s, $f_x(s)$ is analytic and it has a removable singularity at $s = 1 + it_0$. The Residue Theorem calculation and (B.7.1.11) show

$$f_x(s) = \frac{1}{2\pi i}\left(\int_{-\delta-iT}^{c-iT} - \int_{-\delta-iT}^{-\delta+iT} - \int_{-\delta+iT}^{c+iT} \frac{\zeta_F'(s-it_0+w)}{\zeta_F(s-it_0+w)}\,\frac{x^w}{w}\,\mathrm{d}w\right) + O\left(\frac{x^c}{Tc}\right) + O_F\left(\frac{(\log x)^2}{T}\right) + \begin{cases} O_F\left(\frac{\log x}{T|x-N|}\right), & x \notin \mathbf{Z} \\ O_F\left(\frac{c\log x}{Tx}\right), & x \in \mathbf{Z}. \end{cases}$$

Now choose $c = 1/\log(x)$ and $T = (\log(x))^3$, so

$$\delta = \frac{\alpha C}{B + \log(T - t_0 + 3)}$$

for some constants $B, C > 0$. Plugging these into (B.7.1.11), (B.7.1.13), and (B.7.1.14), (compare with [**T2**, Ch. III, §3.13, p. 63]), we claim $f_x(s) \to 0$ as $x \to +\infty$ with x varying through J_b and $s = 1 + it$ for t in an arbitrary but fixed compact subset of $\mathbf{R} - \{t_0\}$. This requires some additional work for the decay of the upper bound in (B.7.1.14) as $x \to +\infty$. That bound is

$$O_{F,t_0,K}((\log T)^3 x^{-\delta}) = O_{F,t_0,K}\left(\frac{(\log(\log x))^3}{x^{M/\log(T)}}\right) = O_{F,t_0,K}\left(\frac{(\log(\log x))^3}{x^{M'/\log(\log x)}}\right)$$

for constants M and M' depending on t_0 and α. Thus, we just need to verify $\log(\log x)/x^{A/\log(\log x)} \to 0$ as $x \to +\infty$ for any $A > 0$. For $y = \log(x) \to +\infty$, this is

$$\frac{logy}{e^{Ay/\log(y)}} \leq \frac{\log(y)}{e^{Ay^{1-\varepsilon}}} \to 0$$

for any $\varepsilon > 0$. This gives the desired decay on $f_x(s)$.

We may write

$$\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} - \frac{x^{1+it_0-s}}{1+it_0-s} = \frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} - \frac{1}{1+it_0-s} + \frac{1}{1+it_0-s} - \frac{x^{1+it_0-s}}{1+it_0-s} \tag{B.7.1.16}$$

and observe that both functions

$$\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} - \frac{1}{1+it_0-s}, \quad \frac{1}{1+it_0-s} - \frac{x^{1+it_0-s}}{1+it_0-s}$$

extend by continuity across the singularity $s = 1 + it_0$ to analytic functions over $1 + i\mathbf{R}$. Upon letting s tend to $1 + it_0$ in (B.7.1.8) we obtain[30] (B.7.1.9) using L'Hopital's rule. The implied constant in (B.7.1.9) is absolute and amounts to the sum of $\|f_x(s)\|_u$ from (B.7.1.8) and the limit as $s \to 1 + it_0$ over $1 + i\mathbf{R}$ of the leftmost function in (B.7.1.16).

(2) : The same method, resting on Proposition B.6.1.1, gives (B.7.1.10) upon observing that the integrand

$$\frac{L_f'(s+w\chi)}{L_f(s+w,\chi)}\,\frac{x^w}{w} \tag{B.7.1.17}$$

as a function of w has only one pole, at $w = 0$, whose residue is $L_f'(s,\chi)/L_f(s,\chi)$ (in particular, no pole $w = 1 - s$ when $s \neq 1$). □

B.7.2. Euler products on the 1-line. In this section, we analyze the Euler product for the finite Hecke L-function $L_f(s,\chi)$, on the 1-line, i.e. the vertical line $\{\mathrm{Re}(s) = 1\}$.

[30] Maybe something needs to be said about this process escaping from any choice of compact K in (1)?

B.7.2.1. *Mode of convergence of partial Euler products on subsets of the* 1*-line.* Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. In this subsection we analyze the mode of convergence of reciprocal partial Euler products of the finite Hecke L-function $L(s,\chi)$ on the line $\sigma = 1$ in $\mathbf{C}$.

THEOREM B.7.2.1 (Euler products on the 1-line - I). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places at which χ is ramified.*

(1) *If $\chi|_{C_F^1}$ is trivial (so $\chi = |\cdot|_F^{-it_0}$ for a unique real number t_0) then for every compact $K \subset \mathbf{R} - \{t_0\}$ and $s \in 1 + iK$ and M the smallest integer $\geq x$, for every $\varepsilon > 0$ there exists a real number $x_0 > 1$ (depending on K and ε) such that for all $x \geq x_0$ with $d(x, \mathbf{Z}) \geq b > 0$ we have on $1 + iK$ that:*

$$\sum_{\mathfrak{p}\subset \mathcal{O}_F : N(\mathfrak{p}^m)\geq x} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} = -\frac{g_{M-1}(s)}{\log(M-1)} + F_x(s) - \sum_{n\geq x} g_{n-1}(s)\left(\frac{1}{\log(n)} - \frac{1}{\log(n-1)}\right). \tag{B.7.2.1}$$

where the series in the left (defined in Remark B.7.2.2 *below) converges uniformly on $1 + iK$, the functions $g_n(s)$ for $n \geq x$ are defined as*

$$g_n(s) := -\frac{1 - n^{1+it_0-s}}{1 + it_0 - s}, \tag{B.7.2.2}$$

and the function $F_x(s)$ is continuous over $1 + iK$ and satisfies $\|F_x|_{1+iK,u}| \leq 2\varepsilon/\log(x)$ for $x \geq x_0$.

(2) *Fix a compact subset $K \subset \mathbf{R}$. If $\chi|_{C_F^1}$ is non-trivial then for every $s \in iK$ and M the smallest integer $\geq x$, for every $\varepsilon > 0$ there exists a real number $x_0 > 1$ (depending on K and ε) such that for all $s \in 1 + iK$ and $x \geq x_0$ we have:*

$$\sum_{\mathfrak{p}\subset \mathcal{O}_F : N(\mathfrak{p}^m)\geq x} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} = F_x(s) \tag{B.7.2.3}$$

where the series in the left (defined in Remark B.7.2.2 *below) converges uniformly over $1 + iK$ and the function $F_x(s)$ is continuous over $1 + iK$ with $\|F_x\|_{1+iK,u} < 2\varepsilon/\log(x)$ for all $x \geq x_0$.*

REMARK B.7.2.2. In (B.7.2.1) and (B.7.2.3), the sum over $\mathfrak{p}$'s on the left side means *by definition*

$$\sum_{n\geq x}\left(\sum_{N(\mathfrak{p}^m)=n} \frac{\chi(\mathfrak{p}^m)}{m}\right)\frac{1}{n^s}.$$

PROOF. We set $b_n := \sum_{0\neq I\subset \mathcal{O}_F : N(I)=n} \chi(I)\Lambda_F(I)/n^s$, and $a_n := 1/\log(n)$ for $n > 1$, and $a_0 = a_1 := 0$. Upon defining

$$B_x := \sum_{0\leq k\leq x} b_k,$$

for any real number $X > 0$ and $N := \lfloor X \rfloor$ the largest integer satisfying $\leq X$, partial summation gives

$$\sum_{\mathfrak{p}\subset \mathcal{O}_F : N(\mathfrak{p}^m)\leq X} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} = \sum_{\mathfrak{p} : N(\mathfrak{p}^m)\leq X} \frac{\chi(\mathfrak{p}^m)\log N\mathfrak{p}}{N(\mathfrak{p}^m)^s \log(N(\mathfrak{p}^m)} = \sum_{0\leq n\leq X} a_n b_n \tag{B.7.2.4}$$

$$= a_N B_N - \sum_{0\leq n\leq N-1} B_n(a_{n+1} - a_n). \tag{B.7.2.5}$$

In particular, applying this to $X = M-1$ for $M = \lceil x \rceil$ the smallest integer $\geq x$ and subtracting it from (B.7.2.4) yields

$$\sum_{\mathfrak{p}\subset \mathcal{O}_F : x \leq N(\mathfrak{p}^m) \leq X} \sum_{m \geq 1} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} = \sum_{M \leq n \leq N} a_n b_n \tag{B.7.2.6}$$

$$= a_N B_N - a_{M-1} B_{M-1} - \sum_{x \leq n \leq X} B_{n-1}(a_n - a_{n-1}). \tag{B.7.2.7}$$

(1) : From Theorem B.7.1.2, for every real number $\varepsilon > 0$, there exists a large enough real number $x_0 > 1$ such that for every $x > x_0$ we have from (B.7.1.8) that

$$B_x = \sum_{0 \neq I \subset \mathcal{O}_F : N(I) \leq x} \frac{\Lambda_{F,\chi}(I)}{N(I)^s} = -\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{x^{1+it_0-s}}{1+it_0-s} + f_x(s) \tag{B.7.2.8}$$

where $f_x(s)$ is an analytic function over $1+i\mathbf{R}$ satisfying $\|f_x\|_{1+iK,u} \leq \varepsilon$ when $d(x,\mathbf{Z}) \geq b > 0$ or $x \in \mathbf{Z}$.

The real number x_0 depends only on ε and K and on a choice of $\alpha \in (0,1)$ made at the beginning as per the proof of Theorem B.7.1.2(1), but not on $s \in 1+i\mathbf{R}$. We deduce that for every $X > x > x_0$ and $N = \lfloor X \rfloor$, $M = \lceil x \rceil \geq 2$, by (B.7.2.6):

(B.7.2.9)

$$\begin{aligned}\sum_{\mathfrak{p}\subset \mathcal{O}_F : x \leq N(\mathfrak{p}^m) \leq X} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} &= \frac{1}{\log(N)}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{N^{1+it_0-s}}{1+it_0-s} + f_N(s)\right\} \\ &- \frac{1}{\log(M-1)}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{(M-1)^{1+it_0-s}}{1+it_0-s} + f_{M-1}(s)\right\} \\ &- \sum_{x \leq n \leq X}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{(n-1)^{1+it_0-s}}{1+it_0-s} + f_{n-1}(s)\right\}\left(\frac{1}{\log(n)} - \frac{1}{\log(n-1)}\right).\end{aligned}$$

For every integer $m \geq x_0$, each of the analytic functions over $1+i(\mathbf{R}-\{t_0\})$:

$$H_m(s) := \frac{1}{\log(m)}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{m^{1+it_0-s}}{1+it_0-s} + f_m(s)\right\} \tag{B.7.2.10}$$

extends to an analytic function over $1+i\mathbf{R}$, since we may write

$$H_m(s) = \frac{1}{\log(m)}\{h(s) + g_m(s) + f_m(s)\} \tag{B.7.2.11}$$

where

$$h(s) := -\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{1}{1+it_0-s}, \quad g_m(s) := -\frac{1-m^{1+it_0-s}}{1+it_0-s} \tag{B.7.2.12}$$

and h extends to an analytic function across the removable singularity at $s = 1+it_0$, as does the function g_m, and $f_m(s)$ is analytic over $1+i\mathbf{R}$ as per Theorem B.7.1.2.

As $N \to +\infty$, $f_N(s) \to 0$ uniformly over $1+iK$, and $(1/\log(N))h \to 0$ uniformly on the compact subsets of $1+i\mathbf{R}$. The function $(1/\log(N))g_N$ tends to zero uniformly on $1+iK$ as $N \to +\infty$ since it even tends to zero uniformly on all subsets of the form $1+i((-\infty, t_0-A] \cup [t_0+A, +\infty))$ for every real number $A > 0$. (At $s = 1+it_0$, it is exactly -1.)

We now analyze the sum

$$\sum_{x \leq n \leq X}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)} + \frac{(n-1)^{1+it_0-s}}{1+it_0-s} + f_{n-1}(s)\right\}\left(\frac{1}{\log(n)} - \frac{1}{\log(n-1)}\right)$$

as $X \to +\infty$. We write

(B.7.2.13)

$$\sum_{x\le n\le X}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)}+\frac{(n-1)^{1+it_0-s}}{1+it_0-s}+f_{n-1}(s)\right\}\left(\frac{1}{\log(n)}-\frac{1}{\log(n-1)}\right)=h(s)\left(\frac{1}{\log(N)}-\frac{1}{\log(M-1)}\right)$$
$$+\sum_{x\le n\le X}g_{n-1}(s)\left(\frac{1}{\log(n)}-\frac{1}{\log(n-1)}\right)$$
$$+\sum_{x\le n\le X}f_{n-1}(s)\left(\frac{1}{\log(n)}-\frac{1}{\log(n-1)}\right).$$

For $x\ge x_0+1$, we have $\|f_n(s)\|_u\le\varepsilon$ for $s\in 1+i\mathbf{R}$ for all $n\ge x$, and so the series $\sum_{n\ge x}f_{n-1}(s)(1/\log(n)-1/\log(n-1))$ is absolutely and uniformly convergent to a bounded analytic function $F_x(s)$ over $1+i\mathbf{R}$ such that $F_x(s)\to 0$ as $x\to+\infty$, uniformly over $1+i\mathbf{R}$. Indeed, we have $|F_x(s)|\le\varepsilon/\log(x)$ for every $s\in 1+i\mathbf{R}$.

As $N\to+\infty$, the functions $(1/\log(N)-1/\log(M-1))h(s)$ tend to $-h(s)/\log(M-1)$ uniformly on $1+iK$. Since the $|g_n|$'s are uniformly bounded on $1+iK$ for $n\ge x$, the series $\sum_{n\ge x}g_{n-1}(s)(1/\log(n)-1/\log(n-1))$ also converges uniformly over $1+iK$.

We may therefore send $X\to+\infty$ in (B.7.2.13) and deduce from (B.7.2.9) the following identity on $1+iK$:

(B.7.2.14)

$$\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p}^m)\ge x}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}=-\frac{1}{\log(M-1)}\left\{-\frac{\zeta_F'(s-it_0)}{\zeta_F(s-it_0)}+\frac{(M-1)^{1+it_0-s}}{1+it_0-s}+f_{M-1}(s)\right\}$$
$$+\frac{h(s)}{\log(M-1)}-F_x(s)-\sum_{n\ge x}g_{n-1}(s)\left(\frac{1}{\log(n)}-\frac{1}{\log(n-1)}\right)$$

where the left side is *defined* in Remark B.7.2.2. Upon renaming $-f_{M-1}(s)/\log(M-1)-F_x(s)$ as $F_x(s)$, and noting that $|f_{M-1}(s)/\log(M-1)|\le\varepsilon/\log(x)$ for $s\in 1+iK$ and $M:=\lceil x\rceil$ big enough, we see that part (1) is established.

(2) : The same method as in the proof of (1) gives (B.7.2.3) upon applying Theorem B.7.1.2(2) in place of Theorem B.7.1.2(1), and upon renaming $-f_{M-1}(s)/\log(M-1)-F_x(s)$ as $F_x(s)$. □

B.7.2.2. *Uniform convergence away from the singularity.*

Corollary B.7.2.3 (Uniform convergence away from the singularity - I). *Let F be a number field, $\chi\in\widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places at which χ is ramified. As $x\to\infty$, the functions*

(B.7.2.15)

$$\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})<x}\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}$$

of s converge locally uniformly over $1+i\mathbf{R}$ if $\chi|_{C_F^1}$ is non-trivial, and locally uniformly over $1+i(\mathbf{R}-\{t_0\})$ if $\chi=|\cdot|_F^{-it_0}$ for $t_0\in\mathbf{R}$ (the case $\chi|_{C_F^1}=1$).

Proof. We may write, for every real number $X>1$:

(B.7.2.16)

$$\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})<X}\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}=\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p}^m)<X}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}-\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p}^m)<X}\sum_{m\ge 2}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}$$

where, as $X\to+\infty$, the partial sums

$$\sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p}^m)<X}\sum_{m\ge 2}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}$$

converge absolutely and uniformly over $1+i\mathbf{R}$.

From (B.7.2.16) and Theorem B.7.2.1, we deduce that the partial sums in (B.7.2.15) converge in the desired manner because for $x < X$ we have

$$G_X(s) - G_x(s) = \sum_{N(\mathfrak{p}^m)\geq x} \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s} - \sum_{N(\mathfrak{p}^m)\geq X} \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}$$

with each sum defined as in Remark B.7.2.2. □

B.7.2.3. *Partial Euler products on the* 1*-line.* Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places of F at which χ is ramified. Let $x > 1$ be an integer. We define

$$E_{f,x}(s,\chi) := \prod_{\mathfrak{p}\subset\mathcal{O}_F : N(\mathfrak{p})<x} (1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-s}). \tag{B.7.2.17}$$

THEOREM B.7.2.4 (Euler products on the 1-line - II). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places at which χ is ramified.*

(1) *If $\chi|_{C_F^1}$ is trivial, so $\chi = |\cdot|_F^{-it_0}$ for a unique real number t_0, for a compact subset $K \subset \mathbf{R} - \{t_0\}$ and every $\varepsilon > 0$ there exists $x_0 > 1$ and for every $x \geq x_0$ a bounded continuous function F_x on K satisfying $\|F_x\|_u \leq 2\varepsilon/\log(x)$, such that*

$$E_{f,x}(1+it,\chi)L_f(1+it,\chi) = \exp\left(F_x(t) + G_x(t) + H_x(1+it)\right) \tag{B.7.2.18}$$

for every $t \in K$, where $G_x(t)$ is the analytic function over $\mathbf{R} - \{t_0\}$ defined as

$$G_x(t) := \frac{1-(M-1)^{i(t_0-t)}}{i\log(M-1)(t-t_0)} + \sum_{n\geq x} \frac{1-(n-1)^{i(t_0-t)}}{i(t_0-t)} \left(\frac{1}{\log(n)} - \frac{1}{\log(n-1)}\right) = O_K\left(\frac{1}{\log(x)}\right) \tag{B.7.2.19}$$

with $M \geq 2$ the least integer $\geq x$, the series in (B.7.2.19) *converging uniformly on K, and*

$$H_x(s) := -\sum_{N\mathfrak{p}<x, N(\mathfrak{p}^m)\geq x} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p}^m)^s} = O\left(\sum_{m\geq 2, N(\mathfrak{p}^m)\geq x} \frac{1}{mN(\mathfrak{p}^m)^s}\right)$$

bounded by the tail of the Dirichlet series $\sum_{m\geq 2,\mathfrak{p}} 1/(mN(\mathfrak{p}^m)^s)$ that is absolutely convergent on $\{\sigma > 1/2\}$ (so $H_x(1+it) \to 0$ uniformly on $\mathbf{R}$ as $x \to \infty$).

In particular, we have

$$\lim_{x\to+\infty} E_{f,x}(s,\chi)L_f(s,\chi) = 1 \tag{B.7.2.20}$$

uniformly over $1 + iK$.

(2) *If $\chi|_{C_F^1}$ is non-trivial, for any compact subset $K \subset \mathbf{R}$ and $\varepsilon > 0$ there exists $x_0 > 1$ and for every $x \geq x_0$ a bounded continuous function F_x on K satisfying $\|F_x\|_u \leq 2\varepsilon/\log(x)$ such that*

$$E_{f,x}(1+it,\chi)L_f(1+it,\chi) = \exp\left(F_x(t) - H_x(1+it)\right) \tag{B.7.2.21}$$

for every $t \in K$. In particular, we have

$$\lim_{x\to+\infty} E_{f,x}(s,\chi)L_f(s,\chi) = 1 \tag{B.7.2.22}$$

uniformly over $1 + iK$.

PROOF. Over $\mathbf{C}_{>0}$,

$$\log\left(1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-s}\right)^{-1} = \sum_{m\geq 1} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}$$

where the series is absolutely convergent. We will use Theorem B.7.2.1 to deduce

$$\log(E_{f,x}(s,\chi)L_f(s,\chi)) = \sum_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})\geq x}\sum_{m\geq 1}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} = \sum_{N(\mathfrak{p}^m)\geq x}\frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}} - H_x(1+it) \tag{B.7.2.23}$$

as functions over $1+i(\mathbf{R}-\{t_0\})$ when $\chi = |\cdot|_F^{-it_0}$ and over $1+i\mathbf{R}$ if $\chi|_{C_F^1}$ is non-trivial.

The identity in (B.7.2.23) holds over the half-plane $\mathbf{C}_{>1}$ of absolute convergence of the Dirichlet series in the right side of (B.7.2.23), and by Theorem B.7.2.1 the right side converges as a Dirichlet series on $\sigma = 1$ (away from $1+it_0$ in part (1)). Thus, by Abel's Theorem for Dirichlet series and continuity of the left side of (B.7.2.23) on $\sigma \geq 1$ we see (B.7.2.23) holds pointwise on the region $\sigma \geq 1$. □

B.7.2.4. *Convergence of partial Euler products on the* 1*-line away from the singularity.*

COROLLARY B.7.2.5 (Convergence of partial Euler products on the 1-line). *Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character. Let S_χ be the finite set of finite places at which χ is ramified.*

The partial Euler product

$$\prod_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})<x}(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1} \tag{B.7.2.24}$$

converges to $L_f(s,\chi)$ locally uniformly over $1+i\mathbf{R}$ if $\chi|_{C_F^1} \neq 1$, and locally uniformly over $1+i(\mathbf{R}-\{t_0\})$ if $\chi = |\cdot|_F^{-it_0}$ for $t_0 \in \mathbf{R}$ (the case $\chi|_{C_F^1} = 1$).

PROOF. Indeed, the partial Euler product in (B.7.2.24) is the reciprocal of $E_{f,x}(s,\chi)$ from (B.7.2.17), and hence $E_x \to 1/L_f$ locally uniformly by Theorem B.7.2.4 because $L_f(s,\chi)$ is non-vanishing on $\sigma = 1$. Reciprocating then gives the result. □

REMARK B.7.2.6. Let F be a number field, $\chi \in \widehat{C_F}$ a continuous character of the form $|\cdot|_F^{-it_0}$ for $t_0 \in \mathbf{R}$. We have

$$\prod_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})<x}(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1-it_0}) = \prod_{\mathfrak{p}\subset\mathcal{O}_F:N(\mathfrak{p})<x}\left(1-\frac{1}{N(\mathfrak{p})}\right) = \frac{e^{-\gamma}}{\overline{\mu}_1(C_F^1)\log(x)}(1+o_x(1)) \tag{B.7.2.25}$$

as $x \to +\infty$. This is Mertens' Theorem for the number field F.

The estimate in (B.7.2.25) follows from (B.7.1.9) of Theorem B.7.1.2. The argument to deduce (B.7.2.25) from (B.7.1.9) is the same as in the proof of [**HW**, p. 351, Thm. 429]; see also [**GL**, Thm. 1,(C1)].

REMARK B.7.2.7. For $\chi \in \widehat{C_F}$ of the form $\chi = |\cdot|_F^{-it_0}$ for $t_0 \in \mathbf{R}$, the reciprocal partial Euler products $E_{f,x}(s,\chi)$ converge to $1/L_f(s,\chi)$ pointwise over $1+i\mathbf{R}$, and locally uniformly on $1+i(\mathbf{R}-\{t_0\})$.

The convergence over any interval containing t_0 in its interior is *not* uniform. Indeed, the partial products indexed by the integers $x \leq n \leq N$ contributing in the limit to $e^{-G_x(t)}$ oscillate too much and can only be kept bounded by a uniform constant when t is kept within (t_0-A, t_0+A) as A decreases towards zero at least as fast as $1/\log(N)$.

APPENDIX C

Preparations on locally convex topological vector spaces

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C.1. Locally convex topological vector spaces

Throughout this first section we provide initial guidance through the vast literature on locally convex topological vector spaces, highlighting the concepts and results that will be used throughout this work, indicating where to find them, and introducing the conventions that will be regularly in force. Later sections give an overview of more advanced notions, such as topologies on duals and tensor products, nuclearity (of spaces and of linear maps), integration valued in locally convex spaces satisfying a completeness condition (this goes far beyond the more familiar Banach case, and is a generality we require crucially), and holomorphicity/meromorphicity for functions valued in suitable locally convex spaces.

Fourier theory for tempered distributions on Hausdorff locally compact abelian groups G, and topological properties of the Schwartz-Bruhat space $\mathcal{S}(G)$ as a locally convex topological vector space, will be a primary focus (with a view towards the intended applications) in Appendix D building upon the present Appendix.

C.1.1. References on basic concepts. A vector space E is a *topological* vector space if and only if it is a topological space over a topological field, and the operations defining the linear structure are continuous. The notation *tvs* is an abbreviation for *topological vector space*. The class of topological vector spaces we will be working with is that of *locally convex* topological vector spaces over $\mathbf{C}$.

We recall the basic terminology below, in the form of a roadmap to the literature, to facilitate the reader's access to it. The references [**Bou-TVS**, Ch. I], [**Scha**, Ch. I], [**Jar**, Ch. 2], [**Kothe-I**, Ch. 3, §15], and [**NB**, Ch. 4] provide basic general background on topological vector spaces over $\mathbf{C}$. Similar and simpler concepts for topological $\mathbf{R}$-vector spaces can be found in [**Bou-TVS**, Ch. II], and indication of how to adapt the results there to the case of $\mathbf{C}$-vector spaces is given in [**Bou-TVS**, Ch. II, §8].

Here are some basic concepts to keep in mind for a tvs E over $\mathbf{C}$ and a subset $C \subset E$:

(1) We say C is *convex* if for all $x, y \in C$, the line segment $\{t\,x + (1-t)y : t \in [0,1]\}$ joining x and y is contained in C. See [**Scha**, Ch. II, §1] for basic stability properties of convex subsets.

The intersection of all convex subsets of E containing a subset $A \subset E$ is called the *convex hull* of A. We will denote it $\mathrm{Ch}(A)$. A direct description of $\mathrm{Ch}(A)$ is as the set of all *finite* $\mathbf{R}$-linear combinations $\sum_{k\geq 1} \lambda_k x_k$ of elements x_k of A, subject to the constraint that $\lambda_k \geq 0$ for every k and $\sum_{k\geq 1} \lambda_k = 1$.

(2) We say C is *circled* if $\lambda C \subset C$ for all λ in the closed unit disc $D \subset \mathbf{C}$. The smallest circled subset of E containing a subset $A \subset E$ is called the *circled hull* of A, denoted $\mathrm{ch}(A)$.

(3) We say C is *absolutely convex* if it is both convex and circled. This is a way to define the concept of a "disc" (or "ball") in a $\mathbf{C}$-vector space without having a normed structure: the absolutely convex subsets are regarded as the "discs" in E.

The *absolutely convex hull*, also called the *convex circled hull*, of a subset $A \subset E$ is the convex hull of the circled hull; it is denoted $\mathrm{cch}(A)$ (in [**Scha**, p. 39] and [**Kothe-I**, p. 173] it is denoted $\Gamma(A)$). We use the terminology "convex circled" rather than "absolutely convex".

We refer the reader to [**NB**, Def. 4.2.7, Thm. 4.2.8] for a simple criterion for C to be its own convex circled hull in E; see also [**Jar**, §6.1, Prop. 1] (for $r = 1$) and [**NB**, Thm. 4.2.11].

(4) We say C is *absorbing* if for every $x \in E$ there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbf{C}$ with $|\lambda| \leq \varepsilon$, $\lambda x \in C$. In other words, each point of E is carried into C ("absorbed" by C) by *all* sufficiently small scaling factors. Absorbing subsets are also called *radial* (see [**Scha**, p. 11] and [**NB**, Def.4.1.2(b)]).

Remark C.1.1.1. In some references the concept of circled subset is referred to as "balanced" (e.g., see [**Tre**, Def. 3.2] and [**NB**, Def. 4.1.2(a)]), and the term "circled" is sometimes reserved for those subsets C of a topological $\mathbf{C}$-vector space E such that $\lambda C \subset C$ for $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ (i.e., C is preserved under "rotations"). We will avoid this latter concept altogether; "circled" and "balanced" are synonymous for us.

A topological vector space E is called *locally convex* if it has a neighborhood base at 0 consisting of convex subsets. Due to the linear structure being continuous in a topological vector space, every point x of E then

has a neighborhood base given by x-translates of convex circled subsets. We further refer the reader to [**Scha**, Ch. I, §1 §6; Ch. II, §1, §4] for the basic generalities on the construction of linear topologies and locally convex topologies.

C.1.1.1. *Basic constructions.* For a subset $A \subset E$ of a topological vector space, we can form closures of the hulls discussed in the foregoing. These are:

(1) the closure $\overline{\mathrm{Ch}}(A)$ of the convex hull of A, called the *closed convex hull*;
(2) the closure $\overline{\mathrm{ch}}(A)$ of the circled hull of A, called the *closed circled hull*;
(3) the closure $\overline{\mathrm{cch}}(A)$ of the convex circled hull of A, called the *closed convex circled hull*.

REMARK C.1.1.2. The closure of a circled hull and the circled hull of a closure are not interchangeable. An example over $\mathbf{R}$ is the hyperbola $A = \{xy = 1\} \subset \mathbf{R}^2$ as noted just after the proof of [**Jar**, §2.2, Prop. 1] (the circled hull $\mathrm{ch}(A) = \mathrm{ch}(\overline{A})$ is the region $\{0 < xy \leq 1\} \cup \{(0,0)\}$, and its closure is $\{0 \leq xy \leq 1\}$).

C.1.2. On the topology of locally convex topological vector spaces. We recall some basic facts about the topology of a topological $\mathbf{C}$-vector space, then specializing to the locally convex case. Every topological vector space E has a neighborhood base at 0 consisting of closed circled subsets, as well as open circled subsets (since $\mathrm{int}(A)$ is circled when A is). Via convex hull, these can be taken to be convex too when E is locally convex [**Scha**, Ch. I, Cor. to 1.3]. The *convexity* assumption on the members of a neighborhood base of 0 is what distinguishes locally convex topological vector spaces from arbitrary topological vector spaces.

REMARK C.1.2.1. A useful structure with which any tvs is endowed (but we won't need) is that of a *canonical uniformity*. We refer to [**Bou-GT**, Ch. II, §1.1, Def. 1] for the definition of uniform structure and the topology associated to such a structure, and [**Scha**, Ch. I, 1.4] for relating the topology on a topological vector space to a unique translation-invariant uniform structure. We further refer the reader to [**Kothe-I**, Ch. 1, §5], [**Bou-TVS**, Ch. I, §1.4] for more on uniform structures and the canonical uniformity of a topological vector space. For our purposes, the concept of uniformity can be avoided since (as we shall discuss) *completion* for Hausdorff topological vector spaces can be made using Cauchy nets rather than going through the theory of completion for uniform spaces.

C.1.2.1. *Local convexity and fundamental families of continuous seminorms.* An equivalent way to describe the topology on locally convex topological vector spaces is through continuous seminorms (as developed in [**Scha**, Ch. II, §1, §4], [**Bou-TVS**, Ch. II, §1, §4], [**NB**, Ch. 5]).

The linear span of every absorbing convex circled subset $A \subset E$ is equipped with a real-valued function

$$p_A(x) := \inf_{x \in \lambda A, \lambda \in \mathbf{R}_{>0}} \lambda \tag{C.1.2.1}$$

called its *Minkowski functional* (or gauge functional). Under the above assumptions on A, p_A is a seminorm [**Scha**, Ch. II, 1.4]. The Minkowski functional can be used to translate the datum of a neighborhood base at 0 of open, or closed, circled convex subsets into a family of seminorms, and conversely [**Scha**, Ch. II, 1.5-1.6, §4]. For instance, continuity of a seminorm p is equivalent to $\{p < 1\}$ being a neighborhood of 0.

Given a collection of seminorms $\{p_i\}$ on E, if we define

$$U_i := \{x \in E : p_i(x) \leq 1\}$$

then the collection of subsets $\{n^{-1} U\}$ as n varies over all the positive integers and U varies over all finite intersections $U_{i_1} \cap \cdots \cap U_{i_m}$ is a neighborhood base at 0 for a unique locally convex topology, and every locally convex topology on E arises in this way. The topology is Hausdorff exactly when the only $x \in E$ such that $p_i(x) = 0$ for all $i \in I$ is $x = 0$; we then say $\{p_i\}_{i \in I}$ is *separating*.

C.1.3. Convergence and completeness. A Hausdorff topological vector space E is metrizable if and only if it has a countable base of 0-neighborhoods, in which case there is a translation-invariant metric [**Scha**,

Ch. I, 6.1] (this provides a pseudo-norm $|\cdot| : E \to \mathbf{R}_{\geq 0}$ such that the metric $d(x,y) = |x-y|$ induces the topology). However, examples of much importance in number theory and in this work are non-metrizable, such as Schwartz spaces of non-archimedean local fields and Schwartz spaces of adele rings and of idele class groups of number fields. Nonetheless, even in the absence of a metric, there are good notions of completion and completeness as we shall now discuss.

In [**Bou-GT**, Ch. II, §3.3, Def. 3], the notion of completeness (for uniform spaces) is defined through the convergence of Cauchy *filters*. For the concept of filter and its use to define convergence in topological spaces, we refer to [**Bou-GT**, Ch. I, 6, 7]. We will use *nets* instead of filters (to be discussed below). For the correspondence between nets and filters (and their equivalent encoding of convergence) we refer the interested reader to [**Sm**] and to [**Cl**, §6] (where the results of [**Sm**], building on work of Bartle in [**Ba**], are further spelled out and the two convergence concepts are related in [**Cl**, §6. Thm. 6.4]), and to [**NB**, Ch. 1, §1.1.2], with emphasis on [**NB**, Thms. 1.1.8, 1.1.9].

Definition C.1.3.1. Let E be a topological vector space.

(1) A *net* in E is a map $f : J \to E$ from a directed set J (also denoted $\{f_j\}_{j\in J}$).
(2) A net $f : J \to E$ *converges* in E if there exists $x \in E$ such that for every 0-neighborhood $U \subset E$ there exists j_0 so that $x - f_j \in U$ for all $j \geq j_0$ (this x is unique when E is Hausdorff).
(3) A net $\{f_j\}_{j\in J}$ in E is called *Cauchy* if and only if for every 0-neighborhood $U \subset E$ there exists a $j_0 \in J$ such that for all $j, j' \geq j_0$ we have $f_j - f_{j'} \in U$. Two Cauchy nets $\{x_j\}_{j\in J}$ and $\{y_{j'}\}_{j'\in J'}$ in E are *equivalent* if and only if the net $\{x_j - y_{j'}\}_{(j,j')\in J\times J'}$ converges to 0; equivalently, for every 0-neighborhood $U \subset E$ there exist $j_0 \in J$ and $j_0' \in J'$ such that $x_j - y_{j'} \in U$ for all $j \geq j_0$, $j' \geq j_0'$ (this is clearly an equivalence relation).
(4) A subset $A \subset E$ is called *complete* when it is Hausdorff and every Cauchy net in E valued in A converges in A.
(5) The topological vector space E is called *Fréchet* when it is locally convex, complete, and metrizable.

In the metrizable case, completeness in the classical sense of Cauchy *sequences* guarantees completeness in the broader sense of nets because there is a countable base of 0-neighborhoods. Indeed, if $\{x_j\}_{j\in J}$ is a Cauchy net in a metrizable E, say with $|\cdot| : E \to \mathbf{R}_{\geq 0}$ a pseudo-norm for which the metric $|x-y|$ induces the topology [**Scha**, Ch. I, 6.1], then for all $n \geq 1$ there exists $j_n \in J$ with $|x_j - x_{j'}| < 1/n$ for all $j, j' \geq j_n$. We can arrange that $j_{n+1} \geq j_n$ for all n, so $\{y_n := x_{j_n}\}$ is a Cauchy sequence in E such that $|x_j - y_n| < 1/n$ for all $j \geq j_n$. Thus, if the limit $\ell := \lim_{n\to\infty} y_n \in E$ exists then we see that $x_j \to \ell$.

For any Hausdorff topological vector space E, there is an initial object $(\widehat{E}, i)$ among complete Hausdorff topological vector spaces equipped with a continuous linear map from E, with the canonical map $i : E \to \widehat{E}$ a topological embedding (in particular, injective) and a base of closed 0-neighborhoods in $\widehat{E}$ given by closures in $\widehat{E}$ of 0-neighborhoods in E. We call $(\widehat{E}, i)$ the *completion* of E. The construction of completion for metric spaces via equivalence classes of Cauchy sequences can be adapted to Cauchy nets to build $(\widehat{E}, i)$, as is done in [**?**]. The universal mapping property, topological embedding property of i, and 0-neighborhood base description in $\widehat{E}$ are all seen from that construction.

Remark C.1.3.2. The main technical difficulty in the construction (in contrast with the metric setting) is that the directed set J can vary a lot (especially when it is uncountable); e.g., in the definition of Cauchy net $\{f_j\}_{j\in J}$ there is no constraint on the cardinality of J and so there is no "set of all nets valued in E". But all we need is a set of all *equivalence classes* of nets valued of E, and this does exist because we claim every net $f : J \to E$ is equivalent to a net $x : J' \to E$ where J' has cardinality bounded in term of E.

Such set-theoretic control is not addressed in [**?**], and it can be handled as follows. For each $j \in J$, let $F_j \subset E$ be the subset of points f_i for all $i \geq j$; this throws away the indexing information on points provided by $f : J \to E$. Each F_j is a non-empty subset of E and any two contain a third, so the set $\mathscr{F}$ of subsets

$F_j \subset E$ is directed by reverse inclusion; beware that a containment $F_j \subset F_{j'}$ (and even an equality) does not imply $j, j' \in J$ are comparable (nor equal when $F_j = F_{j'}$). The Cauchy hypothesis implies that for every 0-neighborhood $U \subset E$ there exists $F \in \mathscr{F}$ so that any two points in F have difference contained in U. Thus, if we define a net $x : \mathscr{F} \to E$ by letting $x_F \in F$ be a choice of point for each $F \in \mathscr{F}$ then $\{x_F\}_{F \in \mathscr{F}}$ is Cauchy and equivalent to the given Cauchy net $\{f_j\}_{j \in J}$: for a choice of 0-neighborhood $U \subset E$ there exists $j_0 \in J$ so that $f_j - f_{j'} \in U$ for all $j, j' \geq j_0$, so if $F_0 := F_{j_0}$ then $x_F - f_j \in U$ for all $F \subset F_0$ and all $j \geq j_0$. Since $\mathscr{F}$ has cardinality bounded by that of the power set of E, we have the desired set-theoretic control.

In most other references (e.g., [**Tre**, Ch. 5]), arguments based on Cauchy filters are used to construct $\widehat{E}$. The relation between completeness by Cauchy-net convergence and completeness by Cauchy-filter convergence is discussed in [**NB**, Thm. 3.1.6] (in the wider setting of commutative topological groups). Another approach is given in [**NB**, §3.7, §4.8], which uses a notion of completeness that makes sense for non-Hausdorff E (in which case completions can fail to be unique even up to linear topological isomorphism [**NB**, Ex. 3.7.3]).

REMARK C.1.3.3. We have opted in favor of nets over filters because for our purposes arguments via nets foster reasoning by analogy with sequential convergence in corresponding classical situations. Since the topology of locally convex vector spaces is generated by a family of seminorms, nets are convenient for studying the convergence of vector-valued integrals (often valued in *non-normable* complete locally convex topological vector spaces) and other settings that shall arise in this work.

The role of nets rather than just sequences is crucial, because many Hausdorff locally convex spaces of importance in this work do *not* have a countable base of 0-neighborhoods (equivalently are not metrizable [**Scha**, Ch. I, 6.1]). This happens for the Schwartz-Bruhat space $\mathcal{S}(G)$ where G is the adele ring $\mathbf{A}_F$ and idele class group C_F of a number field F as well as a non-archimedean local field L (even though $\mathcal{S}(L)$ has countable $\mathbf{C}$-dimension); such spaces $\mathcal{S}(G)$ are studied in depth in Appendix D.

To explain this non-metrizability, we first note that §C.1.5 will discuss a natural Hausdorff locally convex topology on direct limits $E = \varinjlim E_n$ where the E_n's are Hausdorff locally convex topological vector spaces and the continuous linear transition maps $E_n \to E_{n+1}$ are topological embeddings (this is generally strictly coarser than the "inductive limit topology" as in general topology: the continuous bijection from the topological inductive limit to the locally convex inductive limit is usually not a homeomorphism). When the E_n's are all complete then such E is complete [**Scha**, Ch. II, 6.6] but if the E_n's are all Fréchet and $E_n \neq E_{n+1}$ for all n then E cannot be metrizable because otherwise it would be Fréchet (by completeness) and so Lemma C.1.5.4 below applied to the identity map $E \to \varinjlim E_n$ would force $E = E_n$ for all sufficiently large n, a contradiction. The topology on $\mathcal{S}(G)$ for the number-theoretic examples G mentioned above is always of such type $\varinjlim E_n$ that we have just seen is never metrizable.

REMARK C.1.3.4. Much as the quotient of a Banach space by a closed subspace is again Banach (using an associated "residue norm"), the quotient E/M of a Fréchet space E modulo a closed subspace M is Fréchet: it inherits local convexity from E, and by [**Scha**, Ch. I, 6.3] it is complete and metrizable. However, beyond the metrizable setting the quotient of a complete Hausdorff locally convex E modulo a closed subspace can *fail to be complete*; examples are given in [**Scha**, Exer. 11, p. 192].

C.1.4. Boundedness and quasi-completeness. For a topological vector space E, a subset $A \subset E$ is called *bounded* if for every neighborhood U of 0 in E there exists $\lambda \in \mathbf{C}^\times$ such that $\lambda A \subset U$ (equivalently, $A \subset \mu U$ for some $\mu \in \mathbf{C}$). Observe that this notion is intrinsic to the topological vector space structure and is generally *not* the same as metric-boundedness when E is metrizable (since one can always find a bounded metric giving the same topology), but it is the same as norm-boundedness when E is normable. Boundedness is an important notion in this work.

In the absence of a norm it may seem difficult to verify boundedness in examples, so it is useful to have a some criteria for boundedness. In a Hausdorff locally convex vector space E whose topology is generated by a

family of seminorms $\{p_i\}_{i\in I}$, a subset $B \subset E$ is bounded if and only if either of the following (equivalent) conditions holds:

(1) $p_i(B) \subset \mathbf{R}$ is bounded for every $i \in I$ (this restates the definition of boundedness since the p_i's define the topology of E),
(2) for any *sequence* $\{b_k\}_{k\geq 1} \subset B$ and sequence $\{\lambda_k\}_{k\geq 1}$ in $\mathbf{C}$ tending toward 0, $\lambda_k b_k \to 0$ [**Scha**, Ch. I, 5.3].

A useful alternative criteria from [**Scha**, Ch. IV, 3.3] identifies boundedness with "weak boundedness":

PROPOSITION C.1.4.1. *Let E be a Hausdorff locally convex topological vector space. A subset $B \subset E$ is bounded if and only if the image $\ell(B) \subset \mathbf{C}$ is bounded for every continuous linear functional $\ell : E \to \mathbf{C}$.*

DEFINITION C.1.4.2. A subset A of a Hausdorff topological vector space E is called *relatively compact* if its closure in E is compact, and it called *pre-compact* if its closure in the completion $\widehat{E}$ is compact (i.e., pre-compactness means relative compactness in $\widehat{E}$).

Relatively compact subsets of E are pre-compact since a compact subset of E is certainly also closed in $\widehat{E}$. By [**Tre**, Prop. 6.9] (whose proof uses self-contained arguments with filters, ultrafilters, and Cauchy filters, the latter of which can be used to make Cauchy nets by picking a point in each member of the filter), pre-compactness for a subset $A \subset E$ is equivalent to *total boundedness* of A in E: A is covered by finitely many translates of each 0-neighborhood $V \subset E$. Totally bounded subsets $A \subset E$ are bounded (in particular, pre-compact subsets of E are bounded), but the converse is false; e.g., the closed unit ball in any infinite-dimensional Hilbert space is bounded but non-compact and so is not pre-compact, hence not totally bounded.

REMARK C.1.4.3. We caution the reader that pre-compactness is defined differently in various references. For example, total boundedness in E is taken as the definition of pre-compactness in [**NB**, Def. 3.3.1], in which case the work shifts to proving total boundedness in $\widehat{E}$ implies relative compactness. In [**Scha**, p. 8] pre-compactness for a subset $A \subset E$ is defined as compactness for the completion of A in the sense of uniform structures, but it is noted in [**Scha**, p. 25] that this is equivalent to compactness of the closure in $\widehat{E}$ and to total boundedness (these equivalences are sufficient for all use of pre-compactness in [**Scha**]). For the purpose of our literature citations related to pre-compactness, the equivalence of total boundedness of $A \subset E$ and compactness of the closure of A in $\widehat{E}$ [**Tre**, Prop. 6.9] is sufficient.

Many locally convex topological vector spaces, some of interest to this work, *are not complete*. Here are two related examples arising from any infinite-dimensional Hilbert space H.

EXAMPLE C.1.4.4. The locally convex weak topology $\sigma(H, H')$ on H of convergence under continuous functionals (i.e., a base of 0-neighborhoods is given by $\{|\langle\cdot, x_1\rangle| < \varepsilon, \ldots, |\langle\cdot, x_n\rangle| < \varepsilon\} \subset H$ for $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$) is not complete. Indeed, if $\ell \in H^*$ is a discontinuous linear functional on H then for each finite-dimensional subspace $V \subset H$ we can use the Hahn-Banach Theorem to build a continuous $\ell_V = \langle\cdot, x_V\rangle \in H' \simeq H$ extending $\ell|_V$ [**Scha**, Ch. II, 4.2]. Ordering the set of such V's via inclusion, we see that $\{x_V\}_V$ is a Cauchy net in H (since for any $y_1, \ldots, y_n \in H$ we have $\langle x_V - x_W, y_i\rangle = 0$ when V, W contain all y_i's), but $\{x_V\}$ has no limit $x \in H$ for the weak topology. Indeed, if $x_V \to x$ weakly then $\langle y, x\rangle = \lim\langle y, x_V\rangle = \lim \ell_V(y) = \ell(y)$ for all $y \in H$, which is to say $\ell = \langle\cdot, x\rangle$, contradicting that ℓ is discontinuous.

EXAMPLE C.1.4.5. Equip the space $\mathscr{B}(H)$ of bounded linear operators on H with the locally convex strong operator topology of pointwise convergence (rather than the operator-norm topology). A discontinuous linear functional on H can be used (similarly to Example C.1.4.4) to construct a Cauchy net in $\mathscr{B}_{\mathrm{SOT}}(H)$ that is not bounded (in contrast with Cauchy sequences, which are bounded in any topological vector space); this implies $\mathscr{B}_{\mathrm{SOT}}(H)$ is not complete. See Remark C.4.3.4 for details on the construction.

In both of the preceding examples, the failure of completeness has a salvage: those spaces are *quasi-complete*: closed bounded subsets are complete (in the sense of convergence of Cauchy nets). Some references call this *bounded completeness*. It is immediate from the definitions that any complete E is quasi-complete.

Since all Cauchy nets in a compact subset of a Hausdorff topological vector space clearly converge, quasi-completeness of $H_{\sigma(H,H')}$ in Example C.1.4.4 is a consequence of two facts: (i) Proposition C.1.4.1 and (ii) all bounded subsets of such a Hausdorff locally convex topological vector space E are pre-compact relative to the weak topology $\sigma(E, E')$ (essentially Alaoglu's theorem: [**Scha**, Ch. IV, Cor. 2 to 5.5]). Quasi-completeness of $\mathscr{B}_{\mathrm{SOT}}(H)$ in Example C.1.4.5 is deduced from the Uniform Boundedness Principle for Banach spaces: see Lemma C.2.1.7 (applied with $E = F = H$ and $\mathfrak{S}$ the set of finite subsets of H).

Quasi-completeness is a consequential concept throughout this work, insofar as there is a good integration theory for functions valued in quasi-complete Hausdorff locally convex topological complex vector spaces. This goes far beyond the Banach-valued theory of Bochner that is inapplicable for our purposes (since we need to work with infinite-dimensional nuclear Fréchet spaces, and those are never normable [**Tre**, Cor. 2 to Prop. 50.2]). The key stability property of quasi-completeness that enables integration theory to take off is [**Scha**, Ch. II, Cor. to 4.3]: the closed convex circled hull of a pre-compact subset of a quasi-complete Hausdorff locally convex topological $\mathbf{C}$-vector space is compact. More on quasi-completeness and its role will be discussed in §C.4 on integration in the sense of Pettis and §C.5 on holomorphy and meromorphy for vector-valued functions on a Riemann surface (such as the surface $\Omega(C_F)$ of quasi-characters of the idele class group C_F of a number field F).

Example C.1.4.6. Of main interest to this work are the following classes of locally convex spaces:

(1) separable Hilbert spaces: they are locally convex since their topology is even normed, and are complete (so quasi-complete).

(2) Banach spaces: specifically, the Banach space of bounded linear operators on a Hilbert space equipped with the operator-norm topology. By completeness, they are quasi-complete too.

(3) The spaces $\mathcal{S}(G)$ of Schwartz-Bruhat functions on second-countable locally compact abelian groups G, especially those G of arithmetic origin such as the adele ring of a number field, the idele class group of a number field, and their local analogs. As was discussed in Remark C.1.3.3, these have natural locally convex topologies that are often non-metrizable but complete.

    Much deeper is that such spaces $\mathcal{S}(G)$ are nuclear (a concept reviewed in §C.2.4) and Montel (this means compactness of closed bounded subsets and reflexivty in the sense that the continuous double-duality map $E \to (E'_\beta)'_\beta$ for the strong-dual topologies [**Scha**, pp. 144-5] is a topological isomorphism). These further properties are established in Theorem D.1.5.7.

(4) LF-spaces: As will be discussed in §C.1.5, for any inductive system $\{E_\alpha\}$ of locally convex topological vector spaces there is a natural locally convex topology on $\varinjlim E_\alpha$ making it have the expected universal mapping property [**Scha**, Ch. II, p. 54]. By construction the natural continuous linear map $\oplus E_\alpha \to \varinjlim E_\alpha$ is a topological quotient map, so a base for the topology in such inductive limits is given by the image of a base for the topology in the corresponding locally convex direct sum $\oplus E_\alpha$; one such base for the latter is given in (C.1.5.1) below.

    The most important cases of interest are *Hausdorff* locally convex *countable* inductive limits $E = \varinjlim E_n$ of Fréchet spaces E_n, which are called *LF-spaces* as in [**Gro-f**, Def. 4 p. 146]. We call such E a *strict LF-space* when we can choose the transition maps $E_n \to E_{n+1}$ to be topological embeddings, in which case it is automatic (i.e., does not need to be imposed as an additional requirement) that E is Hausdorff and $E_n \to E$ is a topological embedding for all n [**Scha**, Ch. II, 6.4]. The strictness hypothesis is assumed for all work with inductive limits in [**Scha**] (after the initial construction) and in [**Tre**] (see [**Tre**, p. 126]).

    There is a powerful Open Mapping Theorem for LF-spaces without a strictness assumption (see Theorem C.1.5.5 below); its proof ultimately bootstraps from the more familiar Fréchet case. This result is

often used in the present work to deduce that an LF-space not yet known to be strict is actually so. From the definitions one sees that any quotient of an LF-space is an LF-space, but beware that a quotient of a strict LF-space can fail to be strict. Indeed, since $E = \varinjlim E_n$ is always a topological quotient of $\oplus_{n\geq 1} E_n$ that is itself strict (as $\varinjlim F_N$ for $F_N = \oplus_{1\leq n\leq N} E_n$ with the evident transition maps), any non-strict LF-space provides such an example and [**SN**, §2, Ex. 2] gives a class of such examples.

REMARK C.1.4.7. The vast majority of the Hausdorff locally convex topological vector spaces of interest to this work are nuclear Montel LF-spaces. In fact, they will generally arise for us as a strict countable locally convex inductive limit of nuclear Fréchet spaces (such inductive limits are always nuclear and Montel by [**Scha**, Ch. II, 6.6, Cor. to 7.2; Ch. III, Cor. to 7.4] and [**Tre**, Cor. 3 to Prop. 50.2]), but sometimes the verification of strictness rests on the Open Mapping Theorem for general LF-spaces. Good properties of strict inductive limits are provided throughout [**Scha**], which is our main reference on the matter (but again, we emphasize the power of Open Mapping Theorem for LF-spaces without a strictness hypothesis).

We will provide guidance on where to find the required foundational results on locally convex topological vector spaces as the need arises. Our main references are [**Scha**, Ch. I–IV] (especially for duality and some basic aspects of nuclearity) and [**Tre**] (especially for topological tensor products and deeper aspects of nuclearity), along with [**Gro-b**] for Grothendieck's work on topological tensor products and nuclear spaces (particularly the inductive tensor product topology that we need rather crucially and does not seem to be systematically treated apart from [**Gro-b**]). Some other useful references are [**Bou-TVS**, Ch. II, IV], [**Jar**, Ch. 2, 6, 8], [**Cnw-a**, Ch. IV], and [**NB**, Ch. 4, 5, 8],

C.1.5. Inductive limits and detection of limit subspaces. In this subsection we discuss some complements around inductive limits in the category of locally convex topological vector spaces (over $\mathbf{C}$), with a focus on the special case of a sequence $\{E_k\}_{k\geq 0}$ of Hausdorff locally convex topological $\mathbf{C}$-vector spaces equipped with continuous injective linear maps $f_k : E_k \to E_{k+1}$ for each $k \geq 0$.

If $\{E_\alpha, f_{\alpha\beta}\}$ is an inductive system of locally convex topological vector spaces over $\mathbf{C}$, we can put a locally convex topology on $E := \varinjlim E_\alpha$ having the desired universal property (namely, a linear map $g : E \to F$ to a locally convex topological vector space F is continuous if and only if the composite linear maps $E_\alpha \to E \to F$ are all continuous) provided that this can be done for the direct sum $\oplus E_\alpha$. Indeed, if direct sums can be equipped with such a locally convex topology then the quotient topology on E from the direct sum via the usual algebraic construction of E will do the job for E due to the universal property of the quotient topology on V/W for any locally convex V and linear subspace W (since the topology on V/W is also locally convex).

For an arbitrary algebraic direct sum $E = \oplus E_\alpha$ of locally convex E_α's, a locally convex topology on E having the categorical mapping property for a direct sum is the one with a base of 0-neighborhoods given by the sets

$$U = \left\{ \sum \lambda_\alpha x_\alpha : x_\alpha \in U_\alpha, \sum |\lambda_\alpha| \leq 1 \right\} \tag{C.1.5.1}$$

where U_α is a convex circled 0-neighborhood in E_α (in each sum $\sum \lambda_\alpha x_\alpha$ all but finitely many x_α's vanish and so only the corresponding λ_α's matter; i.e., $\sum |\lambda_\alpha|$ may be assumed to have only finitely many non-zero terms). Since each U_α is circled, it is equivalent to require the λ_α's to be real numbers in $[0, 1]$, which is to say U is the convex hull of choices of convex circled 0-neighborhoods $U_\alpha \subset E_\alpha$ over all α. Note that each $E_{\alpha_0} \to \oplus E_\alpha$ is a topological embedding.

In the special case of a locally convex countably infinite direct sum E of *normed* spaces E_k ($k \geq 1$), we can scale each member of a *countably infinite* collection of 0-neighborhoods in E so the rescaled 0-neighborhoods all contain a common 0-neighborhood, as recorded below. This will be useful in our later discussion of how countable locally convex inductive limits interact with the "projective" tensor product topology.

PROPOSITION C.1.5.1. *Let E be the locally convex topological vector space $\bigoplus_{k\geq 1} E_k$ for a sequence of normed vector spaces $E_1, E_2, \ldots$ over $\mathbf{C}$. For any sequence of 0-neighborhoods $V_1, V_2, \cdots \subset E$ there are scalars $\rho_i \in \mathbf{R}_{>0}$ such that $\bigcap_{i\geq 1} \rho_i V_i$ is a 0-neighborhood.*

PROOF. It is harmless to shrink each V_i, so via (C.1.5.1) we can assume V_i is the convex circled hull of $\bigoplus_{k\geq 1} U_{ik}$ for 0-neighborhoods $U_{ik} \subset E_k$ for all $k \geq 1$:

$$V_i = \left\{ \sum_{k\geq 1} \lambda_k u_{ik} : u_{ik} \in U_{ik}, \sum |\lambda_k| \leq 1 \right\}.$$

Since each E_k is normed, we can further shrink each U_{ik} to be bounded in E_k. Letting $\rho_1 = 1$, we recursively pick ρ_n for all $n \geq 1$ by requiring

$$\rho_n U_{nk} \supset \rho_i U_{ik} \text{ inside } E_k \text{ for all } 1 \leq k \leq n, 1 \leq i < n$$

(as we may do each each $U_{ik} \subset E_k$ is a bounded 0-neighborhood).

By design, for all $k \geq 1$ and $i > k$ we have $\rho_i U_{ik} \supset \rho_k U_{kk}$. Thus, for all $k \geq 1$ we have

$$\bigcap_{i\geq 1} \rho_i U_{ik} \supset \bigcap_{1\leq i\leq k} \rho_i U_{ik} =: W_k$$

(since for $i > k$ the ith term in the intersection on the left contains $\rho_k U_{kk}$). Since W_k is an intersection of finitely many 0-neighborhoods in E_k, it is also a 0-neighborhood in E_k. By design, for each $i \geq 1$ and all $k \geq 1$ we have $\rho_i U_{ik} \supset W_k$, so

$$\rho_i V_i \supset \left\{ \sum \lambda_k w_k : w_k \in W_k, \sum |\lambda_k| \leq 1 \right\} =: W$$

for all i. This says that the 0-neighborhood W in E is contained in every $\rho_i V_i$, so $\bigcap_{i\geq 1} \rho_i V_i$ is a 0-neighborhood in E as desired. □

Let us now focus on the special case that E is Hausdorff and is the locally convex inductive limit of a countable inductive system $\{(E_k, f_k)\}_{k\geq 0}$ for which the transition maps $f_k : E_k \to E_{k+1}$ are injective. The best-behaved situation is when each f_k is moreover a topological embedding and the E_k's are all Hausdorff; such cases are called *strict*. In the strict case (as we noted in Example C.1.4.6(4)), automatically such an inductive limit E is Hausdorff and each $E_k \to E$ is a topological embedding. (Beware that in [**Scha**] the terminology "locally convex topological vector space" includes a Hausdorff hypothesis but the terminology "locally convex topology" does not). The following is given in [**Gro-f**, p. 146].

DEFINITION C.1.5.2. A locally convex topological vector space E is called an *LF-space* if it is Hausdorff and $E = \varinjlim E_k$ for a sequence of continuous linear maps $f_k : E_k \to E_{k+1}$ between Fréchet spaces.

Letting $i_k : E_k \to E$ be the natural continuous linear map, by the universal property of E we see that for the formation of E it is harmless to replace each E_k with $E_k/\ker(i_k)$ (which is also Fréchet since $\ker(i_k)$ is closed due to continuity of i_k and the Hausdorff hypothesis on E; see Remark C.1.3.4). In other words, we lose no generality in Definition C.1.5.2 by assuming the f_k's are injective. The following description of boundedness in the strict case under a closedness hypothesis on the maps $E_k \to E_{k+1}$ (as holds when all E_k's are complete, for example) is [**Scha**, Ch. II, 6.5] and is very useful:

PROPOSITION C.1.5.3. *If $E = \varinjlim E_k$ is strict with each $E_k \to E_{k+1}$ a closed embedding, a subset $B \subset E$ is bounded if and only if B is a bounded subset of some E_k.*

Remarkably, if $E = \varinjlim E_k$ with each $E_k \to E_{k+1}$ injective between Fréchet spaces and *without* assuming such maps are topological embeddings (or equivalently that each $i_k : E_k \to E$ is a topological embedding), the Open Mapping Theorem for Fréchet spaces yields a continuous factorization property through the E_k's:

LEMMA C.1.5.4. *Let E be a Hausdorff topological vector space equipped with continuous linear injections $i_k : E_k \to E$ from Fréchet spaces for $k \geq 1$ such that $i_k(E_k) \subset i_{k+1}(E_{k+1})$ for all k and $E = \bigcup_k i_k(E_k)$. Any continuous linear map $u : F \to E$ from a Fréchet space F factors continuously through some $i_k : E_k \to E$.*

PROOF. This result is [**Gro-f**, Thm. 1, p. 147], and for the convenience of the reader we provide the proof. Let $H_k = \{(x, y) \in F \times E_k : u(x) = i_k(y)\}$, a subspace of the Fréchet $F \times E_k$ that is closed since E is Hausdorff (as the diagonal of E is then closed in $E \times E$), so H_k is Fréchet. The linear projection $p_k : H_k \to F$ to the first factor is injective since i_k is injective, with $p_k(H_k) \subset p_{k+1}(H_{k+1})$ for all k. By hypothesis F is the union of the rising countable sequence of linear subspaces $p_k(H_k)$ (since each $u(x) \in E$ lies in some $i_k(E_k)$), so the Fréchet property of F implies via the Baire category theorem that one of the linear subspaces $p_{k_0}(H_{k_0})$ has closure in F with non-empty interior, so that closure is equal to F. In other words, the continuous linear injection $p_{k_0} : H_{k_0} \to F$ has dense image for some k_0.

We claim p_{k_0} is surjective. Since H_{k_0} is a complete metric space, by [**Gro-f**, Lemma, p. 39] a sufficient criterion for surjectivity is that the closure $\overline{p_{k_0}(U)} \subset F$ is a 0-neighborhood for every 0-neighborhood $U \subset H_{k_0}$. Pick any 0-neighborhood $U \subset H_{k_0}$, and then choose a 0-neighborhood $V \subset H_{k_0}$ so that $V - V \subset U$. Closures in F are obtained via sequential limits since F is metrizable, so $\overline{p_{k_0}(V)} - \overline{p_{k_0}(V)} \subset \overline{p_{k_0}(V - V)} \subset \overline{p_{k_0}(U)}$. Thus, it suffices to show that differences among $\overline{p_{k_0}(V)}$ exhaust a 0-neighborhood in F.

Since $\bigcup_{n \geq 1} nV = H_{k_0}$, the countable collection $np_{k_0}(V)$ of subsets of F has union $p_{k_0}(H_{k_0})$ that is dense in F. Thus, by the Baire category theorem some $n_0 p_{k_0}(V)$ has closure in F with non-empty interior. Hence, $p_{k_0}(V)$ has closure in F with non-empty interior. Differences among pairs from any non-empty open subset of F exhaust a 0-neighborhood, so $\overline{p_{k_0}(V)} - \overline{p_{k_0}(V)}$ is a 0-neighborhood in F as desired.

By the Open Mapping Theorem for complete metrizable topological vector spaces (see [**KG**, pp. 53-54], where the existence of a translation-invariant metric [**Scha**, Ch. I, Thm. 6.1(iv)] is used near the end of the proof), the continuous linear bijection p_{k_0} is a topological isomorphism and so its continuous inverse $F \to H_{k_0}$ followed by the continuous projection $H_{k_0} \to E_{k_0}$ defines a *continuous* linear map $v : F \to E_{k_0}$ such that $i_{k_0} \circ v = u$. □

The preceding lemma is used in the proof of the Open Mapping Theorem for LF-spaces which does not require strictness. One even gets a more general result:

THEOREM C.1.5.5 (Open Mapping Theorem). *Let $q : F \to E$ be a surjective continuous linear map between Hausdorff topological vector spaces with E an LF-space and F exhausted by a rising sequence of images of continuous linear maps $u_n : F_n \to F$ with Fréchet F_n's (e.g., F also an LF-space). Then q is open, or equivalently $F/\ker(q) \to E$ is a topological isomorphism.*

In particular, a continuous linear surjection between LF-spaces is open.

Although there is no strictness hypothesis, it is assumed each of E and F is Hausdorff. This result is a special case of [**Gro-f**, Thm. 2(1), p. 148], which works with a wider class of spaces E that Grothendieck called "strictly bornological" [**Gro-f**, p. 148] (later called "ultrabornological": see [**NB**, Def. 13.2.3(b), Ex. 13.2.8(d), Thm. 13.2.11]). For the reader's convenience, we give Grothendieck's proof specialized to the context as stated in Theorem C.1.5.5 that suffices for our needs.

PROOF. We write $E = \varinjlim E_k$ for an inductive system $\{E_k\}$ of Fréchet spaces, and since E is Hausdorff each $f_k : E_k \to E$ has closed kernel. The quotient E'_k of E_k by $\ker(f_k)$ is therefore also Fréchet, and by consideration of the universal property of $\varinjlim E_k$ we see that the natural map $\varinjlim E'_k \to E$ is an isomorphism. Thus, we may and do replace E_k with E'_k so that each continuous linear map $f_k : E_k \to E$ is injective. We can likewise replace F by the Hausdorff $F/\ker(q)$ (note that $\ker(q)$ is closed since E is Hausdorff), so q is now a linear bijection, and we can then replace each F_n by $F_n/\ker(u_n)$ so that each u_n is injective (beware that

the resulting continuous linear bijection $\varinjlim F_n \to F$ might not be a homeomorphism; we don't even know if the locally convex topology on $\varinjlim F_n$ is Hausdorff).

Our aim is to show that the linear inverse of the continuous linear bijection $q : F \to E = \varinjlim E_k$ is continuous. By the universal property of such a locally convex inductive limit, we just have to show that each linear map $E_k \to E \overset{q^{-1}}{\simeq} F$ is continuous. Since E is exhausted by the rising sequence of subspaces $q(F_n)$ with $q : F_n \to E$ a continuous linear injection with Fréchet F_n, by Lemma C.1.5.4 each $E_k \to E$ factors continuously through some $F_{n_k} \to E$. In other words, $q(F_{n_k}) \supset E_k$ with the resulting linear map $E_k \to F_{n_k}$ *continuous*. But then the composite map $E_k \to F_{n_k} \overset{u_{n_k}}{\to} F$ is continuous, and this is exactly $q^{-1}|_{E_k}$. □

[1]

C.1.5.1. *Acyclicity and limit subspaces of LF-spaces.* A technically important but rather subtle aspect of LF-spaces is centered around the following concept:

DEFINITION C.1.5.6. Let E be an LF-space, and let $f_k : E_k \to E_{k+1}$ be a sequence of injective linear maps (not necessarily a topological embedding) between Fréchet spaces such that $E \simeq \varinjlim E_k$. A closed subspace $F \subset E$ is called a *limit subspace* if for $F_k = F \cap E_k$ equipped with with the subspace topology from E_k, the continuous linear bijection $\varinjlim F_k \to F$ a topological isomorphism.

The preceding concept is independent of the choice of defining sequence $\{E_k\}$ for E. Indeed, if $\{E'_k\}$ is a second such defining sequence then $E_k \to E$ factors continuously through some $E'_{k'} \to E$ by Lemma C.1.5.4 and likewise each $E'_{k'} \to E$ factors continuously through some $E_k \to E$. This ensures that for $F_k = F \cap E_k$ and $F'_{k'} = F \cap E'_{k'}$ the natural linear bijection between the countable inductive limits $\varinjlim F_k$ and $\varinjlim F'_{k'}$ (via their identifications with F) is a topological isomorphism.

Closed subspaces $F \subset E$ in LF-spaces are not generally limit subspaces, by they are under some favorable conditions. We provide a sufficient criterion below, recorded here for ease of access. This rests on the concept of *acyclicity* that was developed by Palamodov [**Pa**, §7, p. 38] (resting on homological techniques) and Retakh [**Ret**], and further studied by Vogt (see [**Vogt-1**, §2], and specifically [**Vogt-1**, §2, Thm. 2.10, p. 67], where Retakh's characterization of acyclic LF-spaces is reproved avoiding homological methods).

Strict LF-spaces are acyclic, as we will be discussing below, and we will see in Theorem C.1.5.11 that if each of the members E_k of a defining countable strict inductive system is nuclear then any closed subspace $F \subset E$ for which $q : E \to E/F$ lifts bounded subsets (i.e., each bounded $B \subset E/F$ has the form $q(B')$ for bounded $B' \subset E$) is a limit subspace and so is itself a strict LF-space. These conditions will always be fulfilled for the applications discussed throughout this work.

For the following example due to Grothendieck, we refer the reader to the discussion in the second half of [**Gro-e**, §II.3, p. 98]. It provides an example of a closed subspace F of a strict locally convex inductive limit E of an increasing sequence of Fréchet spaces E_k such that F is not an LF-space.

EXAMPLE C.1.5.7 (A closed subspace of a strict LF-space which is not a limit subspace). There exists a countable locally convex direct sum E of Banach spaces and a closed subspace $F \subset E$ such that F is not an LF-space with respect to the subspace topology. In particular, F is not a limit subspace. This answers a question of Dieudonné and Schwartz from 1954 [**DS**, §15, no. 2, p. 97].

To be specific, pick $1 < p, q < \infty$ with $1/p + 1/q = 1$ and let $E = \bigoplus_{n \geq 1} \ell^q(\mathbf{N})$, so the topological dual E' of E is identified as a vector space with $\prod_{n \geq 1} \ell^p(\mathbf{N})$. By [**Scha**, Ch. IV, 3.4, 4.3(2)], the locally convex Mackey topology $\tau(E', E)$ on E' is the product topology. There is a closed subspace $F' \subset E'$ that is a Köthe sequence

[1] Adjust cross-references all throughout the document to refer to Theorem C.1.5.5 as opposed to the literature, for the Open Mapping Theorem for LF-spaces. Say somewhere appropriate what Köthe exactly proved, as opposed to Grothendieck's version.

space[2] such that the two natural strong topologies[3] on the closed subspace $F := F'^{\perp} \subset E$ are distinct. The subspace topology on F is not an LF-space.[4]

DEFINITION C.1.5.8. A countable inductive system $\{E_k\}_{k\geq 0}$ of Hausdorff locally convex topological $\mathbf{C}$-vector spaces with transition maps $f_k : E_k \to E_{k+1}$ is called *acyclic* if and only if the natural algebraically short exact sequence of $\mathbf{C}$-linear maps

$$0 \to \bigoplus_{k\geq 0} E_k \xrightarrow{\sigma} \bigoplus_{k\geq 0} E_k \xrightarrow{q} E := \varinjlim E_k \to 0 \tag{C.1.5.2}$$

(with q topologically a quotient map) has σ a closed topological embedding, where σ is the $\mathbf{C}$-linear map defined by

$$\sigma((x_k)_{k\geq 0}) := (x_k - x_{k-1})_{k\geq 0},\ x_{-1} := 0 \tag{C.1.5.3}$$

(also writing x_k also for the image of x_{k-1} into E_k under f_{k-1}, where $f_{-1} := 0$).

In the setup of Definition C.1.5.8, q is the linear map $q : (x_k)_{k\geq 0} \mapsto \sum_{k\geq 0} x_k$ (where the sum is finite) and σ admits a linear algebraic inverse $\sigma^{-1} : \ker(q) \to \bigoplus_{k\geq 0} E_k$ defined by

$$\sigma^{-1}((x_k)_{k\geq 0}) := (y_k)_{k\geq 0} \tag{C.1.5.4}$$

where $y_k := \sum_{0\leq j\leq k} x_k$ is the kth partial sum of the finite sum $\sum_{k\geq 0} x_k$. Indeed,

$$\sigma(\sigma^{-1}((x_k)_{k\geq 0})) = (\sum_{0\leq j\leq k} x_j - \sum_{0\leq j\leq k-1} x_j)_{k\geq 0} = (x_k)_{k\geq 0}$$

and $\sigma^{-1}(\sigma((x_k)_{k\geq 0})) = \sigma((x_k - x_{k-1})_{k\geq 0})$ with $x_{-1} := 0$, which reads $\sigma^{-1}(\sigma((x_k)_{k\geq 0})) = (x_k)_{k\geq 0}$. Acyclicity of $\{(E_k, f_k)\}_{k\geq 0}$ is therefore equivalent to σ^{-1} being continuous.

By [**Vogt-1**, Cor. 1.3], which rests on Grothendieck's Factorization Theorem (Lemma C.1.5.4) and the Open Mapping Theorem for LF-spaces (Theorem C.1.5.5), Definition C.1.5.8 with Fréchet E_k's depends only on the Hausdorff inductive limit LF-space $E = \varinjlim E_k$: it is independent of the chosen defining sequence of Fréchet spaces with Hausdorff inductive limit E. Moreover, as a special case of [**Vogt-1**, Thm. 2.10], a *strict* LF-space is always acyclic.[5]

The acyclicity of all strict LF-spaces leads to a useful sufficient condition stated in Theorem C.1.5.11 below to recognize limit subspaces within strict locally convex countable inductive limits of *nuclear* Fréchet spaces. Before we state the condition, we introduce some general concepts:

DEFINITION C.1.5.9. For a continuous linear map $f : E \to F$ between Hausdorff locally convex topological $\mathbf{C}$-vector spaces, we say a bounded subset $B \subset F$ can be *f-lifted* if and only if there exists a bounded subset $B' \subset E$ such that $B \subset f(B')$ (it is clearly equivalent to require $f(B') = B$). If this condition holds for all bounded subsets in F then we say f *lifts bounded subsets* or satisfies the *bounded lifting property*.

A class of locally convex spaces E for which the bounded lifting property is satisfied by every quotient map $q : E \to E/F$ modulo a closed subspace was identified by Grothendieck in [**Gro-e**, II.4, Thm. 12, p. 119]: the Fréchet spaces E that are also *Schwartz*, a concept introduced in [**Gro-e**, II.4, Def. 5, p. 117]. (The result is more precise: every bounded subset of E/F is contained in $q(C)$ for a compact subset $C \subset E$.)

One of the equivalent definitions of being Schwartz (for a locally convex topological vector space E) is that for every closed convex circled 0-neighborhood $U \subset E$ there exists a 0-neighborhood $V \subset E$ with pre-compact

[2] should say more: what *exactly* is F'? Calling it simply "a Köthe sequence space" is not helpful to a reader.

[3] What are these? Does $F' = \prod_{n\geq 1} F'_n$ for some closed $F'_n \subset \ell^q(\mathbf{N})$ and $F = \oplus F_n$ for $F_n = (F'_n)^{\perp} \subset \ell^p(\mathbf{N})$? And is this just saying the subspace topology on $F \subset E$ is not the locally convex direct sum topology that is LF?

[4] Give reference for why F as subspace is not LF.

[5] The array of references originally given was incomprehensible; e.g., [**Vogt-1**, Lemma 2.4, p. 163]. Admittedly the reference now given is no better if one looks under the hood.

image in the normed space E_U (the maximal Hausdorff quotient of E defined by the semi-norm associated to the gauge $p_U : E \to \mathbf{R}_{\geq 0}$ of U); i.e., the image of V in the Banach space completion $\widehat{E_U}$ has compact closure. For our purposes, nuclear spaces are the key source of locally convex spaces that are Schwartz:

EXAMPLE C.1.5.10. If a locally convex topological vector space E is a nuclear then we claim it is Schwartz. By [**Scha**, Ch. III, 7.2], for every convex circled 0-neighborhood $U \subset E$ (in particular, such U that are closed) there exists a convex circled 0-neighborhood $V \subset U$ such that the natural map of Banach spaces $T : \widehat{E_V} \to \widehat{E_U}$ is nuclear. But any nuclear map between Banach spaces is a norm-topology limit of finite-rank operators [**Scha**, p. 98] and hence is compact. Since V maps into the closed unit ball of $\widehat{E_V}$, its image under T has compact closure. Taking U to be closed, such V fulfills the requirement in the definition of E to be Schwartz.

The examples of the bounded lifting property that arise throughout this work are quotient maps $q : E \twoheadrightarrow G$ where G is Hausdorff and E is a strict countable inductive limit of nuclear Fréchet spaces (it is not apparent yet that all such q satisfy the bounded lifting property since such an E is generally not Fréchet, but in Corollary C.1.5.13 we will see that they all do). Note that any such E is nuclear, hence Schwartz, and $G = E/F$ for the subspace $F = \ker(q) \subset E$ that is closed since G is Hausdorff. Such quotients arise in the following result.

THEOREM C.1.5.11 (Recognizing limit subspaces of strict LF-spaces). *Let E be a strict countable inductive limit of nuclear Fréchet spaces. For any closed subspace $F \subset E$, the following are equivalent:*

(1) *F is a limit subspace; i.e., for $\{E_m\}_{m\geq 0}$ a strict defining increasing sequence of nuclear Fréchet spaces, the subspace topology on F equals the locally convex inductive limit topology of the inductive system $\{F \cap E_m\}_{m\geq 0}$ where each $F \cap E_m$ is given the subspace topology from E_m;*
(2) *the quotient map $q : E \to E/F$ satisfies the bounded lifting property; i.e., for every bounded subset $B \subset E/F$, there exists a bounded subset $B' \subset E$ such that $B \subset q(B')$.*

The result is actually true more broadly, when F is merely assumed to be *step-wise closed*: $F \cap E_m$ is closed in E_m for every integer $m \geq 0$. This is wider generality because there are examples of non-closed linear subspaces of strict LF-spaces that are step-wise closed: the discussion at the end of [**Gro-e**, §II.3, p. 98] gives a strict LF-space E and a non-closed subspace $F \subset E$ such that the intersection of F with each Fréchet subspace $E_0 \subset E$ is closed in E_0. The equivalence in Theorem C.1.5.11 with F merely assumed to be step-wise closed is [**Wngr-2**, §3, Cor. 3.4, p. 254], where each Fréchet E_m is assumed merely to be Schwartz (which implies nuclearity by Example C.1.5.10).

REMARK C.1.5.12. If E is a strict countable inductive limit of nuclear Fréchet spaces E_m as in Theorem C.1.5.11(1) then *every* strict defining sequence $\{E'_m\}$ for E has all terms E'_m nuclear since (i) each $E'_m \to E$ is a topological embedding by strictness, (ii) nuclearity is inherited by both strict countable inductive limits (so E is nuclear) and subspaces (such as $E'_m \subset E$) by [**Scha**, Ch. III, 7.4].

COROLLARY C.1.5.13. *Let E be a strict countable inductive limit of nuclear Fréchet spaces, and $F \subset E$ a closed subspace. Assume E/F (with the quotient topology) is also a strict countable inductive limit of a countable sequence of nuclear Fréchet spaces.*

The quotient map $q : E \to E/F$ lifts bounded subsets, so the closed subspace $\ker(q) \subset E$ *is a limit subspace and hence* $\ker(q)$ *is a strict countable inductive limit of nuclear Fréchet spaces.*

The proof of [**FlM**, §5.5, Corollary, p. 110] inspires the argument given below.

PROOF. Once it is shown that q satisfies the bounded lifting property, the rest follows from Theorem C.1.5.11. By hypothesis, the Hausdorff quotient E/F is the locally convex inductive limit of an increasing sequence of nuclear Fréchet subspaces $\{Q_m\}_{m\geq 0}$, and E is the locally convex inductive limit of an increasing sequence of nuclear Fréchet spaces $\{E_m\}_{m\geq 0}$. Define $P_m := E_m/(\ker(q) \cap E_m)$ equipped with the quotient topology inherited from E_m.

We do not (yet) know if the continuous linear injections $P_m \to P_{m+1}$ are topological embeddings, and in the absence of such strictness for $\{P_m\}$ we don't (yet) know the locally convex topological vector space $P := \varinjlim P_m$ is Hausdorff (a property that is required in the definition of "LF-space"). But we can directly verify the Hausdorff property: the natural linear bijection $f : P \to E/F$ is continuous and has Hausdorff target, so $f^{-1}(0) = \{0\}$ is closed in P, ensuring the topological group P is Hausdorff. Thus, P is an LF-space, so the Open Mapping Theorem for LF-spaces (namely Theorem C.1.5.5) can be applied to conclude that f is a topological isomorphism. Hence, for each $k \geq 0$ we can apply Grothendieck's Factorization Theorem (namely Lemma C.1.5.4) to $Q_k \hookrightarrow E/F \simeq P$ to conclude that it factors continuously through $P_{m(k)} \to P \simeq E/F$ for some $m(k) \geq 0$.

Let $B \subset E/F$ be a bounded subset. Since E/F is the strict inductive limit of the complete Q_k's (with each topological embedding $Q_k \to Q_{k+1}$ closed due to completeness of the Fréchet spaces Q_k), for sufficiently large k we have $B \subset Q_k$ with B is bounded in Q_k [**Scha**, Ch. II, 6.5]. The image of B under the continuous $Q_k \to P_{m(k)}$ (which we also denote as B) is therefore bounded in $P_{m(k)}$. Since nuclear spaces are Schwartz (Example C.1.5.10), such as the Fréchet E_m's, by [**Gro-e**, II.4, Thm. 12, p. 119] for any closed subspace $M \subset E_{m(k)}$ the quotient map $E_{m(k)} \to E_{m(k)}/M$ lifts bounded subsets. Applying this to $M = \ker(q) \cap E_{m(k)} = F \cap E_{m(k)}$ (closed in $E_{m(k)}$ since $F \to E$ is a closed embedding and each $E_m \to E$ is a topological embedding by strictness of $\{E_m\}$), there exists a bounded subset $B' \subset E_{m(k)}$ whose image in $E_{m(k)}/M = P_{m(k)}$ contains B. The image of B' under the continuous map $E_{m(k)} \to E$ is bounded, and that bounded subset of E clearly has image in E/F containing the bounded $B \subset E/F$, so we are done. □

C.1.6. Duality and inductive limits. We record here for later use throughout this work a basic fact concerning the topological dual of certain inductive limits:

LEMMA C.1.6.1 (Topological duals of inductive limits). *Let $\{j_{m,m'} : E_m \to E_{m'}\}$ be a countable inductive system of Hausdorff locally convex topological* **C***-vector spaces, with locally convex inductive limit E and each $j_{m,m'}$ injective. Assume E is Hausdorff (as is automatic when the given inductive system is strict; i.e., the maps $j_{m,m'}$ are topological embeddings).*

(1) *The natural linear map $q : E' \to \varprojlim_{m \geq 0}(E_m)'$ is a linear topological isomorphism with respect to the weak topology $\sigma(E', E)$ on the source, and the projective limit topology of the weak topologies $\sigma(E'_m, E_m)$ on the target.*

(2) *If for every bounded subset $B \subset E$ there exists an integer $m \geq 0$ such that $B \subset E_m$ inside E and B is bounded inside E_m (so the same then holds for all $m' \geq m$) then q in* (1) *is a linear topological isomorphism when using the strong dual topologies.*

The hypothesis in (2) holds whenever $\{E_m\}$ is strict with each $E_m \to E_{m+1}$ a closed embedding (such as happens when each E_m is complete), by [**Scha**, Ch. II, 6.5]. By definition, the strong topology $\beta(F', F)$ on the topological dual F' of a Hausdorff locally convex topological vector space is uniform convergence on bounded subsets of F.

PROOF. (1) : The bijectivity of q is immediate from the universal mapping property of locally convex inductive limits, and q is continuous by the universal property of locally convex projective limits and the functoriality of $\sigma(F', F)$ in Hausdorff locally convex topological vector spaces F. To show q is a homeomorphism, by the Hausdorff condition on source and target it suffices to show that q identifies convergent nets on both sides (a subset U of a Hausdorff topological space T is open if and only if for every convergent net $\{t_i\}$ in T with limit in U there exists i_0 such that $t_i \in U$ for all $i \geq i_0$).

More specifically, if $\{\ell_i\}$ is a net in E' whose image under q converges for the projective limit of weak topologies on the duals E'_m then we want to show $\{\ell_i\}$ converges in E' with respect to $\sigma(E', E)$. Letting $\ell \in E'$ correspond to that limit, we can pass to $\ell_i - \ell$ so that $\ell_i|_{E_m} \to 0$ weakly in E'_m for all m and we want

to conclude that $\ell_i \to 0$ weakly in E'. In other words, for each $x \in E$ we want $\ell_i(x) \to 0$. But $x \in E_{m_0}$ for some m_0, so since $\ell_i|_{E_{m_0}} \to 0$ we are done.

(2) : Functoriality of the strong topology ensures that the linear bijection $q : E'_\beta \to \varprojlim (E'_m)_\beta$ is continuous. Exactly as we handled the weak topology in the proof of (1) using that each point in E occurs inside some E_m, the hypothesis that every bounded subset of E arises from a bounded subset of some E_m allows us to use the same argument as in (1) to conclude (the role of the choice of $x \in E$ is replaced by the choice of a bounded subset $B \subset E$). □

REMARK C.1.6.2. If in Lemma C.1.6.1(2) the inductive system were to satisfy the weaker condition that each bounded subset $B \subset E$ is contained in the closure of a bounded subset of some E_m then the conclusion still holds (the same proof goes through), and conversely the conclusion implies that this weaker condition on bounded subsets of E must hold: see [**BKMcK**, §2, Theorem, p. 190].

C.2. Complements on mapping spaces, topological tensor products, and nuclearity

Topological tensor products and topologized spaces of continuous linear maps are used to build many constructions of interest, so we discuss many results and concepts related to such ideas in this section for use in the main text.

C.2.1. Mapping spaces. We fix topological $\mathbf{C}$-vector spaces E and F. For any set X, we denote by F^X the set-theoretic product

$$F^X := \prod_{u \in X} F = \mathrm{Map}(X, F)$$

where $\mathrm{Map}(X, F)$ denotes the set of all set-functions $X \to F$. We will ultimately be most interested in the subset of F^E consisting of continuous linear mappings $E \to F$, but the generality of F^X is convenient at the outset.

As in [**Scha**, Ch. III, §3] we denote by $\mathfrak{S}$ a collection of subsets of E that is cofiltered under upwards inclusion. For any $T \in \mathfrak{S}$ and subset $V \subset F$, define

$$U(T, V) := \{f \in F^E : f(T) \subset V\}. \tag{C.2.1.1}$$

As explained in [**Scha**, Ch.III, §3, p. 79,(*)], if $\mathfrak{B}$ is neighborhood base at 0 for F then the collection

$$\mathcal{U}(\mathfrak{S}, \mathfrak{B}) := \{U(T, V)\}_{(T,V) \in \mathfrak{S} \times \mathfrak{B}} \tag{C.2.1.2}$$

is a neighborhood base at 0 for a unique topology on F^E that moreover makes translation by each element of F is continuous, and this topology is independent of the choice of $\mathfrak{B}$. This is called the $\mathfrak{S}$-*topology* on F^E, or the *topology of uniform convergence on members of* $\mathfrak{S}$.

The $\mathfrak{S}$-topology on F^E is clearly unaffected by passage to a co-final subcollection within $\mathfrak{S}$. By [**Scha**, Ch.III, 3.1], the $\mathfrak{S}$-*topology* on F^E promotes a vector subspace $L \subset F^E$ to a topological vector space if and only if the image $f(T) \subset F$ is bounded for every $f \in L$ and every $T \in \mathfrak{S}$. The following result is [**Scha**, Ch.III, 3.2], recalling that "locally convex topological vector space" in [**Scha**] is always assumed to be Hausdorff by definition (see [**Scha**, Ch.II, §4].

LEMMA C.2.1.1. *Let F be a Hausdorff locally convex topological $\mathbf{C}$-vector space, X a topological space, and $\mathfrak{S}$ a cofiltered family of subsets of X whose union is dense in X. If a linear subspace $L \subset F^X$ satisfies*

(1) *every $f \in L$ is continuous as a function $f : X \to F$,*
(2) *for every $T \in \mathfrak{S}$, $f(T)$ is bounded in F (equivalently, L is a topological vector space for the natural F-invariant topology on F^X)*

then L is a Hausdorff locally convex topological $\mathbf{C}$-vector space.

The following special case is [**Scha**, Ch.III, Cor. to 3.2].

LEMMA C.2.1.2. *Let E and F be Hausdorff topological $\mathbf{C}$-vector spaces with F locally convex, and let $\mathfrak{S}$ a family of bounded subsets of E whose union linearly generates a dense subspace of E. The $\mathfrak{S}$-topology on the $\mathbf{C}$-vector space $\mathscr{L}(E,F)$ of continuous linear maps from E to F is a Hausdorff locally convex $\mathbf{C}$-vector space.*

We denote by

$$\mathscr{L}_{\mathfrak{S}}(E,F) \tag{C.2.1.3}$$

the $\mathbf{C}$-linear space $\mathscr{L}(E,F)$ topologized with the $\mathfrak{S}$-topology. Its local convexity is made explicit as follows: if $\{\nu_i\}_{i\in I}$ is a family of seminorms generating the Hausdorff locally convex topology on F then the family of seminorms on $\mathscr{L}(E,F)$ defined by

$$q_{T,i}(f) := \sup_{x\in T} \nu_i(f(x)) \tag{C.2.1.4}$$

for $(T,i) \in \mathfrak{S} \times I$ generates the $\mathfrak{S}$-topology on $\mathscr{L}(E,F)$: see [**Scha**, Ch.III, p. 81]. Note in particular that (in contrast with the theory of Banach spaces) there is *no preferred topology* on $\mathscr{L}(E,F)$, not even when $F = \mathbf{C}$.

EXAMPLE C.2.1.3. Topologies on continuous linear mapping spaces of the form discussed above are ubiquitous and underlie duality theory in locally convex topological $\mathbf{C}$-vector spaces.

DEFINITION C.2.1.4. For a Hausdorff locally convex topological $\mathbf{C}$-vector space E, the space $\mathscr{L}(E,\mathbf{C})$ endowed with the $\mathfrak{S}$-topology for a cofiltered collection $\mathfrak{S}$ of bounded subsets of E linearly generating a dense subspace of E is denoted $E'_{\mathfrak{S}}$ and is often called the $\mathfrak{S}$-*dual* of E. For instance:

- If $\mathfrak{S}$ is the collection of all finite subsets of E then $E'_{\mathfrak{S}}$ is the dual space E' equipped with the weak topology $\sigma(E',E)$ and is denoted E'_σ; this is called the *weak dual* of E.
- If $\mathfrak{S}$ is the collection of all pre-compact subsets of E (those with compact closure in $\widehat{E}$) then $E'_{\mathfrak{S}}$ is denoted E'_γ.
- $\mathfrak{S}$ is the collection of all bounded subsets of E then $E'_{\mathfrak{S}}$ is denoted E'_β, called the *strong dual* of E.

Strong duals are ubiquitous in the theory of distributions.

A very useful fact is that the weak closure and the topological closure of any convex subset of E coincide [**Scha**, Ch. IV, 3.1] More generally, we record below some terminology and notation.

DEFINITION C.2.1.5. Let E,F be Hausdorff locally convex topological $\mathbf{C}$-vector spaces, and $\mathfrak{S}$ a family of bounded subsets of E whose union linearly generates a dense subset of E. We denote by $\mathscr{L}_{\mathfrak{S}}(E,F)$ the $\mathbf{C}$-vector space $\mathscr{L}(E,F)$ of continuous linear maps equipped with the $\mathfrak{S}$-topology. As special cases:

(1) $\mathscr{L}_\sigma(E,F)$ if $\mathfrak{S}$ is the collection of all finite subsets of E. The $\mathfrak{S}$-topology on $\mathscr{L}_\sigma(E,F)$ is called the topology of *simple* (or *pointwise*) *convergence*.
(2) $\mathscr{L}_\gamma(E,F)$ if $\mathfrak{S}$ is the collection of all pre-compact subsets of E. The $\mathfrak{S}$-topology on $\mathscr{L}_\gamma(E,F)$ is called the topology of *pre-compact convergence*.
(3) $\mathscr{L}_\beta(E,F)$ if $\mathfrak{S}$ is the collection of all bounded subsets of E. The $\mathfrak{S}$-topology on $\mathscr{L}_\beta(E,F)$ is called the topology of *bounded convergence*.

Case (2) is relevant to traces for nuclear operators between Hausdorff locally convex spaces, as discussed in §C.2.4.5.

REMARK C.2.1.6. If E and F are normed spaces then the norm

$$\|f\| := \sup_{\|u\|_E \le 1} \|f(u)\|_F < \infty$$

generates the topology of bounded convergence on $\mathscr{L}_\beta(E,F)$. When $\mathfrak{S}$ is chosen to be the collection of all finite subsets of E then the $\mathfrak{S}$-topology of pointwise convergence on $\mathscr{L}(E,F)$ is the locally convex topology

generated by the subfamily of the seminorms from (C.2.1.4) given by $q_{\{u\},i}$ for $u \in E$ and $\{\nu_i\}_{i\in I}$ a family of seminorms generating the locally convex topology on F. If $E = F$ is a Hilbert space then this is called the *strong operator topology* on $\mathscr{B}(E)$. In other words,

$$\mathscr{L}_\sigma(E) := \mathscr{L}_\sigma(E,E) = \mathscr{B}_{\mathrm{SOT}}(E)$$

when E is a Hilbert space.

An important property of $\mathfrak{S}$-topologies is that a subset of $\mathscr{L}(E,F)$ that is bounded with respect to the topology of pointwise convergence is bounded with respect to any $\mathfrak{S}$-topology on $\mathscr{L}(E,F)$ so long as E is quasi-complete and $\mathfrak{S}$ is a cofiltered collection of bounded subsets of E whose union is E (i.e. not merely linearly generating a dense subspace): see [**Scha**, Ch.III, Cor. to 3.4]. The following result (see [**Scha**, Ch.III, Cor. to 4.4]) is important for later use.[6]

LEMMA C.2.1.7 (Quasi-completeness of mapping spaces). *Let E, F be Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces. Assume:*

(1) *E is barreled.*
(2) *F is quasi-complete.*
(3) *$\mathfrak{S}$ is a cofiltered collection of bounded subsets of E whose union is E.*

Then $\mathscr{L}_{\mathfrak{S}}(E,F)$ is quasi-complete.

REMARK C.2.1.8. If $E = F = H$ is a Hilbert space and $\mathfrak{S}$ is the collection of all finite subsets of H respectively the collection of all bounded subsets of H then Lemma C.2.1.7 implies quasi-completeness of $\mathscr{B}_{\mathrm{SOT}}(H)$ respectively $\mathscr{B}_{\mathrm{NT}}(H)$ (operator-norm topology). Indeed, every Hilbert space (and even every Fréchet space) is barrelled [**Scha**, Ch.II, Cor. to 7.1]) and complete. Of course, as is well-known, $\mathscr{B}_{\mathrm{NT}}(H)$ is even complete.

Let E, F be two Hausdorff locally convex topological $\mathbf{C}$-vector spaces. On $\mathscr{L}(E,F)$, the topology of uniform convergence on the bounded subsets of E will be of particular importance, because it is a flexible topology to define distributions. Specifically, E will be chosen to be a "familiar" locally convex topological linear space (of functions), and F will be chosen to be an operator algebra with topologies of varying degrees of strength; $\mathscr{L}_\beta(E,F)$ will be a space of "F-valued distributions".

In order to study continuous linear mapping spaces topologized with a suitable $\mathfrak{S}$-topology, we need two topologies on tensor products of Hausdorff locally convex topological $\mathbf{C}$-vector spaces. In order to discuss the first, we now recall some terminology on equicontinuity. A subset $H \subset \mathscr{L}(E,F)$ is called *equicontinuous* if and only if for every 0-neighborhood $V \subset F$ there exists a 0-neighborhood $U \subset E$ such that $f(U) \subset V$ for every $f \in H$. We refer the reader to [**Scha**, Ch. III, §4] for more on equicontinuity and the Uniform Boundedness Principle. Important properties of equicontinuous subsets in $\mathscr{L}(E,F)$ are the following (see [**Scha**, Ch. III, Cor. to 4.1, 4.5]):

LEMMA C.2.1.9. *Let E, F be Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces, $H \subset \mathscr{L}(E,F)$ an equicontinuous subset.*

(1) *H is bounded in $\mathscr{L}_{\mathfrak{S}}(E,F)$ for $\mathfrak{S}$ any collection of bounded subsets of E whose union linearly generates a dense subspace of E.*
(2) *The subspace topologies on H inherited from either $\mathscr{L}_\sigma(E,F)$ or $\mathscr{L}_\gamma(E,F)$ are the same.*

[6] Give some brief indications of where or how it is "important": for some Pettis integration or some weak-compactness result or what?

C.2.2. Topologies of a duality and the equicontinuous compactology. We will say a few words on bornologies and on topologies associated to certain bilinear pairings (in effect, convergence conditions via linear functionals), explaining the role that the main results on them serve throughout our work.

C.2.2.1. *Generation of polar topologies: bornologies and topologies of a duality.* Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. A *bornology on* E is a non-empty collection of subsets of E, cofiltered by upward-inclusion, such that each of its members is bounded with respect to the $\sigma(E, E')$-topology. Note that boundedness in the given topology on E and in the $\sigma(E, E')$-topology are the same [**Scha**, Ch. IV, 3.3]. The defining condition of a bornology on E is that for each $B \in \mathfrak{S}$ and $\lambda \in \mathbf{C}^\times$ there exists $C \in \mathfrak{S}$ such that $\lambda B \subset C$. All the $\mathfrak{S}$-topologies defined in Definition C.2.1.5 are induced by bornologies.

As in [**Scha**, Ch. IV, §1], we now consider a bi-linear pairing

$$\langle \cdot, \cdot \rangle : E \times F \to \mathbf{C} \tag{C.2.2.1}$$

with E, F two Hausdorff locally convex topological $\mathbf{C}$-vector spaces. We assume that the pairing $\langle \cdot, \cdot \rangle$ is *separating*: if $x \in E$ and $\langle x, \cdot \rangle = 0$ as a linear functional on F then $x = 0$, and similarly for $\langle \cdot, y \rangle$ as a linear functional on E. The notation $\langle E, F \rangle$ refers to the datum $(E, F, \langle \cdot, \cdot \rangle)$.

The datum of a separating bi-linear pairing $g : E \times F \to \mathbf{C}$ leads to the definition of the weak topology $\sigma(E, F)$ on E and the weak topology $\sigma(F, E)$ on F, respectively defined as the topology of pointwise convergence with respect to the family of linear maps $\{g(\cdot, y) : y \in F\}$ on E and $\{g(x, \cdot) : x \in E\}$ on F respectively. This leads to the concepts of weak boundedness and weak compactness in E, F, which underlie other topologies associated to $\langle E, F \rangle$ via the following construction.

For $A \subset F$ a subset, we define its *polar set* in E with respect to the duality $\langle E, F \rangle$ to be

$$A^\circ := \{x \in E : |\langle x, y \rangle_{E,F}| \leq 1, \text{ for all } y \in A\} \subset E \tag{C.2.2.2}$$

(this is called the *absolute polar* of A in [**Scha**, Ch. IV, p. 125], where *polar* is defined using $\mathrm{Re}(\langle x, y \rangle)$ as is also done in [**Bou-TVS**, Ch. II, §8, No. 4, Def. 1, p. II.64]; the two concepts agree when A is circled), and similarly for any subset $A \subset E$ we define its *polar set* in F with respect to the duality $\langle E, F \rangle$ to be

$$A^\circ := \{y \in F : |\langle x, y \rangle_{E,F}| \leq 1, \text{ for all } x \in A\} \subset F. \tag{C.2.2.3}$$

Polars provide a general recipe to produce 0-neighborhoods in either one of the two linear spaces E, F in a duality $\langle E, F \rangle$. Note that if $V \subset E$ a vector subspace, the subset $V^\circ \subset F$ is the orthogonal (annihilator) $V^\perp = \mathrm{ann}_F(V)$. The reader may want to compare the construction and use of polars with that of orthogonals/annihilators from the structure theory of Hausdorff locally compact abelian groups, which is used in Appendix D to construct and study Schwartz-Bruhat spaces of functions on Hausdorff locally compact abelian groups. When taking F to be the algebraic dual E^* of E we denote by $A^\bullet$ the polar sets in either direction with respect to the pairing $\langle \cdot, \cdot \rangle_{E,E^*}$.

We can use polars in E^* of 0-neighborhoods of E to identify the continuous dual of E within E^*:

$$E' = \bigcup_{U \in \mathcal{U}_0(E)} U^\bullet \subset E^* \tag{C.2.2.4}$$

where $\mathcal{U}_0(E)$ is a 0-neighborhood base in E. The meaning of (C.2.2.4) is that a linear functional on E is continuous if and only if it is bounded on a 0-neighborhood in E.

The datum of a duality $\langle E, F \rangle$ leads to an axiomatization of the robust theory of Banach spaces and their duals, with generalization of the main concepts (such as reflexivity) to the case of general Hausdorff locally convex topological $\mathbf{C}$-vector spaces. A key basic result is:

THEOREM C.2.2.1 (Bi-polar Theorem). *Let $\langle E, F \rangle$ be a duality. Let $A \subset E$ be a subset. Then:*

$$(A^\circ)^\circ = \overline{\mathrm{cch}}(A \cup \{0\}) \tag{C.2.2.5}$$

where the right side of (C.2.2.5) *is the* $\sigma(E,F)$*-closure of the convex circled hull of* A *in* E*.*

Our convention has been to define *polar* of $A \subset E$ to be $\{y \in F : |\langle x, y\rangle| \leq 1 \text{ for all } x \in A\}$ which some references call the *absolute polar*, in contrast with *real polar* $\{y \in F : \mathrm{Re}\,|\langle x, y\rangle \leq 1 \text{ for all } x \in A\}$. Since the absolute polar A° agrees with the real polar of the circled $A_b = D \cdot A$ for $D \subset \mathbf{C}$ the closed unit disk, Theorem C.2.2.1 for real polars as in [**Scha**, Ch. IV, 1.5] yields the result for absolute polars as stated above.

An efficient account of the basic properties of polars is given in [**Scha**, Ch. IV, §1] and [**NB**, Thms. 8.3.6, 8.3.7]. From the definition one sees that the polar set $A^\circ \subset F$ of a subset $A \subset E$ is always convex, circled and $\sigma(F,E)$-closed, and is absorbing (i.e. radial) if and only if A is $\sigma(E,F)$-bounded. This paves the way to defining 0-neighborhood bases within E respectively F via polars of subsets of F respectively E, and then using the Bi-polar Theorem as a dictionary to establish basic properties of such "polar" topologies.

To a directed collection of $\sigma(F,E)$-bounded subsets of F, we assign the corresponding collection of polars in E. The idea is to think of a $\sigma(F,E)$-bounded subset $B \subset F$ as a replacement for the unit ball in the normed dual of a normed space, and define the function:

$$p_B(x) := \sup_{y \in B} |\langle x, y\rangle_{E,F}|$$

a seminorm on E. The "unit ball" with respect to p_B is exactly the polar set $U := B^\circ$, and so long as the chosen class of $\sigma(F,E)$-bounded subsets of F satisfies basic stability properties, this construction leads to a 0-neighborhood base of a linear topology on E, with the polar sets B° playing the role of unit-balls. Polar topologies are therefore defined by using *polarity* to indicate which subsets are supposed to serve as unit-balls. This is the origin of the concept of *bornology*, whose definition we recalled at the beginning of this subsection.

C.2.2.2. *Consistent topologies.* We conclude this account of the main elements of the construction of polar topologies by highlighting a great advantage of the local convexity assumption on linear topologies of topological vector spaces. It affords flexibility on the concepts of boundedness, closure, and denseness in terms of refining the polar topology on E induced by a duality $\langle E, F\rangle$.

Specifically, certain properties of subsets of E defined in terms of a polar topology using F are independent of the choice of such topology so long as the polar topology is *consistent* with $\langle E, F\rangle$ in the sense that the space of continuous linear functionals on E with respect to that topology is exactly F. For example, the weak topology $\sigma(E,F)$ on E is always consistent with a given separating duality $g : E \times F \to \mathbf{C}$ (as is a given Hausdorff locally convex topology on E when $F = E'$ is the topological dual and g is the evaluation pairing).

A first important example of this phenomenon is the equivalence of boundedness and weak boundedness, as well as formation of closure, respectively weakly closure for convex sets in any Hausdorff locally convex topological $\mathbf{C}$-vector space: see [**Scha**, Ch. IV, 3.1, 3.3].

Although consistency is defined on all linear topologies on E, for Hausdorff locally convex topologies the extremes of such consistent topologies are known. This is the content of the Mackey-Arens Theorem [**Scha**, Ch. IV, 3.2-3.3]. It rests crucially on the local convexity assumption and ultimately on separations theorems [**Scha**, Ch. II, §9] relying upon the Hahn-Banach Theorem. The statement of the result is that such topologies are precisely those which are finer than $\sigma(E,F)$ and coarser than the *Mackey topology* $\tau(E,F)$ that is the $\mathfrak{S}$-topology on E for the collection $\mathfrak{S}$ of all subsets of F that are convex, circled and $\sigma(F,E)$-compact. (In [**Jar**], this topology is denoted using the letter μ rather than τ.)

In the absence of local convexity, control on polar topologies is lost: we refer the reader to the example of $\ell^p(\mathbf{N})$ for $0 < p < 1$ (see [**Voigt**, Ch. 5, Ex. 5.5(a), p. 38]). Most notable among the polar topologies that are not necessarily consistent is the strong topology $\beta(E,F)$ that is always stronger than the Mackey topology. By [**Scha**, Ch. IV, 5.2], the condition under which the strong-dual topology $\beta(E',E)$ agrees with the Mackey topology $\tau(E',E)$ is *barreledness* of E [**Scha**, Ch. II, §7]: this means that the barrels of E (closed, convex, circled and absorbing subsets, examples of which are provided by the polars of weakly bounded subsets) are

0-neighborhoods in E. Under such an additional condition, a tighter relation between equicontinuous and bounded subsets is attained in the form of a general version of the Uniform Boundedness Principle (see [**Scha**, Ch. III, 4.2-4.3]).

Further classes of Hausdorff locally convex topological **C**-vector spaces will be discussed as the need arises.

C.2.2.3. *Uniform convergence on the weakly closed convex and circled equicontinuous subsets of E'.* With an eye towards the discussion of ε-tensor products, and the proofs of several "kernel theorems" in Appendix D, we record the following result for which we refer the reader to [**Scha**, Ch. IV, Cor. 3 to 1.5] for a proof.

LEMMA C.2.2.2 (Topology of uniform convergence on the equicontinuous subsets of E'). *Let E be a Hausdorff locally convex topological* **C***-vector space, and $\mathcal{E}$ the collection of all equicontinuous subsets of E' that are convex, circled, and closed with respect to the $\sigma(E', E)$-topology on E'. Then the family of subsets of E*

$$\mathcal{U} := \{U^\circ : U \in \mathcal{E}\} \tag{C.2.2.6}$$

(where U° is the polar of U in E with respect to the duality $\langle E, E'\rangle$) is a 0-neighborhood base for the given topology on E.

The collection $\mathcal{E}$ from Lemma C.2.2.2 is usually referred to as the *equicontinuous compactology* on E'. The weak-closedness and convexity assumptions on U considered in (C.2.2.6) are harmless because every subset $U \subset E$ satisfies $U^\circ = U^{\circ\circ\circ}$ by [**Scha**, Ch. IV, Cor. 1 to 1.5] with $U^{\circ\circ}$ weakly closed and convex as described by the Bi-Polar Theorem.

The collection $\mathcal{E}$ will underlie the definition of the injective tensor product topology $\otimes_\varepsilon$ in §C.2.3.1 but via Lemma C.2.2.2 it also leads to a tool underpinning proofs of "kernel theorems" in use throughout this work (these are results which say that certain function spaces on $X \times Y$ coincide with a suitably completed tensor product of corresponding function spaces on X and on Y) and comparisons among multiple topologies on tensor products. In the presence of favorable conditions (such as nuclearity of at least one of the tensor factors) this includes criteria under which there is equality between the injective tensor product topology $\otimes_\varepsilon$ and the projective tensor product topology $\otimes_\pi$ (see §C.2.3.1 and §C.2.3.2). Such equality is a powerful feature because the injective topology is well-behaved for topological embeddings and the projective topology is well-behaved for quotient maps.

A prototypical such application occurs in the proof of the kernel theorem for the Schwartz space of Euclidean spaces: that the continuous linear map $\mathcal{S}(\mathbf{R}^n) \otimes_\pi \mathcal{S}(\mathbf{R}^m) \to \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ extends to a linear topological isomorphism $\mathcal{S}(\mathbf{R}^n) \widehat{\otimes}_\pi \mathcal{S}(\mathbf{R}^m) \to \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$. Nuclearity of the Schwartz spaces identifies injective and projective topologies on the tensor product, and then Lemma C.2.2.2 is the additional ingredient used to show that a net in $\mathcal{S}(\mathbf{R}^n) \otimes_\pi \mathcal{S}(\mathbf{R}^m)$ converges in $\mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ only if it converges in $\mathcal{S}(\mathbf{R}^n) \otimes_\pi \mathcal{S}(\mathbf{R}^m)$.

Among the advantages of the local convexity condition (most of which rest on the separation theorems which constitute forms of the Hahn-Banach Theorem) is the stability of important concepts and properties of subsets of Hausdorff locally convex topological **C**-vector spaces with respect to change of topology compatibly with a duality in the presence of local convexity (the bounds on such topologies being described by the Mackey-Arens Theorem). Another is the description of locally convex topologies by means of fundamental families of seminorms (see the discussion in §C.1.2), and the characterization of the given topology as the topology of uniform convergence on the closed convex circled equicontinuous subsets of the continuous dual.

C.2.3. Topological tensor products. Throughout this subsection, we provide an overview about three Hausdorff locally convex topologies on $E \otimes F$ for any two Hausdorff locally convex topological **C**-vector spaces E and F. These are the injective tensor product (also called the "ε-tensor product"), the projective tensor product (also called the "π-tensor product"), and the inductive tensor product. When $E \otimes F$ is equipped with such topologies, we denote the resulting locally convex topological vector space as $E \otimes_\varepsilon F$, $E \otimes_\pi F$, and

$E \otimes_\iota F$ respectively. The corresponding completions are denoted $E\widehat{\otimes}_\varepsilon F$, $E\widehat{\otimes}_\pi F$, and $E\overline{\otimes}F$ respectively. The inductive tensor product is the least-encountered in the published literature, but it serves an important role in our work.

C.2.3.1. *Injective tensor products by bi-equicontinuous convergence.* We start with the ε-topology, also called the "injective tensor product" because it interacts well with subspace inclusions [**Tre**, Prop. 43.7] (as we shall discuss). For Hausdorff locally convex topological vector spaces E_1 and E_2, we write $B(E_1, E_2)$ for the space of continuous bilinear forms $E_1 \times E_2 \to \mathbf{C}$ and $\mathscr{B}(E_1, E_2)$ for the space of separately continuous bilinear forms $E_1 \times E_2 \to \mathbf{C}$ (i.e., continuous in each entry when the other is fixed). By [**Tre**, Prop. 42.4], the canonical linear map

$$\rho : E \otimes F \to B(E'_\sigma, F'_\sigma) \tag{C.2.3.1}$$

uniquely determined on elementary tensors $x \otimes y$ by

$$\rho(x \otimes y) : (\varphi_E, \varphi_F) \mapsto \varphi_E(x)\varphi_F(y) \tag{C.2.3.2}$$

is a linear isomorphism.

Let $E' = \mathscr{L}(E, \mathbf{C})$ denote the topological dual of a Hausdorff locally convex topological vector space E, on which we have the *Mackey topology* $\tau = \tau(E', E)$ of uniform convergence on convex, circled, weakly-compact subsets of E.

Remark C.2.3.1. The Mackey topology on E' can be described in another way in an important class of examples. If E has the Heine-Borel property that all closed bounded subsets are compact (conversely, compact subsets in any Hausdorff topological vector space are always closed and bounded) then E'_τ clearly agrees with the topology E'_γ of pre-compact convergence. The most interesting class of E with the Heine-Borel property are *Montel spaces* [**Scha**, Ch. IV, p. 147], which (as is explained in Appendix D) includes many function spaces of interest to us, such as Schwartz spaces on the adelic ring and idele class group of a number field. All barreled quasi-complete nuclear spaces are Montel (by [**Tre**, Cor. 1 to Prop. 50.4] and [**Scha**, Ch. IV, 5.6]; nuclearity will be reviewed in §C.2.4).

For general Hausdorff locally convex topological $\mathbf{C}$-vector spaces E, F, we have a linear identification

$$\mathscr{L}(E'_\tau, F) = \mathscr{B}(E'_\sigma, F'_\sigma) \tag{C.2.3.3}$$

via the inverse operations

$$f \mapsto ((x', y') \mapsto y'(f(x'))), \quad B \mapsto (x' \mapsto B(x', \cdot) \in (F'_\sigma)' = F)$$

(that the second of these yields a continuous linear map $E'_\tau \to F$ is seen by considering the inclusion $E \otimes F \simeq B(E'_\sigma, F'_\sigma) \hookrightarrow \mathscr{B}(E'_\sigma, F'_\sigma)$). We endow the right side of (C.2.3.3) with the topology of uniform convergence on the products of equi-continuous subsets of E'_σ and F'_σ, and [**Tre**, Prop. 42.2] directly describes the corresponding topology on the left side (which it also identifies with $\mathscr{L}(E'_\sigma, F_\sigma)$); this topological vector space is denoted $\mathscr{L}_\varepsilon(E'_\tau, F)$. We call this topology on either side of (C.2.3.3) the topology of *bi-equicontinuous convergence*, and denote the space $\mathscr{B}(E'_\sigma, F'_\sigma)$ equipped with that topology as $\mathscr{B}_\varepsilon(E'_\sigma, F'_\sigma)$.

Following [**Tre**, Def. 43.1], we define

$$E \otimes_\varepsilon F \tag{C.2.3.4}$$

to be the $\mathbf{C}$-vector space $E \otimes F$ topologized with the subspace topology via $E \otimes F \overset{\rho}{\simeq} B(E'_\sigma, F'_\sigma) \subset \mathscr{B}_\varepsilon(E'_\sigma, F'_\sigma)$. The topological vector space $E \otimes_\varepsilon F$ is Hausdorff and locally convex by [**Scha**, Ch. III, Cor. to 3.2] applied to $\mathscr{L}_\varepsilon(E'_\tau, F)$. We denote by $E\widehat{\otimes}_\varepsilon F$ the completion of $E \otimes_\varepsilon F$.

REMARK C.2.3.2. [7] An alternative but equivalent way to define the ε-topology is used in [**Jar**, §16.3, pp. 343-4], so we discuss that here since for some results[8] on the ε-topology we refer to proofs in [**Jar**]. Let μ be the topology on E' given by uniform convergence on compact subsets of the completion $\widehat{E}$. If E is complete then μ is the topology γ of pre-compact convergence on E'. Thus, when E is complete and metrizable then by the Banach-Dieudonné Theorem [**Scha**, Ch. IV, 16.3] the topology μ on E' is the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on all equicontinuous subsets.

We endow the space of continuous linear maps $\mathscr{L}(E'_\mu, F)$ with the $\mathcal{E}$-topology where $\mathcal{E}$ is the collection of equicontinuous subsets of E' that are weakly-closed, convex and circled (the same $\mathcal{E}$ that features in Lemma C.2.2.2: the polar sets of whose members generate the initial topology on E). The canonical linear map

$$\rho_0 : E \otimes F \to \mathscr{L}_{\mathcal{E}}(E'_\mu, F) \tag{C.2.3.5}$$

uniquely determined on elementary tensors $x \otimes y$ by the assignment

$$\rho_0(x \otimes y)(\varphi) := \langle x, \varphi\rangle_{E,E'} \cdot y \tag{C.2.3.6}$$

is a linear injection and realizes $E \otimes F$ as the subspace of $\mathscr{L}(E'_\mu, F)$ of all finite-rank operators that are continuous with respect to the μ-topology on the source, and the $\sigma(F, F')$-topology on the target. This injection defines $E \otimes_\varepsilon F$ as the subspace topology in [**Jar**, 16.3].

REMARK C.2.3.3 (On the equicontinuous weak-$*$ topology). The topology μ from Remark C.2.3.2 is referred to as the *equicontinuous weak-$*$ topology*, and it enables the general treatment of those Hausdorff locally convex topological $\mathbf{C}$-vector spaces E whose weak dual E'_σ has its closed subsets characterized exactly as for the convex subsets of the weak dual of a Fréchet space in the Krein-Smulian Theorem [**Scha**, Ch. IV, 6.4]), leading to the class of Ptak (or B-complete) spaces ([**Scha**, Ch, IV, §8], [**Jar**, §9.5]). There is an Open Mapping Theorem for B-complete spaces [**Scha**, Ch. IV, Cor. 1 to 8.3].

We now describe the Hausdorff locally convex topology on $E \otimes_\varepsilon F$ in a more direct way, using the interpretation of equicontinuous subsets of E' in terms of polars U° for U varying through a 0-neighborhood base in E and similarly for F' and F. Let $\mathcal{U}, \mathcal{V}$ be 0-neighborhood bases in E, F, respectively. We may and do assume the members of $\mathcal{U}$ and $\mathcal{V}$ are closed convex and circled. For any $U \in \mathcal{U}$, $V \in \mathcal{V}$, we use (C.2.3.1) to define the following family of seminorms on the algebraic tensor product $E \otimes F$:

$$\varepsilon_{U,V}(f) := \sup\{|\rho(f)(\varphi, \psi)| : \varphi \in U^\circ \subset E', \psi \in V^\circ \subset F'\} < \infty \tag{C.2.3.7}$$

(finite since $\rho(f) \in B(E'_\sigma, F'_\sigma)$ with U° and V° each weakly bounded by design). By the definition of the ε-topology, the collection $\{\varepsilon_{U,V}\}$ as (U, V) varies over $\mathcal{U} \times \mathcal{V}$ is a family of seminorms on $E \otimes F$ generating the locally convex topology on $E \otimes_\varepsilon F$.

EXAMPLE C.2.3.4. Assume E and F are normed spaces, with norms $\|\cdot\|_E$ and $\|\cdot\|_F$. The linear inclusion

$$E \otimes F \simeq B(E'_\sigma, F'_\sigma) \subset \mathscr{B}(E'_\sigma, F'_\sigma) \simeq \mathscr{L}(E'_\tau, F) \tag{C.2.3.8}$$

is $x \otimes y \mapsto (\varphi \mapsto \varphi(x)y)$. For the closed unit balls $U = B_E$ and $V = B_F$, $\varepsilon_{U,V}$ is the norm induced on $E \otimes F$ under (C.2.3.8) by the operator norm on $\mathscr{L}(E'_\tau, F)$ (it is finite on members of this space of linear mappings since the strong-dual topology is finer than the Mackey topology). This norm on $E \otimes F$ therefore defines the topology on $E \otimes_\varepsilon F$.

We now collect some properties of the injective tensor product for later use. By (C.2.3.7) (or [**Tre**, Prop. 43.6]), if E_1, E_2 and F_1, F_2 are pairs of Hausdorff locally convex topological $\mathbf{C}$-vector spaces and $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are continuous linear maps, then linear map $T_1 \otimes T_2$ is continuous with respect

[7] This and the next Remark should ideally be removed, much too heavy on a reader. The purpose, if retained, must be much more clear.

[8] Really so necessary?

to the injective tensor product topology on source and target. We denote this continuous linear map whose underlying algebraic linear map is $T_1 \otimes T_2$ by

$$T_1 \otimes_\varepsilon T_2, \tag{C.2.3.9}$$

and denote by $T_1 \mathbin{\widehat{\otimes}}_\varepsilon T_2$ the unique continuous linear extension of $T_1 \otimes_\varepsilon T_2$ to the completions:

$$T_1 \mathbin{\widehat{\otimes}}_\varepsilon T_2 : E_1 \mathbin{\widehat{\otimes}}_\varepsilon E_2 \to F_1 \mathbin{\widehat{\otimes}}_\varepsilon F_2. \tag{C.2.3.10}$$

LEMMA C.2.3.5. *Let E_1, E_2 and F_1, F_2 be Hausdorff locally convex topological $\mathbf{C}$-vector spaces and let $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ be continuous linear maps.*

(1) *If T_1 and T_2 are topological embeddings then so are $T_1 \otimes_\varepsilon T_2$ and $T_1 \mathbin{\widehat{\otimes}}_\varepsilon T_2$. In particular, if E_1, E_2, F_1 and F_2 are normed and T_1, T_2 are isometric embeddings then so are $T_1 \otimes_\varepsilon T_2$ and $T_1 \mathbin{\widehat{\otimes}}_\varepsilon T_2$.*
(2) *If T_1, T_2 are topological embeddings with dense range then so is $T_1 \otimes_\varepsilon T_2$. In particular, the canonical continuous linear map*

$$E_1 \mathbin{\widehat{\otimes}}_\varepsilon E_2 \to \widehat{E}_1 \mathbin{\widehat{\otimes}}_\varepsilon \widehat{E}_2 \tag{C.2.3.11}$$

is a topological linear isomorphism.
(3) *The formation of injective tensor products is topologically associative.*

PROOF. For (1) it suffices to treat the case of completed tensor products, which is [**Tre**, Prop. 43.7]. The assertion in (3) is seen via the seminorm description in (C.2.3.7).

For (2), first observe that by (1) and Hahn-Banach, it suffices to check that a continuous linear form φ on $F_1 \widehat{\otimes}_\varepsilon F_2$ that vanishes on $E_1 \widehat{\otimes}_\varepsilon E_2$ must be 0. By continuity of φ and density of the image of any Hausdorff locally convex topological vector space in its completion, it suffices to prove the same on the uncompleted tensor product spaces. To carry out the latter, we note that for Hausdorff topological vector spaces E and F, it is explained at the start of [**Tre**, Ch. 49] that $(E \otimes_\varepsilon F)'$ is naturally a linear subspace of the space $B(E, F)$ of continuous bilinear forms $E \times F \to \mathbf{C}$. We want to show that any continuous linear $\ell : F_1 \otimes_\varepsilon F_2 \to \mathbf{C}$ vanishing on $E_1 \otimes E_2$ must be 0, and the bilinear $B : F_1 \times F_2 \to \mathbf{C}$ corresponding to ℓ is continuous yet vanishes on $E_1 \times E_2$, so clearly $B = 0$. □

LEMMA C.2.3.6. *Let I, J be sets, E_i for $i \in I$, F_j for $j \in J$, Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Then*

(1) *If I and J are finite then the canonical continuous linear map*

$$\left(\prod_{i\in I} E_i\right) \otimes_\varepsilon \left(\prod_{j\in J} F_j\right) \to \prod_{(i,j)\in I\times J} E_i \otimes_\varepsilon F_j \tag{C.2.3.12}$$

is a linear topological isomorphism.
(2) *The canonical continuous linear map*

$$\left(\prod_{i\in I} E_i\right) \mathbin{\widehat{\otimes}}_\varepsilon \left(\prod_{j\in J} F_j\right) \to \prod_{(i,j)\in I\times J} E_i \mathbin{\widehat{\otimes}}_\varepsilon F_j \tag{C.2.3.13}$$

is a linear topological isomorphism.

PROOF. The assertion in (1) is immediate from the definition of $\otimes_\varepsilon$ and the passage of finite products through weak-duals and spaces of separately continuous bilinear forms. For (2), see [**Jar**, §16.3,Thm.1].[9] □

[9]The proof of this looks short; can a proof be given here to save the reader from [**Jar**]?

C.2.3.2. *Projective tensor products.* Another topology on tensor products of Hausdorff locally convex topological **C**-vector spaces we need is the projective tensor product topology, also called the π-topology. For E, F two Hausdorff locally convex complex vector spaces, there exists on the algebraic tensor product $E \otimes F$ a unique locally convex topology such that for any locally convex complex vector space G, continuous linear maps $E \otimes F \to G$ correspond bijectively to *continuous* (i.e. not merely separately continuous) bilinear maps $E \times F \to G$. This topology is referred to as the *projective* tensor product topology on $E \otimes F$, and when $E \otimes F$ is equipped with this topology it is denoted

$$E \otimes_\pi F. \tag{C.2.3.14}$$

The space of $B(E, F)$ of continuous bilinear forms on $E \times F$ is linearly identified with the continuous dual of $E \otimes_\pi F$.

By [**Tre**, Ch. 43] (or [**Scha**, Ch. III, pp. 93-94]), a fundamental system of neighborhoods of 0 for this topology on $E \otimes F$ is given by the convex circled hulls of the sets of the form

$$U_i \otimes V_j = \{u_i \otimes v_j : u_i \in U_i, v_j \in V_j\} \tag{C.2.3.15}$$

where U_i and V_j vary through fundamental systems of neighborhoods of 0 in E, respectively F. This description of a base of 0-neighborhoods clearly implies the associativity of the π-topology for tensor products of finitely many Hausdorff locally convex spaces.

The projective tensor product of two Hausdorff locally convex topological **C**-vector spaces is Hausdorff [**Tre**, Prop. 43.3]. The completion of $E \otimes F$ with respect to the projective topology is denoted $E \mathbin{\widehat{\otimes}}_\pi F$. Its formation is compatible with completion of the factors by [**Tre**, Prop. 43.9] and the universal property of completion.

The analog of Lemma C.2.3.5 for projective tensor products holds by [**Jar**, §15.4, Thm. 1, p. 332]).[10]

By [**Scha**, Ch. III, 6.4], in the special case when the tensor factors E, F of the projective tensor product $E \mathbin{\widehat{\otimes}}_\pi F$ are Fréchet spaces, the general element of $E \mathbin{\widehat{\otimes}}_\pi F$ can be written in the form

$$\sum_{k \geq 1} \lambda_k \, x_k \otimes y_k \tag{C.2.3.16}$$

where $\{\lambda_k\}$ is an absolutely summable sequence in **C** and $\{x_k\}_{k \geq 1}$, $\{y_k\}_{k \geq 1}$ are sequences in E, F respectively that converge to 0 (it is readily checked that conversely all such infinite series converge within $E \mathbin{\widehat{\otimes}}_\pi F$). This result due to Grothendieck is extremely useful when carrying out calculations, as we shall see (for example) in Appendix D.

Of great importance is the good behavior of $\otimes_\pi$ with respect to quotient maps. If $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are quotient maps between Hausdorff locally convex topological vector spaces then $T_1 \otimes_\pi T_2$ is clearly a quotient map by (C.2.3.15), but the analogue for the completions is more subtle [**Tre**, Prop. 43.9]:

PROPOSITION C.2.3.7. *If $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are quotient maps between Hausdorff locally convex topological vector spaces then $T_1 \mathbin{\widehat{\otimes}}_\pi T_2$ is a quotient map onto its image (given the subspace topology), and it is surjective when E_1 and E_2 are metrizable.*

The surjectivity of $T_1 \mathbin{\widehat{\otimes}}_\pi T_2$ can fail more generally (when it is not assumed that E_1 and E_2 are metrizable). The projective tensor product topology is finer than the injective tensor product topology: the identity map $\rho : E \otimes_\pi F \to E \otimes_\varepsilon F$ on $E \otimes F$ is continuous [**Tre**, Prop. 43.5]. When one of E or F is nuclear (a topic to be discussed in §C.2.4) then the linear bijection $\rho : E \otimes_\pi F \to E \otimes_\varepsilon F$ is a homeomorphism, so in such cases $E \mathbin{\widehat{\otimes}}_\pi F$ inherits the good properties of $E \mathbin{\widehat{\otimes}}_\varepsilon F$ with respect to subspace inclusions in F.

REMARK C.2.3.8 (On the equicontinuous subsets of $(E \otimes_\varepsilon F)'$). We now record a useful observation from the paragraph following [**Tre**, Def. 49.1, p.500] that provides a description of equicontinuous subsets of the

[10] Is this sentence really necessary? If so, why? If not, remove.

topological dual of $E \otimes_\varepsilon F$ for Hausdorff locally convex topological $\mathbf{C}$-vector spaces E and F. By definition, a 0-neighborhood base of $E \otimes_\varepsilon F$ is given by polars $(A' \otimes B')^\circ$ for $A' \otimes B' := \{a' \otimes b' : a' \in A', b' \in B'\}$ with equicontinuous subsets $A' \subset E'_\sigma$ and $B' \subset F'_\sigma$. An equicontinuous subset of $(E \otimes_\varepsilon F)'$ is therefore exactly a subset of the *bipolar* $(A' \otimes B')^{\circ\circ}$ for equicontinuous $A' \subset E'_\sigma$ and $B' \subset F'_\sigma$.

This description of the equicontinuous subsets of $(E \otimes_\varepsilon F)'$ will be used when proving "kernel theorems" in Appendix D. (By Lemma C.2.2.2, the given topology on a Hausdorff locally convex topological $\mathbf{C}$-vector space is the same as the topology of uniform convergence on the weakly closed convex circled equicontinuous subsets of the topological dual. When applied to $E \otimes_\varepsilon F$ that gives us nothing new via the preceding bipolar description since $M^{\circ\circ\circ} = M^\circ$ for any subset M of a Hausdorff locally convex topological vector space [**Scha**, Ch IV, Cor. 1 to 1.5].)

The proofs of Lemma D.1.3.7, Lemma D.2.2.2, and [**Tre**, Thm. 51.6(51.3), p. 530] are typical arguments resting on the preceding description. It is particularly useful when E or F is nuclear because the resulting topological equality $E \otimes_\pi F = E \otimes_\varepsilon F$ in such cases (as noted above, and to be discussed further in §C.2.4) then relates the projective tensor product topology to a function space on a product.

Before we continue on and discuss a third (and final) tensor topology – the inductive topology – for which the only reference we are aware of in the literature is Grothendieck's AMS memoir [**Gro-b**] (see [**Gro-b**, Ch. I, §3, No. 1, p. 73]), we record an important result on the interaction of projective tensor products with inductive limits in one tensor factor provided the other tensor factor is normable.

PROPOSITION C.2.3.9 (Normability and preservation of inductive limits under projective tensor products). *Let I be a set, $\{E_i\}_{i\in I}$ an I-indexed family of Hausdorff locally convex topological $\mathbf{C}$-vector spaces, F a Hausdorff locally convex topological $\mathbf{C}$-vector space. Assume F is normable. Then*

(1) *the canonical continuous linear bijection*

$$\bigoplus_{i\in I}(E_i \otimes_\pi F) \to \left(\bigoplus_{i\in I} E_i\right) \otimes_\pi F \tag{C.2.3.17}$$

is a linear topological isomorphism. In particular, it extends to a linear topological isomorphism

$$\bigoplus_{i\in I}(E_i \mathbin{\widehat{\otimes}}_\pi F) \to \left(\bigoplus_{i\in I} E_i\right) \mathbin{\widehat{\otimes}}_\pi F. \tag{C.2.3.18}$$

(2) *If I is filtered then the canonical continuous linear bijection*

$$\varinjlim_{i\in I}(E_i \otimes_\pi F) \to \left(\varinjlim_{i\in I} E_i\right) \otimes_\pi F \tag{C.2.3.19}$$

is a linear topological isomorphism.

The second assertion in (1) follows from the first by passage to completions since an arbitrary locally convex direct sum of complete spaces is complete [**Scha**, Ch. II, 6.2]. The assertion in (2) is immediate from the first assertion in (1) via the construction of locally convex inductive limits as quotients of locally convex direct sums. The first assertion in (1) is [**Jar**, §15.5, Cor. 4(a), p. 334], and earlier [**Gro-b**, Ch. I, §3, No. 1, Cor. p. 47]. It rests on [**Gro-b**, Ch. I, §1, No. 3, Prop. 6(2), p. 46], the proof of which is a simple verification that the continuous duals of $\bigoplus_{i\in I}(E_i \otimes_\pi F)$ and $(\bigoplus_{i\in I} E_i) \otimes_\pi F$ have the same equicontinuous parts. This in turn comes down to the explicit description of a 0-neighborhood base for locally convex direct sums as given in [**Scha**, Ch. II, p. 55] (combined with the fact that since F is normable, the unit ball B is a bounded 0-neighborhood): that description enables proving that an I-indexed product of equicontinuous parts of continuous bi-linear forms on the $E_i \times F$'s is equicontinuous in the continuous dual of $(\bigoplus_{i\in I} E_i) \otimes_\pi F$ as well (by testing it on the closed convex circled hull of the union of $\{V_i \otimes B\}$ in $(\bigoplus_{i\in I} E_i) \otimes_\pi F$, with V_i a 0-neighborhood in E_i).

We will encounter throughout this work examples where even if F is nuclear Fréchet of infinite dimension (so never normable [**Tre**, Cor. 2 to Prop. 50.2]) and $I = \mathbf{N}$ with each E_i nuclear Fréchet and the transition maps are linear topological embeddings (hence with closed range), the continuous linear bijection (C.2.3.19) is not a topological isomorphism. This is a primary reason we will need to discuss the inductive tensor topology that is better-behaved with respect to inductive limits.

In the presence of normability, projective tensor products do behave well with respect to *countable* inductive limits:[11]

PROPOSITION C.2.3.10 (Countable inductive systems of normable tensor factors). *Let $\{E_k\}_{k\geq 0}$ and $\{F_k\}_{k\geq 0}$ be sequences of normable Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Let E and F be their respective locally convex inductive limits. The canonical continuous linear bijection*

$$\varinjlim_{(k,k')\geq(0,0)} E_k \otimes_\pi F_{k'} \to E \otimes_\pi F \tag{C.2.3.20}$$

(with $\mathbf{N}\times\mathbf{N}$ partially ordered via component-wise comparison) is a linear topological isomorphism.

For the convenience of the reader, we provide a direct argument below to bypass additional technical concepts that arise in various proofs one finds in the literature. It rests on a simple construction with 0-neighborhoods in locally convex direct sums that we gave in Lemma C.1.5.1.

PROOF OF PROPOSITION C.2.3.10. Since $\otimes_\pi$ takes quotient maps to quotient maps, in view of the construction of locally convex inductive limits as quotients of the corresponding direct sums it suffices to prove the analogous result for countable direct sums. By symmetry between the E_k's and $F_{k'}$'s in the normability hypotheses, it suffices to show that projective tensor product against $F := \bigoplus_{k\geq 0} F_k$ passes through countable locally convex direct sums of normable spaces.

In other words, we want to show that the inverse of the natural continuous linear bijection

$$\bigoplus_{k\geq 0}(E_k \otimes_\pi F) \to E \otimes_\pi F$$

is continuous. It is enough to show that members of a base of 0-neighborhoods of the source have images that are 0-neighborhoods in the target. We use the base of 0-neighborhoods $W = \mathrm{cch}(\bigcup_{k\geq 0}(U_k \otimes V_k))$ for 0-neighborhoods $U_k \subset E_k$ and $V_k \subset F$ for each $k \geq 0$. By Proposition C.1.5.1 applied to the V_k's in F, we can rewrite each $U_k \otimes V_k$ as $(1/\rho_k)U_k \otimes (\rho_k V_k)$ for suitable $\rho_k > 0$ so that all $\rho_k V_k$'s contain a common 0-neighborhood V in F. Thus, for $U_k' := (1/\rho_k)U_k$ we see that W contains $\mathrm{cch}(U' \otimes V)$ for the 0-neighborhood $U = \mathrm{cch}(\cup_{k\geq 0} U_k') \subset E$, so we are done. □

C.2.3.3. *Metrizability, and separately continuous and continuous bi-linear forms.* We next record a criterion to determine when a separately sequentially continuous bilinear map $E \times F \to G$ between Hausdorff locally convex topological $\mathbf{C}$-vector spaces E, F, G is sequentially continuous. When sequences are sufficient to probe the topologies of E and F, this will provide a mechanism for sometimes verifying that a given separately continuous bilinear mapping is continuous. For example, sometimes this can be used to show continuity of linear maps from projective tensor products, in view of the universal mapping property of projective tensor products. We refer the reader to the proof of [**Bou-TVS**, Ch. III, §5, no. 2, Prop. 2, p. III.29; Cor. 1, p. III.30] for similar arguments resting on the Uniform Boundedness Principle.

PROPOSITION C.2.3.11 (Sequential continuity of separately sequentially continuous bi-linear mappings). *Let E, F, G be Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Let:*

$$A : E \times F \to G \tag{C.2.3.21}$$

[11] Why is this result recorded, since it is never cited anywhere? Is it implicit somewhere else?

be a separately sequentially continuous bi-linear mapping. If E is Fréchet then A is sequentially continuous.

PROOF. Let $\{u_m\}_{m\geq 0} \subset E$ be a null-sequence in E (i.e., converging to 0), and $\{v_m\}_{m\geq 0} \subset F$ a convergent sequence in F. It suffices to show $A(u_m, v_m) \to 0$. By separate sequential continuity of A and E being a metric space, the linear operators $A(\cdot, v_m) : E \to G$ are continuous. Moreover, for every $u \in E$, the subset $\{A(u, v_m)\}_{m\geq 0} \subset G$ is bounded since $A(u, \cdot) : F \to G$ is continuous.

Thus, the sequence $\{A(\cdot, v_m)\}_{m\geq 0} \subset \mathscr{L}(E, G)$ is pointwise bouinded, so since E is Baire (as it is Fréchet), we see by [**Scha**, Ch. III, 4.2] that $\{A(\cdot, v_m)\}_{m\geq 0} \subset \mathscr{L}(E, G)$ is equicontinuous. Let $U \subset G$ be any 0-neighborhood. There therefore exists a 0-neighborhood $U_0 \subset E$ such that $A(U_0, v_m) \subset U$ for every integer $m \geq 0$. Being $\{u_m\}_{m\geq 0}$ a null-sequence, there exists $m_0 \geq 0$ such that for every integer $m \geq m_0$, we have $u_m \in U_0$. It follows that for every $m \geq m_0$, $A(u_m, v_m) \in U$, and hence $\lim_{m\to+\infty} A(u_m, v_m) = 0$. □

C.2.3.4. *Inductive tensor products.* One more locally convex topology on algebraic tensor products of Hausdorff locally convex topological **C**-vector spaces features throughout this work: the inductive tensor product topology. As literature references, we mention [**Scha**, Ch. III, p. 96] and [**Gro-b**, Ch. I, §3, No. 1, Def. 3, p. 74].

For E, F two Hausdorff locally convex topological **C**-vector spaces, we define on $E \otimes F$ the locally convex topology that is finest among those such that for any Hausdorff locally convex topological **C**-vector space G, the bilinear maps

$$E \times F \to G$$

that are separately continuous make the corresponding linear map $E \otimes F \to G$ be continuous. That such a Hausdorff locally convex topology on $E \otimes F$ indeed does exist is the content of [**Gro-b**, Ch. I, §3, No. 1, Prop. 13, p. 73].[12] We denote by

$$E \otimes_\iota F \tag{C.2.3.22}$$

the algebraic tensor product topologized in this manner, and refer to it as the *inductive tensor product* of E and F.

We claim $E \otimes_\iota F$ is Hausdorff. It suffices to show 0 is a closed point. The identity map on $E \otimes F$ is continuous as a bijective map of topological groups $E \otimes_\iota F \to E \otimes_\pi F$ because the continuous bilinear pairing $E \times F \to E \otimes_\pi F$ is separately continuous. Thus, the Hausdorff property of $E \otimes_\pi F$ gives the desired closedness of $\{0\}$ in $E \otimes_\iota F$. With the Hausdorff property established, there is a good notion of completion for $E \otimes_\iota F$, which we denote as

$$E \mathbin{\overline{\otimes}} F. \tag{C.2.3.23}$$

As a consequence of Proposition C.2.3.11 and universal mapping properties of the inductive and projective tensor product topologies, we have the following:

COROLLARY C.2.3.12. *Let E, F be two Fréchet spaces. Then the canonical continuous linear map*

$$E \otimes_\iota F \to E \otimes_\pi F \tag{C.2.3.24}$$

is a linear topological isomorphism. In particular, the induced continuous linear map $E \mathbin{\overline{\otimes}} F \to E \mathbin{\widehat{\otimes}}_\pi F$ between completions is a topological isomorphism.

We collect properties of inductive tensor products into the following:

PROPOSITION C.2.3.13 (Permanence properties of inductive tensor products). *Formation of inductive tensor products satisfies the following properties:*

[12] Maybe can insert a more readable explanation than Grothendieck gives?

(1) *Let I, J be sets, E_i for $i \in I$, F_j for $j \in J$, Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Assume that F is a Fréchet space. Then the canonical continuous linear map*

$$\left(\prod_{i\in I} E_i\right) \mathbin{\overline{\otimes}} F \to \prod_{i\in I}(E_i \mathbin{\overline{\otimes}} F) \tag{C.2.3.25}$$

is a linear topological isomorphism.

(2) *If I and J are filtered sets, and $\{E_i\}_{i\in I}$ and $\{F_i\}_{i\in J}$ two directed families of Hausdorff locally convex topological $\mathbf{C}$-vector spaces. We define $E := \varinjlim_{i\in I} E_i$ and $F := \varinjlim_{j\in J} F_j$ endowed with their locally convex inductive limit topologies, which we assume are Hausdorff.*[13]

The canonical continuous linear map

$$\varinjlim_{(i,j)\in I\times J} E_i \otimes_\iota F_j \to E \otimes_\iota F \tag{C.2.3.26}$$

is a linear topological isomorphism. The subspace topology inherited from $E \mathbin{\overline{\otimes}} F$ by the linear subspace linearly generated by the images of each $E_i \mathbin{\overline{\otimes}} F_j$ into $E \mathbin{\overline{\otimes}} F$ is the locally convex inductive limit of the $E_i \mathbin{\overline{\otimes}} F_j$'s. In other words, the canonical continuous linear map

$$\varinjlim_{(i,j)\in I\times J} E_i \mathbin{\overline{\otimes}} F_j \to E \mathbin{\overline{\otimes}} F \tag{C.2.3.27}$$

is a topological embedding (with dense image via (C.2.3.25)*).*

If I, J are countable and totally ordered, E_i, F_j are nuclear Fréchet spaces, and the resulting inductive systems are strict, then (C.2.3.27) *is a linear topological isomorphism.*

(3) *For Hausdorff locally convex topological $\mathbf{C}$-vector spaces E and F, if they are barrelled respectively bornological then so is $E \mathbin{\overline{\otimes}} F$.*

Nuclearity will be discussed in §C.2.4.

PROOF. Assertion (1) and all but the final assertion in (2) are [**Gro-b**, Ch. I, §3, No. 1, Prop. 14, p. 76]. For the last assertion in (2), we first note that since the topological embedding (C.2.3.27) has dense image, it suffices to check that the countable inductive limit on left side is complete. Since a countable strict inductive limit of complete Hausdorff locally convex spaces is Hausdorff and complete [**Scha**, Ch. II, 6.6], it suffices for the final assertion in (2) to verify that the countable inductive system $\{E_i \mathbin{\overline{\otimes}} F_j\}$ is strict. By Corollary C.2.3.12 it agrees with the countable inductive system $E_i \mathbin{\widehat{\otimes}}_\pi F_j$, and the nuclearity hypothesis ensures (by [**Tre**, Thm. 50.1(a)⇔(f)]) that this agrees with the inductive system $E_i \mathbin{\widehat{\otimes}}_\varepsilon F_j$. The latter is strict by Lemma C.2.3.5(1).

The proof of (3) is a consequence of the definitions (see also [**Gro-b**, Ch. I, §3, No. 2, p. 78]).[14] □

C.2.3.5. *Linear topological embeddings and injective tensor products in the presence of nuclearity.* The following result provides a sufficient condition for completed inductive tensor products to commute with linear topological embeddings (the analogous assertion for completed injective tensor products is Lemma C.2.3.5(1), which then gives the same for completed projective tensor products when one of the tensor factors is nuclear because in such cases $\mathbin{\widehat{\otimes}}_\varepsilon = \mathbin{\widehat{\otimes}}_\pi$ by [**Tre**, Thm. 50.1(a)⇔(f)] (nuclearity will be reviewed at length in §C.2.4; this aspect of nuclearity is all we will need about it before then, in addition to the fact that nuclearity is inherited by Hausdorff quotients [**Scha**, Ch. III, 7.4]).

[13] Inserted hypothesis here that E and F are Hausdorff, which should be harmless in practice, right?

[14] Must say more about an actual proof; not clear how it follows from "the definitions", and the reference to [**Gro-b**] merely says it is obvious.

PROPOSITION C.2.3.14 (Inductive tensor products and linear topological embeddings). *Let E_1, E_2, F be Hausdorff locally convex topological $\mathbf{C}$-vector spaces, and $E_1 \to E_2$ a linear topological embedding. Assume:*

(1) *E_1, E_2 are nuclear Fréchet spaces (so the given topological embedding $E_1 \to E_2$ is closed),*
(2) *F is a strict locally convex countable inductive limit of nuclear Fréchet spaces F_k.*

Then the continuous linear map

$$E_1 \mathbin{\overline{\otimes}} F \to E_2 \mathbin{\overline{\otimes}} F \tag{C.2.3.28}$$

is a linear topological embedding.

Beware that the result for the F_k's in place of F doesn't formally imply the result for F via passage to a locally convex inductive limit via the final assertion in Proposition C.2.3.13(2) because passage to locally convex inductive limits doesn't generally preserve the property of maps being topological embeddings (even when the given maps are closed embeddings at each layer). Such failure underlies the phenomenon of closed subspaces of LF-spaces that are not limit subspaces, as discussed in Example C.1.5.7. Thus, the proof has to use features specific to the given circumstances.

PROOF. We let $\{F_k\}_{k\geq 0}$ be a defining sequence of nuclear Fréchet spaces, with $F_k \subset F_{k+1}$ inclusions and linear topological embeddings for each integer $k \geq 0$, such that $\varinjlim_{k\geq 0} F_k = F$ linearly and topologically. Each $E_i \mathbin{\overline{\otimes}} F_k$ is a nuclear Fréchet space by Corollary C.2.3.12 combined with [**Tre**, (50.17)] and [**Scha**, Ch. III, 7.5; Ch. I, 6.1; Ch. IV, 5.2; Ch. II, Cor. to 7.1]. By the end of Proposition C.2.3.13(2) applied to $I = \{*\}$, $J = \mathbf{N}$, with $E_* = E_1, E_2$ respectively, we see that $E_i \mathbin{\overline{\otimes}} F = \varinjlim_k (E_i \mathbin{\overline{\otimes}} F_k)$ as a *strict* countable inductive limit of nuclear Fréchet spaces.

Each map $E_1 \mathbin{\overline{\otimes}} F_k \to E_2 \mathbin{\overline{\otimes}} F$ is a topological embedding because it is a composition

$$E_1 \mathbin{\overline{\otimes}} F_k \simeq E_1 \mathbin{\widehat{\otimes}_\pi} F_k \simeq E_1 \mathbin{\widehat{\otimes}_\varepsilon} F_k \hookrightarrow E_2 \mathbin{\widehat{\otimes}_\varepsilon} F_k = E_2 \mathbin{\overline{\otimes}} F_k \hookrightarrow E_2 \mathbin{\overline{\otimes}} F$$

(first isomorphism using Corollary C.2.3.12, the second using nuclearity of E_1, the middle embedding using Lemma C.2.3.5(1), and the final embedding as a general feature of strict countable locally convex inductive limits [**Scha**, Ch. II, 6.4]). We need to show that the continuous linear inclusion of the strict countable inductive limit of the terms on the left side into the right side is also a topological embedding.

For every integer $k \geq 0$, the complex of $\mathbf{C}$-linear spaces

$$0 \to E_1 \mathbin{\widehat{\otimes}_\pi} F_k \xrightarrow{j_k} E_2 \mathbin{\widehat{\otimes}_\pi} F_k \xrightarrow{q_k} (E_2/E_1) \mathbin{\widehat{\otimes}_\pi} F_k \to 0 \tag{C.2.3.29}$$

is topologically exact:

- the continuous linear map j_k is a topological embedding due to nuclearity of F_k identifying both $\widehat{\otimes}_\pi$'s with $\widehat{\otimes}_\varepsilon$'s (to which Lemma C.2.3.5(1) may be applied), and j_k has closed range by completeness of $E_1 \mathbin{\widehat{\otimes}_\pi} F_k$,
- the continuous linear map q_k is a surjective quotient map by Proposition C.2.3.7 since E_2/E_1 is metrizable and even Fréchet [**Scha**, Ch. I, 6.3] (and F_k is metrizable by hypothesis),
- the middle exactness follows from the Hahn-Banach theorem and mapping properties of completions and the projective tensor product (if the closed subspace $(\ker q)/(E_1 \mathbin{\widehat{\otimes}_\pi} F_k) \subset (E_2 \mathbin{\widehat{\otimes}_\pi} F_k)/(E_1 \mathbin{\widehat{\otimes}_\pi} F_k)$ is nonzero then some nonzero continuous linear functional ℓ on $E_2 \mathbin{\widehat{\otimes}_\pi} F_k$ vanishes on $E_1 \mathbin{\widehat{\otimes}_\pi} F_k$ and does *not* factor through $(E_2/E_1) \mathbin{\widehat{\otimes}_\pi} F_k$, and this corresponds to a continuous bilinear form $B : E_2 \times F_k \to \mathbf{C}$ that vanishes on $E_1 \times F_k$ yet doesn't factor through $(E_2/E_1) \times F_k$, an absurdity).

Passing to the inductive limit on the j_k's, we see that the continuous linear map $E_1 \mathbin{\overline{\otimes}} F \to E_2 \mathbin{\overline{\otimes}} F$ is injective (not yet known to be a topological embedding).

Just as we showed strictness for each $\{E_i \mathbin{\widehat{\otimes}_\pi} F_k\}_{k\geq 0}$, the inductive system $\{(E_2/E_1) \mathbin{\widehat{\otimes}_\pi} F_k\}_{k\geq 0}$ is strict (as the Fréchet E_2/E_1 is nuclear too). The description of a base of 0-neighborhoods for the locally convex topology

of an inductive limit via the corresponding direct sum as in (C.1.5.1) shows that for a compatible system of linear quotient maps $f_i : V_i \to \overline{V}_i$ the resulting map $\varinjlim V_i \to \varinjlim \overline{V}_i$ is also a quotient map. Thus, $\mathrm{coker}(j : E_1 \mathbin{\overline{\otimes}} F \to E_2 \mathbin{\overline{\otimes}} F)$ equipped with its quotient topology from $E_2 \mathbin{\overline{\otimes}} F$ is linearly and topologically identified with the strict locally convex inductive limit $(E_2/E_1) \mathbin{\overline{\otimes}} F$ of the nuclear Fréchet spaces $(E_2/E_1) \mathbin{\widehat{\otimes}}_\pi F_k$.

We deduce that the continuous linear map $q : E_2 \mathbin{\overline{\otimes}} F \to (E_2/E_1) \mathbin{\overline{\otimes}} F$ between strict LF-space spaces is a quotient map whose closed kernel $\ker(q)$ received a continuous linear bijection from $E_1 \mathbin{\overline{\otimes}} F$. Since the source and target of q are strict inductive limits of nuclear Fréchet spaces, Corollary C.1.5.13 (whose proof ultimately rests on the Open Mapping Theorem for LF-spaces) tells us that the closed subspace $\ker(q)$ is a limit subspace of $E_2 \mathbin{\overline{\otimes}} F$. More specifically, since $\ker(q)$ meets each layer $E_2 \mathbin{\overline{\otimes}} F_k = E_2 \mathbin{\widehat{\otimes}}_\pi F_k$ of $E_2 \mathbin{\overline{\otimes}} F$ in exactly $E_1 \mathbin{\overline{\otimes}} F_k = E_1 \mathbin{\widehat{\otimes}} F_k$, Corollary C.1.5.13 gives that $\varinjlim_k E_1 \mathbin{\overline{\otimes}} F_k \to \ker(q)$ is a topological isomorphism. This latter inductive limit is topologically $E_1 \mathbin{\overline{\otimes}} F$, so the continuous injective linear map (C.2.3.28) is a topological embedding as desired. □

C.2.4. Nuclear spaces. The two topological tensor products $\otimes_\pi$ and $\otimes_\varepsilon$ of Hausdorff locally convex topological $\mathbf{C}$-vector spaces E, F are related: the projective topology $\otimes_\pi$ on $E \otimes F$ is finer than the injective topology $\otimes_\varepsilon$, which is to say the identity map

$$E \otimes_\pi F \to E \otimes_\varepsilon F \tag{C.2.4.1}$$

is continuous. This is deduced from their respective universal properties (see [**Tre**, Prop. 43.5]).

We are interested in two aspects relating to these two locally convex topologies on $E \otimes F$:

(1) to discuss when (C.2.4.1) is a topological isomorphism;
(2) to illustrate when the natural map $E' \otimes F \to \mathscr{L}_{\mathfrak{S}}(E, F)$ has dense image with respect to suitable $\mathfrak{S}$-topologies, and which such topologies qualify.

These questions were the subject of a significant portion of Grothendieck's doctoral dissertation, and we refer the reader particularly to [**Gro-b**, Ch. I, §5; Ch. II, §3].

C.2.4.1. *Characterizations of nuclearity.* The first of the two questions at hand, namely when the continuous linear bijection $E \otimes_\pi F \to E \otimes_\varepsilon F$ is a homeomorphism for Hausdorff locally convex topological $\mathbf{C}$-vector spaces E, F, is best phrased as asking when for a *fixed* Hausdorff locally convex topological $\mathbf{C}$-vector space E the natural transformation $E \otimes_\pi (\cdot) \to E \otimes_\varepsilon (\cdot)$ of functors from Hausdorff locally convex topological $\mathbf{C}$-vector spaces to Hausdorff locally convex topological $\mathbf{C}$-vector spaces is an isomorphism; i.e., when is (C.2.4.1) a topological isomorphism for *all* F?

The second part of Grothendieck's doctoral dissertation is devoted to answering and studying this question, leading to the definition of *nuclear* spaces. The following is [**Gro-b**, Ch. II ,§2.1, Def. 4].

Definition C.2.4.1 (Nuclear Hausdorff locally convex topological $\mathbf{C}$-vector spaces). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. We say E is *nuclear* if and only if the identity map $E \otimes_\pi F \to E \otimes_\varepsilon F$ is a linear topological isomorphism for every Hausdorff locally convex topological $\mathbf{C}$-vector space F.

It is equivalent to work with the completed tensor products in this definition, since the uncompleted versions are Hausdorff (and so topologically embed into their completions). There is a different definition of nuclearity given in [**Scha**, Ch. III, p. 100], but this is equivalent to the above definition by combining [**Scha**, Ch. III, 7.2] with a long list of equivalents to Definition C.2.4.1 given in [**Tre**, Thm. 50.1].

As simple as Definition C.2.4.1 appears, the theory of nuclearity cannot take off the ground without concrete and usable characterizations. These all rest on the concept of nuclear operator between Banach spaces, which we will be recalling throughout §C.3. We briefly recall for now a concrete characterization (which however is not the definition) (see [**Scha**, Ch. III, Rem. to 7.1]): for E, F Banach spaces, a continuous linear operator $T : E \to F$ is *nuclear* if and only if there exists an absolutely summable sequence $\{\lambda_k\}_{k \geq 0}$ in $\mathbf{C}$,

a sequence $\{f_k\}_{k\geq 0} \subset E'$ converging to 0 (in the usual Banach-dual topology), and $\{x_k\}_{k\geq 0} \subset F$ converging to 0 such that

$$T = \sum_{k\geq 0} \lambda_k(\langle \cdot, f_k\rangle_{E,E'})x_k \tag{C.2.4.2}$$

where any such sum always converges pointwise on E. In particular, finite-rank linear maps are nuclear.

With this concept in mind, nuclearity of a Hausdorff locally convex topological $\mathbf{C}$-vector space E is characterized as follows (see [**Scha**, Ch. III, 7.2(b)⇔(c)] and [**Tre**, Thm.50.1] for the proof):

THEOREM C.2.4.2 (Characterizations of nuclearity - I). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. The following conditions are equivalent:*

(1) *E is nuclear.*
(2) *For any convex circled 0-neighborhood $U \subset E$, there exists a convex circled 0-neighborhood $V \subset U$ such that the canonical continuous linear map $\widehat{E}_V \to \widehat{E}_U$ is nuclear, where $\widehat{E}_U$ is the completion of E for the gauge semi-norm p_U associated to U as in* (C.1.2.1) *and likewise for V.*
(3) *For any Banach space F, the canonical continuous linear map $E \mathbin{\widehat{\otimes}}_\pi F \to E \mathbin{\widehat{\otimes}}_\varepsilon F$ is a topological isomorphism.*
(4) *For any Hausdorff locally convex topological vector space F, the canonical continuous linear map $E \mathbin{\widehat{\otimes}}_\pi F \to E \mathbin{\widehat{\otimes}}_\varepsilon F$ is topological isomorphism.*

Upon defining the general concept of nuclear maps (not only for continuous linear maps between Banach spaces), nuclearity for E as in the preceding theorem is further characterized by the following property of E: for any Banach space F, any continuous linear map $E \to F$ is nuclear. For more characterizations of nuclearity of spaces, see [**Tre**, Thm. 50.1].

C.2.4.2. *Continuous dual of $E \otimes_\varepsilon F$.* The proof of equivalences of the many characterizations of nuclearity in [**Tre**, Thm.50.1] rests on a description of the continuous dual of $E \otimes_\varepsilon F$ (the uncompleted injective tensor product of two Hausdorff locally convex topological $\mathbf{C}$-vector spaces) provided in the fundamental [**Tre**, Prop.49.1] in terms of *integral* bilinear forms on $E \times F$ that we now review.

Integral bilinear forms and their use in describing the continuous dual of $E \otimes_\varepsilon F$ also underlie further characterizations of nuclearity we will not need, such as combinatorial control over the equicontinuous subsets of the continuous dual of a nuclear space resting on integration over compact (or weakly compact) Choquet simplices, which turns out to characterize nuclearity and for which we refer the reader to [**Jar**, §21.4, Thm. 3, p. 490].

The description of the continuous dual of $E \otimes_\varepsilon F$, which we will be recalling from [**Tre**, Prop.49.1] shortly rests on the following construction that we wish to further explain before referring the reader to [**Tre**, Ch. 49, p. 501] and the discussion contained therein. Recall that for any Hausdorff locally convex topological $\mathbf{C}$-vector space E and any convex circled and absorbing subset U (not necessarily a 0-neighborhood), we topologize the linear subspace $E_U := \bigcup_{m\in\mathbf{N}} mU \subset E$ by the Minkowski functional p_U as in (C.1.2.1), a seminorm on E_U. The separated completion $\widehat{E}_U$ of the semi-normed space (E_U, p_U) is a Banach space which we refer to as the *local Banach space* of E at U. By [**Scha**, Ch. II, 1.5], $\widehat{E}_U$ only depends on the closure of U.

REMARK C.2.4.3. The local Banach spaces $\widehat{E}_U$ are denoted in [**Scha**, Ch. III, p. 97] as $\widetilde{E}_U$ and in [**Jar**, §6.8, p. 116] as $\widetilde{E_{(U)}}$.

Many conditions and structural results on classes of Hausdorff locally convex topological $\mathbf{C}$-vector spaces E reduce to an analysis in the special case of Banach spaces via the local Banach spaces of E. We begin with the following observation, which provides a class of nuclear spaces:

LEMMA C.2.4.4 (Local Banach spaces of weak duals). *Let E be a Hausdorff locally convex topological* **C***-vector space. For any convex circled 0-neighborhood $U \subset E'_\sigma$, the local Banach space $\widehat{(E'_\sigma)}_U$ is finite-dimensional. In particular, the weak dual E'_σ of E is always nuclear.*

PROOF. Since U is a 0-neighborhood for the weak topology on E', there exist finitely many elements $\{x_1, \ldots, x_N\} \subset E$ and a real number $\varepsilon > 0$ such that

$$U \supset U_\varepsilon(x_1, \ldots, x_N) := \bigcap_{1 \le k \le N} \{\varphi \in E' : |\langle x_k, \varphi \rangle_{E,E'}| \le \varepsilon\}. \tag{C.2.4.3}$$

Consider the linear subspace $V := \bigcap_{1 \le k \le N} \ker(\iota_E(x_k)) \subset E'$, where $\iota_E : E \to (E')^*$ is the evaluation linear map. Noting that each $\iota_E(x_k)$ is continuous on E' with respect to the weak topology, V is closed in E'_σ and of finite codimension. It is contained in $U_\varepsilon(x_1, \ldots, x_N)$, and hence in U. In particular, for $F := E'$ we see that $F_U/p_U^{-1}(0)$ is a quotient of F/V and so is finite-dimensional. In particular, $\widehat{F}_U = F_U/p_U^{-1}(0)$ is finite-dimensional. □

We will be using below the following observation: by Lemma C.2.1.7, if E is barrelled then E'_σ is quasi-complete, and hence the (weakly-)closed convex circled hull of a weakly compact subset of E'_σ is weakly compact. This is a consequence of the preservation of pre-compactness (i.e., compactness of closure in the completion) under formation of the convex circled hull in any Hausdorff locally convex topological **C**-vector space [**Scha**, Ch. II, 4.3]. Indeed, in a quasi-complete Hausdorff locally convex topological vector space F any pre-compact subset $C \subset F$ is bounded in $\widehat{F}$ (as compactness implies boundedness) and hence is bounded in F. Thus, the closure $\overline{C} \subset F$ is bounded [**Scha**, Ch. I, 5.1] and thus is complete in F (as F is quasi-complete), so $\overline{C}$ is also closed in $\overline{F}$. By pre-compactness of C in $\widehat{F}$, this gives the compactness of $\overline{C}$ as desired.

REMARK C.2.4.5. The barreledness of E above as a sufficient condition for quasi-completeness of E'_σ cannot be entirely dropped: there are examples of Hausdorff locally convex topological **C**-vector spaces E for which E'_σ is not quasi-complete.

We refer the reader to [**Gro-e**, II, §1, p. 88] for an example of a Fréchet space F whose strong dual $E := F'_\beta$ is not bornological (equivalently, not barrelled; see [**Scha**, Ch. IV, 6.6]) and whose weak dual E'_σ is not quasi-complete. To see this example is not quasi-complete, we assume to the contrary that the nuclear space E'_σ is quasi-complete and seek a contradiction. [15]

We now return to the task at hand. The following lemma is outlined in [**Tre**, pp. 500-501]. We provide here below the details of the argument, in light of its importance in the description of the continuous dual of $E \otimes_\varepsilon F$ for E, F Hausdorff locally convex topological **C**-vector spaces:

LEMMA C.2.4.6 (Local representation of continuous forms). *Let E be a Hausdorff locally convex topological* **C***-vector space, and $K \subset E$ a compact subset. For any $x \in \overline{\mathrm{ch}}(K)$ (the closed convex hull of K in E) there exists a positive Radon measure μ on K satisfying $\mu(K) = 1$ such that the $(E')^*$-valued function $f : K \to (E')^*$ defined as $f(y) := \iota_E(y)$ is μ-integrable in the sense of Pettis, and we have*

$$\iota_E(x) = \int_K f(y)\, \mathrm{d}\mu(y) \tag{C.2.4.4}$$

as continuous linear forms on E'_σ (so $\ell(x) = \int_K \ell(y)\, \mathrm{d}\mu(y)$ for all $\ell \in E'$).

We will be discussing the integral in the sense of Pettis in §C.4. We record the proof of Lemma C.2.4.6 below for ease of access, referring to the relevant results in §C.4 on vector-valued integration.

PROOF. Using weak topologies, the natural linear map $\iota_E : E_\sigma \to (E')^*_\sigma$ is continuous. Since closure agrees with weak closure for convex subsets of any Hausdorff locally convex topological vector space [**Scha**,

[15] How do we get a contradiction? Can it be via [**Tre**, Cor. 1 or Cor. 3 to Prop. 50.2]?

Ch. IV, 3.1], it follows that the $\sigma((E')^*, E')$-closure of the convex hull of $\iota_E(K)$ contains the image under ι_E of the closed convex hull of K in E. By Theorem C.4.3.2, such $\sigma((E')^*, E')$-closure is itself compact in $(E')^*$, and it equals the set $\{\int_K f \, \mathrm{d}\mu\}$ as μ varies over all the probability measures on K. For any $x \in \overline{\mathrm{ch}}(K)$, there therefore exists a probability measure μ on K such that $\iota_E(x) = \int_K \iota_E(y) \, \mathrm{d}\mu(y)$. (This integral a-priori belongs to $((E')^*_\sigma)'^*$.) □

Remark C.2.4.7. The construction discussed in [**Tre**, Prop.49.1] of the Radon measure μ associated to $x \in \overline{\mathrm{ch}}(K)$ as in the statement of Lemma C.2.4.6 contains a point that, we believe, warrants clarification. We therefore thought it helpful to the reader to give a proof of Lemma C.2.4.6 above, so as to render the Lemma applicable as intended throughout the proof of [**Tre**, Prop.49.1].

The issue is continuity of the extension by linearity of the continuous inverse of $\delta : K \to C(K)'_\sigma$ that sends each $x \in K$ to the point-evaluation functional $\delta_x \in C(K)'$ (a homeomorphism onto its image by compactness of K) on the linear span V of $\delta(K)$ which would, if true (with E endowed with its given topology), lead to a continuous linear map $\overline{V} \to \widehat{E}$.

Then since $C(K)'_\sigma$ is nuclear with finite-dimensional local Banach spaces (by Lemma C.2.4.4),'so likewise for $\overline{V}$, we would obtain that such a continuous map has finite-dimensional linear image in E containing K, which would force K to be contained in a finite-dimensional linear subspace of E. Such linear extension of the inverse to the homeomorphism δ, cannot be continuous as soon as E is an infinite-dimensional Banach space and K is not contained within any finite-dimensional subspace (such K exists, such as $\{0\} \cup \{x_n\}_{n\geq 1}$ for a sequence of linearly independent vectors converging to 0).

On the other hand, the construction being outlined in [**Tre**, Prop.49.1] is correct because Lemma C.2.4.6 exhibits the desired measure μ in [**Tre**, (49.1)].

The following result is [**Tre**, Prop.49.1], and the reader can now go through its proof as written in loc.cit., using Lemma C.2.4.6 providing the proof of [**Tre**, (49.1)] upon which the proof of the result rests.

Proposition C.2.4.8 (Integral bi-linear forms). *A continuous bi-linear form $u : E \times F \to \mathbf{C}$ is integral, i.e. an element of $\mathscr{I}(E, F) \subset \mathscr{B}(E, F)$, if and only if there exist weakly closed equicontinuous subsets $A' \subset E'$ and $B' \subset F'$ and a positive Radon measure μ on the weakly compact subset $A' \times B' \subset E'_\sigma \times F'_\sigma$ such that $\mu(A' \times B') \leq 1$ and for all $(x, y) \in E \times F$, we have*

$$u(x, y) = \int_{A' \times B'} \langle x, \varphi \rangle_{E,E'} \langle y, \psi \rangle_{F,F'} \, \mathrm{d}\mu(\varphi, \psi). \tag{C.2.4.5}$$

C.2.4.3. *Integral mappings and characterizations of nuclearity.* Using integral continuous bilinear forms, we can extend the list of equivalent characterizations of nuclearity for Hausdorff locally convex topological $\mathbf{C}$-vector spaces from Theorem C.2.4.2 as follows (from [**Tre**, Rem. 50.1], using [**Tre**, Def. 49.2]):[16]

Theorem C.2.4.9 (Characterizations of nuclearity - II). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. The following conditions are equivalent:*

(1) *E is nuclear.*
(2) *For any convex circled 0-neighborhood $U \subset E$, there exists a convex circled 0-neighborhood $V \subset U$, such that the canonical continuous linear operator $\widehat{E}_V \to \widehat{E}_U$ is integral in the sense that the associated continuous bi-linear form on $\widehat{E}_V \times (\widehat{E}_U)'_\beta$ is integral.*

The idea behind the equivalence in Theorem C.2.4.9 is that integral and nuclear maps between Hilbert spaces are the same class of continuous linear maps (see [**Tre**, Prop. 49.5, Thm. 49.6] and that integral maps to a complete target can be factored through a Hilbert space (see [**Tre**, Prop. 49.7].

[16] Is this result ever needed? If not, remove all of §C.2.4.3. Too heavy on the reader, need not be told "everything". Personally, I found [**Tre**, Rem. 50.1] to be more exhausting than convincing.

C.2.4.4. *Permanence properties.* Nuclearity satisfies the following permanence properties, contained within [**Scha**, Ch. III, 7.4, 7.5].

LEMMA C.2.4.10 (Permanence properties). *Nuclearity of Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces is stable under passage to:*

(1) *linear topological subspaces, Hausdorff quotients by linear subspaces;*
(2) *linear topological products indexed by an arbitrary set, countable topological direct sums;*
(3) *binary projective tensor products (i.e. if two Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces* E, F *are nuclear then* $E \otimes_\pi F$ *is nuclear).*

We further recall the following nuclearity criterion from [**Tre**, Prop. 50.6]:[17]

LEMMA C.2.4.11 (Duality and nuclearity for Fréchet spaces). *Let* E *be a Fréchet space. Then* E *is nuclear if and only if so is* E'_β.

We will use the following permanence properties that follow from Lemma C.2.4.10(1, 2):

LEMMA C.2.4.12 (Nuclearity of inductive and projective limits). *A Hausdorff locally convex topological* $\mathbf{C}$*-vector space* E *is nuclear if any of the following conditions hold:*

(1) E *is the locally convex inductive limit of a sequence of nuclear Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces.*
(2) E *is a projective limit, indexed by an arbitrary co-filtered set, of nuclear Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces.*

REMARK C.2.4.13. The Hausdorff assumption on E is also assumed[18] in [**Gro-b**, Ch. II, §2, no. 2, Cor. 1] insofar as it is necessary to deduce nuclearity from presenting the locally convex inductive limit as a *separated* quotient of a nuclear locally convex topological direct sum. The assumption is needed since separatedness is not generally inherited by locally convex inductive limits. We have specialized part (1) to the case the inductive system of nuclear spaces is comprised of Hausdorff spaces too.

REMARK C.2.4.14. We further refer the reader to [**Scha**, Ch.III, §7] for a discussion of the above results from the second part of Grothendieck's dissertation, noting that the Hausdorff property is part of the *definition* of "locally convex topological vector space" in [**Scha**, Ch. II, p. 47].

C.2.4.5. *Approximation Property.* We consider the injective linear map

$$\lambda : E' \otimes F \to \mathscr{L}(E, F) \tag{C.2.4.6}$$

uniquely determined by

$$\lambda(\varphi \otimes f) := \langle \cdot, \varphi \rangle_{E,E'} \cdot f. \tag{C.2.4.7}$$

The following result is [**Scha**, Ch. III, 9.1] (in which we use the notation $\mathscr{L}_\gamma$ from Definition C.2.1.5(2)):

LEMMA C.2.4.15 (Approximation property). *Let* E *be a Hausdorff locally convex topological* $\mathbf{C}$*-vector space. The following conditions are equivalent:*

(1) *The identity map in* $\mathscr{L}_\gamma(E, E)$ *is in the closure of the image of* $E' \otimes E$ *under* λ *from* (C.2.4.6).
(2) *The image of* $E' \otimes E$ *in* $\mathscr{L}_\gamma(E, E)$ *under* λ *from* (C.2.4.6) *is dense.*
(3) *For every Hausdorff locally convex topological* $\mathbf{C}$*-vector space* F*, the image of* $E' \otimes F$ *in* $\mathscr{L}_\gamma(E, F)$ *under* λ *from* (C.2.4.6) *is dense.*

[17] This result originally included E that is (DF), but who cares? Removed that.
[18] But nuclearity hasn't even been *defined* beyond the Hausdorff case, so why make this Remark at all?

(4) *For every Hausdorff locally convex topological* $\mathbf{C}$*-vector space* F*, the image of* $F' \otimes E$ *in* $\mathscr{L}_\gamma(F, E)$ *under* λ *from* (C.2.4.6) *is dense.*

The next lemma records two properties of main interest for us about nuclear spaces beyond their defining condition from Definition C.2.4.1. The first of the two properties stated in the lemma will be used systematically throughout this work to infer that certain Fréchet spaces, or strict inductive limits of Fréchet spaces, are reflexive by deducing that they are Montel via [**Tre**, Cor. 3 to Prop. 50.2], which asserts that any nuclear Hausdorff locally convex $\mathbf{C}$-vector space that is also barreled and quasi-complete is Montel (all Fréchet spaces are barreled [**Scha**, Ch. II, Cor. to 7.1]). The second will be used to infer that in the presence of the equivalent conditions from Lemma C.2.4.15, the traces of nuclear operators (to be discussed in §C.3) can be computed as matrix traces relative to a Schauder basis if one exists.

LEMMA C.2.4.16 (Nuclearity and the Approximation Property). *Let* E *be a Hausdorff locally convex topological* $\mathbf{C}$*-vector space. Assume* E *is nuclear.*

(1) *Every bounded subset of* E *is pre-compact. In particular, for any Hausdorff locally convex topological* $\mathbf{C}$*-vector space* F*,* $\mathscr{L}_\gamma(E, F) = \mathscr{L}_\beta(E, F)$*.*
(2) *The image of* $E' \otimes E$ *under* λ *from* (C.2.4.6) *is dense in* $\mathscr{L}_\gamma(E, E)$ *(so all four properties in Lemma* (C.2.4.15) *hold).*

PROOF. Part (1) is [**Scha**, Ch. III, Cor. 2 to 7.2], and part (2) is [**Scha**, Ch. III, Cor. 2 to 9.1]. □

REMARK C.2.4.17. As a consequence of Lemma C.2.4.16, for E, F two Hausdorff locally convex topological $\mathbf{C}$-vector spaces with E nuclear, the image of $\lambda : E' \otimes F \to \mathscr{L}_\beta(E, F)$ is dense (i.e., the finite-rank operators are dense) where the target has the topology of convergence on the bounded subsets of E.

We also note that the first part of the lemma says bounded subsets of nuclear spaces E have compact closure in $\widehat{E}$. It does *not* say their closure in E is compact; if E is also quasi-complete (in addition to being nuclear) then bounded subsets of E do have compact closure [**Tre**, Cor. 1 to Prop. 50.2].

C.2.4.6. *Kernel Theorems.* Nuclearity ensures the existence of kernels for certain classes of continuous operators. We recall the statement of the Schwartz Kernel Theorem in its general form (see [**Scha**, Ch. IV, 9.5, p. 172]):

THEOREM C.2.4.18 (Schwartz Kernel Theorem). *Let* E, F *be Hausdorff locally convex topological* $\mathbf{C}$*-vector spaces. Assume* E *is nuclear. For each continuous bilinear forms* $B : E \times F \to \mathbf{C}$*, there exist equicontinuous sequences* $\{\varphi_m\}_{m\geq 0} \subset E'$ *and* $\{\psi_m\}_{m\geq 0} \subset F'$*, and an absolutely summable sequence of complex numbers* $\{\lambda_m\}_{m\geq 0}$*, such that*

$$B = \sum_{m\geq 0} \lambda_m \langle \cdot, \varphi_m \rangle_{E,E'} \cdot \langle \cdot, \psi_m \rangle_{E,E'} \tag{C.2.4.8}$$

where the convergence is with respect to the topology (on the space of continuous bilinear forms) of uniform convergence on the bounded subsets of $E \times F$*.*

We refer the reader to [**Scha**, Ch. III, §5, p. 91] for a discussion of the topology (on the space of continuous bilinear forms) of uniform convergence on the bounded subsets of $E \times F$, particularly that it is a Hausdorff locally convex topological $\mathbf{C}$-vector space structure (the notation of loc. cit.,writes this space as $\mathscr{B}(E, F; \mathbf{C})$ whereas [**Tre**, Ch. 42] denotes it as $B(E, F)$). Continuity of $B : E \times F \to \mathbf{C}$ implies that the image $B(S, T) \subset \mathbf{C}$ is bounded for any bounded subset $S \times T \subset E \times F$.

We now specialize to the case of a complete nuclear Hausdorff locally convex topological $\mathbf{C}$-vector space E, and a continuous linear map $T : E \to E'_\beta$, so the bilinear form on $E \times E$ defined by

$$(x, y) \mapsto \langle x, Ty \rangle_{E,E'} \tag{C.2.4.9}$$

is continuous. By Theorem C.2.4.18, there exist equicontinuous sequences $\{\varphi_m\}_{m\geq 0}, \{\psi_m\}_{m\geq 0} \subset E'$ and an absolutely summable sequence of complex numbers $\{\lambda_m\}_{m\geq 0}$ such that

$$\langle x, Ty\rangle_{E,E'} = \sum_{m\geq 0} \lambda_m \langle x, \varphi_m\rangle_{E,E'} \cdot \langle y, \psi_m\rangle_{E,E'}.$$

We will be using the Schwartz Kernel Theorem in special forms (of which Theorem C.2.4.18 is an extension) as given in [Tre, Thm. 51.6] (e.g., $\mathcal{S}(\mathbf{R}^n) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(\mathbf{R}^m) \simeq \mathcal{S}(\mathbf{R}^{n+m})$ as topological vector spaces). We record one such version here for reference:[19]

THEOREM C.2.4.19 (Schwartz Kernel Theorem for Schwartz-Bruhat spacees). *Let G_1, G_2 be Hausdorff locally compact abelian groups. Assume G_1 and G_2 are second-countable. Let $\mathcal{S}(G_1), \mathcal{S}(G_2)$ be the Hausdorff locally convex topological $\mathbf{C}$-vector spaces of Schwartz-Bruhat functions on G_1, respectively G_2. The canonical continuous linear map*

$$\mathcal{S}(G_1)'_\beta \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G_2) \to \mathscr{L}_\beta(\mathcal{S}(G_1), \mathcal{S}(G_2)) \tag{C.2.4.10}$$

is a topological isomorphism.

We refer the reader to Appendix D for a discussion of Schwartz-Bruhat spaces $\mathcal{S}(G)$ in general. Lemma D.1.8.1 gives further variants of the result.

PROOF. By Theorem D.1.5.7, the spaces $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ as well as their strong duals are nuclear, complete, and Montel, where the latter means "closed bounded subsets of E are compact, and E is reflexive (in the sense of the strong double-dual)" [Scha, p. 147]. Reflexivity implies barreledness [Scha, Ch. IV, 5.6], so we can apply [Tre, Prop. 50.5] which gives that the natural continuous linear map

$$E'_\beta \mathbin{\widehat{\otimes}}_\pi F \to \mathscr{L}_\beta(E, F)$$

is a topological isomorphism for any complete Hausdorff locally convex topological vector spaces E and F such that E is barreled and E'_β is nuclear and complete. □

C.2.5. Complements on the finest locally convex topology. Finite-dimensional $\mathbf{C}$-vector spaces have a unique Hausdorff topological vector space structure [Scha, Ch. I, 3.2]. Building on that, as a piece of algebra, there is the notion of *finest locally convex topology* on a $\mathbf{C}$-vector space E. This can be described in two ways (which must therefore coincide): it is defined by the family of all seminorms on E, and it is the locally convex inductive limit of the collection of finite-dimensional subspaces. This will be used only in countable-dimensional cases, so such spaces are then even strict LF-spaces.

One example is the Schwartz-Bruhat function space $\mathcal{S}(F_v) = C_c^\infty(F_v)$ for F a number field and v a finite place of F. Another such example is $\mathcal{S}(\prod_{v\in S} F_v)$ for S a finite set of *finite* places of F, as are $\mathcal{S}(\mathbf{A}_F^f)$ (for $\mathbf{A}_F^f$ the finite adeles of F) and $\mathcal{S}(\widehat{\prod}_{v\notin S}(F_v, \mathcal{O}_{F_v}))$ for S any finite set of places of F containing the Archimedean places of F. In this subsection we record some basic facts about this topology, most of which are elementary.

PROPOSITION C.2.5.1 (Properties of the finest locally convex topology). *Let E be a $\mathbf{C}$-vector space, τ its finest locally convex topology.*

(1) *The topology τ is defined by the family of all seminorms on E, and is the locally convex inductive limit of all finite-dimensional subspaces of E.*
(2) *For any Hausdorff locally convex topological $\mathbf{C}$-vector space M, any linear map $T : E \to M$ is continuous. In particular, the inclusion $(E, \tau)' \subset E^*$ is an equality of vector spaces.*
(3) *Every $\mathbf{C}$-linear subspace of E is τ-closed and topologically complemented, so τ is inherited by all $\mathbf{C}$-linear subspaces and quotients of E. Moreover, (E, τ) is complete.*

[19] This result is [Tre, Prop. 50.5] without needing any of the extra generalities above, none of which are therefore ever cited, so why include any of the stuff above?

(4) *A subset $B \subset E$ is τ-bounded if and only if it is contained in a finite-dimensional subspace of E and bounded with respect to the Euclidean topology on such a subspace.*
(5) *A subset $C \subset E$ is τ-sequentially closed if and only if its intersection with each finite-dimensional $\mathbf{C}$-linear subspace of E is closed.*
(6) *If E is has countable dimension then (E, τ) is a nuclear strict LF-space. In particular, with respect to τ any such E is complete, nuclear and Montel, and its strong dual E'_β is a nuclear Fréchet space.*

PROOF. We discussed (1) already (namely both descriptions are finest among all locally convex topologies, so they must coincide). The assertion in (2) is immediate from the description as a locally convex inductive limit. To see (3), if $E_0 \subset E$ is a $\mathbf{C}$-linear subspace and $E_1 \subset E$ is a linear complement then for the respective finest locally convex topologies τ_j on E_j we see that $E_0 \oplus E_1 \to E$ is a topological isomorphism since every finite-dimensional subspace of E is contained in $F_0 \oplus F_1$ for finite-dimensional subspaces $F_j \subset E_j$.

The completeness of (E, τ) is immediate from yet another description of τ as a locally convex direct sum of lines, since completeness is inherited by arbitrary locally convex direct sums [**Scha**, Ch. II, 6.2]. This description via a direct sum of lines yields the assertion in (4) from a general property of bounded subsets of locally convex direct sums [**Scha**, Ch. II, 6.3]. Since convergent *sequences* in a Hausdorff topological vector space are bounded, the sufficiency in (5) is immediate from (4). The necessity in (5) is clear.

Finally, in the countable-dimensional case we can express E as a strict countable inductive limit of finite-dimensional spaces, so E is a nuclear strict LF-space since finite-dimensional spaces are certainly nuclear and strict countable inductive limits preserve nuclearity [**Scha**, Ch. III, Cor. to 7.4]. Likewise the Montel property is preserved by such inductive limits [**Scha**, p. 147], so E is Montel. Nuclearity of E'_β is inherited from that of E because E is a nuclear Fréchet space [**Tre**, Prop. 50.6]. Completeness of E'_β is a consequence of the bornological property of E [**Scha**, Ch. IV, 6.1], where the bornological property is inherited through inductive limits from the finite-dimensional case [**Scha**, Ch. II, Cor. 1 to 8.2].

It remains to show in (6) that E'_β is metrizable. Since E is a locally convex countable direct sum of lines, which in turn can be viewed as the strict locally convex direct limit of finite direct sums among such lines, Lemma C.1.6.1(2) yields that the strong dual is the locally convex countable product of (the duals of such) lines. The topology on such a product is the product topology, so the metrizability is immediate from the countability of the product. □

One final permanence property of the finest locally convex topology concerns stability under tensor products in the countable-dimensional case:

LEMMA C.2.5.2. *Let E_1, E_2 be two Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Assume E_1, E_2 are countably $\mathbf{C}$-linearly generated, and carry the finest locally convex topology. Then so do $E_1 \otimes_\pi E_2$, $E_1 \otimes_\varepsilon E_2$ and $E_1 \otimes_\iota E_2$, so all three topologies on $E_1 \otimes E_2$ coincide and are complete.*

PROOF. The completeness of the finest locally convex topology on any $\mathbf{C}$-vector space is part of Proposition C.2.5.1(3). Since E_1, E_2 are nuclear, the continuous linear map $E_1 \otimes_\varepsilon E_2 \to E_1 \otimes_\pi E_2$ given by the identity on $E_1 \otimes E_2$ is a topological isomorphism by Theorem C.2.4.2(4) (this gives isomorphism after passing to completions, but both topological tensor products topologically embed into their completions and so we get the assertion without passing to completions). Thus, it suffices to analyze the projective topology and the inductive topology on $E_1 \otimes E_2$.

By (C.2.3.26), $E_1 \otimes_\iota E_2$ carries the finest locally convex topology on $E_1 \otimes E_2$. Since E_1 and E_2 are each topologically a locally convex direct sum of lines (with their unique Hausdorff topological vector space structure), and lines are normable, by Proposition C.2.3.9(1) it follows that $E_1 \otimes_\pi E_2$ is also a locally convex direct sum of countably many lines; i.e., this projective tensor product topology is the finest locally convex topology on $E_1 \otimes E_2$. □

C.3. Nuclear operators

In this section we collect results on nuclear operators. We will be first discussing a few complements on nuclear operators on general Hausdorff locally convex topological **C**-vector spaces, with emphasis on a factorization principle enabling the computation of nuclear traces of nuclear operators equipped with a factorization through a continuous injective linear map between Hausdorff locally convex topological **C**-vector spaces. This has appeared before in [**Me1**, §4, Cor. 4.7, p. 28], and we discuss it here in detail for ease of access.

We will then focus on the case of nuclear operators on Hilbert spaces, and methods to compute their traces. We will first discuss some complements on the strong operator topology (i.e., pointwise convergence) on the **C**-algebra of continuous endomorphisms on a Hilbert space, then recall the basic properties of Hilbert-Schmidt and nuclear operators on a Hilbert space and several convergence results of compositions with sequences of orthogonal projectors and more general self-adjoint operators. We include proofs of some of these results for ease of access.

We will then briefly discuss the Schatten-von Neumann classes of index $p = 1, 2$ for operators between two (possibly distinct) Hilbert spaces. We conclude this section on nuclear operators with a discussion of a special class of nuclear operators: those associated to a measurable integral kernel on Hilbert spaces of the special form $L^2(X, \mu)$, for X a Hausdorff locally compact topological space and μ a Radon measure on X. We will discuss how to "smoothen" their integral kernel along the diagonal, and use that smoothening to compute nuclear traces.

C.3.1. Preparations. Let E, F be Hausdorff locally convex topological vector spaces. For Banach E, the strong dual E'_β for E Banach has its topology defined by the dual norm $\|\varphi\| = \sup_{\|x\|\leq 1} |\varphi(x)|$. In general there is a canonical continuous linear map

$$E'_\beta \otimes_\pi F \to \mathscr{L}_\beta(E, F). \tag{C.3.1.1}$$

The target has the topology of uniform convergence on bounded sets, so for Banach E and F it is the operator-norm topology and so in such cases the target is complete. Thus, for Banach E and F, (C.3.1.1) uniquely extends by continuity to a continuous linear map

$$E'_\beta \mathbin{\widehat{\otimes}}_\pi F \to \mathscr{L}_\beta(E, F). \tag{C.3.1.2}$$

For Banach E and F, nuclear maps in $\mathscr{L}_\beta(E, F)$ are defined to be those in the range of (C.3.1.2); these have an explicit description as in (C.2.4.2).

Remark C.3.1.1. Beyond the Banach setting there is another class of sufficient conditions for completeness of the target of (C.3.1.1), with a stronger conclusion: if E and F are complete, E is barreled, and E'_β is nuclear and complete (these hold when E and F are Banach with E nuclear; see [**Tre**, Prop. 50.6] for the nuclearity of E'_β when E is nuclear Fréchet), [**Tre**, Prop. 50.5] gives that the target of (C.3.1.1) is complete and the resulting continuous linear extension in (C.3.1.2) is a topological isomorphism.

To extend the concept of nuclearity for continuous linear maps $T : E \to F$ beyond the Banach setting, consider a convex circled absorbing subset $U \subset E$ (e.g., a convex circled 0-neighborhood). By the absorbing property, U spans E linearly. The separated completion of the seminormed space (E, p_U), with $p_U : E \to \mathbf{R}_{\geq 0}$ the Minkowski functional of U from (C.1.2.1), is denoted $\widehat{E}_U$ and is a Banach space.

Given instead a *bounded* convex circled subset $B \subset E$, the linear span $E_B = \cup_{m\geq 1} mB$ of B in E equipped with its Minkowski functional $p_B : E_B \to \mathbf{R}_{\geq 0}$ is also a normed space (i.e. its semi-normed topology is already separated). If (E_B, p_B) is a Banach space already before completion, we refer to B as a *Banach disc*.

For E, F two locally convex topological **C**-vector spaces, a continuous linear operator $T : E \to F$ is termed *nuclear* if and only if there exists a convex circled 0-neighborhood $U \subset E$ and a Banach disc B in F, such that

$T(U) \subset B$, and the induced linear map $\overline{T} : \widehat{E}_U \to F_B$ is nuclear within $\mathscr{L}(\widehat{E}_U, F_B)$ between Banach spaces; i.e., it is of the form recalled in (C.2.4.2).

Every nuclear operator between two Banach spaces E, F is in particular a limit in the topology of bounded convergence (and so in any coarser $\mathfrak{S}$-topology) of finite-rank operators since (C.3.1.1) is continuous. This holds more generally for nuclear maps between Hausdorff locally convex topological $\mathbf{C}$-vector spaces, with the explicit description (C.2.4.2) in the Banach case replaced by a mildly generalized description as follows (see [**Scha**, Ch. III, 7.1, Remark] for a proof):

LEMMA C.3.1.2. *Let E, F be Hausdorff locally convex topological $\mathbf{C}$-vector spaces. A linear map $T \in \mathscr{L}(E, F)$ is nuclear if and only if there exist:*

(1) *a sequence $\{\lambda_n\}_{n\geq 1} \subset \mathbf{C}$ such that $\sum_{n\geq 1} |\lambda_n| < \infty$,*
(2) *an equicontinuous sequence $\{\varphi_m\}_{m\geq 1} \subset E'$,*
(3) *a sequence $\{x_m\}_{m\geq 1} \subset B$ for a bounded B that is a Banach disc of F*

such that

$$T = \sum_{n\geq 1} (\lambda_n \langle \cdot, \varphi_m \rangle_{E,E'}) x_m \tag{C.3.1.3}$$

as $\mathbf{C}$-linear operators on E. It can be further arranged that $\varphi_m \to 0$ uniformly on some 0-neighborhood $U \subset E$ and $x_m \to 0$ in F_B (so also $x_m \to 0$ in F).

Sums as on the right side of (C.3.1.3) are always convergent in the topology of uniform convergence on the bounded subsets of E. It follows from the lemma that nuclear maps u are compact in the sense that the closure $\overline{u(U)} \subset F$ is compact for some 0-neighborhood $U \subset E$.

C.3.2. Nuclear operators picking up their nuclear trace from a subspace. The following result asserts that nuclear endomorphisms of a Hausdorff locally convex topological $\mathbf{C}$-vector space with the approximation property have a well-defined trace:

PROPOSITION C.3.2.1. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. Assume E has the approximation property as in Lemma* C.2.4.15. *Let $A \in \mathscr{L}(E)$ be a nuclear operator, and let*

$$A = \sum_{m\geq 0} \lambda_m \langle \cdot, \varphi_m \rangle_{E,E'} x_m \tag{C.3.2.1}$$

be a representation for A as in (C.3.1.3). *Then the absolutely convergent sum*

$$\mathrm{Trace}(A) := \sum_{m\geq 0} \lambda_m \langle x_m, \varphi_m \rangle_{E,E'} \tag{C.3.2.2}$$

is well posed; i.e., it does not depend on the choices of $\{\lambda_m\}_{m\geq 0} \subset \mathbf{C}$, $\{\varphi_m\}_{m\geq 0} \subset E'$ and $\{x_m\}_{m\geq 0} \subset E$, but only on A.

If $T \in \mathscr{L}(E)$ is a linear topological automorphism then $T^{-1}AT$ is nuclear and

$$\mathrm{Trace}(A) = \mathrm{Trace}(T^{-1} \circ A \circ T). \tag{C.3.2.3}$$

PROOF. This is [**Lit-I**, §4.2, Thm. 1] (which uses the approximation property). We now prove the identity in (C.3.2.3). Let $B \subset E$ be a Banach disc containing $\{x_m\}_{m\geq 0}$. Then $T^{-1}(B) \subset E$ is clearly a Banach disc, and since T^{-1} induces a continuous isomorphism $E_B \to E_{T^{-1}(B)}$ with inverse is induced by T (so $E_{T^{-1}(B)}$ is a Banach space).

The set $\{T'(\varphi_m)\}_{m\geq 0} \subset E'$ is clearly equicontinuous. It follows that

$$T^{-1} \circ A \circ T = \sum_{m\geq 0} \lambda_m \langle \cdot, T'(\varphi_m) \rangle_{E,E'} T^{-1}(x_m)$$

is a nuclear presentation for $T^{-1} \circ A \circ T$ on E, and by (C.3.2.2) we have

$$\begin{aligned} \text{(C.3.2.4)} \qquad \mathrm{Trace}(T^{-1} \circ A \circ T) &= \sum_{m \geq 0} \lambda_m \langle T^{-1}(x_m), T'(\varphi_m) \rangle_{E,E'} \\ &= \sum_{m \geq 0} \lambda_m \langle T T^{-1}(x_m), \varphi_m \rangle_{E,E'} \\ &= \sum_{m \geq 0} \lambda_m \langle x_m, \varphi_m \rangle_{E,E'} \\ &= \mathrm{Trace}(A). \end{aligned}$$

□

We refer to $\mathrm{Trace}(A)$ as defined in (C.3.2.2) to the *nuclear trace* of A. The following result records some stability properties of the nuclear trace of certain nuclear operators.

LEMMA C.3.2.2 (Picking up traces through continuous linear injections). *Let E_1, E_2 be two Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Let $A : E_2 \to E_2$ be a continuous endomorphism. Assume:*

(1) *E_1, E_2 both have the approximation property,*
(2) *there exists an injective continuous operator $T : E_1 \to E_2$,*
(3) *there exists a continuous linear operator $A_0 : E_2 \to E_1$ with $A = T \circ A_0$,*
(4) *A_0 is nuclear.*

Then $A_1 := A_0 \circ T : E_1 \to E_1$ is nuclear and

$$\text{(C.3.2.5)} \qquad \mathrm{Trace}(A_1) = \mathrm{Trace}(A).$$

To explain the intuition behind this lemma before we carry out the proof (which uses an argument adapted from [**Me1**, §4, Cor. 4.7]), we give a commutative diagram summarizing some of the information (with T injective):

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ {\scriptstyle A_1}\downarrow & \swarrow_{A_0} & \downarrow{\scriptstyle A} \\ E_1 & \xrightarrow[T]{} & E_2 \end{array}$$

PROOF. Assume first that A is of finite rank, so $A(E_2)$ is topologically complemented in E_2 by [**Scha**, Ch. II, Cor. 2 to 4.2]. There exist finitely many elements $\{x_1, \ldots, x_N\} \subset A(E_2) \subset T(E_1) \subset E_2$, continuous $\{\varphi_1, \ldots, \varphi_N\} \subset E_2'$, and complex numbers $\lambda_1, \ldots, \lambda_N \in \mathbf{C}$, such that

$$\text{(C.3.2.6)} \qquad A = \sum_{1 \leq m \leq N} \lambda_m \langle \cdot, \varphi_m \rangle_{E_2, E_2'} \, x_m.$$

Since T is injective, there therefore exist unique $y_1, \ldots, y_N \in E_1$ such that $T(y_m) = x_m$ for $1 \leq m \leq N$ and we have

$$\text{(C.3.2.7)} \qquad T \circ A_1 = A \circ T = \sum_{1 \leq m \leq N} \lambda_m \langle \cdot, T'(\varphi_m) \rangle_{E_1, E_1'} \, x_m = T\left(\sum_m \lambda_m \langle \cdot, T'(\varphi_m) \rangle_{E_1, E_1'} \, y_m \right),$$

so

$$A_1 = \sum_{1 \leq m \leq N} \lambda_m \langle \cdot, T'(\varphi_m) \rangle_{E_1, E_1'} \, y_m.$$

Thus,

$$\begin{aligned}\mathrm{Trace}(A_1) &= \sum_{1\leq m\leq N} \lambda_m \langle y_m, T'(\varphi_m)\rangle_{E_1,E_1'} \\ &= \sum_{1\leq m\leq N} \lambda_m \langle T(y_m), \varphi_m\rangle_{E_2,E_2'} \\ &= \sum_{1\leq m\leq N} \lambda_m \langle x_m, \varphi_m\rangle_{E_2,E_2'} \\ &= \mathrm{Trace}(A).\end{aligned}$$

This means that the identity in (C.3.2.5) holds when A has finite rank.

Since E_2 has the approximation property (see Lemma C.2.4.15), we may approximate the map $A_0 \in \mathscr{L}(E_2, E_1)$ (obtained as the unique continuous linear $E_2 \to E_1$ such that $A = T \circ A_0$) by a net $\{A_k\}_{k\in I}$ in the image of $E_1 \otimes E_2'$ with respect to the topology of uniform convergence on pre-compact subsets (see Definition C.2.1.5(2)).

Both nets $A_k \circ T$ and $T \circ A_k$ converge in $\mathscr{L}_\gamma(E_1)$, respectively $\mathscr{L}_\gamma(E_2)$; i.e., with respect to the topology of uniform convergence on pre-compact subsets. The limit of the former is $A_0 \circ T = A_1$, and the limit of the latter is $T \circ A_0 = A$, and since $\mathrm{Trace}(A_k \circ T) = \mathrm{Trace}(T \circ A_k)$ by the foregoing, which establishes this identity for every $k \in I$ since $T \circ A_k$ is of finite rank, we have that (C.3.2.5) holds by adapting the method of proof of [**Lit-I**, §4.2,Thm.1] (which only considers endomorphisms, rather than intervening maps such as T that are not endomorphisms), as can be done since A and A_1 inherit nuclearity from A_0 (by [**Scha**, Ch. III, Cor. 2, to 7.1]) and the topology of uniform convergence on all absolutely convex compact subsets is coarser than the topology of uniform convergence on all pre-compact subsets. □

C.3.3. Topologies on the endomorphism algebra of a Hilbert space. We recall that a Hausdorff locally convex topological complex vector space E is *complete* if and only if it is complete as a uniform space (every topological vector space has a canonical uniform structure, and it is with respect to this structure that completeness is defined in [**Scha**, Ch. I, pp. 6-10]), equivalently if and only if every Cauchy net converges to an element of E. We further refer the reader to [**NB**, §3.1] for more on completeness, and to [**NB**, Thm. 3.1.6] for the dictionary between Cauchy nets and Cauchy filter-bases.

Completeness for a Hausdorff locally convex topological vector space E that is already quasi-complete is a genuinely stronger condition than quasi-completeness. The key issue is that completeness entails every Cauchy net converging to an element of E, while quasi-completeness is the condition that any *bounded* Cauchy net is convergent. Cauchy *sequences* are bounded, but generally not Cauchy nets.

For example, for a Fréchet space E (so it is barreled [**Scha**, Ch. II, Cor. to 7.1]), the weak topology E'_σ on E' is quasi-complete by Lemma C.2.1.7 but the topological embedding $E'_{\sigma(E',E)} \hookrightarrow E^*_{\sigma(E^*,E)}$ is identified with the completion (see Lemma C.4.1.2 for a self-contained argument via some literature references). The algebraic dual E^* strictly contains E' when E is infinite-dimensional, so in such cases E'_σ is quasi-complete but not complete. In *metrizable* locally convex topological vector spaces, it was explained after Definition C.1.3.1 that the concepts of completeness and metric (equivalently sequential) completeness coincide, and these both agree with quasi-completeness because Cauchy sequences in a Hausdorff topological vector space are always bounded.

For H a complex Hilbert space, we denote by $\mathscr{B}(H)$ the $\mathbf{C}$-vector space of all bounded linear operators on H. Recall that the strong operator topology, denoted "SOT", on $\mathscr{B}(H)$ is the topology of pointwise convergence, generated by the family of seminorms

$$q_u(T) := \|T(u)\|_H,\ u \in H \tag{C.3.3.1}$$

where $\|\cdot\|_H$ is the norm on H. The operator-norm of a bounded linear operator $T : H \to H$ is defined as

$$\|T\| := \sup_{u\in H, \|u\|_H \leq 1} \|T(u)\| \tag{C.3.3.2}$$

and promotes $\mathscr{B}(H)$ to a Banach space. We denote it by $\mathscr{B}_{\mathrm{SOT}}(H)$ when endowed with the strong operator topology, and $\mathscr{B}_{\mathrm{NT}}(H)$ when endowed with the operator-norm topology.

LEMMA C.3.3.1 (Properties of the strong-operator topology). *Let H be a complex Hilbert space.*

(1) *If H is separable then every bounded subset in $\mathscr{B}_{\mathrm{SOT}}(H)$ is metrizable.*
(2) *$\mathscr{B}_{\mathrm{SOT}}(H)$ is Hausdorff locally convex and quasi-complete.*

PROOF. (1) : This is [**Cnw-a**, Ch. IX, Prop. 1.3(e)], where "bounded" is meant for the norm topology on $\mathscr{B}(H)$, which is the same as boundedness in $\mathscr{B}_{\mathrm{SOT}}(H)$ by [**Scha**, Ch. III, 3.4].

(2) : The locally convex strong operator topology on $\mathscr{B}(H)$ is clearly Hausdorff. Let $\{T_i\}_{i\in I}$ be a bounded Cauchy net in $\mathscr{B}_{\mathrm{SOT}}(H)$, with I a directed set. Since the evaluation maps $e_\phi : \mathscr{B}(H) \to H$ sending T to $e_\phi := T(\phi)$ for every $\phi \in H$ are continuous with respect to the strong operator topology, the net $\{T_i(\phi)\}_{i\in I}$ is Cauchy and hence it converges in H since H is complete. We call $T(\phi)$ the limit in H. By the Uniform Boundedness Principle [**Scha**, Ch. III, Cor. to 4.2], $\sup_{i\in I} \|T_i\|$ is finite. Thus, T is a bounded linear operator on H and T_i converges to T in the strong operator topology by design. This proves that every bounded Cauchy net of $\mathscr{B}_{\mathrm{SOT}}(H)$ converges in $\mathscr{B}_{\mathrm{SOT}}(H)$. □

We further refer the reader to Remark 4.3.1.6, for more on the topology of $\mathscr{B}_{\mathrm{SOT}}(H)$ specifically, and on topological properties of the topological group of unitary operators on a Hilbert space in the special case $H = L^2(G, \mu)$ for a second-countable Hausdorff locally compact abelian group G and a choice of Haar measure μ on G.

C.3.4. Trace-class operators on Hilbert spaces. Hilbert spaces have the approximation property by [**Scha**, Ch. III, Cor. 1 to 9.2]. Nuclear endomorphisms of a Hilbert space are exactly the trace-class operators (see [**Kub**, Thm.6.1]), and serve as a familiar example.

C.3.4.1. *Trace-class operators: definitions and basic properties.* In the special case of Hilbert spaces, we favor the notation $\mathscr{B}_{\mathrm{NT}}(H)$ over $\mathscr{L}_\beta(H)$, and $\mathscr{B}_{\mathrm{SOT}}(H)$ over $\mathscr{L}_\sigma(H)$, the former being endowed with the norm topology, the latter with the strong operator topology.

LEMMA C.3.4.1. *Let H be a Hilbert space. Let Σ be an orthonormal basis of H. If $T \in \mathscr{B}_{\mathrm{NT}}(H)$ is nuclear (equivalently, trace-class) then:*

(1) *the infinite sum $\sum_{\phi\in\Sigma} |\langle T(\phi), \phi\rangle_H|$ converges;' i.e., $\langle T(\phi), \phi\rangle_H$ vanishes for all but countably many ϕ, and the sum of the countably many nonzero ones is absolutely convergent.*
(2) *The value of the sum $\sum_{\phi\in\Sigma} |\langle T(\phi), \phi\rangle_H|$ is independent of the choice of Σ.*
(3) *The value of the absolutely convergent sum $\sum_{\phi\in\Sigma} \langle T(\phi), \phi\rangle_H$ is independent of the choice of Σ.*

We refer the reader to [**Cnw-c**, Ch. 3, Prop. 18.9] for the proof of Lemma C.3.4.1. The following is [**Cnw-c**, Ch. 3, Def. 18.10]:

DEFINITION C.3.4.2. Let H be a Hilbert space, and $T \in \mathscr{B}_{\mathrm{NT}}(H)$ a nuclear operator. We define:

(1) the trace $\mathrm{Trace}(T)$ of T to be the value of the sum $\sum_{\phi\in\Sigma} \langle T(\phi), \phi\rangle_H$ for any choice of orthonormal basis Σ of H.
(2) The trace-norm of T to be $\|T\|_{\mathrm{Tr}} := \sum_{\phi\in\Sigma} \langle |T|(\phi), \phi\rangle_H$, where $|T| := \sqrt{T^*T}$ via the Spectral Theorem for T^*T (finiteness of $\|T\|_{\mathrm{Tr}}$ is the definition of trace-class).

We denote by $\mathrm{N}(H)$ the subspace of $\mathscr{B}_{\mathrm{NT}}(H)$ of nuclear endomorphisms of H.

Lemma C.3.4.1 implies that the concepts defined in Definition C.3.4.2 are well-posed. We will now briefly review their properties, collecting the ones required, below.

LEMMA C.3.4.3. *Let H be a Hilbert space, A a compact operator on H, $Q \in \mathscr{B}(H)$. Let $\{Q_m\}_{m\geq 0}$ be a sequence of bounded operators on H such that $Q_m \to Q$ in $\mathscr{B}_{\mathrm{SOT}}(H)$. Then $Q_m A \to QA$ with respect to the operator-norm topology, i.e. in $\mathscr{B}_{\mathrm{NT}}(H)$.*

PROOF. By [**Cnw-a**, Ch. II, Thm. 4.4(c)], there exists a sequence $\{A_k\}_{k\geq 0}$ of operators on H of finite rank, such that $\|A_k - A\| \to 0$ as $k \to +\infty$. Let $\varepsilon > 0$ and choose $k_0 \geq 0$ so large that we have $\|A_k - A\| < \varepsilon/2$ for all $k \geq k_0$.

We have $\|Q_m(A_k - A)\| \leq \|Q_m\|\,\|A_k - A\|$, which is bounded above by $\frac{\varepsilon}{2}\|Q_m\|$ for all $k \geq k_0$. Since $\{Q_m\}_{m\geq 0} \subset \mathscr{B}_{\mathrm{SOT}}(H)$ is a convergent Cauchy sequence, hence bounded, by the Uniform Boundedness Principle there exists a positive real number $L > 0$ such that $\sup_{m\geq 0}\|Q_m\| < L$. It follows that we have, for every $k \geq k_0$ and every integer $m \geq 0$:

$$\|Q_m(A_k - A)\| \leq \frac{L\varepsilon}{2}. \tag{C.3.4.1}$$

We deduce the inequality, for every pair of integers $m, m' \geq 0$ and every $k \geq k_0$:

$$\|(Q_m - Q_{m'})A\| \leq \|(Q_m - Q_{m'})A - (Q_m - Q_{m'})A_k)\| + \|(Q_m - Q_{m'})A_k\| \leq L\varepsilon + \|(Q_m - Q_{m'})A_k\|. \tag{C.3.4.2}$$

Being $\varepsilon > 0$ arbitrary and independent of $m, m' \geq 0$, it is enough to replace A with A_{k_0}, so we are reduced to the case A is of finite rank.

Since A is therefore a finite sum of rank-1 operators, we may and do assume A is an operator of rank 1; i.e., a continuous functional on H. Thus, upon choosing $\phi_0 \neq 0$ in the range of A, there exists a unique $\phi \in H$ such that $A = \langle \phi, \cdot \rangle_H \, \phi_0$.

We have:

$$Q_m A = \langle \phi, \cdot \rangle_H \, Q_m \phi_0$$

and since $Q_m - Q$ tends to zero with respect to the strong operator topology, we have $Q_m\phi_0 \to Q\phi_0$ with respect to the norm topology on H, and hence $Q_m A \to QA$ with respect to the operator-norm. □

LEMMA C.3.4.4. *Let H be a Hilbert space. Then:*

(1) *$\mathrm{N}(H)$ is a two-sided ideal of the $\mathbf{C}$-algebra $\mathscr{B}(H)$, and $\|\cdot\|_{\mathrm{Tr}}$ is a norm on $\mathrm{N}(H)$, promoting $\mathrm{N}(H)$ to a normed space.*

(2) *For any $T \in \mathrm{N}(H)$ and $A \in \mathscr{B}(H)$, we have*

$$\|AT\|_{\mathrm{Tr}} = \|TA\|_{\mathrm{Tr}} \leq \|A\|\,\|T\|_{\mathrm{Tr}}. \tag{C.3.4.3}$$

(3) *The linear map $\mathrm{Trace} : \mathrm{N}(H) \to \mathbf{C}$ satisfies $|\mathrm{Tr}(A)| \leq \|A\|_r$, so it is continuous with respect to the $\|\cdot\|_{\mathrm{Tr}}$-topology.*

(4) *$\mathrm{N}(H)$ contains the range of $H' \otimes_\pi H \to \mathscr{B}_{\mathrm{NT}}(H)$ as a dense linear subspace. In particular, $\mathrm{N}(H)$ is separable as a normed space if H is separable as a Hilbert space.*

(5) *Let $\{P_i\}_{i\in I} \subset \mathscr{B}(H)$ be a net of bounded operators strongly converging to zero. Let $F \in \mathscr{B}(H)$ be a finite-rank operator. Then $\{P_iF\}_{i\in I}$ converges to zero in $\mathrm{N}(H)$, i.e. with respect to the trace-norm topology.*

PROOF. Since the concepts of nuclear operator and of trace-class operator are equivalent for endomorphisms of a Hilbert space (see [**Kub**, Thm.6.1]), we use the theory of trace-class operators.

(1) : This is [**Cnw-c**, Ch. 3, Thm. 18.11(a)].

(2) : This is [**Cnw-c**, Ch. 3, Thm. 18.11(g, e)].

(3) : This is [**Cnw-c**, Ch. 3, Thm. 18.11(c)] (applied to $T = \mathrm{Id}$).

(4) : This is [**Cnw-c**, Ch.3, Thm.18.11(d)].

(5) : Let m be the rank of F, and let $\{\alpha_{i,1}, \ldots, \alpha_{i,k_i}\}$ be (without multiplicity) the eigenvalues of $|P_iF|$, which is a positive, compact, self-adjoint operator. Note that since P_iF is of finite rank, so is $(P_iF)^* \cdot (P_iF)$, and hence so is its square-root $|P_iF|$. The ranks of each are bounded above by m, and hence $k_i \leq m$, and each $\alpha_{i,n}$ is a nonnegative real number. We may order the $\alpha_{i,n}$ increasingly, and assume α_{i,k_i} is their maximum. Since F is finite-rank and $P_i \to 0$ in $\mathscr{B}_{\mathrm{SOT}}(H)$, then $P_iF \to 0$ in norm. The norm of each P_iF is α_{i,k_i}, and hence the net $\{\alpha_{i,k_i}\}$ tends to zero.

The sum $\sum_{1 \leq n \leq k_i} \alpha_{i,n}$ is the trace-norm of P_iF for each $i \in I$, and since $k_i \leq m$ for each $i \in I$, we have $\|P_iF\|_{\mathrm{Tr}} \leq m\alpha_{i,k_i} \to 0$, and so $P_iF \to 0$ in $\mathrm{N}(H)$. □

C.3.5. Convergence properties of nuclear operators on Hilbert spaces. The following Proposition enables to reduce trace calculations for nuclear operators on a Hilbert space to calculations on closed subspaces.

Proposition C.3.5.1 (Nuclear convergence of left-compositions with orthogonal projectors). *Let H be a separable Hilbert space, $\{H_i\}_{i \in I}$ a set of nonzero closed subspaces such that for every $i, i' \in I$, with $i \neq i'$, H_i and $H_{i'}$ are orthogonal in H, so I is countable.*

Let $\mathcal{F}_I$ be the countable set of finite subsets of I, partially ordered by inclusion.

Let:

$$P_i : H \to H_i \tag{C.3.5.1}$$

be the orthogonal projection operator onto H_i. Let $T \in \mathrm{N}(H)$ be a nuclear operator.

Assume that the net $\{\sum_{i \in \Sigma} P_i\}_{\Sigma \in \mathcal{F}_I}$ converges in $\mathscr{B}_{\mathrm{SOT}}(H)$ to the identity (i.e., $\widehat{\oplus} H_I = H$).

(1) *For any cofinal subsequence $\{\Sigma_k\}_{k \geq 0} \subset \mathcal{F}_I$, the sequence*

$$\{\sum_{i \in \Sigma_k} P_iT\}_{k \geq 0} \subset \mathrm{N}(H) \tag{C.3.5.2}$$

converges in $\mathrm{N}(H)$ to T, i.e. with respect to the normed topology induced by the trace-norm $\|\cdot\|_{\mathrm{Tr}}$ on $\mathrm{N}(H)$.

(2) *Let $\{\Sigma_k\}_{k \geq 0} \subset \mathcal{F}_I$ be a cofinal subsequence. The trace of the nuclear operator T is the limit of the sequence of complex numbers:*

$$\left\{\sum_{i \in \Sigma_k} \mathrm{Trace}(P_iT)\right\}_{k \geq 0}. \tag{C.3.5.3}$$

Remark C.3.5.2. We observe that given a Hilbert space H, a set I and a collection $\{H_i\}_{i \in I}$ of closed subspaces such that for every pair of indeces $i, i' \in I$, with $i \neq i'$, H_i is orthogonal to $H_{i'}$, requiring that the orthogonal projectors $P_i : H \to H_i$ satisfy, upon defining for every finite subset $\Sigma \subset I$, $P_\Sigma := \sum_{i \in \Sigma} P_i$, that the net $\Sigma \mapsto P_\Sigma$ on the set of all finite subsets of I, converges to the identity with respect to the strong operator topology on $\mathscr{B}(H)$ is equivalent to requiring that $\bigoplus_{i \in I} H_i \subset H$ is dense.

Indeed, for every $\phi \in (\bigoplus_{i \in I} H_i)^\perp \subset H$, we have $P_\Sigma \phi \to \phi$ in H, while $P_\Sigma \phi = 0$ for every finite subset $\Sigma \subset I$, and hence $\phi = 0$. By [**Cnw-a**, Ch. I, Cor. 2.11], $\bigoplus_{i \in I} H_i$ is dense as a linear subspace of H.

Conversely, for any $\phi \in H$, we have $P_\Sigma \phi - \phi \in (\bigoplus_{i \in \Sigma} H_i)^\perp$, and we may find, by denseness of $\bigoplus_{i \in I} H_i$ in H and [**Cnw-a**, Ch. I, Cor. 2.11], for each real number $\varepsilon > 0$, a large enough finite subset $\Sigma_\varepsilon \subset I$ such that $\|P_{\Sigma_\varepsilon} \phi - \phi\|_H$ is smaller than ε, which shows that $\{P_\Sigma\}_{\Sigma \in \mathcal{F}_I}$ converges in $\mathscr{B}_{\mathrm{SOT}}(H)$ to the identity.

C.3.5.1. *Lemmas on singular values.* Before proving Proposition C.3.5.1, we need two lemmas on singular values.

Lemma C.3.5.3 (Min-Max formulas for singular values). *Let H be a Hilbert space, A a compact operator on H. Let $\{\sigma_m(A)\}_{m \geq 1}$ be its sequence of singular values, labelled decreasingly with multiplicity. Then:*

(1) *For every integer* $1 \leq m \leq \dim(H)$*, we have* $\sigma_m(A) = \sigma_m(A^*)$*. (The operator* A^* *is compact by* [**Cnw-a**, Ch. II, Thm. 4.4(a)].)
(2) $\sigma_1(A) = \|A\|$.
(3) *The following formulas hold for all integers* $1 \leq m \leq \dim(H)$*:*

$$\sigma_m(A) = \min_{F \in H_{m-1}} \max_{\phi \in F^\perp, \|\phi\|_H = 1} \|A\phi\|_H = \max_{F \in H_m} \min_{\phi \in F, \|\phi\|_H = 1} \|A\phi\|_H \tag{C.3.5.4}$$

where H_m *is the set of all* m*-dimensional subspaces of* H*.*

PROOF. The singular values of A are the eigenvalues of the compact and positive operator $|A|$, which is self-adjoint, being positive on a Hilbert space. We have $\|A\phi\|_H = \||A|\phi\|_H$, and hence we may replace A with $|A|$ in (2) and (3), with (2) contained in (3) as the special case $m = 1$.

Since $|A|$ is compact, positive, and self-adjoint, by the Spectral Theorem [**Cnw-a**, Ch. II, Thm. 5.1, Cor. 5.3] there exists a monotonically decreasing sequence of nonnegative real numbers $\{\lambda_m\}_{m \geq 1}$ tending to 0 such that upon denoting by P_m the orthogonal projector onto $\ker(\mathbf{C}e_m)$ for a unit λ_m-eigenvector e_m of $|A|$, we have

$$|A| = \sum_{m \geq 1} \lambda_m P_m = \sum_{m \geq 1} \lambda_m \langle \cdot, e_m \rangle_H \, e_m. \tag{C.3.5.5}$$

The convergence of both sums in (C.3.5.5) is with respect to the operator-norm topology on $\mathscr{B}(H)$.

Let $F \subset \mathbf{C}$ be a finite-dimensional subspace of dimension $m \leq \dim(H)$, and call $V \subset H$ the closure of the linear subspace linearly generated by the $\lambda_{m'}$-eigenspaces for all $m' \geq m$. Then V has co-dimension $m - 1$ in H, and hence $V \cap F \neq 0$. We let $\phi \in V \cap F$ be a unit vector.

Since $\phi \in V$, then $\langle A\phi, \phi \rangle_H \leq \lambda_m$. Indeed, we can write $\phi = \sum_{m' \geq m} c_{m',k} e_{m'}$ for complex numbers $c_{m'}$ such that $\sum_{m' \geq m} |c_{m'}|^2 = 1$. Then

$$\langle A\phi, \phi \rangle_H = \left\langle \sum_{m' \geq m} \lambda_{m'} c_{m'} e_{m'}, \sum_{m' \geq m} c_{m'} e_{m'} \right\rangle_H = \sum_{m' \geq m} \lambda_{m'} |c_{m'}|^2 \tag{C.3.5.6}$$

and the right side of (C.3.5.6) is bounded above by $\lambda_m \sum_{m' \geq m} |c_{m'}|^2 = \lambda_m$. It follows that

$$\inf_{\phi \in F, \|\phi\|_H = 1} \langle A\phi, \phi \rangle_H \leq \lambda_m. \tag{C.3.5.7}$$

The function $f : H \to \mathbf{C}$ defined by $f(\phi) := \langle A\phi, \phi \rangle_H$ is weakly continuous, since A is compact. Indeed, for $\{\phi_k\}_{k \geq 0} \subset H$ a sequence converging weakly to ϕ, so $\{\phi_k\}$ is weakly bounded and hence bounded, we have $A\phi_k \to A\phi$ strongly (this is clear for A of finite rank, and then for general compact A as a uniform limit of finite-rank operators), and so $f(\phi_k) \to f(\phi)$ (since $f(\phi_k) - f(\phi) = \langle A\phi_k - A\phi, \phi_j \rangle_H = \langle A\phi, \phi_k - \phi \rangle_H$); i.e., f is continuous with respect to the $\sigma(H, H^*)$-topology on the source.

By [**Cnw-a**, Ch. V, Thm. 4.2(d)], the closed unit ball B in F is weakly compact in H, so $F \cap B$ is too, so f attains a maximum and a minimum over $F \cap B$. We may therefore replace the infimum in (C.3.5.7) by a minimum.

We deduce that we have, for every integer $m \geq 1$:

$$\sup_{F \in H_m} \min_{\phi \in F, \|\phi\|_H = 1} \langle A\phi, \phi \rangle_H \leq \lambda_m. \tag{C.3.5.8}$$

Upon choosing F to be the span of $e_1, \ldots, e_m$, the value of the the supremum in (C.3.5.7) is λ_m.

For A replaced by $|A|^2$ and λ_m by $\sigma_m(A)^2$, since $\|A\phi\|_H^2 = \langle |A|^2 \phi, \phi \rangle_H$ we have:

$$\max_{F \in H_m} \min_{\phi \in F, \|\phi\|_H = 1} \|A\phi\|_H^2 = \max_{F \in H_m} \min_{\phi \in F, \|\phi\|_H = 1} \langle |A|^2 \phi, \phi \rangle_H = \sigma_m(A)^2. \tag{C.3.5.9}$$

But $\max_{F\in H_m}\min_{\phi\in F,\|\phi\|_H=1}\|A\phi\|_H^2=\left(\max_{F\in H_m}\min_{\phi\in F,\|\phi\|_H=1}\|A\phi\|_H\right)^2$, as the squaring map on $\mathbf{R}_{\geq 0}$ is monotonically non-decreasing, so the outer terms in (C.3.5.4) coincide.

The remaining identity in (C.3.5.4) is proved in a similar manner, using the weak continuity of f and the weak compactness of closed and bounded subsets of H, to replace the supremum with a maximum over the elements of $\phi\in F^\perp$ satisfying $\|\phi\|_H=1$. This settles (2) and (3).

It remains to prove (1). We write $A=\sum_{m\geq 1}\sigma_m(A)\langle\cdot,\phi_m\rangle_H\,\psi_m$ for orthonormal sets $\{\phi_m\}_{m\geq 1},\{\psi_m\}_{m\geq 1}$, which can be done by applying [**Cnw-a**, Ch. II, Cor. 5.4] to $|A|$ and using Polar Decomposition of A: the existence of $U\in\mathscr{B}(H)$ which is isometric on the range of A and zero on its orthogonal, such that $A=U|A|$. Namely, upon defining $\{\phi_m\}_{m\geq 1}$ to be an orthonormal basis for $\ker(|A|)^\perp$ in H, diagonalizing $|A|$ as per [**Cnw-a**, Ch. II, Cor.5.3(b)] with eigenvalues $\sigma_m(A)$ for each $m\geq 1$ respectively, we then define $\psi_m:=U\phi_m$.

Then we have $A^*=\sum_{m\geq 1}\sigma_m(A)\langle\psi_m,\cdot\rangle_H\,\phi_m$. This implies that we have:

$$AA^*=\sum_{m\geq 1}\sigma_m(A)^2\langle\cdot,\psi_m\rangle_H\,\psi_m \tag{C.3.5.10}$$

and hence the singular values $\{\sigma_m(A^*)\}_{m\geq 1}$ of A^* being the nonnegative square roots of the eigenvalues of AA^* are exactly the nonnegative real numbers $\sigma_m(A)$, for each $m\geq 1$ and counted with multiplicity. □

LEMMA C.3.5.4 (Estimates on singular values). *Let H be a Hilbert space. Let $A,B\in\mathscr{B}(H)$ be bounded endomorphisms of H. Assume that A is compact. Then:*

(1) *AB and BA are compact,*
(2) *upon denoting by $\{\sigma_m(A)\}_{m\geq 1}$, $\{\sigma_m(B)\}_{m\geq 1}$ and $\{\sigma_m(BA)\}_{m\geq 1}$, $\{\sigma_m(AB)\}_{m\geq 1}$ the respective sequences of singular values ordered decreasingly, we have:*

$$\sigma_m(AB)\leq\sigma_m(A)\,\|B\|,\quad \sigma_m(BA)\leq\sigma_m(A)\,\|B\| \tag{C.3.5.11}$$

for every integer $1\leq m\leq\dim(H)$.

PROOF. (1) : This is classical: see [**Cnw-a**, Ch. II, Prop. 4.2(c)].

(2) : By the first of the two formulas in (C.3.5.4) from Lemma C.3.5.3 for the singular values of the compact operators A,AB,BA, we have, for each integer $1\leq m\leq\dim(H)$:

$$\sigma_m(BA)=\inf_{F\in H_{m-1}}\sup_{\phi\in F^\perp,\|\phi\|_H=1}\|BA\phi\|_H \tag{C.3.5.12}$$

where H_{m-1} is the set of all $(m-1)$-dimensional $\mathbf{C}$-linear subspaces of H.

Upon fixing $F\in H_{m-1}$, we have:

$$\sup_{\phi\in F^\perp,\|\phi\|_H=1}\|BA\phi\|_H\leq\|B\|\sup_{\phi\in F^\perp,\|\phi\|_H=1}\|A\phi\|_H. \tag{C.3.5.13}$$

Passing to the infimum over all $F\in H_{m-1}$, we get:

$$\sigma_m(BA)=\inf_{F\in H_{m-1}}\sup_{\phi\in F^\perp,\|\phi\|_H=1}\|BA\phi\|_H\leq\|B\|\inf_{F\in H_{m-1}}\sup_{\phi\in F^\perp,\|\phi\|_H=1}\|A\phi\|_H=\sigma_m(A)\,\|B\|. \tag{C.3.5.14}$$

The analog of the inequality in (C.3.5.14) with BA in place of AB holds too, since A^* is compact:

$$\sigma_m(AB)=\sigma_m(B^*A^*)\leq\|B^*\|\,\sigma_m(A^*)=\|B\|\,\sigma_m(A). \tag{C.3.5.15}$$

The first and last identities in (C.3.5.15) use Lemma C.3.5.3(1) and that $\|B\|=\|B^*\|$. □

We can now prove Proposition C.3.5.1, bearing in mind from [**Kub**, Thm.6.1] that the concepts of nuclear and trace-class operator on a Hilbert space coincide.

PROOF. (1) : Each P_iT is nuclear since it is of finite rank. We observe that for any *finite* subset $\Sigma\subset I$, the subspace topology on $\bigoplus_{i\in\Sigma}H_i$ is a closed subspace of H.

Upon calling $P_\Sigma := \sum_{i\in\Sigma} P_i$, then P_Σ is itself an orthogonal projector onto $\bigoplus_{i\in\Sigma} H_i$. In particular, it satisfies $\|P_\Sigma\| \leq 1$. By the same token, $\mathrm{Id} - P_\Sigma$ also satisfies $\|\mathrm{Id} - P_\Sigma\| \leq 1$, because $\mathrm{Id} - P_\Sigma$ is itself an orthogonal projector onto $(\oplus H_i)^\perp$.

Since, T and $P_i T$ are compact for every $i \in I$, we deduce that for every integer $m \geq 1$, we have by (C.3.5.11) from Lemma C.3.5.4(2):

$$0 \leq \sigma_m(T - P_\Sigma T) \leq \|\mathrm{Id} - P_\Sigma\|\, \sigma_m(T) \leq \sigma_m(T)$$

and since T is nuclear we have by [**Cnw-c**, Ch. 3, Thm. 18.11(b)] that the series $\sum_{m\geq 1} \sigma_m(T)$ converges (absolutely, since each $\sigma_m(T)$ is nonnegative). By Dominated Convergence, we have:

$$\lim_{k\to+\infty} \sum_{m\geq 1} \sigma_m(T - P_{\Sigma_k} T) = \sum_{m\geq 1} \lim_{k\to+\infty} \sigma_m(T - P_{\Sigma_k} T). \tag{C.3.5.16}$$

By Lemma C.3.5.3(2), we have $\sigma_m(T - P_{\Sigma_k} T) \leq \sigma_1(T - P_{\Sigma_k} T) = \|T - P_{\Sigma_k} T\|$, and by Lemma C.3.4.3 we have $\lim_{k\to+\infty} \|T - P_{\Sigma_k} T\| = 0$, since $P_{\Sigma_k} \to \mathrm{Id}$ with respect to the strong operator topology, and T is nuclear, hence compact by [**Scha**, Ch. III, Cor. 1 to 7.1].

By (C.3.5.16), we therefore have:

$$\lim_{k\to+\infty} \|T - P_{\Sigma_k} T\|_{\mathrm{Tr}} = \sum_{m\geq 1} \lim_{k\to+\infty} \sigma_m(T - P_{\Sigma_k} T) = 0 \tag{C.3.5.17}$$

This says the sequence of nuclear operators $\{P_{\Sigma_k} T\}_{k\geq 0}$ converges to T with respect to the topology on $\mathrm{N}(H)$ induced by the trace-norm.

(2) : The assertions follow from continuity of the trace functional on $\mathrm{N}(H)$ with respect to $\|\cdot\|_{\mathrm{Tr}}$, which follows from [**Cnw-c**, Ch. 3, Thm. 18.11(e)] (with $T = \mathrm{id}_H$). □

Corollary C.3.5.5 (SOT-exhaustive filtrations and nuclear convergence). *Let H be a separable Hilbert space. Let $\{H_i\}_{i\geq 1}$ be a countable set of closed linear subspaces such that for every $i' \geq i$ we have $H_i \subset H_{i'}$.*

Let $T \in \mathrm{N}(H)$ be a nuclear operator, and $P_i : H \to H_i$ the orthogonal projector onto H. Assume $\bigcup_{i\geq 1} H_i$ is dense in H. Then:

(1) *$P_i \to \mathrm{Id}$ with respect to the strong operator topology.*
(2) *The sequences of nuclear operators $\{P_i T\}_{i\geq 1}$ and $\{T P_i\}_{i\geq 1}$ both converge to T with respect to the topology induced by the trace-norm $\|\cdot\|_{\mathrm{Tr}}$ on $\mathrm{N}(H)$.*
(3) *We have:*

$$\lim_{i\to+\infty} \mathrm{Trace}(P_i T) = \lim_{i\to+\infty} \mathrm{Trace}(T P_i) = \mathrm{Trace}(T). \tag{C.3.5.18}$$

Proof. (1) : For any $\phi \in \bigcup_{i\geq 1} H_i$, there exists $i_0 \geq 0$ large enough so as to have $\phi \in H_{i_0}$, and hence we have:

$$\lim_{i\to+\infty} \|(\mathrm{Id} - P_i)\phi\|_H^2 = \lim_{i\to+\infty} \langle (\mathrm{Id} - P_i)\phi, \phi\rangle_H = 0 \tag{C.3.5.19}$$

since $(\mathrm{Id} - P_i)\phi$ is orthogonal to H_{i_0} for every $i \geq i_0$.

Let $\phi \in H$ be arbitrary, and fix a real number $\varepsilon > 0$. Let $\phi_0 \in \bigcup_{i\geq 1} H_i$ be such that $\|\phi - \phi_0\| < \varepsilon$. There exists an integer $i_0 \geq 1$ large enough so as to have $\phi_0 \in H_{i_0}$. We have:

$$\|(\mathrm{Id} - P_i)(\phi - \phi_0) + (\mathrm{Id} - P_i)\phi_0\|_H \leq \|(\mathrm{Id} - P_i)(\phi - \phi_0)\|_H + \|(\mathrm{Id} - P_i)\phi_0\|_H \tag{C.3.5.20}$$

with $\lim_{i\to+\infty} \|(\mathrm{Id} - P_i)\phi_0\|_H = 0$ and $\|(\mathrm{Id} - P_i)(\phi - \phi_0)\|_H \leq \|\phi - \phi_0\|$ since $\|\mathrm{Id} - P_i\| \leq 1$, so

$$\overline{\lim}_{i\to+\infty} \|(\mathrm{Id} - P_i)\phi\|_H < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\lim_{i\to+\infty} \|(\mathrm{Id} - P_i)\phi\|_H = 0. \tag{C.3.5.21}$$

It follows that indeed $P_i \to \mathrm{Id}$ in $\mathscr{B}_{\mathrm{SOT}}(H)$ as $i \to +\infty$. This settles (1).

Assertion (3) follows from (2) by continuity of the trace functional on $\mathrm{N}(H)$ with respect to $\|\cdot\|_{\mathrm{Tr}}$, so it remains to prove (2). We define $Q_0 := P_1$ and Q_i for $i \geq 1$ to be the orthogonal projection operator onto $H_{i+1} \cap H_i^{\perp}$ in H. Since $H_{i+1} = H_i \oplus (H_{i+1} \cap H_i^{\perp})$ as Hilbert subspaces of H, we have $P_{i+1} = \sum_{0 \leq m \leq i} Q_m$ as elements of $\mathscr{B}(H)$. By hypothesis T is nuclear, and (by [**Kub**, Thm.6.1]) nuclear and trace-class operators are equivalent concepts on $\mathscr{B}(H)$, so $Q_m T$ is nuclear for every $m \geq 0$ by [**Cnw-b**, Ch. 3, Thm. 18.11(a)]. Hence,

$$\mathrm{Trace}(P_{i+1}T) = \sum_{0 \leq m \leq i} \mathrm{Trace}(Q_m T) \tag{C.3.5.22}$$

for every integer $i \geq 0$.

By Proposition C.3.5.1(1) applied with the projectors Q_i, we have $P_{i+1}T = \sum_{0 \leq m \leq i} Q_m T \to T$ with respect to the normed topology on $\mathrm{N}(H)$ induced by the trace-norm. By Proposition C.3.5.1(2), (C.3.5.18) also holds.

We have

$$\|TP_i - T\|_{\mathrm{Tr}} = \sum_{m \geq 1} \sigma_m(TP_i - T) = \sum_{m \geq 1} \sigma_m((TP_i - T)^*) \tag{C.3.5.23}$$

by Lemma C.3.5.3(1). Note that since T is compact, by [**Cnw-a**, Ch. II, Thm. 4.4(b)] so are TP_i, $T - TP_i$, and the adjoints $(T - TP_i)^*$, T^*, $P_i^* T^* = (TP_i)^*$ for each integer $i \geq 1$. Their singular values are therefore defined, and the Lemma applies to provide the identities in (C.3.5.23).

We likewise see that $(TP_i - T)^*$ is nuclear and its trace-norm agrees with $\|TP_{i_k} - T\|_{\mathrm{Tr}}$. Thus,

$$\|TP_i - T\|_{\mathrm{Tr}} = \sum_{m \geq 1} \sigma_m(P_i^* T^* - T^*) = \sum_{m \geq 1} \sigma_m(P_i T^* - T^*) = \|P_i T^* - T^*\|_{\mathrm{Tr}} \tag{C.3.5.24}$$

where we are using self-adjointness of P_i. Since T^* is nuclear, by Proposition C.3.5.3(1) we obtain from the previous case that $\|P_i T^* - T^*\|_{\mathrm{Tr}} \to 0$ as $i \to +\infty$, so $\lim_{i \to +\infty} \|TP_i - T\|_{\mathrm{Tr}} = 0$. □

C.3.6. Schatten-von Neumann classes of index $p = 1, 2$. In this subsection we continue to discuss nuclear operators on Hilbert spaces, in certain special cases of interest: integral operators acting on L^2-spaces of homogeneous spaces.

We let E, F be Hilbert spaces, and denote by $\mathscr{K}(E, F)$ the $\mathbf{C}$-linear space of compact operators $T : E \to F$ (namely, $\overline{T(U)} \subset F$ is compact for some 0-neighborhood $U \subset E$). For $H := E = F$, $\mathscr{K}(H)$ is an ideal of operators and is closed under the subspace topology inherited from $\mathscr{B}_{\mathrm{N}}(H) = \mathscr{L}_{\beta}(H)$; i.e., the operator-norm topology (see [**Cnw-a**, Ch. II, Prop. 4.2(b)]). In general, the formation of $\mathscr{K}(E, F)$ is compatible with composition of operators on both sides (by the definition). , by [**Jar**, Part III,Ch.17,§17.1,Prop.1(c)].

Compact operators are characterized as follows:

THEOREM C.3.6.1 (Structure of compact operators on Hilbert spaces). *Let E, F be Hilbert spaces, $T \in \mathscr{L}(E, F)$. The following conditions on T are equivalent:*

(1) *T is compact.*
(2) *There exist a sequence $\{\lambda_k\}_{k \geq 1} \subset \mathbf{R}_{\geq 0}$ monotonically decreasing with limit 0 and orthonormal sequences $\{e_k\}_{k \geq 1} \subset E$, $\{f_k\}_{k \geq 1} \subset F$, such that*

$$T = \sum_{k \geq 1} \lambda_k \langle \cdot, e_k \rangle_E \, f_k \tag{C.3.6.1}$$

where the sum is convergent with respect to the topology of simple convergence.

If T is compact, then the sum in (C.3.6.1) *converges uniformly on the bounded subsets of E; i.e., for the operator norm. The operator-norm of T is λ_1.*

PROOF. Passing to the endomorphism of $E \oplus F$ given by

$$\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

reduces the implication (1) $\Rightarrow$ (2) with operator-norm convergence to the case $E = F$ that we saw via Polar Decomposition for T and the Spectral Theorem for $|T|$ in the proof of Lemma C.3.5.3(3).

We note that the norm of Tx for each $x \in E$ satisfies

$$\|Tx\|_F^2 = \sum_{k\geq 0} |\lambda_k \langle x, e_k\rangle_E|^2 \leq |\lambda_k| \, \|x\|_E^2$$

for every $k \geq 1$, and hence $\|T\|_F \leq |\lambda_1| = \lambda_1$. The upper bound is attained upon choosing $x = e_1$.

The operator-norm convergence under the pointwise hypothesis in (2) follows from the inequality $\|T - \sum_{1\leq k\leq N} \lambda_k \, \langle \cdot, e_k\rangle_E \, f_k\|_F \leq |\lambda_{N+1}|$. Likewise, (2) implies T is a norm-limit of finite-rank operators, hence compact. □

We further recall the following, concerning the decreasing sequence $\{\lambda_k\}_{k\geq 1} \subset \mathbf{R}_{\geq 0}$. It is uniquely determined by T, and does not depend on the orthonormal sequences $\{e_k\}_{k\geq 0}, \{f_k\}_{k\geq 0}$ from Theorem C.3.6.1. This is the content of [**Jar**, §20.2, Prop. 1]. For $E = F$, the real numbers λ_k agree with the singular values of T, counted with multiplicity and ordered decreasingly. In general, these are the *n-th approximation numbers* given in [**Jar**, §19.8]. We denote them by $\sigma_k(T)$, for each integer $k \geq 1$. By definition, for any continuous map $T \in \mathscr{L}(E, F)$,

$$\sigma_k(T) := \inf\{\|A - T\|_{\mathscr{L}_\beta(E,F)} : \dim(\mathrm{Im}(A)) \leq k - 1\}. \tag{C.3.6.2}$$

DEFINITION C.3.6.2 (Schatten-von-Neumann classes). Let E, F be Hilbert space spaces. Let $p \geq 1$ be a real number. We define:

(1) The *Schatten-von Neumann class of index p* to be the $\mathbf{C}$-linear subspace:

$$S_p(E, F) \subset \mathscr{K}(E, F) \tag{C.3.6.3}$$

of those compact operators such that $\sum_{k\geq 1} \sigma_k(T)^p$ is convergent. We denote by $\|\cdot\|_{S_p}$ the function on $S_p(E, F)$ defined as

$$\|T\|_{S_p} := \left(\sum_{k\geq 1} \sigma_k(T)^p\right)^{1/p}. \tag{C.3.6.4}$$

(2) The space of *Hilbert-Schmidt operators* $E \to F$, to be the $S_2(E, F)$.
(3) The space of *trace-class operators* $E \to F$, to be $S_1(E, F)$.

We record here a generalization of Lemma C.3.5.4 for singular values of continuous operators $E \to F$ when E and F are general Hilbert spaces, allowing $E \neq F$.

LEMMA C.3.6.3 (Compositions and singular values). *Let E, F, G be Hilbert spaces, and $m, n \geq 1$ be integers. We have:*

(1) $\sigma_m(A + B) \leq \sigma_m(A) + \sigma_m(B)$, *for any* $A, B \in \mathscr{L}(E, F)$.
(2) $|\sigma_m(A) - \sigma_m(B)| \leq \|A - B\|_{\mathscr{L}_\beta(E,F)}$, *for any* $A, B \in \mathscr{L}(E, F)$.
(3) $\sigma_{m+n}(AB) \leq \sigma_m(A)\sigma_n(B)$, *for any* $A \in \mathscr{L}(F, G)$, $B \in \mathscr{L}(E, F)$.

PROOF. This is [**Jar**, §19.8, Prop. 1]. □

C.3.6.1. *Nuclear, Hilbert-Schmidt and compact operators as topological tensor products.* For X, Y two Hausdorff locally compact second countable topological spaces, and μ, ν two Radon measures on X, respectively Y,

$$S_2(L^2(X,\mu), L^2(Y,\nu)) = L^2(X \times Y, \mu \otimes \nu) \tag{C.3.6.5}$$

i.e. every Hilbert-Schmidt operator $T : L^2(X,\mu) \to L^2(Y,\nu)$ is obtained by integration against a kernel function $k \in L^2(X \times Y, \mu \otimes \nu)$:

$$(T\phi)(y) = \int_X k(x,y)\phi(x)\,\mathrm{d}\mu(x) \tag{C.3.6.6}$$

for almost all $y \in Y$. The typical example we will be encountering throughout this work, for $S_p(E,F)$, is the cases $p = 1, 2$ and for $E = L^2(X,\mu)$, $F = L^2(Y,\nu)$.

We will need the following:

PROPOSITION C.3.6.4. *Let E, F be Hilbert spaces, $p \geq 1$ a real number. Then:*

(1) *the canonical injective linear map:*

$$\chi : E' \otimes F \to \mathscr{L}(E,F) \tag{C.3.6.7}$$

uniquely determined by $\phi \otimes f \mapsto \phi \cdot f$, is valued in $S_p(E,F)$ for every $p \geq 1$.

(2) *The subspace topology induced on $E' \otimes F$ by $\|\cdot\|_{S_2}$ is equivalent to the linear topology induced by the inner product structure on $E' \otimes F$ uniquely determined on elementary tensors by:*

$$\langle \varphi_1 \otimes f_1, \varphi_2 \otimes f_2 \rangle_\sigma := \langle \varphi_1, \varphi_2 \rangle_{E'} \langle f_1, f_2 \rangle_F. \tag{C.3.6.8}$$

Upon denoting by:

$$E' \widehat{\otimes}_\sigma F \tag{C.3.6.9}$$

the completion of $E' \otimes F$ with respect to the normed topology induced by the norm $\|\cdot\|_\sigma$ associated to the inner product assigned by (C.3.6.8)*, the continuous linear map $E' \widehat{\otimes}_\sigma F \to S_2(E,F)$ is an isometry.*

(3) *The injective linear map χ from* (C.3.6.7) *extends by continuity with respect to both the projective and injective tensor product topologies, and the continuous linear operators $\chi_\pi : E' \widehat{\otimes}_\pi F \to \mathscr{L}(E,F)$ and $\chi_\varepsilon : E' \widehat{\otimes}_\varepsilon F \to \mathscr{L}(E,F)$ obtained by continuous extension are linear isometries onto $S_1(E,F)$, respectively $\mathscr{K}(E,F)$.*

PROOF. (1) : This is clear since the image consists of finite-rank maps.

(2) and (3) : This is a consequence of the definitions. Upon denoting by $\|\cdot\|_\sigma$ the norm on $E' \otimes F$ induced by the inner product $\langle \cdot, \cdot \rangle_\sigma$ assigned in (C.3.6.8), $\|\cdot\|_\sigma$ is a "reasonable tensor norm" in the sense of [**Ryan**, §6.1,p.127], so its norm-topology on $E' \otimes F$ is finer than the ε-topology and weaker than the π-topology (see [**Ryan**, Prop. 6.1(a)]). The completion of $E' \otimes F$ with respect to the ε-topology is isometric to the closure in $\mathscr{L}_\beta(E,F)$ of $\chi(E' \otimes F)$, which is $\mathscr{K}(E,F)$ since Hilbert spaces have the metric Approximation Property (see [**Ryan**, Cor. 4.13]).

That χ_π is an isometry onto $S_1(E,F)$ is the content of [**Ryan**, Prop. 7.24] for $p = 1$ (see also [**Ryan**, Prop. 6.6(c)]). □

C.3.6.2. *Closure properties of Schatten-von Neumann classes.* We collect here generalizations of Proposition C.3.5.1 and Corollary C.3.5.5, for the Schatten-von Neumann ideals $S_p(E,F)$, with E, F Hilbert spaces.

PROPOSITION C.3.6.5 (Closure properties of Schatten-von Neumann classes). *Let E, F be Hilbert spaces. Let $p \geq 1$ be a real number. Pick $A \in S_p(E,F)$ and let $\{B_k\}_{k\geq 0} \subset \mathscr{L}(E)$ and $\{C_j\}_{j\geq 0} \subset \mathscr{L}(F)$ be sequences of continuous operators converging strongly (i.e., pointwise) to $B \in \mathscr{L}(E)$ respectively $C \in \mathscr{L}(F)$.*

Assume that B_k is self-adjoint for each $k \geq 0$ (so $B = B^*$). Then the following sequences:

$$\{C_j A\}_{j\geq 0},\ \{AB_k\}_{k\geq 0},\ \{C_j AB_k\}_{j,k\geq 0} \tag{C.3.6.10}$$

converge respectively in $S_p(E,F)$ to CA as $j \to \infty$, AB as $k \to \infty$, and CAB as $j,k \to \infty$ together.

PROOF. Assume first that A is of finite rank $N \geq 0$. We may write

$$A = \sum_{1\leq r\leq N} \lambda_r \langle \cdot e_r\rangle_E f_r \tag{C.3.6.11}$$

for positive real numbers numbers $\lambda_1, \ldots, \lambda_N \leq \|A\|$, and sets of unit vectors $\{e_r\}_{1\leq r\leq N} \subset E$ and $\{f_r\}_{1\leq r\leq N} \subset F$.

We have:

(C.3.6.12)
$$C_j A = \sum_{1\leq r\leq N} \lambda_r \langle \cdot, e_r\rangle_E C_j f_r,\ AB_k = \sum_{1\leq r\leq N} \lambda_r \langle \cdot, B_k^* e_r\rangle_E f_r,\ C_j AB_k = \sum_{1\leq r\leq N} \lambda_r \langle \cdot, B_k^* e_r\rangle_E C_j f_r.$$

Since B_k is self-adjoint for each $k \geq 0$, we have $B_k^* = B_k$. Since $B_k \to B$ and $C_k \to C$ strongly, we deduce $B_k^* e_r \to Be_r$ and $C_j f_r \to Cf_r$, and we claim $C_j A \to CA$, $AB_k \to AB$, and $C_j AB_k \to CAB$ within $S_p(E,F)$; i.e., with respect to the topology on $S_p(E,F)$ induced by the norm $\|\cdot\|_{S_p}$ from (C.3.6.4). Indeed, we have:

$$\|C_j A - CA\|_{S_p} \leq \sum_{1\leq r\leq N} |\lambda_r| \|C_j f_r - C_j f_r\|_F,\ \|AB_k - AB_k\|_{S_p} \leq \sum_{1\leq r\leq N} |\lambda_r| \|B_k e_r - Be_r\|_E \tag{C.3.6.13}$$

where we have used the triangle inequality for $\|\cdot\|_{S_p}$, which follows from Lemma C.3.6.3(1) combined with $\sigma_m(\langle \cdot, e_r\rangle_E C_j f_r) = 0 = \sigma_m(\langle \cdot, B_k^* e_r\rangle_E f_r)$ for $m > 1$ (see [**Jar**, §19.8, Prop. 2] for $n = 1$), which gives $\|\langle \cdot, e_r\rangle_E (C_j f_r - Cf_r)\|_{S_p} = \|\langle \cdot, e_r\rangle_E (C_j f_r - Cf_r)\| \leq \|C_j f_r - Cf_r\|$, and $\|\langle (B_k^* e_r - B_k^* e_r, \cdot\rangle_E f_r\| \leq \|B_k e_r - Be_r\|$. The inequalities in (C.3.6.13) imply that we have $C_j A \to CA$ and $AB_k \to AB$ in $S_p(E,F)$.

For any $j,k \geq 0$ we have

(C.3.6.14)
$$\|C_j AB_k - CAB\|_{S_p} \leq \|C_j AB_k - CAB_k\|_{S_p} + \|CAB_k - CAB_k\|_{S_p} \leq \|C_j A - CA\|_{S_p} \sup_{i\geq 0} \|B_k\| + \|CAB_k - CAB_k\|_{S_p}$$

where we used the inequality $\|C_j AB_k - CAB_k\|_{S_p} \leq \|C_j A - CA\|_{S_p} \|B_k\|$ by (C.3.5.11) and Lemma C.3.6.3(2). Finiteness of $\sup_{k\geq 0} \|B_k\|$ holds by the Uniform Boundedness Principle since $B_k \to B$ strongly and $\mathscr{B}_{\mathrm{SOT}}(E)$ is Hausdorff.

We conclude that $\|C_j AB_k - CAB\|_{S_p} \to 0$ as $j,k \to \infty$ together, which proves the Proposition in case A is of finite rank.

We now discuss the general case. By [**Jar**, §19.8, Thm.3], for every real number $\varepsilon > 0$ there exists a finite rank operator $A_0 \in \mathscr{L}(E,F)$ such that $\|A - A_0\|_{S_p} \leq \varepsilon$. We have:

$$\|C_j A - CA\|_{S_p} \leq \|C_j A_0 - CA_0\|_{S_p} + (\|C_j\| + \|C\|)\, \|A - A_0\|_{S_p} < \|C_j A_0 - CA_0\|_{S_p} + (\|C_j\| + \|C\|)\, \varepsilon. \tag{C.3.6.15}$$

Since $C_j \to C$ strongly and $\mathscr{B}_{\mathrm{SOT}}(F)$ is Hausdorff, we have that $\sup_{j\geq 0} \|C_j\|$ is finite by the Uniform Boundedness Principle.

We deduce from (C.3.6.15) and the settled case of finite-rank operators that

$$\overline{\lim}_{j\to+\infty} \|C_j A - CA\|_{S_p} \leq (\sup_{j\geq 0} \|C_j\| + \|C\|)\, \varepsilon. \tag{C.3.6.16}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $C_j A \to CA$ in $S_p(E,F)$.

Likewise

$$\|AB_k - AB\|_{S_p} \leq \|A_0 B_k - A_0 B_k\|_{S_p} + (\|B_k\| + \|B\|)\, \|A - A_0\|_{S_p} \leq \|A_0 B_k - A_0 B\|_{S_p} + (\|B_k\| + \|B\|)\, \varepsilon. \tag{C.3.6.17}$$

That $\sup_{k\geq 0}\|B_k\|$ is finite was noted earlier. We deduce from (C.3.6.17) that

$$\overline{\lim}_{k\to+\infty}\|AB_k - AB\|_{S_p} \leq (\sup_{k\geq 0}\|B_k\| + \|B\|)\,\varepsilon. \tag{C.3.6.18}$$

Since $\varepsilon > 0$ is arbitrary, $AB_k \to AB$ in $S_p(E,F)$.

Finally,

$$\|C_jAB_k - CAB\|_{S_p} \leq \|C_jAB_k - CAB_k\|_{S_p} + \|CAB_k - CAB\|_{S_p} \leq \|C_jA - CA\|_{S_p} \sup_{k\geq 0}\|B_k\| + \|CAB_k - CAB\|_{S_p} \tag{C.3.6.19}$$

where we used the inequality $\|C_jAB_k - CAB_k\|_{S_p} \leq \|C_jA - CA\|_{S_p}\,\|B_k\|$ explained already. and finiteness of $\sup_{k\geq 0}\|B_k\|$ noted already. We deduce from (C.3.6.19) that $\|C_jAB_k - CAB\|_{S_p} \to 0$ as $j,k\to\infty$ together. □

Remark C.3.6.6. For $p = 1$, Proposition C.3.6.5 contains Proposition C.3.5.1 and Corollary C.3.5.5 as a special case. Indeed, for any cofinal sequence $\{\Sigma_k\}_{k\geq 0} \subset \mathcal{F}_I$ as in Proposition C.3.6.5(2), we may call $C_k := \sum_{i\in\Sigma_k} P_i$ and $A := T$, to obtain Proposition C.3.5.1 from Proposition C.3.6.5.

Corollary C.3.5.5 is obtained from Proposition C.3.6.5 upon choosing $B_k := P_{i_k}$, a self-adjoint operator converging to the identity strongly by assumption.

C.3.7. Integral kernels. We further recall here a result describing a class of examples of nuclear operators, and how to compute their trace. For X a Hausdorff locally compact second-countable topological space, and μ a Radon measure on X, we consider the Hilbert space $L^2(X,\mu)$ of μ-measurable square μ-integrable functions on X, topologized with the L^2-norm topology.

A nuclear operator A on $E := L^2(X,\mu)$ is, in particular, Hilbert-Schmidt (see [**Cnw-c**, Prop.18.8(c)]); i.e., $A \in S_p(E)$ for $p = 1, 2$, and it therefore corresponds to a unique element of $E'\,\widehat{\otimes}_\sigma E$ (see Proposition C.3.6.4) which, since $E'\,\widehat{\otimes}_\sigma E$ is isometric to $L^2(X\times X, \mu\otimes\mu)$, corresponds to a unique $(\mu\otimes\mu)$-measurable function $k(x,y) \in L^2(X\times X, \mu\otimes\mu)$ such that the effect of A on $\phi \in L^2(X,\mu)$ is by convolution, i.e. we have

$$(A\phi)(x) = \int_X k(x,y)\phi(y)\,\mathrm{d}\mu(y)$$

as μ-measurable and square μ-integrable functions on X.

Characterizing nuclearity of A in terms of the integral kernel k is a non-trivial task. In the special case A is positive, this is indeed possible. Using an analog of the Hardy-Littlewood Maximal Theorem and an analog of Lebesgue's Differentiation Theorem (for which we refer the reader to the discussion in §A.5.1 and §A.5.2, §A.5.3 and §A.5.4 of Appendix A.5 in the special case $X = G$ is a Hausdorff locally compact abelian group equipped with a D'-sequence, which serves the same purpose of the open balls in Euclidean spaces through the proof of the inequalities of the Maximal Theorem, and their application to the Differentiation Theorem), which the reader will find discussed in an axiomatized manner using martingales in [**Bris-a**, §2, pp. 231-2], and in [**Bris-b**, §2, p. 1182] in the case of Euclidean spaces (and more general subsets of Euclidean spaces), we can "smoothen" the kernel function $k(x,y)$ along the diagonal in $X\times X$ (which is a $(\mu\otimes\mu)$-null set, on whose points the kernel function may be undefined), to define a new μ-measurable function $\widetilde{k}(x,x)$ out of $k(x,y)$. It turns out (see [**Bris-a**, Thm. 4.2, p. 238]) that A is nuclear if and only if $\widetilde{k}(x,x)$ is absolutely μ-integrable as a μ-measurable function of $x \in X$ defined μ-almost everywhere.

Beyond the case of positive operators this characterization breaks down, as illustrated by means of an example (see [**Bris-b**, Example 3.2, p. 1186] for the well known example of the Volterra operator on $L^2([0,1],\lambda)$, with λ the normalized Lebesgue measure on $\mathbf{R}$, which is a Hilbert-Schmidt operator whose kernel has an absolutely integrable smoothening along the diagonal).

It is always possible to factorize a nuclear operator on a Hilbert space as the composition of two Hilbert-Schmidt operators, and it turns out that the convolution of their own Hilbert-Schmidt kernels makes sense along the diagonal (i.e., away from a subset of the diagonal of μ-measure zero, it agrees with the smoothening

of the kernel function associated to the nuclear operator itself, viewed as a Hilbert-Schmidt operator; see [**Bris-a**, Thm. 3.3, p. 238]).

This observation is used, for example, in [**Arthur**, §3,Lemma 3.8,p.353] (where the kernels H' and H'' are Hilbert-Schmidt kernels, whose convolution is the kernel H whose associated operator is nuclear and whose trace is being computed in the Lemma) in the process of proving instances of the classical Arthur-Selberg trace formula.

If the operator A is assumed to be nuclear, then the smoothening of its Hilbert-Schmidt kernel along the diagonal (in whichever way it is computed, whether through Lebesgue's Differentiation or by convolution of two factor Hilbert-Schmidt kernels) can be used to compute its nuclear trace. This is an important result, which generalizes a classical Theorem of Mercer (see [**Bris-b**, Prop. 1.2, p. 1182]) asserting that when the Hilbert-Schmidt kernel of the nuclear operator A is continuous as a function of $(x, y) \in X \times X$ then its integral along the diagonal as a function of (x, x) and with respect to the Radon measure μ on X equals the nuclear trace of A.

THEOREM C.3.7.1 (Trace formula for integral nuclear operators). *Let X be a Hausdorff locally compact second countable topological space, μ a Radon measure on X. Let $k \in L^2(X \times X, \mu \otimes \mu)$ be a square-integrable function. Then:*

(1) *the integral operator:*

$$(T\phi)(x) := \int_X k(x, y)\phi(y)\, \mathrm{d}\mu(y) \tag{C.3.7.1}$$

is Hilbert-Schmidt, with Hilbert-Schmidt norm given by $\|k\|_{L^2(X\times X,\mu\otimes\mu)}$.

(2) *Assume T is nuclear. Then:*

   (a) if $k(x, y)$ is continuous almost everywhere along the diagonal in $X \times X$ then it is absolutely integrable and

$$\mathrm{Trace}(T) = \int_X k(x, x)\, \mathrm{d}\mu(x). \tag{C.3.7.2}$$

   (b) The kernel function $k(x, y)$ admits a smoothening $\widetilde{k}(x, x)$ along the diagonal such that the trace formula in (C.3.7.2) *holds with $\widetilde{k}(x, x)$ replacing $k(x, x)$.*

We refer to [**Bris-a**, Cor. 3.2, pp. 237] for a proof of Theorem C.3.7.1(2.a), and to [**Bris-a**, Thm. 3.1, p. 235] for a proof Theorem C.3.7.1(2.b).

Throughout the trace calculation that is the main focus of this work, we will be showing how to define the smoothening of the appropriate integral kernel directly, using the analogs of the Hardy-Littlewood Maximal Theorem and Lebesgue's Differentiation Theorem discussed in Appendix A.5 to avoid necessitating the reader to familiarize themself with martingales and their use in [**Bris-a**].

C.4. Integration in the sense of Pettis

One essential tool to analyze holomorphic scalar-valued functions is contour integration. We now discuss its counterpart for vector-valued functions, with a view towards the applications of interest, with values in a locally convex space that is generally not Banach. We have selected the integral in the sense of Pettis, and we will be collecting the necessary foundations throughout this section. The material in this section will be used for integration of certain linear operators varying continuously over a Hausdorff locally compact abelian topological group.

All locally convex topological vector spaces throughout this section will be Hausdorff and over the complex numbers.

C.4.1. Weak completion. We begin by recalling a few facts about the relation of a Hausdorff locally convex topological $\mathbf{C}$-vector space E, and the algebraic dual $(E')^*$ of its continuous dual E'.

The algebraic dual of a complex vector space V is simply the complex vector space V^* of all $\mathbf{C}$-linear forms on it. If V and W are two such spaces equipped with a bilinear pairing

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbf{C}$$

yielding linear *injections* $V \hookrightarrow W^*$ and $W \hookrightarrow V^*$ then we say "V and W are in duality" and write $\langle V, W \rangle$ to denoted this data (as in [**Scha**, Ch. IV, §1]).

We recall that on a Hausdorff locally convex topological complex vector space E, we find the locally convex weak topology $\sigma(E, E')$ generated by the family of seminorms

$$q_\varphi : E \to \mathbf{R}_{\geq 0}$$

defined by $q_\varphi(u) := |\varphi(u)|$ as φ varies over E'. By the Hahn-Banach Theorem, $\sigma(E, E')$ is Hausdorff. We sometimes refer to the given topology on E as the *initial topology*, as in [**Bou-TVS**, p. IV.4] (where $\sigma(E, E')$ is referred to as the *weakened topology* on E).

Among the Hausdorff locally convex topologies $\mathscr{T}$ on E, a special class are those for which E' is the subspace of E^* consisting of $\mathscr{T}$-continuous linear functionals; these are called *polar topologies*, or *consistent* with the pairing between E and E'. The weak topology is polar, as is the *Mackey topology* $\tau(E, E')$ defined as the topology of uniform convergence in E on the weakly-compact convex circled subsets in E'. Among the topologies such that the continuous dual of E remains E', $\sigma(E, E')$ is the coarsest and $\tau(E, E')$ the finest (and the initial topology sits between them); see [**Scha**, Ch. IV, §3-§4] for more on such topologies. We will now look into $\sigma(E, E')$ more closely.

Given a locally convex topological complex vector space E, there exists a canonical $\mathbf{C}$-linear map

$$\iota_E : E \to (E')^* \tag{C.4.1.1}$$

sending $u \in E$ to the $\mathbf{C}$-linear form $\iota_E(u)$ on $(E')^*$ taking a continuous linear form $\varphi \in E'$ to $\varphi(u)$. This is injective by the Hahn-Banach Theorem [**Scha**, Ch. III, Cor. 1 to 4.2].

The map ι_E enjoys the following naturality property:

LEMMA C.4.1.1. *Let E, E_1, E_2 be Hausdorff locally convex topological $\mathbf{C}$-vector spaces. Then ι_E from* (C.4.1.1) *is injective and the $\sigma((E')^*, E')$-topology on $(E')^*$ induces the $\sigma(E, E')$-topology on E as the subspace topology. Moreover:*

(1) *The $\mathbf{C}$-linear injection $\iota_E : E \to (E')^*$ is continuous with respect to the initial topology on the source and the $\sigma((E')^*, E')$-topology on the target.*
(2) *For any continuous $\mathbf{C}$-linear operator $T : E_1 \to E_2$, the algebraic dual map $T^* : E_2^* \to E_1^*$ restricts to a $\mathbf{C}$-linear map $T' : E_2' \to E_1'$ and this is continuous with respect to the $\sigma(E_2', E_2)$-topology on the source and the $\sigma(E_1', E_1)$-topology on the target. Moreover, the algebraic dual $(T')^* : (E_1')^* \to (E_2')^*$ of the topological dual T' is continuous with respect to $\sigma((E_1')^*, E_1')$ and $\sigma((E_2')^*, E_2')$, and the following diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ {\scriptstyle \iota_{E_1}}\downarrow & & \downarrow{\scriptstyle \iota_{E_2}} \\ (E_1')^* & \xrightarrow[(T')^*]{} & (E_2')^* \end{array} \tag{C.4.1.2}$$

commutes.

PROOF. Assertion (1) is a matter of unraveling definitions (continuity even holds using $\sigma(E, E')$ on E). Then (2) is also definition-chasing. □

LEMMA C.4.1.2. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. The completion of E with respect to the $\sigma(E, E')$-topology is isomorphic to $(E')^*$. Moreover:*

(1) *If $C \subset E$ is convex then its closure in E for the given topology is the same as for $\sigma(E, E')$, and if that closure is compact for the given topology on E then it is also the closure in $(E')^*$ for $\sigma((E')^*, E')$.*
(2) *If E is a normed space then the linear map $\iota_E : E \to (E')^*$ in* (C.4.1.1) *factors through the natural embedding $c_E : E \to E'' := (E'_\beta)'$ that is isometric, where E' is equipped with the usual dual-norm topology.*

The dual-norm topology on E' in (2) is familiar from the theory of Banach spaces (applied to $\widehat{E}$ whose topological dual agrees with E') and agrees with the strong-dual topology $\beta(E', E)$ of convergence on bounded (or equivalently weakly bounded) subsets of E; see [**Scha**, p. 142].

PROOF. Density of E in $(E')^*$ with respect to $\sigma((E')^*, E')$ holds by [**Scha**, Ch. IV, 1.3] (with $G = (E')^*$, $F = E'$ the notation there). Completeness of $(E')^*$ with respect to $\sigma((E')^*, E')$ is [**NB**, Thm. 8.2.9].

(1) : The closure assertions follow from [**Scha**, Ch. IV, 3.1] (with $F = E$, $G = E'$ there), and for the compactness aspect use that $\sigma((E')^*, E')|_E = \sigma(E, E')$ and that $\sigma((E')^*, E')$ is Hausdorff.

(2) : The map ι_E factors as a linear map through $c_E : E \to E''$ as explained near the start of [**Bou-TVS**, Ch. IV, §1, p. IV.14], and by [**Bou-TVS**, Ch. IV, §1, Prop.8(i)] it is an isometric embedding. □

REMARK C.4.1.3. The locally convex topological complex vector space $(E')^*$ with the topology $\sigma((E')^*, E')$ is referred to as the *weak completion* of E in [**Gro-a**, §2].

C.4.2. Vector-valued measurable and integrable functions. We recall that for a complex Borel measure μ on a locally compact Hausdorff space X, μ is a Radon measure if and only if the total variation $|\mu|$ of μ is a Radon measure (see [**Fol**, Thm. 7.16]). Moreover, $\|\mu\| := |\mu|(X)$ is a norm on the complex vector space of complex bounded Radon measures on X. For X that is also second-countable, every complex Borel measure finite on compact subsets is Radon by [**Fol**, Thm. 7.8].

DEFINITION C.4.2.1. Let X be a locally compact Hausdorff space and μ a Radon measure on X. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space and $f : X \to E$ a function. We say

(1) f is *weakly measurable* if and only if for every $\varphi \in E'$, the function $\varphi \circ f : X \to \mathbf{C}$ is μ-measurable.
(2) f is *weakly μ-integrable* if and only if it is weakly measurable and for every $\varphi \in E'$, $\varphi \circ f : X \to \mathbf{C}$ is μ-integrable. For a weakly μ-integrable function $f : X \to E$, we denote by $\int_X f \, \mathrm{d}\mu$ the unique element of $(E')^*$ defined by

$$\left\langle \int_X f \, \mathrm{d}\mu, \varphi \right\rangle_{(E')^*, E'} := \int_X \langle f(x), \varphi \rangle_{E, E'} \, \mathrm{d}\mu(x), \ \varphi \in E'. \tag{C.4.2.1}$$

In other words, $\int_X f \, \mathrm{d}\mu \in (E')^*$ is $\varphi \mapsto \int_X (\varphi \circ f) \mathrm{d}\mu$ by definition.

(3) f is *μ-integrable*, or *integrable in the sense of Pettis*, if and only if it is weakly μ-integrable (so in particular weakly μ-measureable) and the $\mathbf{C}$-linear form $\int_X f \, \mathrm{d}\mu$ on E' defined in (C.4.2.1) is in the image of the canonical $\mathbf{C}$-linear injection $\iota_E : E \to (E')^*$. We denote by $\int_X f \, \mathrm{d}\mu$ again the unique element of E that maps to the linear form on E' from (C.4.2.1) under ι_E, if any.

REMARK C.4.2.2. When E is a Banach space, there is Bochner's theory of E-valued integration that can be developed in a style similar to scalar-valued integration, as is done for example in [**Lang-d**, Ch. VI]. Any $f : X \to E$ that is integrable in the sense of Bochner is also integrable in the sense of Pettis with the same value for $\int_X f \mathrm{d}\mu \in E$, since the formation of the Bochner integral commutes with continuous linear mappings $E \to F$ (such as a continuous linear functional $E \to \mathbf{C}$).

The relation between the two theories is often not needed in this work because (as was noted §C.1.4) the Banach-valued theory is often not applicable due to infinite-dimensional nuclear spaces E never being normable [**Tre**, Cor. 2 to Prop. 50.2]. Nonetheless, the Bochner integral often agrees with the integral in the

sense of Pettis when E is a Banach space. Specifically, we may deduce (from [**Za**, Ch. 6, §31, p. 222,(e)], for instance) as a consequence of Pettis' Separable Measurability Theorem (see [**DU**, Ch. II, §1, Thm. 2]) that the two integrals coincide when the function is continuous and weakly integrable on a locally compact Hausdorff Radon space and valued in a separable Banach space. For all the intended purposes of this work, the two theories of integration will coincide when both are available.

C.4.3. Integrability criteria and permanence properties. We recall that $C(X)$, the $\mathbf{C}$-vector space of continuous complex-valued functions on the topological space X, is a Banach space with respect to the uniform norm if X is a compact Hausdorff topological space. The $\mathbf{C}$-linear space $\mathscr{M}(X)$ of complex Radon measures on X is its continuous dual [**Fol**, Thm. 7.1, Cor. 7.18], with $\|\mu\|$ as the dual norm on $\mathscr{M}(X)$. In particular, for such compact Hausdorff X, by Agaoglu's Theorem [**Scha**, Ch. III, Cor. to 4.3; Ch. IV, 3.1] the subset $\mathscr{M}^{\leq 1}(X)$ of such measures μ with $\|\mu\| \leq 1$ is weakly-compact (i.e., compact with respect to $\sigma(\mathscr{M}(X), C(X))$). We refer to the weak topology on $\mathscr{M}(X)$ (i.e., the $\sigma(\mathscr{M}(X), C(X))$-topology) as the *vague topology* as in [**Fol**, p. 223].

C.4.3.1. *Vector-valued Mean Value Theorems.* The following Lemma indicates one avenue that leads to a criterion for integrability.

LEMMA C.4.3.1 (Mean-Value Theorem). *Let X be a locally compact Hausdorff space and μ a finite complex Radon measure on X. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space and $f : X \to E$ a weakly μ-integrable continuous function. Then we have*

$$\int_X f \, \mathrm{d}\mu \in \|\mu\| \cdot \overline{\mathrm{cch}}(f(X)) \tag{C.4.3.1}$$

where $\overline{\mathrm{cch}}(f(X))$ is the weak-closure within $(E')^*$ of the convex circled hull (also called the absolutely convex hull) of the image of f.

The formation of the convex circled hull of a subset $C \subset E$ commutes with continuous linear maps to another Hausdorff locally convex topological $\mathbf{C}$-vector space. But when applied to ι_E, beware that the map ι_E is generally not closed and so the closure of $\mathrm{cch}(C)$ in E with respect to the given topology on E and in $(E')^*$ with respect to the $\sigma((E')^*, E')$-topology may differ (when the closure in E is not compact).

We refer the reader to [**Bou-I**, Ch. VI ,§1, no. 2, p. VI.5, Prop. 5] for the special case of Lemma C.4.3.1 when μ is real-valued and positive.

PROOF. For any $\varphi \in E'$, we have

$$\left| \int_X \langle f(x), \varphi \rangle_{E,E'} \, \mathrm{d}\mu(x) \right| \in [0, |\mu|(X)] \cdot \overline{|\langle f(X), \varphi \rangle_{E,E'}|} \subset \mathbf{R}.$$

By design of $\sigma((E')^*, E')$ as an inverse-image topology (see the discussion at the bottom of [**Bou-TVS**, p. II.42]) from the product topology on $\prod_{\varphi \in E'} \mathbf{C}$ along the map $u \mapsto (\langle u, \varphi \rangle_{(E')^*,E'})_{\varphi \in E'}$, the average $\|\mu\|^{-1} \int_X f \, \mathrm{d}\mu$ is contained in the bi-polar set $(f(X)^\circ)^\bullet$ of $f(X)$ (because on any φ in the (absolute) polar $f(X)^\circ \subset E'$, the absolute value of $\|\mu\|^{-1} \int_X \langle f(x), \varphi \rangle_{E,E'} \, \mathrm{d}\mu(x)$ is bounded above by 1), which by [**DS**, §2, Prop. 1, p. 64] is $\overline{\mathrm{cch}}(f(X))$ (the closure being formed with respect to the $\sigma((E')^*, E')$-topology, and the duality being $\langle E, E' \rangle$ for the formation of the polar $f(X)^\circ$ and being $\langle E', (E')^* \rangle$ for the formation of the second polar $(\cdot)^\bullet$). □

The following is an existence result for Radon measures on a compact Hausdorff topological space X characterizing the closed convex hull of the set of values of a continuous vector-valued function:

THEOREM C.4.3.2 (Weakly closed convex hull of the value set of a continuous vector-valued function). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, X a compact Hausdorff topological space. Let $f : X \to E$ be a continuous function. Let $\mathscr{M}(X)$ be the $\mathbf{C}$-linear space space of complex Radon measures on*

X, $\mathscr{M}_+^1(X)$ the subset of the positive Radon measures on X satisfying $\|\mu\| = 1$, $\mathscr{M}^{\leq 1}(X)$ the compact subset of those complex Radon measures satisfying $\|\mu\| \leq 1$.

(1) *We have, as subsets of $(E')^*$:*

$$\left\{ \int_X f(x)\,\mathrm{d}\mu(x), \mu \in \mathscr{M}_+^1(X) \right\} = \overline{\mathrm{Ch}}(f(X)) \tag{C.4.3.2}$$

the closure of the right side of (C.4.3.2) *being formed in $(E')^*$ topologized with the $\sigma((E')^*, E')$-topology. Moreover, both sets in* (C.4.3.2) *are compact with respect to the $\sigma((E')^*, E')$-topology.*

(2) *The assertions in* (1) *hold upon replacing $\mathscr{M}_+^1(X)$ with $\mathscr{M}^{\leq 1}(X)$ and the closed convex hull with the closed convex circled hull.*

PROOF. (1) : This is [**Voigt**, Prop. 14.4] since we work over $\mathbf{C}$ ([**Bou-I**, Ch. III, §3, Prop. 5] is over $\mathbf{R}$).

(2) : Let A be the convex circled hull of the subset of $(E')^*$ given by the values $\int_X f(x)\,\mathrm{d}\mu(x)$ as μ varies over the complex Radon measures on X with *finite support* satisfying $\|\mu\| \leq 1$. Then by definition $A = \mathrm{cch}(f(X))$. By Lemma C.4.3.1, the convex circled superset

$$B := \left\{ \int_X f(x)\,\mathrm{d}\mu(x), \mu \in \mathscr{M}^{\leq 1}(X) \right\} \subset (E')^*$$

of the values $\int_X f(x)\,\mathrm{d}\mu(x)$ as μ varies over $\mathscr{M}^{\leq 1}(X)$, which contains A, is contained in the $\sigma((E')^*, E')$-closure of $\mathrm{cch}(f(X))$. This proves that the $\sigma((E')^*, E')$-closure of the set B is $\overline{\mathrm{cch}}(f(X))$. It remains to show that B is itself $\sigma((E')^*, E')$-compact, in which case it is $\sigma((E')^*, E')$-closed and hence equals $\overline{\mathrm{cch}}(f(X))$.

The assignment $\mu \mapsto \int_X f(x)\,\mathrm{d}\mu(x)$ on $\mu \in \mathscr{M}(X)$ is continuous with respect to the $\sigma(\mathscr{M}(X), C(X))$-topology on each bounded subset of $\mathscr{M}(X)_\sigma$ (such as the compact subset $\mathscr{M}^{\leq 1}(X)$) and the $\sigma((E')^*, E')$-topology on $(E')^*$. (Note that formation of $\int_X f(x)\,\mathrm{d}\mu(x)$ is not generally continuous with respect to $\mu \in \mathscr{M}(X)$ using the $\sigma(\mathscr{M}(X), C(X))$-topology; it is continuous on the convex cone of positive measures, and on all $\sigma(\mathscr{M}(X), C(X))$-bounded subsets by [**Bou-I**, Ch. III, §4, No. 4, Corollary, p. III.40], which is what we are using here.) But $\mathscr{M}^{\leq 1}(X)$ is compact with respect to the $\sigma(\mathscr{M}(X), C(X))$-topology, and hence so is B with respect to the $\sigma((E')^*, E')$-topology. □

REMARK C.4.3.3. In the proof of Theorem C.4.3.2, we used properties of the $\sigma(\mathscr{M}(X), C(X))$-topology for X a compact Hausdorff topological space.

Vague continuity on the cone of positive Radon measures can be thought of as a generalization of an analog of the Monotone Convergence Theorem, while continuity on weakly bounded subsets of $\mathscr{M}(X)$ can be thought as being in a similar vein as the Dominated Convergence Theorem.

We are going to be routinely making two assumptions on E throughout our work, aside from E being locally convex: that E be Hausdorff, and *quasi-complete*. In our situations of interest, these will always be satisfied. This and Lemma C.4.3.1 provide an integrability criterion for continuous compactly supported functions (see Theorem C.4.3.5 below).

A priori, the linear form $\int_X f\,\mathrm{d}\mu$ is only contained in the weak completion $(E')^*$ of E (see the start of Lemma C.4.3.1). For the purposes of this work, there will be two avenues to decide when a weakly integrable function $f : X \to E$ is integrable (in the sense of Definition C.4.2.1(3)). Quasi-completeness of E is one of them. The definition of quasi-completeness is that every *bounded* closed subset of E is complete. (Here we are using boundedness in the sense of locally convex topological vector spaces.) This is equivalent to saying that every *bounded* Cauchy net in E converges in E. It is therefore also referred to as *bounded completeness*. We refer the reader to [**Scha**, Ch. I, 5.6; Ch. II, Cor. to 4.3] for some properties of quasi-complete Hausdorff locally convex topological vector spaces.

REMARK C.4.3.4 (Cautionary remark on boundedness and net-convergence). For the purposes of checking convergence or continuity in a Hausdorff topological vector space with nets, we are allowed to test such

conditions upon replacing an I-indexed net $\{x_i\}_{i\in I}$ with the resulting I_0-indexed net obtained by restriction to $I_0 \subset I$, with $I_0 := \{i \in I : i \geq i_0\}$, for some fixed arbitrary $i_0 \in I$; we refer to $\{x_i\}_{i\geq i_0}$ as a *tail* of the original net. It is elementary that every Cauchy sequence in a topological $\mathbf{C}$-vector space E is bounded (in the sense of topological vector spaces), but in metrizable locally convex topological $\mathbf{C}$-vector spaces even a convergent net can fail to be "eventually bounded" (i.e., no tail $\{x_i\}_{i\geq i_0}$ is bounded).

To make examples, we begin with the simpler construction of an unbounded convergent net. Let $I = \mathbf{R}_{>0}$ equipped with its natural ordering. Define $z_a := a^{-1} \in \mathbf{C}$ for each $a \in I$. The net of complex numbers $\{z_a\}_{a\in I}$ is convergent to $0 \in \mathbf{C}$, but the net is not bounded in $\mathbf{C}$. However, this net is eventually bounded; e.g., $\{z_a\}_{a\geq 1}$ is bounded.

To make a net converging to 0 in the meterizable case such that the net is not eventually bounded, consider an infinite-dimensional nuclear metrizable locally convex space F (such as the Schwartz space $\mathcal{S}(\mathbf{R})$) and pick a decreasing base of convex 0-neighborhoods $U_1 \supset U_2 \supset \cdots$ (this exists by metrizability). For each *nonzero* $x \in U_1$, define $n(x)$ to be the largest positive integer N such that $x \in U_N$, and let J be the set $U_1 - \{0\}$ with the partial ordering $x \geq y$ precisely when $n(x) \geq n(y)$. Defining $x_j = j \in U_1$, the net $\{x_j\}_{j\in J}$ converges to 0 and we claim no tail $\{x_j\}_{j\geq j_0}$ is bounded. Suppose to the contrary that some such tail is bounded, so the convex 0-neighborhood $U_{n(x_{j_0})}$ is bounded. Then F is normable [**Scha**, Ch. II,, 2.1], but a nuclear normable locally convex space is finite-dimensional [**Tre**, Cor. 2 to Prop. 50.2], a contradiction.

In the non-metrizable locally convex case, to make a Cauchy set that is not eventually bounded (as will also help to make examples that are quasi-complete but not complete, a distinction that does not arise in the metrizable setting), let H be an infinite-dimensional separable Hilbert space and $E := \mathscr{B}_{\mathrm{SOT}}(H)$ be the $\mathbf{C}$-vector space of bounded endomorphisms of H equipped with the strong operator topology (i.e., pointwise convergence). By Lemma C.3.3.1, this E is Hausdorff, locally convex, and quasi-complete. We will construct a Cauchy net in E that has no limit (so E is not complete) so by quasi-completeness of E such a Cauchy net cannot be eventually bounded (as otherwise it would have a limit, by quasi-completeness). This also shows E is not metrizable, since quasi-completeness implies completeness in the metrizable case.

To make the desired Cauchy net, let $\varphi \in H^* - H'$ be a discontinuous linear functional on H (as exists since H is infinite-dimensional). Then by extending the restriction of φ on each finite-dimensional subspace to a continuous linear functional on H using the Hahn-Banach Theorem (in the form given in [**Scha**, Ch. II, 4.2]) and partially ordering the set I of finite-dimensional subspaces of H by inclusion, we obtain a net $\{\varphi_i\}_{i\in I} \subset H'$ of continuous linear functionals converging to φ "pointwise": for every $h \in H$, we have $\varphi_i(h) \to \varphi(h)$ as a net of complex numbers.

Upon fixing $h \in H$ nonzero, we obtain a net $\{A_i\}_{i\in I} \subset E$ by defining $A_i(u) := \varphi_i(u) \cdot h$ for each $u \in H$, and $A_i \to A$ pointwise on H for $A(u) := \varphi(u) \cdot h$ but A is not a bounded operator on H. Explicitly, we have $\|A_i(u) - A(u)\|_H = |\varphi_i(u) - \varphi(u)|\,\|h\|_H \to 0$. In particular, $\{A_i\}_{i\in I} \subset E$ is a Cauchy net. But this net has no limit in E, since a limit (if one exists) would have to agree pointwise with the discontinuous A.

C.4.3.2. *Stability properties of vector-valued integrals and integrability criteria.* Quasi-completeness of a Hausdorff locally convex topological vector space E is a useful criterion for integrability in favorable cases: if f is continuous and compactly supported, then it is integrable in the E-valued sense.

THEOREM C.4.3.5 (Pettis' Integrability Criterion). *Let X be a locally compact Hausdorff space and μ a finite complex Radon measure on X. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space and $f : X \to E$ a compactly supported continuous function. If E is quasi-complete then f is μ-integrable.*

This is [**Gro-a**, Lemme 2], where the assumptions of the Lemma are met because convex hulls interact well with pre-compactness in the quasi-complete case: in a Hausdorff quasi-complete locally convex topological $\mathbf{C}$-vector space, the convex circled hull of any pre-compact subset is pre-compact [**Scha**, Ch. II, Cor. to 4.3].

PROOF. For closed $Z \subset X$ with $\mathrm{supp}(f) \subset Z$, we have $\int_Z f\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu$ in E' by the respective definitions of these functionals. By Lemma C.4.3.1 applied to $\mathrm{supp}(f)$, we have

$$\int_X f\,\mathrm{d}\mu \in \mu(\mathrm{supp}(f)) \cdot \overline{\mathrm{cch}}(f(X)) \tag{C.4.3.3}$$

where $\overline{\mathrm{cch}}f(X)$ is the closure of the convex circled hull (also called the absolutely convex hull) of the image of f, formed within the weak completion $(E')^*$ of E. But $\mathrm{supp}(f)$ is compact, so $f(X)$ is also compact, and the quasi-completeness of E then ensures such closure is contained in E by Lemma C.4.1.2(1). There therefore exists an element of E representing the integral $\int_X f\,\mathrm{d}\mu$. □

REMARK C.4.3.6. We refer the reader to [**Bou-I**, §3.3, Prop. 7, p.III.37] for another criterion for the linear form $\varphi \mapsto \int_X \langle f(x), \varphi\rangle\,\mathrm{d}\mu(x)$ (an element of $(E')^*$) to be contained in E.

In our case, X will either be a closed positively oriented simple parametric curve within the complex plane, equipped with its own finite complex Radon measure induced by the parametrization, or a compact subset of a locally compact Hausdorff abelian topological group equipped with a normalized Haar measure, or a fibral Radon measure in a product factorization of a Haar measure induced by fibering such group over another locally compact Hausdorff abelian topological group.

Quasi-completeness of E in Theorem C.4.3.5 is one avenue to ensure integrability. Another such criterion leverages duality:

LEMMA C.4.3.7. *Let X be a locally compact Hausdorff space, and μ a finite complex Radon measure on X. Let B be a complex Banach space, and $f : X \to B$ a weakly measurable function. Assume that f is weakly μ-integrable and B is reflexive. Then f is μ-integrable.*

PROOF. Let B' be the strong dual of B (this is the usual dual Banach space). By [**DU**, Ch. II, §3, Lemma 1], applied to $E = X$ in the notation of loc.cit., there exists $\lambda \in B''$ such that

$$\langle \lambda, \varphi\rangle_{B'',B'} = \int_X \langle f(x), \varphi\rangle_{B,B'}\,\mathrm{d}\mu(x) \tag{C.4.3.4}$$

for every $\varphi \in B'$ (i.e., the right side is continuous in $\varphi \in B'$). Since B is reflexive by hypothesis, $\iota_B : B \to (B')^*$ maps B onto (and in fact topologically isomorphically) B'' and hence there exists $b \in B$ such that $\iota_B(b) = \lambda$. It follows that f is μ-integrable. □

Before discussing a consequence of Lemma C.4.3.7, we require the following two properties of the integral in the sense of Pettis.

LEMMA C.4.3.8. *Let $T : E_1 \to E_2$ be a continuous linear map between Hausdorff locally convex complex vector spaces, and (X, μ) a locally compact Hausdorff topological space with μ a finite complex Radon measure. Let $f : X \to E_1$ be a weakly μ-integrable function. Then*

$$\int_X (T \circ f)\,\mathrm{d}\mu = (T')^* \left(\int_X f\,\mathrm{d}\mu \right) \tag{C.4.3.5}$$

as elements of $(E_2')^*$. If f is μ-integrable then so is $T \circ f$, and then

$$\int_X (T \circ f)\,\mathrm{d}\mu = T \left(\int_X f\,\mathrm{d}\mu \right) \tag{C.4.3.6}$$

as elements of E_2.

Note that the integrals in (C.4.3.5) are simply the algebraic linear forms $\varphi \mapsto \int_X \langle f(x), \varphi\rangle_{E_1,E_1'}\,\mathrm{d}\mu(x)$ on E_1' and $\varphi \mapsto \int_X \langle T(f(x)), \varphi\rangle_{E_2,E_2'}\,\mathrm{d}\mu(x)$ on E_2', i.e. elements of $(E_1')^*$ and $(E_2')^*$ only, respectively. The Lemma is then a diagram chase for (C.4.3.5) with (C.4.1.2) then yielding (C.4.3.6).

PROOF. We let $\varphi \in E_2'$ be arbitrary, and compute

$$\left\langle \int_X (T \circ f)\,\mathrm{d}\mu, \varphi \right\rangle_{(E_2')^*, E_2'} = \int_X \langle T(f(x)), \varphi \rangle_{E_2, E_2'}\,\mathrm{d}\mu(x) \tag{C.4.3.7}$$

Since T is continuous, then by [**Bou-TVS**, Ch.II,§6.4,Prop.5] we have $\langle T(f(x)), \varphi\rangle_{E_2,E_2'} = \langle f(x), T'(\varphi)\rangle_{E_1,E_1'}$ for every $x \in X$ and $\varphi \in E_2'$, and so the right side of (C.4.3.7) equals

$$\int_X \langle f(x), T'(\varphi)\rangle_{E_1,E_1'}\,\mathrm{d}\mu(x) = \left\langle \int_X f\,\mathrm{d}\mu, T'(\varphi) \right\rangle_{(E_1')^*, E_1'} \tag{C.4.3.8}$$

We now observe that since T' is continuous as an operator $E_2' \to E_1'$ with respect to the $\sigma(E_2', E_2)$-topology on the source and the $\sigma(E_1', E_1)$-topology on the target, then it is also continuous with respect to the $\sigma(E_2', (E_2')^*)$-topology on the source and the $\sigma(E_1', (E_1')^*)$-topology on the target by [**Bou-TVS**, Ch.II,§6.2,Cor.2], because $E_k \subset (E_k')^*$ for $k = 1, 2$, since E_1, E_2 are Hausdorff. By [**Bou-TVS**, Ch.II,§6.4,Prop.5], $(T')^*$ exists and is continuous and the right side of (C.4.3.8) further equals

$$\left\langle (T')^* \left(\int_X f\,\mathrm{d}\mu \right), \varphi \right\rangle_{(E_1')^*, E_1'}$$

This proves the identity in (C.4.3.5), being $\varphi \in E_2'$ arbitrary. Compatibility of $\iota_{E_k} : E_k \to (E_k')^*$ for $k = 1, 2$ with T and $(T')^*$, provided by Lemma C.4.1.1, gives that if f is μ-integrable, i.e. $\int_X f\,\mathrm{d}\mu$ is in the image of ι_{E_1}, then $\int_X (T \circ f)\,\mathrm{d}\mu$ is in the image of ι_{E_2}, and so $T \circ f$ is also μ-integrable, and (C.4.3.6) holds in E_2 by injectivity of ι_{E_2}. □

LEMMA C.4.3.9. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, and (X, μ) a locally compact Hausdorff topological space with μ a finite complex Radon measure. Let $f : X \to E$ be a μ-integrable function. For every continuous seminorm ν on E such that $x \mapsto \nu(f(x))$ is μ-measurable, we have*

$$\nu\left(\int_X f\,\mathrm{d}\mu \right) \leq \int_X \nu(f(x))\,\mathrm{d}\mu(x). \tag{C.4.3.9}$$

A variant is given in [**Bou-I**, Ch. VI, §1, no. 2, Prop. 6].

REMARK C.4.3.10. The measurability assumption on $\nu(f(x))$ as a function of $x \in X$ ensures that the right side of (C.4.3.9) makes sense as a Lebesgue integral. The integral on the right side of (C.4.3.9) may still be infinite.

PROOF. For every $u \in E$, there exists $\varphi \in E'$ such that $\varphi(u) = \nu(u)$ and for all $w \in E$ we have $|\varphi(w)| \leq \nu(w)$. This is a consequence of the Hahn-Banach Theorem.Upon choosing $u := \int_X f\,\mathrm{d}\mu$ and $w = f(x)$ for any $x \in X$, we obtain

$$\left| \int_X \langle f(x), \varphi\rangle_{E,E'}\,\mathrm{d}\mu(x) \right| \leq \int_X |\langle f(x), \varphi\rangle_{E,E'}|\,\mathrm{d}\mu(x) \leq \int_X \nu(f(x))\,\mathrm{d}\mu(x) \tag{C.4.3.10}$$

where the left side is $|\varphi(u)| = \nu(u)$. □

LEMMA C.4.3.11. *Let E be a complex Hilbert space, and $\mathscr{B}(E)$ the vector space of continuous $\mathbf{C}$-linear operators on E. Let X be a locally compact Hausdorff space, and μ a finite complex Radon measure on X. Let $f : X \to \mathscr{B}(E)$ be a function. Assume*

(1) *The function $f(u) : x \mapsto f(x)(u) \in E$ is weakly measurable for every $u \in E$.*
(2) *The function $x \mapsto \|f(x)\|$ is μ-measurable and μ-integrable.*

Then for every $u \in E$, the function $x \mapsto f(x)(u)$ is μ-integrable, and the assignment

$$u \mapsto \int_X f(u)\,\mathrm{d}\mu \tag{C.4.3.11}$$

is a bounded **C***-linear operator on* E*.*

PROOF. Let $\varphi \in E'$ be an arbitrary continuous linear form on E. Then we have, upon denoting $\|\cdot\|_{E'}$ the operator norm on E', by $\|\cdot\|_E$ the norm on E, and $\|\cdot\|$ the operator norm on $\mathscr{B}(E)$,

$$|\langle f(x)(u), \varphi\rangle_{E,E'}| \leq \|\varphi\|_{E'}\, \|f(x)(u)\|_E \leq \|\varphi\|_{E'}\, \|f(x)\|\, \|u\|_E \tag{C.4.3.12}$$

By assumption, the function $x \mapsto \|f(x)\|$ is μ-measurable. Since $x \mapsto f(x)(u)$ is weakly measurable, the integral of $\langle f(x)(u), \varphi\rangle_{E,E'}$ over X is meaningful to contemplate. We integrate the outermost sides of (C.4.3.12) over X and obtain

$$\int_X |\langle f(x)(u), \varphi\rangle_{E,E'}|\, \mathrm{d}\mu(x) \leq \|\varphi\|_{E'}\, \|u\|_E \int_X \|f(x)\|\, \mathrm{d}\mu(x). \tag{C.4.3.13}$$

Since $x \mapsto \|f(x)\|$ is μ-integrable, we deduce from (C.4.3.13) that for every $u \in E$, $x \mapsto f(x)(u)$ is weakly μ-integrable. Since E is a Hilbert space, it is reflexive. Lemma C.4.3.7 therefore applies and gives that $x \mapsto f(x)(u)$ is μ-integrable for every $u \in E$. The topology on E is normed, and by Lemma C.4.3.9 applied with $\nu = \|\cdot\|_E$, we have

$$\left\| \int_X f(u)\, \mathrm{d}\mu \right\|_E \leq \int_X \|f(u)\|_E\, \mathrm{d}\mu \leq \|u\|_E \int_X \|f(x)\|\, \mathrm{d}\mu(x) \tag{C.4.3.14}$$

showing boundedness of the linear endomorphism $u \mapsto \int_X f(u)\, \mathrm{d}\mu$ of E. □

C.4.3.3. *Continuity criterion.* For X a locally compact Hausdorff space and E a Hausdorff locally convex topological **C**-vector space, a net $\{f_i\}_{i\in I}$ of continuous functions on X valued in E converges to $f : X \to E$ *uniformly on compact subsets*, or *locally uniformly*, if for every compact subset $K \subset X$, the net $\{f_i|_K\}_{i\in I}$ converges to $f|_K$ *uniformly*. In turn, by this we mean that for every continuous seminorm ν on E, $\lim_{i\in I} \sup_{x\in K} \nu(f_i - f) = 0$. In other words, the terminology for uniform and locally uniform convergence of nets of continuous scalar-valued functions retains the same meaning for continuous E-valued functions. Such convergence can be equivalently tested on a generating family of continuous seminorms on E.

LEMMA C.4.3.12. *Let E be a Hausdorff locally convex topological* **C***-vector space, and (X, μ) a locally compact Hausdorff topological space with μ a finite complex Radon measure. Let I be a directed set, and $\{f_i\}_{i\in I}$ a net of continuous, μ-integrable functions $f_i : X \to E$ converging uniformly to the continuous μ-integrable function $f : X \to E$. Then*

$$\lim_{i\in I} \int_X f_i\, \mathrm{d}\mu = \int_X f\, \mathrm{d}\mu. \tag{C.4.3.15}$$

PROOF. We call $e_i := \int_X f_i\, \mathrm{d}\mu$ (all elements of E by Theorem C.4.3.5). Let $\nu : E \to \mathbf{R}_{\geq 0}$ be an arbitrary continuous seminorm on E. By Lemma C.4.3.9, we have

$$\nu\left(\int_X (f_i - f)\, \mathrm{d}\mu\right) \leq \int_X \nu(f_i - f)(x)\, \mathrm{d}\mu(x) \tag{C.4.3.16}$$

for every $i \in I$. Since μ is finite Radon and the convergence of $\{f_i\}_{i\in I}$ is uniform, the right side of (C.4.3.16) is bounded above by $|\mu(X)| \sup_{x\in X} \nu(f_i - f)$ and so it tends to zero. □

REMARK C.4.3.13. We will specialize the concepts and results in this subsection to two classes of E: they are Hilbert spaces, and the **C**-vector space of bounded linear endomorphism of a Hilbert space equipped with the locally convex strong operator topology, on the one hand, and with the operator-norm topology (which makes it into a Banach space), on the other. Hilbert spaces, as complete metric spaces, are quasi-complete as a consequence of the definition. As for $E := \mathscr{B}_{\mathrm{SOT}}(H)$, the space of bounded linear endomorphisms of a Hilbert space H endowed with the strong operator topology, E is indeed quasi-complete, as a consequence of the Uniform Boundedness Principle: see Lemma C.3.3.1.

C.4.4. Fubini's Theorem and Weil's Integral Formula. We require a version of Fubini's Theorem for the integral in the sense of Pettis.

LEMMA C.4.4.1. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. Let X, Y be two locally compact Hausdorff σ-compact topological spaces, and μ, ν complete* [20] *Radon measures on X, Y, respectively. Let $f : X \times Y \to E$ be a weakly $(\mu \otimes \nu)$-integrable function. Assume:*

(1) *the function $f(x, \cdot)$ is ν-integrable for every $x \in X$,*
(2) *the function $f(\cdot, y)$ is μ-integrable for every $y \in Y$.*

Then $x \mapsto \int_Y f(x,y)\,\mathrm{d}\nu(y) \in E$ and $y \mapsto \int_X f(x,y)\,\mathrm{d}\mu(x) \in E$ are weakly μ-integrable on X and weakly ν-integrable on Y respectively and as elements of $(E')^*$ we have the identity

$$\int_{X\times Y} f\,\mathrm{d}(\nu\otimes\mu) = \int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x). \tag{C.4.4.1}$$

PROOF. By definition, for every $\varphi \in E'$ we have

$$\varphi\left(\int_X f\,\mathrm{d}(\mu\otimes\nu)\right) = \int_X (\varphi\circ f)\,\mathrm{d}(\mu\otimes\nu) = \int_X \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}(\mu\otimes\nu) \tag{C.4.4.2}$$

Since f is weakly $(\mu\otimes\nu)$-integrable by assumption, then by the $\mathbf{C}$-valued Fubini theorem we have the following conclusions:

(1) $y \mapsto \langle f(x,y),\varphi\rangle_{E,E'}$ is ν-integrable for almost all $x \in X$,
(2) $x \mapsto \langle f(x,y),\varphi\rangle_{E,E'}$ is μ-integrable for almost all $y \in Y$,
(3) the functions $x \mapsto \int_Y \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\nu(y)$ and $y \mapsto \int_X \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\mu(x)$, defined almost everywhere on X, respectively Y, are integrable,
(4) we have the identity

(C.4.4.3)
$$\int_{X\times Y} \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}(\mu\otimes\nu)(x,y) = \int_Y \left(\int_X \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y) = \int_X \left(\int_Y \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x).$$

By hypothesis the functions $f(x,\cdot)$ and $f(\cdot,y)$ are integrable for *all* $x \in X$, respectively *all* $y \in Y$. The assignments $\varphi \mapsto \int_X \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\mu(x)$ and $\varphi \mapsto \int_Y \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\nu(y)$ are therefore elements of E, as opposed to merely elements of $(E')^*$, denoted $\int_X f(\cdot,y)\,\mathrm{d}\mu$ and $\int_Y f(x,\cdot)\,\mathrm{d}\nu$, and they are so for all $y \in Y$, and all $x \in X$, respectively. Each is weakly ν-, respectively weakly μ-integrable by (3) above. The integrals

$$\int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y), \quad \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x). \tag{C.4.4.4}$$

are therefore both elements of $(E')^*$, and by (C.4.4.3) they equal the left side of (C.4.4.1) as elements of $(E')^*$. □

REMARK C.4.4.2. Upon assuming that f in Lemma C.4.4.1 is continuous and compactly supported and E is quasi-complete, the result of the Lemma relates to the Remark in [**Bou-I**, Ch. III, §4, p. III.43]. Indeed, the image $f(X \times Y)$ in E, endowed with the subspace topology, agrees with $f(\mathrm{supp}(f))$ and is therefore compact by continuity of f. Its closed convex circled hull is therefore also compact, since E is quasi-complete (see [**Scha**, Ch. II,Cor. to 4.3]). In particular, that closed convex circled hull is complete and convex. By [**Bou-I**, Ch. III, §3, Prop. 7], the integral $\int_{X\times Y} f\,\mathrm{d}(\mu\otimes\nu)$ a priori only an element of $(E')^*$ is therefore contained in E (as is also provided by Theorem C.4.3.5). Similarly $\int_Y f(\cdot,y)\,\mathrm{d}\nu(y)$ and $\int_X f(x,\cdot)\,\mathrm{d}\mu(x)$ are μ-integrable and ν-integrable respectively.

[20] Completeness of measures has been added in to ensure $\mu\otimes\nu$ is complete, as may be implicit in the proof; for uses elsewhere should Haar measures be complete by convention, so "Borel set" modified accordingly?

By the Remark in [**Bou-I**, Ch. III, §4, p. III.43] and [**Bou-I**, Ch. III, Lemma 2], the functions $f(x,\cdot)$ and $(\cdot,y)$ are elements of $\widetilde{\mathscr{K}}(Y,E)$, respectively $\widetilde{\mathscr{K}}(X,E)$, in the notation of loc.cit. (see [**Bou-I**, Ch. III, §3, p. III.32]), and these are continuous E-valued functions on Y, respectively X, such that their composition with every $\varphi \in E'$ is compactly supported as a scalar-valued function.

Since $\langle f(x,y),\varphi\rangle_{E,E'}$ is a continuous scalar-valued integrable function on $X\times Y$, [**Bou-I**, Ch. III, §4, Lemma 2] applies and implies that both $y \mapsto \int_X \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\mu(x)$ and $x \mapsto \int_Y \langle f(x,y),\varphi\rangle_{E,E'}\,\mathrm{d}\nu(y)$ are continuous compactly supported functions on Y, respectively X, for every $\varphi \in E'$. By (C.4.3.3) and by compactness of the support of the above functions, the integrals in (C.4.4.1) are contained in E by Theorem C.4.3.5, being E quasi-complete. However, the integrals $\int_X f(x,\cdot)\,\mathrm{d}\nu$ and $\int_X f(\cdot,y)\,\mathrm{d}\mu$ are a priori only elements $(E')^*$. Lemma C.4.4.1, instead, *assumes* that they are elements in $E \subset (E')^*$.

We recall here the following important result on the Haar integral, for which we refer the reader to [**DE**, Thm.1.5.3] beyond the abelian case. The scope is to formulate and prove a form of it for integrals in the sense of Pettis.

THEOREM C.4.4.3. *Let G be a locally compact Hausdorff abelian group, $H \subset G$ a closed subgroup. There exists a Haar measure ν on the quotient G/H uniquely determined by a choice of Haar measure μ_G on G and μ_H on H and such that for every $f \in C_c(G)$, we have*

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/H}\int_H f(xh)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\nu(x). \tag{C.4.4.5}$$

Moreover, for every $f \in L^1(G,\mu_G)$, then the assignment $x \mapsto \int_H f(xh)\,\mathrm{d}\mu_H(h)$ defines μ_G-almost everywhere on G a function that is ν-measurable and absolutely ν-integrable on G/H, and (C.4.4.5) *holds for f.*

As an additional result, to check that a μ_G-measurable function $f : G \to \mathbf{C}$ is (absolutely) μ_G-integrable it is enough to check that the iterated integral in the right side of (C.4.4.5) for $|f|$ converges. This is [**DE**, Cor.1.5.4]:

LEMMA C.4.4.4. *Let G be a locally compact Hausdorff abelian group, $H \subset G$ a closed subgroup, and μ_G and μ_H Haar measures on G and H respectively. Let ν be the unique Haar measure on the quotient G/H provided by Theorem* C.4.4.3. *Let $f : G \to \mathbf{C}$ be a measurable function. Then if*

$$\int_{G/H}\int_H |f(xh)|\,\mathrm{d}\mu_H(h)\,\mathrm{d}\nu(x) \tag{C.4.4.6}$$

converges, f is absolutely μ_G-integrable over G, and (C.4.4.5) *holds.*

We now formulate an analog of Weil's formula (C.4.4.5) when the functions are valued in a Hausdorff locally convex topological $\mathbf{C}$-vector space.[21]

LEMMA C.4.4.5. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. Let G be a locally compact Hausdorff abelian group, $H \subset G$ a closed subgroup, and μ_G and μ_H Haar measures on G and H respectively. Let ν be the Haar measure on the quotient G/H provided by Theorem C.4.4.3. Let $f : G \to E$ be a weakly μ_G-integrable function. Assume that for every $x \in G$, the function $h \mapsto f(xh)$ is μ_H-integrable for every $x \in G$. Then*

(1) *the function $F : G \to E$ defined by*

$$F(x) := \int_H f(xh)\,\mathrm{d}\mu_H(h), \quad x \in G \tag{C.4.4.7}$$

factors through a unique weakly ν-integrable function on $f^H : G/H \to E$.

[21] Does this result require Haar measures to be complete?

(2) *The function f^H is weakly ν-integrable and we have the identity*

$$\int_G f \,\mathrm{d}\mu_G = \int_{G/H} f^H(xH)\,\mathrm{d}\nu(xH) \tag{C.4.4.8}$$

as elements of $(E')^*$.

Proof. (1) : Since the function $h \mapsto f(hx)$ on H is μ_H-integrable for every $x \in G$, then the integral

$$F(x) := \int_H f(xh)\,\mathrm{d}\mu_H(h) \tag{C.4.4.9}$$

is contained in E for every $x \in G$, instead of merely in $(E')^*$. Let x_0h' be the arbitrary element of the left H-coset $x_0H \subset G$. Then we have, for every $\varphi \in E'$

$$\langle F(x_0h'), \varphi\rangle_{E,E'} = \langle F(x_0h'), \varphi\rangle_{(E')^*,E'} = \int_H \langle f(x_0h'h), \varphi\rangle_{E,E'}\,\mathrm{d}\mu_H(h) \tag{C.4.4.10}$$

where we have tacitly identified $\iota_E(F(x_0h'))$ with $F(x_0h')$. Upon calling $h'' := h'h$, the right side of (C.4.4.10) turns into

(C.4.4.11)

$$\int_H \langle f(x_0h''), \varphi\rangle_{E,E'}\,\mathrm{d}\mu_H((h')^{-1}h'') = \int_H \langle f(x_0h), \varphi\rangle_{E,E'}\,\mathrm{d}\mu_H(h) = \langle F(x_0), \varphi\rangle_{(E')^*,E'} = \langle F(x_0), \varphi\rangle_{E,E'}$$

by left-invariance of the Haar measure μ_H on H. We deduce the identity $\langle F(x_0), \varphi\rangle_{E,E'} = \langle F(x_0h'), \varphi\rangle_{E,E'}$, valid for every $\varphi \in E'$. Since E is Hausdorff, by the Hahn-Banach Theorem we deduce $F(x_0h') = F(x_0)$. Thus, F is constant on left H-cosets and hence descends uniquely to a function $f^H : G/H \to E$, defined as $f^H(xH) := F(x)$. By Theorem C.4.4.3 applied to the function $\langle f, \varphi\rangle_{E,E'} \in L^1(G, \mu_G)$, we have that the function $\langle f^H((\cdot)H), \varphi\rangle_{E,E'}$ is ν-integrable on G/H for every $\varphi \in E'$.

(2) : For every $\varphi \in E'$, we have by Lemma C.4.4.4 that $\langle f(x), \varphi\rangle_{E,E'}$ is μ_G-integrable for every $\varphi \in E'$, and by Theorem C.4.4.3

$$\int_G \langle f(x), \varphi\rangle_{E,E'}\,\mathrm{d}\mu_G(x) = \int_{G/H} \langle f^H(xH), \varphi\rangle_{E,E'}\,\mathrm{d}\nu(xH). \tag{C.4.4.12}$$

This defines both $\int_G f\,\mathrm{d}\mu_G$ and $\int_{G/H} f^H\,\mathrm{d}\nu$ as elements of $(E')^*$, and as such they are equal. □

We have now discussed all the required preparations on integration of functions defined on locally compact Hausdorff Radon spaces, and valued in Hausdorff locally convex topological $\mathbf{C}$-vector spaces.

C.5. Meromorphic functions valued in a quasi-complete locally convex tvs

We begin by recalling the classical concept of holomorphy for functions valued in a locally convex topological complex vector space. Our main reference for the concept and basic characterizations of holomorphy for functions valued in a locally convex topological vector space is [**Gro-a**]. We will occasionally use [**Gro-b**, Ch. II, §3] to characterize certain spaces of holomorphic and meromorphic functions as projective tensor products, under favorable conditions that we will find to be met in our situations of interest.

The literature on holomorphic and meromorphic functions valued in locally convex complex vector spaces is extensive. Sources of input for the material in this subsection and the next are [**Kruse**, §3,7], [**Jor-a**, §3], and [**BJM**]. The latter two in particular further build from work of Grosse-Erdmann in [**GE**] on the topologies on spaces of scalar-valued meromorphic functions on complex domains, an analog of which we will be requiring.

The main difference between the existing literature and the material discussed in this section is that we will be allowing Riemann surfaces as domains of definition for meromorphic vector-valued functions, which entails some, while limited, non-trivial amount of additional work.

C.5.1. Vector-valued continuous and differentiable functions. For a Hausdorff locally compact space X, we denote by $\overline{X}$ its one-point compactification, with ∞ the point at infinity, i.e. $\{\infty\} = \overline{X} - X$. The space $\overline{X}$ is compact Hausdorff, and $X \subset \overline{X}$ is an open subset whose subspace topology is the intrinsic topology on X.

We say a continuous complex-valued function f on X *tends to zero at infinity* if and only if it is bounded on X and its (hence existing) continuous extension $F : \overline{X} \to \mathbf{C}$ satisfies $\lim_{x\to+\infty} F(x) = 0$. This limit is in the sense of convergence in general topological spaces, defined by means of Cauchy filter-base or Cauchy net convergence.

We further recall that on $C_b(X)$, the $\mathbf{C}$-vector space of complex valued *bounded* continuous functions on X, we may define the uniform norm

$$\|f\|_u := \sup_{x\in X} |f(x)| \tag{C.5.1.1}$$

which promotes $C_b(X)$ to a complex Banach space. We call the norm-topology on $C_b(X)$ the topology of *uniform convergence*.

Within $C_b(X)$ we further find the $\mathbf{C}$-vector subspace $C_c(X)$ of continuous complex-valued compactly supported functions. This is, algebraically, the filtered colimit of the $\mathbf{C}$-linear subspaces $C(X, K)$ of continuous functions on X whose support is contained within the compact subset $K \subset X$. We denote by $\mathcal{K}_X$ the set of all compact subsets of X. An inclusion of compact subsets $K' \subset K$ induces a $\mathbf{C}$-linear extension-by-zero operator

$$C(X, K') \to C(X, K)$$

realizing $C_c(X)$ as $C_c(X) = \operatorname*{colim}_{K\in\mathcal{K}_X} C(X, K)$ as a $\mathbf{C}$-linear subspace of $C_b(X)$. The uniform norm on $C_b(X)$ endows each $C(X, K)$ with the structure of a complete normed space, since it is a closed subspace of $C(K)$, this latter equipped with the topology of uniform convergence.

We may endow $C_c(X)$ with two topologies: one is the topology of uniform convergence inherited from $C_b(X)$. This is a Hausdorff norm-topology and as such locally convex. We note that the subspace $C_c(X) \subset C_b(X)$ equipped with the topology of uniform convergence is not complete. Its completion is denoted $C_0(X)$, and it is the subspace of $C_b(X)$ of those bounded continuous functions that vanish at ∞.

The other topology on $C_c(X)$ is the locally convex inductive limit topology, upon endowing each $C(X, K)$ with its norm-topology induced by the uniform norm. With this topology, $C_c(X)$ is a locally convex topological $\mathbf{C}$-vector space, and it is Hausdorff because this topology is finer than the topology of uniform convergence, which is Hausdorff. We refer to the locally convex inductive limit topology on $C_c(X)$ as the *inductive limit topology* on $C_c(X)$. Assuming X is second-countable, so there is a *countable* cofinal system of K's with respect to inclusion, the subspace topology on $C(X, K)$ from the inductive limit topology on $C_c(X)$, is its original the topology of uniform convergence by [**Scha**, Ch. II, 6.4] (this uses that the inclusions $C(X, K) \to C(X, K')$ for $K \subset K'$ are not only topological embeddings with respect to the uniform norm but are also closed embeddings since $C(X, K)$ is complete).

When the continuous functions on X are valued in a Hausdorff locally convex topological $\mathbf{C}$-vector space E, similar notions exist and properties hold. Our main reference for the extension of these concepts to the case of E-valued functions is [**Bou-I**, Ch. III, §1].

We fix a Hausdorff locally convex topological $\mathbf{C}$-vector space E. The $\mathbf{C}$-vector space $C(X, E)$ of continuous E-valued functions is endowed with the topology generated by the collection of seminorms

$$q_{K,\nu}(f) := \sup_{x\in K} \nu(f(x)) \tag{C.5.1.2}$$

as ν varies over the set $S(E)$ of all continuous seminorms on E ($\nu \circ f$ is continuous on K, hence bounded) and K varies over the set $\mathcal{K}_X$ of all compact subsets of X.

The family $\{q_{K,\nu}\}_{(K,\nu)\in\mathcal{K}_X\times S(E)}$ is separating because if $q_{K,\nu}(f)=0$ for all $(K,\nu)\in\mathcal{K}_X\times S(E)$ then $f(x)=0$ for all $x\in K$ and all $K\in\mathcal{K}_X$, and hence $f=0$. In particular, endowed with this topology, $C(X,E)$ is Hausdorff. We refer to this locally convex topology on $C(X,E)$ as the topology of *uniform convergence on compact subsets*.

C.5.1.1. *Properties of the spaces of continuous compactly supported E-valued functions.* The following is [**Bou-I**, Ch. III, §1.1, Def. 1].

DEFINITION C.5.1.1. Let X be a Hausdorff locally compact space and E a Hausdorff locally convex topological $\mathbf{C}$-vector space. Let $f:X\to E$ be a continuous E-valued function. We define the *support* of f as the closure of the open subset

$$X_f := X - f^{-1}(0) \tag{C.5.1.3}$$

and denote it by $\mathrm{supp}(f)$. We say f is *compactly supported* if and only if $\mathrm{supp}(f)$ is a compact subset of X. We denote by $C_c(X,E)$ the $\mathbf{C}$-vector space of all compactly supported E-valued continuous functions on X, and for any compact subset $K\subset X$, by $C(X,K;E)$ the linear subspace of $C_c(X,E)$ of all those compactly supported E-valued functions whose support is contained in K. We topologize $C_c(X,E)$ with the locally convex inductive limit of the topologies on $C(X,K;E)$, each induced by the family of seminorms $\{q_{K,\nu}\}_{\nu\in S(E)}$ from (C.5.1.2).

REMARK C.5.1.2. On Definition C.5.1.1, we make the following observations.

(1) The subspace topology on $C_c(X,E)$ inherited from the topology on $C(X,E)$ of uniform convergence on compact subsets is locally convex .Its inductive limit topology (see [**Scha**, Ch. II, §6, pp. 56-60]) is also locally convex. These two topologies are generally *not* the same; we refer the reader to Example C.5.1.5. The former is coarser than the latter (i.e., the bijection $\varinjlim_K C(X,K;E)\to C_c(X,E)$ is continuous when the target has the topology of uniform convergence on compact subsets). When considering $C_c(X,E)$ as a locally convex topological $\mathbf{C}$-vector space, we will always be assuming that the topology it is being endowed with is the locally convex inductive limit topology.

LEMMA C.5.1.3. *Let X be a Hausdorff locally compact space, E a Hausdorff locally convex $\mathbf{C}$-vector space. Then*

(1) *The locally convex topological $\mathbf{C}$-vector space $C_c(X,E)$ of continuous E-valued compactly supported functions on X is Hausdorff. For any compact subset $K\subset X$, the topology induced on $C(X,K;E)$ by $C_c(X,E)$ is the topology of uniform convergence, and each of the linear subspaces $C(X,K;E)$ of $C_c(X,E)$ is closed.*
(2) *If X is the topological disjoint union of an I-indexed family of locally compact Hausdorff spaces $\{X_i\}_{i\in I}$ then the canonical $\mathbf{C}$-linear map*

$$C_c(X,E)\to\bigoplus_{i\in I}C_c(X_i,E) \tag{C.5.1.4}$$

is a linear topological isomorphism.
(3) *If X is paracompact and E is quasi-complete then $C_c(X,E)$ is quasi-complete.*
(4) *If X is paracompact then for every bounded subset $B\subset C_c(X,E)$ there exists a compact subset $K\subset X$ such that $B\subset C(X,K;E)$ (so it is bounded in $C(X,K;E)$ by (1)).*
(5) *$C_c(X,E)\subset C(X,E)$ is a dense subspace with respect to the topology of uniform convergence on compact subsets.*
(6) *If E is quasi-complete, $C(X,E)$ is locally convex, Hausdorff and quasi-complete with respect to the topology of uniform convergence on compact subsets.*

The spaces of E-valued continuous functions discussed in the foregoing and studied in Lemma C.5.1.3 account for many spaces of vector-valued functions throughout this work. By [**Bou-GT**, Ch. I, §9, no. 10,

Thm. 5], X is paracompact if and only if topologically $X = \coprod X_i$ for X_i's that are each covered by countably many compact subsets (so the inductive limit structure on each $C_c(X_i, E)$ in such cases is a countable strict inductive limit).

PROOF. Parts (1) and (2) are [**Bou-I**, Ch. III, §1, Prop. 1(i)] and [**Bou-I**, Ch. III, §1, Prop. 1(iii)] respectively, where $C_c(X, E)$ is denoted $\mathscr{K}(X, E)$. Part (3) is [**Bou-I**, Ch. III, §1, p .III.4, Corollary], part (4) is [**Bou-I**, Ch. III, §1, Prop. 2(ii)], and part (5) is [**Bou-I**, Ch. III, §1, p .III.6, Prop. 4].

(6) : The topology of uniform convergence on compact subsets of X is generated on $C(X, E)$ by the family of seminorms

$$q_{K,i}(f) := \sup_{x \in K} \nu_i(f(x)) \tag{C.5.1.5}$$

where $\{\nu_i\}_{i \in I}$ is a generating family for the locally convex topology on E. Since E is Hausdorff, $\{\nu_i\}_{i \in I}$ is separating. It follows that $q_{K,i}(f) = 0$ for all $i \in I$ implies $f(x) = 0$ for all $x \in K$, for any compact subset $K \subset X$. This gives $f = 0$, so $C(X, E)$ is Hausdorff.

Let $K \subset X$ be a compact subset. Then the restriction maps

$$\rho_K : C(X, E) \to C(K, E)$$

form a projective system, The topology on each $C(K, E)$ is generated the family of seminorms $\{q_{K,i}\}_{i \in I}$ from (C.5.1.5), for K fixed. The topology on $C(X, E)$ is clearly the projective limit of the topologies on $C(K, E)$. By [**Scha**, Ch. II, 5.3], we are thereby reduced to the case X is compact Hausdorff. In particular, $C(X, E) = C_c(X, E)$ and hence is quasi-complete by (3). □

REMARK C.5.1.4. By part (2), we see that part (4) is a special case of [**Scha**, Ch. II, 6.3, 6.5], a theorem of Dieudonné and Schwartz.

EXAMPLE C.5.1.5 (Distinctness of the two topologies on $C_c(X, E)$). Assume X is a Hausdorff, locally compact, paracompact space, and E is a Fréchet space. By Lemma C.5.1.3, $C_c(X, E)$ is a *strict* locally convex inductive limit of the closed topological linear subspaces $C(X, K; E)$ (see [**Scha**, Ch. II, p. 57]). Every Cauchy sequence in $C_c(X, E)$ is bounded and hence contained within $C(X, K; E)$ for some compact subset $K \subset X$ and bounded there. The locally convex topological $\mathbf{C}$-vector space $C(X, K; E)$ is a Fréchet space, and as such, any Cauchy sequence in $C(X, K; E)$ converges in $C(X, K; E)$.

Assume that, for a locally compact Hausdorff abelian group G, X carries a continuous free proper action of G. Let $K \subset X$ be a non-empty compact subset, and $f \in C_c(X, K; E)$ a fixed function. The action of G by left translation provides a family of functions $f_g := L_g(f)$, where $(L_g(f))(x) = f(g^{-1}x)$ for all $x \in X$ and each $g \in G$.

We now specialize to the case $X = \mathbf{R}$ and $G = \mathbf{Z}$ acting on X by additive translation, and $E = \mathbf{C}$. Let K be the unit interval in $\mathbf{R}$ and pick $f \neq 0$. Then the sequence $f_n := L_n(f)$ in $C_c(\mathbf{R})$ does not converge in $C_c(\mathbf{R})$. Indeed, if it did then it would have to be contained and converge in some $C(K', \mathbf{C})$ for some compact subset $K' \subset \mathbf{R}$, which cannot be since $\mathrm{supp}(f_n) \cap K' = \varnothing$ for all integers $n \in \mathbf{Z}$ with $|n| \gg 0$, and hence $\{f_n\}_{n \geq 1}$, a sequence of functions whose supports are non-empty in $\mathbf{R}$, cannot be contained in a single $C(K', \mathbf{C})$. On the other hand, the sequence $\{f_n\}_{n \geq 1}$ tends to the constant function 0 in $C(\mathbf{R})$, and so it is a null sequence in the linear space $C_c(\mathbf{R})$ endowed with the topology of uniform convergence on compact subsets.

For E an arbitrary nonzero Fréchet space, choose a nonzero vector $e \in E$. The sequence $\{\phi_n\}_{n \geq 0} \subset C_c(\mathbf{R}, E)$ defined by $\phi_n := f_n \otimes e$, viewed as a sequence in $C_c(\mathbf{R}, E)$ under the injective linear map

$$C_c(\mathbf{R}) \otimes E \to C_c(\mathbf{R}, E)$$

is also a sequence which does not converge to 0 with respect to the locally convex inductive limit topology but is a null sequence in $C_c(\mathbf{R}, E)$ endowed with the topology of uniform convergence on compact subsets.

A similar example can be made with $\mathbf{R}$ replaced by any Hausdorff locally compact space containing a non-empty open subset which is a topological manifold.

We will be henceforth assuming, unless otherwise stated, that $C_c(X, E)$ is endowed with the locally convex inductive limit topology of the topologies of uniform convergence on each $C(X, K; E)$ as in Definition C.5.1.1. We will be assuming that $C(X, E)$ is endowed, instead, with the topology of uniform convergence on compact subsets (so $C_c(X, E) \to C(X, E)$ is .continuous but generally not a topological embedding).

C.5.1.2. *Vector-valued differentiability.* We let k be either one of the Archimedean topological fields $k = \mathbf{R}, \mathbf{C}$, and E a Hausdorff locally convex topological $\mathbf{C}$-vector space. We will be closely following [**Kruse**, Def. 3.1] in defining three concepts of differentiability for E-valued functions on an open subset $\Omega \subset k^d$ for an integer $d \geq 0$.

The Hausdorff locally convex topological vector spaces throughout this subsection will still be assumed to be complex topological vector spaces even when $k = \mathbf{R}$, in which case for $E = \mathbf{C}$ the differentiable vector-valued functions discussed herein will recover the classical notion of differentiable complex-valued functions on a Euclidean space over the reals. In other words, the topological field k is not the scalar field of the topological vector spaces occurring throughout this subsection: such field is always assumed to be $\mathbf{C}$.

We denote by $|\cdot|$ the Euclidean norm on k, whether real or complex. We further denote by $|\cdot|$ again, the Euclidean norm on k^d for any $d \geq 0$.

Throughout this subsection, occasional references are made to the basic results on the integral in the sense of Pettis from §C.4.3: the Mean-Value Theorem (Theorem C.4.3.1; see also Theorem C.4.3.2), and the Integrability Criterion of Theorem C.4.3.5.

Definition C.5.1.6 (Weak differentiability). Let $d \geq 0$ be an integer, and $\Omega \subset k^d$ an open subset. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, and $f : \Omega \to E$ a function. We say f is *weakly differentiable* over Ω if and only if for every $\varphi \in E'$, the function $\langle f, \varphi\rangle_{E,E'} : \Omega \to \mathbf{C}$ defined by

$$\xi \mapsto \langle f(\xi), \varphi\rangle_{E,E'} \tag{C.5.1.6}$$

is once-differentiable in the ordinary sense (i.e. real-differentiable if $k = \mathbf{R}$, and differentiable in the complex sense if $k = \mathbf{C}$; see Remark C.5.1.7).

For f weakly differentiable over Ω, we define the function $\mathrm{D}f : \Omega \times k^d \to (E')^*$ by requiring that

$$\langle \mathrm{D}f, \varphi\rangle_{E,E'} := \mathrm{D}\langle f, \varphi\rangle_{E,E'} : \Omega \times k^d \to \mathbf{C} \tag{C.5.1.7}$$

be the function of $(\xi_1, \xi_2) \in \Omega \times k^d$ given by $(\xi_1, \xi_2) \mapsto \mathrm{D}_{\xi_1}\langle f, \varphi\rangle_{E,E'}(\xi_2)$, with $\mathrm{D}_{\xi_1}\langle f, \varphi\rangle_{E,E'}$ the differential of the $\mathbf{C}$-valued function $\langle f(\xi), \varphi\rangle_{E,E'}$ of $\xi \in \Omega$ at $\xi_1 \in \Omega$.

We refer to $\mathrm{D}f$ as the *weak differential* of f. If $\mathrm{D}f : \Omega \times k^d \to (E')^*$ is continuous we say f is *weakly continuously differentiable*.

Remark C.5.1.7. For a scalar-valued function $f : \Omega \to \mathbf{C}$, $\Omega \subset k^d$ an open subset, to be differentiable "in the ordinary sense", we mean that there exists a (clearly unique) k-linear map $J_{\xi_0} : k^d \to \mathbf{C}$ such that

$$\lim_{\xi \to 0} \frac{f(\xi_0 + \xi) - f(\xi_0) - J_{\xi_0}(\xi)}{|\xi|} \tag{C.5.1.8}$$

exists and vanishes. Specifically, we are not assuming that $J_{\xi_0}(\xi) \in \mathbf{C}$ is continuous in (ξ_0, ξ); i.e., continuous as a map $\Omega \times k^d \to \mathbf{C}$. Differentiable "in the ordinary sense" does not mean "continuously differentiable". In (C.5.1.8), $J_{\xi_0}(\xi)$ refers to the effect of the k-linear functional J_{ξ_0} on the vector $\xi \in k^d$. If $k = \mathbf{C}$, and f is complex-differentiable, then in particular we have $J_{\xi_0}(\xi) = f'(\xi_0)\xi$.

Much as for the integral in the sense of Pettis in §**??**, we now discuss differentiability for vector-valued functions in the weak sense first and then see when such a condition is equivalent to a stronger one.

DEFINITION C.5.1.8 (Vector-valued differentiability). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 0$ an integer, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their norm-topology. Let $\Omega \subset k^d$ be an open subset and $f : \Omega \to E$ be a function.

(1) f is *continuously differentiable* over Ω *within* $(E')^*$ if and only if:
   (a) for every $\xi_1 \in \Omega$, there exists a (clearly unique) k-linear map $\mathrm{D}_{\xi_1} f : k^d \to (E')^*$, continuous with respect to the Euclidean topology on the source and the $\sigma((E')^*, E')$-topology on the target, and such that for every $\xi_2 \in k^d$, $(\mathrm{D}_{\xi_1} f)(\xi_2)$ is a continuous $(E')^*$-valued function of $\xi_1 \in \Omega$, such that the limit in $(E')^*$ with respect to the $\sigma((E')^*, E')$-topology:

$$\lim_{\xi \to \xi_1, \xi \neq \xi_1} \frac{f(\xi) - f(\xi_1) - (\mathrm{D}_{\xi_1} f)(\xi - \xi_1)}{|\xi - \xi_1|} = 0 \tag{C.5.1.9}$$

   exists as an element of $(E')^*$ and it vanishes, where the limit is formed as $\xi \to \xi_1$ from within the open subset $\{\xi \in \Omega : \xi \neq \xi_1\} = \Omega - \{\xi_1\} \subset k^d$.
   (b) Formation of the assignment $(\xi_1, \xi_2) \mapsto (\mathrm{D}_{\xi_1} f)(\xi_2)$ is continuous as a function $\Omega \times k^d \to (E')^*$ with respect to the product topology on the source, and the $\sigma((E')^*, E')$-topology on the target.

(2) f is *Gateaux-differentiable* over Ω *within* $(E')^*$ if and only if for every $(\xi_1, \xi_2) \in \Omega \times k^d$, the limit in $(E')^*$ with respect to the $\sigma((E')^*, E')$-topology:

$$\lim_{h \to 0, h \in W} \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h} \tag{C.5.1.10}$$

   exists as an element of $(E')^*$, where the limit is formed as $h \to 0$ over the open subset $W := \{h \in k : h \neq 0, \xi_1 + h\,\xi_2 \in \Omega\} \subset k$.

   We denote the value in $(E')^*$ of the limit in (C.5.1.10) as

$$(\partial_{\xi_1} f)(\xi_2) := \lim_{h \to 0, h \in k, h \neq 0} \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h} \tag{C.5.1.11}$$

   and refer to it as the *directional derivative* of f at $\xi_1 \in \Omega$ along $\xi_2 \in k^d$.

(3) f is *continuously differentiable over* Ω if and only if:
   (a) it is continuously differentiable over Ω within $(E')^*$.
   (b) The map

$$\Omega \times k^d \to (E')^* \tag{C.5.1.12}$$

   given by

$$(\xi_1, \xi_2) \mapsto (\mathrm{D}_{\xi_1} f)(\xi_2) \tag{C.5.1.13}$$

   is valued in E and continuous with respect to the Euclidean topology on the source, and the initial topology on E. We call the map in (C.5.1.13) the *differential* of f.

REMARK C.5.1.9. Since k^d in Definition C.5.1.8 is a normed finite dimensional k-vector space, any k-linear map to $(E')^*$ is continuous with respect to the Euclidean topology on the source and the $\sigma((E')^*, E')$-topology on the target.

Definition C.5.1.8 is contained in [**Kruse**, Def. 3.1] with one difference. We are assuming that the linear differential and the limits in Definition C.5.1.8 are valued in $(E')^*$, which is complete with respect to the $\sigma((E')^*, E')$-topology, and agrees with the completion of E with respect to the $\sigma(E, E')$-topology, also referred to as the *weak completion* of E in [**Gro-a**]. We refer the reader to Lemma C.4.1.2 and Remark C.4.1.3.

The symbol $\widehat{E}$ in [**Kruse**, Def.3.1] refers to the completion of E as a topological vector space. Continuity of $E \to (E')^*$ for the given topology on E and the weak topology $\sigma((E')^*, E')$ on the target (which is complete) induces a continuous linear map $\widehat{E} \to (E')^*$ but it may be non-injective or not a topological embedding.

REMARK C.5.1.10. We note that if $k = \mathbf{R}$ and $d = 2$, then upon identifying $\mathbf{R}^2$ with the complex plane, Gateaux-differentiability of a function $f : \mathbf{C} \to \mathbf{C}$ (choosing $E = \mathbf{C}$) does not imply differentiability in the complex sense, i.e. upon rethinking f as a function $k^d \to \mathbf{C}$ with $k = \mathbf{C}$ and $d = 1$. For $k = \mathbf{C}$ and $d = 1$, Gateaux-differentiability amounts to differentiability in the complex sense, which we shall be discussing in §C.5.2.1.

REMARK C.5.1.11. We note that there is no continuity assumption on f in any of the conditions assigned in Definition C.5.1.8.

LEMMA C.5.1.12 (On the equivalence of vector valued differentiability conditions). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 0$ an integer, k either one of the fields $k = \mathbf{R}, \mathbf{C}$ endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function. Then the following conditions are equivalent:*

(1) *f is weakly differentiable over Ω, and the differential $\mathrm{D}f : \Omega \times k^d \to (E')^*$ is continuous with respect to the product topology on the source, and the $\sigma((E')^*, E')$-topology on the target.*
(2) *f is continuously differentiable over Ω within $(E')^*$.*

If f satisfies the above equivalent conditions, the weak differential satisfies:

$$(\mathrm{D}f)(\xi_1, \xi_2) = (\mathrm{D}_{\xi_2} f)(\xi_3) \tag{C.5.1.14}$$

for every $(\xi_1, \xi_2) \in \Omega \times k^d$. Moreover, f is Gateaux-differentiable within $(E')^*$.

The following proof closely follows the proof of [**Kruse**, Rem. 3.2], to which we further refer the reader.

PROOF. (1) $\Rightarrow$ (2) : Assume f is weakly differentiable. The weak differential $\mathrm{D}f$ exists and by assumption it is continuous with respect to the product topology on $\Omega \times k^d$ and the $\sigma((E')^*, E')$-topology on $(E')^*$. In particular, upon denoting by $i_{\xi_1} : k^d \to \Omega \times k^d$ the continuous map sending $\xi_2 \in k^d$ to $(\xi_1, \xi_2) \in \Omega \times k^d$, and by $j_{\xi_2} : \Omega \to \Omega \times k^d$ the continuous map sending ξ_1 to (ξ_1, ξ_2), both maps $(\mathrm{D}f) \circ i_{\xi_1}$ and $(\mathrm{D}f) \circ j_{\xi_2}$ are continuous as $(E')^*$-valued functions.

Moreover, by definition of weak differentiability, we have that the limit in $(E')^*$

$$\lim_{\xi \to \xi_1, \xi \neq \xi_1} \frac{\iota_E(f(\xi)) - \iota_E(f(\xi_1)) - (\mathrm{D}f)(\xi_1, \xi - \xi_1)}{|\xi - \xi_1|} \tag{C.5.1.15}$$

exists in $(E')^*$ with respect to the $\sigma((E')^*, E')$-topology, and it vanishes. This implies that f is continuously differentiable over Ω within $(E')^*$.

(2) $\Rightarrow$ (1) : Suppose f is continuously differentiable over Ω within $(E')^*$. Then there exists a map:

$$(\mathrm{D}_{(\cdot)} f)(\cdot) : \Omega \times k^d \to (E')^* \tag{C.5.1.16}$$

separately continuous over each factor of $\Omega \times k^d$, each endowed with its topology induced from the norm on k^d, and the $\sigma((E')^*, E')$-topology on $(E')^*$. Moreover, the limit

$$\lim_{\xi \to \xi_1, \xi \neq \xi_1} \frac{\iota_E(f(\xi)) - \iota_E(f(\xi_1)) - (\mathrm{D}_{\xi_1} f)(\xi - \xi_1)}{|\xi - \xi_1|} = 0 \tag{C.5.1.17}$$

in the topology of $\sigma((E')^*, E')$-convergence. For every $\varphi \in E'$, we may evaluate $\langle \cdot, \varphi \rangle_{(E')^*, E'}$ on (C.5.1.17), and obtain

$$\lim_{\xi \to \xi_1, \xi \neq \xi_1} \left\langle \frac{f(\xi) - f(\xi_1) - (\mathrm{D}_{\xi_1} f)(\xi - \xi_1)}{|\xi - \xi_1|}, \varphi \right\rangle_{E, E'} = \lim_{\xi \to \xi_1, \xi \neq \xi_1} \frac{\langle f(\xi), \varphi \rangle_{E,E'} - \langle f(\xi_1), \varphi \rangle_{E,E'} - \langle (\mathrm{D}_{\xi_1} f)(\xi - \xi_1), \varphi \rangle_{E,E'}}{|\xi - \xi_1|} \tag{C.5.1.18}$$

so the right side vanishes since the left side does by (C.5.1.17). This gives weak differentiability of f, and hence the weak differential is defined. By uniqueness of the differential for $\mathbf{C}$-valued differentiable functions

on Ω, we have that for every $\varphi \in E'$:

$$\langle (\mathrm{D}f)(\xi_1, \xi_2), \varphi \rangle_{E,E'} = \langle (\mathrm{D}_{\xi_1} f)(\xi_2), \varphi \rangle_{E,E'}$$

and hence (C.5.1.14) follows from the Hahn-Banach Theorem since E is Hausdorff.

To see that f is Gateaux-differentiable, we choose $\xi = \xi_1 + h\,\xi_2$, and rename ξ_2 in (C.5.1.17) as ξ_1, in condition (C.5.1.9), to obtain that condition (C.5.1.10) is satisfied. □

C.5.1.3. *Permanence properties of vector-valued differentiation.* We collect here basic properties of differentiability for E-valued functions over an open subset $\Omega \subset k^d$. Lemma C.5.1.13 below closely follows [**Kruse**, Rem. 3.2] and, in this form, is due to Grothendieck (see the sketch in [**Gro-a**, §2, Footnote 4, p. 39]).

Lemma C.5.1.13 (On weak and strong differentiability). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their norm-topology. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function. Then we have:*

(1) *If f is weakly continuously differentiable (i.e., the weak derivative $\mathrm{D}f : \Omega \times k^d \to (E')^*$ exists and is continuous) then f is Lipschitz-continuous (and hence continuous); i.e., for all $\xi', \xi \in \Omega$, $\xi' \neq \xi$, satisfying $|\xi' - \xi| \leq 1$, the ratios:*

$$\frac{f(\xi') - f(\xi)}{|\xi' - \xi|} \tag{C.5.1.19}$$

are contained in a bounded, equivalently closed convex circled bounded, subset of E.

(2) *If f is weakly continuously differentiable and the weak differential $\mathrm{D}f$ is valued in E and is continuous with respect to the product topology on the source, and the given topology on E then f is continuously differentiable over Ω and for every $(\xi_1, \xi_2) \in \Omega \times k^d$ the limit:*

$$\lim_{h \to 0, h \in W} \frac{f(\xi_1 + h\xi_2) - f(\xi_1)}{h} \tag{C.5.1.20}$$

converges in E to $(\mathrm{D}f)(\xi_1, \xi_2)$ with respect to the given topology on E, for $W \subset k$ the open subset $\{h \in k^\times : \xi_1 + h\xi_2 \in \Omega\}$.

Proof. (1) : Fix $\xi_1, \xi_2 \in \Omega$ near enough so that the closed segment $[\xi_1, \xi_2]$ is contained in Ω, and define $h := |\xi_1 - \xi_2|$. We consider the smooth curve $\gamma : [0, h] \to \xi_1 + tu$ where $u := (\xi_2 - \xi_1)/h$. As a function of t, $F : t \mapsto (\mathrm{D}f)(\gamma(t), \gamma'(t))$ is valued in $(E')^*$, which is complete with respect to the $\sigma((E')^*, E')$-topology, and it is continuous since we assume f is weakly continuously differentiable.

The integrability criterion of Theorem C.4.3.5 (viewing k-valued functions as $\mathbf{C}$-valued, since we have either $k = \mathbf{R} \subset \mathbf{C}$ or $k = \mathbf{C}$) therefore applies to F and $(E')^*$, to give that the integral

$$\int_{[0,h]} (\mathrm{D}f)(\gamma(t), \gamma'(t))\,\mathrm{d}t$$

exists as an element of $(E')^*$, and not merely as an element of $(((E')^*)')^*$. It evaluates to

$$\int_0^h \mathrm{D}_{\gamma(t)} \langle f, \varphi \rangle_{E,E'}(\gamma'(t))\,\mathrm{d}t = \int_0^h \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle f, \varphi \rangle_{E,E'} \circ \gamma \right) \mathrm{d}t = \langle f(\xi_2), \varphi \rangle_{E,E'} - \langle f(\xi_1), \varphi \rangle_{E,E'}$$

since $\langle f, \varphi \rangle_{E,E'}$ is differentiable over Ω as a $\mathbf{C}$-valued function. As elements of $(E')^*$, we therefore have

$$f(\xi_2) - f(\xi_1) = \int_{[0,h]} (\mathrm{D}f)(\gamma(t), \gamma'(t))\,\mathrm{d}t \tag{C.5.1.21}$$

where, however, the left side of (C.5.1.21) is an element of E. Let $K \subset (E')^*$ be the closed convex circled hull of $F([0, h])$. By continuity of F (with respect to the Euclidean topology on $[0, h]$ and the $\sigma((E')^*, E')$-topology

on $(E')^*$), $F([0,h])$ is compact and hence so is K since $(E')^*$ is complete (see [**Scha**, Ch. III, Cor. to 4.3]). In particular, it is bounded in $(E')^*$.

By Theorem C.4.3.1, we therefore have:

$$\int_{[0,h]} (\mathrm{D}f)(\gamma(t),\gamma'(t))\,\mathrm{d}t \in hK \tag{C.5.1.22}$$

and hence:

$$\int_{[0,h]} (\mathrm{D}f)(\gamma(t),\gamma'(t))\,\mathrm{d}t \in h(K\cap E). \tag{C.5.1.23}$$

Since K is bounded in $(E')^*$ with respect to the $\sigma((E')^*,E')$-topology, $K\cap E$ is bounded in E with respect to the $\sigma(E,E')$-topology that is the subspace topology on E from he $\sigma((E')^*,E')$-topology under the linear injection ι_E. Thus, $K\cap E$ is bounded in E for its given topology [**Scha**, Ch. III, 3.3], so f is Lipschtiz-continuous.

(2) : By (1), f is continuous. Moreover, for every $\varphi\in E'$, we have that the limit:

$$\lim_{\xi\to\xi_1,\xi\neq\xi_1} \left\langle \frac{f(\xi)-f(\xi_1)-(\mathrm{D}_{\xi_1}f)\,(\xi-\xi_1)}{|\xi-\xi_1|},\varphi\right\rangle_{E,E'} \tag{C.5.1.24}$$

exists in k and it vanishes. This means that within $(E')^*$ and with respect to the $\sigma((E')^*,E')$-topology, the limit:

$$\lim_{\xi\to\xi_1,\xi\neq\xi_1} \frac{f(\xi)-f(\xi_1)-(\mathrm{D}f)(\xi_1,\xi-\xi_1)}{|\xi-\xi_1|} \tag{C.5.1.25}$$

exists and it vanishes. In particular, f is differentiable over Ω within $(E')^*$. By assumption, $\mathrm{D}f$ is valued in E and continuous, and so f is differentiable over Ω and $\mathrm{D}f$ is its differential.

It remains to show the assertion on the mode of convergence of the limit in (C.5.1.20), which we already know exists in $(E')^*$ and is valued in E by Lemma C.5.1.12, and converges with respect to the given topology on E.

For any distinct $\xi_1,\xi_2\in\Omega$ near enough so that $[\xi_1,\xi_2]\subset\Omega$, and $h=|\xi_1-\xi_2|$, let $\gamma:[0,h]\to\Omega$ be the smooth curve $\gamma(t)=\xi_1+(t/h)(\xi_2-\xi_1)$. Since $\mathrm{D}f$ is continuous with respect to the product topology on $\Omega\times k^d$ and the given topology on E, the function $(\mathrm{D}f)(\gamma(t),\gamma'(t))$ of $t\in[0,h]$ is continuous with respect to the Euclidean topology on $[0,h]$ and the given topology on E.

By Theorem C.4.3.5, the continuous E-valued function $t\mapsto F(t):=(\mathrm{D}f)(\gamma(t),\gamma'(t))-(\mathrm{D}f)(\xi_1,\gamma'(0))$ is integrable over $[0,h]$ in the sense of Pettis, and as in the proof of (1) its integral satisfies

$$\frac{f(\gamma(h))-f(\xi_1)-(\mathrm{D}f)(\xi_1,\xi_2-\xi_1)}{h} = \frac{1}{h}\int_{[0,h]} F(t)\,\mathrm{d}t \in K\cap E \tag{C.5.1.26}$$

where K is the closed convex circled hull of $F([0,|\xi_2-\xi_1|])$ in $(E')^*$.

We deduce from (C.5.1.26) that

$$\frac{f(\xi_2)-f(\xi_1)-(\mathrm{D}f)(\xi_1,\xi_2-\xi_1)}{|\xi_2-\xi_1|} = \frac{1}{|\xi_2-\xi_1|}\int_{[0,|\xi_2-\xi_1|]} F(t)\,\mathrm{d}t \in K\cap E. \tag{C.5.1.27}$$

The subset $K\subset E$ as a closure for the given topology on E by [**Scha**, Ch. IV, 3.1], so K shrinks to 0 in E for its given topology as $\xi_2\to\xi_1$ by continuity of F with respect to the given topology on E. Hence, as $\xi_2\to\xi_1$, the limit of the left side in (C.5.1.27) converges to 0 in E with respect to the initial topology. Replacing ξ_2 with $\xi+h\xi_2$ for nonzero $\xi\in k^d$ and $h\in k^\times$ sufficiently small, the left side of (C.5.1.27) is

$$\frac{h}{|\xi_2|\,|h|}\cdot\frac{f(\xi_1+h\xi_2)-f(\xi_1)}{h}(\mathrm{D}f)(\xi_1,\xi_2).$$

□

We now define partial derivatives of a function $f : \Omega \to E$, with $\Omega \subset k^d$ an open subset, $d \geq 1$ an integer, and E a Hausdorff locally convex topological $\mathbf{C}$-vector space. We further refer the reader to [**Kruse**, Def. 3.1(c)].

DEFINITION C.5.1.14 (Weak partial derivative operators). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer, k either one of the Archimedean topological fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be a non-empty open subset, and $f : \Omega \to E$ a function. We call e_i the i-th unit vector in the standard k-linear basis for k^d, for $1 \leq i \leq d$.

We say f is *weakly differentiable in the e_i-direction* over Ω if and only if for every $\varphi \in E'$, the function

$$\langle f, \varphi \rangle_{E,E'} : \Omega \to \mathbf{C}$$

is differentiable in the e_i-direction in the ordinary sense (i.e. real-differentiable if $k = \mathbf{R}$, and differentiable in the complex sense if $k = \mathbf{C}$).

For f weakly differentiable in the e_i-direction over Ω, we define the function $\mathrm{D}_i f : \Omega \to (E')^*$ by

$$\langle \mathrm{D}_i f, \varphi \rangle_{(E')^*,E'} := \mathrm{D}_i \langle f, \varphi \rangle_{E,E'} : \Omega \to \mathbf{C} \tag{C.5.1.28}$$

where the right side of (C.5.1.28) is the i-th partial derivative of the function $\langle f, \varphi \rangle_{E,E'}$, differentiable in the e_i-direction. We refer to $\mathrm{D}_i f$ as the *i-th weak partial derivative* of f.

LEMMA C.5.1.15 (On weak differentiability). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer, k either one of the Archimedean topological fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function. Then the following conditions are equivalent:*

1. *f is weakly differentiable over Ω, and the weak differential $\mathrm{D}f$ is a continuous function $\Omega \times k^d \to (E')^*$ with respect to the product topology on the source and the $\sigma((E')^*, E')$-topology on the target.*
2. *f satisfies the following conditions:*
   (a) *f is weakly differentiable in the e_i-direction over Ω, for every $1 \leq i \leq d$.*
   (b) *$\mathrm{D}_i f : \Omega \to (E')^*$ is continuous with respect to the Euclidean topology on the source and the $\sigma((E')^*, E')$-topology on the target, for every $1 \leq i \leq d$.*

PROOF. (1) $\Rightarrow$ (2) : By assumption, the weak differential $\mathrm{D}f : \Omega \times k^d \to (E')^*$ is continuous with respect to the product topology on the source and the $\sigma((E')^*, E')$-topology on the target. It then follows that $(\mathrm{D}f)(\cdot, e_i)$ is an $(E')^*$-valued e_i-direction weak derivative on Ω, and by hypothesis in (1) this is continuous with respect to the topology on Ω and the $\sigma((E')^*, E')$-topology on $(E')^*$. This proves that (2.a) and (2.b) hold.

(2)$\Rightarrow$ (1) : If f is weakly differentiable in the e_i-direction for all i with continuous derivatives with respect to the topology on Ω and the $\sigma((E')^*, E')$-topology on $(E')^*$ then for every $\varphi \in E'$ the function $\xi \mapsto \langle f(\xi), \varphi \rangle_{E,E'}$ admits partial derivatives at all points of Ω, and they are each continuous. It is therefore differentiable on Ω. Since $\varphi \in E'$ arbitrary, f is weakly differentiable with e_i-direction weak derivative $(\mathrm{D}f)(\cdot, e_i)$. Thus, for every $\xi_2 = \sum_{1 \leq i \leq d} c_i(\xi_2) e_i \in k^d$ for $c_i(\xi_2) \in k$,

$$\mathrm{D}f(\xi_1, \xi_2) = \sum_{1 \leq i \leq d} c_i(\xi_2) \mathrm{D}_i f(\xi_1) \tag{C.5.1.29}$$

since both elements of $(E')^*$ evaluate the same on all $\varphi \in E'$. Hence, $\mathrm{D}f$ is continuous with respect to the product topology on $\Omega \times k^d$ on the source and the $\sigma((E')^*, E')$-topology on the target. □

LEMMA C.5.1.16 (On Gateaux-differentiability). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function.*

(1) *If f is Gateaux-differentiable over Ω within $(E')^*$ then it is weakly differentiable in the e_i-direction over Ω for every $1 \leq i \leq d$.*

(2) *If f is Gateaux-differentiable over Ω within $(E')^*$ and the limit*

$$\lim_{h \to 0, h \in k, h \neq 0} \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h} \in (E')^* \tag{C.5.1.30}$$

(for each $\xi_2 \in k^d$ and all $h \in k - \{0\}$ such that $\xi_1 + h\,\xi_2 \in \Omega$) is continuous as a function of $(\xi_1, \xi_2) \in \Omega \times k^d$ with respect to the product topology on $\Omega \times k^d$ on the source and the $\sigma((E')^*, E')$-topology on the target then f is weakly differentiable over Ω and the directional derivative satisfies:

$$(\partial_{\xi_1} f)(\xi_2) = (\mathrm{D}f)(\xi_1, \xi_2) \tag{C.5.1.31}$$

as elements of $(E')^*$, for every $(\xi_1, \xi_2) \in \Omega \times k^d$.

(3) *If f is Gateaux-differentiable over Ω within $(E')^*$ and the limit*

$$\lim_{h \to 0, h \in k, h \neq 0} \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h} \tag{C.5.1.32}$$

(for each $\xi_2 \in k^d$ and all $h \in k - \{0\}$ such that $\xi_1 + h\,\xi_2 \in \Omega$) is E-valued and continuous in $\xi_1 \in \Omega$ for fixed ξ_2 as a function with respect to the product topology on k^d and the given topology on E then f is continuously differentiable over Ω and the directional derivative satisfies

$$(\partial_{\xi_1} f)(\xi_2) = (\mathrm{D}_{\xi_1} f)(\xi_2) = (\mathrm{D}f)(\xi_1, \xi_2) \tag{C.5.1.33}$$

in E for every $(\xi_1, \xi_2) \in \Omega \times k^d$.

PROOF. (1) : In (C.5.1.10) from Definition C.5.1.8(2), we choose $\xi_2 = e_i \in k^d \subset \mathbf{C}^d$ for each $1 \leq i \leq d$. Upon evaluating the arbitrary $\varphi \in E'$ on (C.5.1.10), by linearity and continuity of φ we obtain that f is weakly differentiable in the e_i-direction over Ω.

(2) : We let i be an integer with $1 \leq i \leq d$, and choose $\xi_2 = e_i$ in (C.5.1.32). We define:

$$(\partial_i f)(\xi_1) := \lim_{h \to 0, h \in k, h \neq 0} \frac{f(\xi_1 + h\,e_i) - f(\xi_1)}{h}$$

an element of $(E')^*$ whose formation varies continuously with respect to $\xi_1 \in \Omega$. For $\varphi \in E'$ arbitrary, we have:

$$\begin{aligned}
\langle \partial_i f, \varphi \rangle_{E,E'} &= \left\langle \lim_{h \to 0, h \in k, h \neq 0} \frac{f(\xi_1 + h\,e_i) - f(\xi_1)}{h}, \varphi \right\rangle_{E,E'} \\
&= \lim_{h \to 0, h \in k, h \neq 0} \frac{\langle f(\xi_1 + h\,e_i), \varphi \rangle_{E,E'} - \langle f(\xi_1), \varphi \rangle_{E,E'}}{h} \\
&= \partial_i \langle f, \varphi \rangle_{E,E'}
\end{aligned}$$

where $\partial_i \langle f, \varphi \rangle_{E,E'}$ denotes the ith partial derivative of the function $\xi \mapsto \langle f(\xi), \varphi \rangle_{E,E'}$, and we have used continuity of φ to exchange the order of formation of the limit and of the evaluation pairing, and linearity to evaluate the pairing $(f(\xi_1 + h\,e_i) - f(\xi_1)/h$. In other words, the $\mathbf{C}$-valued function $\xi \mapsto \langle f(\xi), \varphi \rangle_{E,E'}$ is differentiable in the e_i-direction with continuous partial derivative, for each $1 \leq i \leq d$. It is therefore differentiable over Ω, and hence f is weakly differentiable over Ω. The identity in (C.5.1.31) follows upon evaluation of both sides on an arbitrary $\varphi \in E'$.

(3) : By (2), f is weakly differentiable over Ω. In particular, the weak differential $\mathrm{D}f$ is defined. We compute, for $\varphi \in E'$ arbitrary:

$$\text{(C.5.1.34)}\quad \left\langle \lim_{h\to 0, h\in k, h\neq 0} \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h}, \varphi \right\rangle_{E,E'} = \lim_{h\to 0, h\in k, h\neq 0} \frac{\langle f(\xi_1 + h\,\xi_2), \varphi\rangle_{E,E'} - \langle f(\xi_1), \varphi\rangle_{E,E'}}{h}$$
$$= \mathrm{D}_{\xi_1}\langle f, \varphi\rangle_{E,E'}(\xi_2)$$
$$= \langle (\mathrm{D}f)(\xi_1, \xi_2), \varphi\rangle_{E,E'}$$

where $\mathrm{D}_{\xi_1}\langle f, \varphi\rangle_{E,E'}(\xi_2)$ is the ordinary differential of the function $\xi \mapsto \langle f(\xi), \varphi\rangle_{E,E'}$ at the point $\xi_1 \in \Omega$ and evaluated on the k-linear tangent vector ξ_2. By the Hahn-Banach Theorem, $(\mathrm{D}f)(\xi_1, \xi_2)$ equals the limit in (C.5.1.33) as an element of $(E')^*$ and is also given by (C.5.1.29) , so the E-valued $\mathrm{D}f$ is continuous using the given topology on E. □

We collect here for later use two important permanence properties of directional derivatives for functions valued in a Hausdorff locally convex topological $\mathbf{C}$-vector space:

LEMMA C.5.1.17 (Permanence properties of vector-valued differentiability). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega_1, \Omega_2 \subset k^d$ be open subsets, and $f : \Omega_2 \to E$ a function. Let $\phi : \Omega_1 \to \Omega_2$ be a differentiable function. Then:*

(1) *If f is weakly differentiable over Ω_2, so is $f \circ \phi$ over Ω_1.*
(2) *If f is differentiable over Ω_2 then so is $f \circ \phi$ and we have*

$$\text{(C.5.1.35)}\qquad \mathrm{D}(f \circ \phi) = (\mathrm{D}f)(\phi, \mathrm{D}\phi) : \Omega_1 \times k^d \to E$$

where $\mathrm{D}\phi : \Omega_1 \times k^d \to k^d$ is the ordinary differential of the smooth function ϕ and (as E-valued functions over $\Omega_1 \times k^d$)

$$\text{(C.5.1.36)}\qquad (\mathrm{D}_{\xi_1}(f \circ \phi))(\xi_2) = (\mathrm{D}_{\phi(\xi_1)}f)((\mathrm{D}_{\xi_1}\phi)(\xi_2)).$$

REMARK C.5.1.18. When writing $(\phi, \mathrm{D}\phi) : \Omega_1 \times k^d \to \Omega_2 \times k^d$, it is understood that the dependency on Ω_1 of ϕ and $\mathrm{D}\phi$ is, for both, through the same copy of Ω_1. That is to say that, as a function $\Omega_1 \times k^d \to \Omega_2 \times k^d$, we write $(\phi, \mathrm{D}\phi)$ to mean $(\xi_1, \xi_2) \mapsto (\phi(\xi_1), (\mathrm{D}\phi)(\xi_1, xi_2)$.

PROOF. (1) : If f is weakly differentiable over Ω_2 then for any $\varphi \in E'$ the function $\langle f, \varphi\rangle_{E,E'} : \Omega \to k$ is differentiable and hence so is the composition $\langle f, \varphi\rangle_{E,E'} \circ \phi = \langle f \circ \phi, \varphi\rangle_{E,E'}$. Since φ is arbitrary, $f \circ \phi$ is weakly differentiable.

(2) : By (1), $f \circ \phi$ is weakly differentiable. The weak differential $\mathrm{D}f$ is therefore defined. Let $\varphi \in E'$ be the arbitrary continuous linear form on E. We compute, for every $(\xi_1, \xi_2) \in \Omega_1 \times k^d$:

$$\langle \mathrm{D}(f \circ \phi)(\xi_1, \xi_2), \varphi\rangle_{E,E'} = \mathrm{D}_{\xi_1}\langle f \circ \phi, \varphi\rangle_{E,E'}(\xi_2)$$
$$= (\mathrm{D}_{\phi(\xi_1)}\langle f, \varphi\rangle_{E,E'})((\mathrm{D}_{\xi_1}\phi)(\xi_2))$$
$$= \langle (\mathrm{D}f)(\phi(\xi_1), (\mathrm{D}_{\xi_1}\phi)(\xi_2)), \varphi\rangle_{E,E'}$$

where the last term equals $(\mathrm{D}f)\circ(\phi, \mathrm{D}\phi) : \Omega_1 \times k^d \to (E')^*$ pointwise on $\Omega_1 \times k^d$. Moreover, since $\mathrm{D}f : \Omega_2 \times k^d \to E$ is E-valued and continuous with respect to the given topology on E and $(\phi, \mathrm{D}\phi) : \Omega_1 \times k^d \to \Omega_2 \times k^d$ is continuous, we deduce that the left side of (C.5.1.35) is E-valued and continuous with respect to the product topology on $\Omega_1 \times k^d$ on the source and the given topology on E. This proves that $f \circ \phi$ is differentiable and that (C.5.1.36) holds. □

In the Lemma below, we denote by $\mathrm{d}t$ the Lebesgue measure on the real t-line, restricted to any interval, normalized so as to give the unit interval $[0, 1]$ volume 1.

LEMMA C.5.1.19 (Vector-valued Fundamental Theorem of Calculus). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $d \geq 1$ be an integer. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function. Let $\gamma : (a_0, b_0) \to \Omega$ a once-differentiable function, with $a_0, b_0 \in \mathbf{R}$, $a_0 < b_0$. Let $[a, b] \subset (a_0, b_0)$ be a non-empty closed interval. Assume:*

(1) *f is continuously differentiable over an open neighborhood of $\gamma([a, b])$.*
(2) *E is quasi-complete.*

Then the E-valued derivative $(f \circ \gamma)'(t) = (\mathrm{D}f)(\gamma(t), \gamma'(t))$ is continuous on $[a, b]$ and integrable, and we have

$$\int_{[a,b]} (\mathrm{D}f)(\gamma(t), \gamma'(t))\,\mathrm{d}t = f(\gamma(b)) - f(\gamma(a)). \tag{C.5.1.37}$$

REMARK C.5.1.20. Recall that "integrable" in the statement of Lemma C.5.1.19 refers to integrability in the sense of Pettis, defined in Definition C.4.2.1.

PROOF. Since f is differentiable over an open neighborhood of $\gamma([a, b])$ within Ω, then function $(\xi_1, \xi_2) \mapsto (\mathrm{D}_{\xi_1} f)(\xi_2)$ is continuous as a function $\Omega \times k^d \to E$, with respect to the product topology on the source and the initial topology on E. Since the function $t \mapsto (\gamma(t), \gamma'(t))$ is also continuous as a function $(a_0, b_0) \to \Omega \times k^d$, as $\gamma : (a_0, b_0) \to \Omega$ is differentiable, it follows that the composition $(\mathrm{D}_{\gamma(t)} f)(\gamma'(t))$ is continuous over (a_0, b_0). We now note that since f is differentiable in an open neighborhood of $\gamma([a, b])$, it is weakly differentiable over an open neighborhood of $\gamma([a, b])$ within Ω, and hence by (C.5.1.14) from Lemma C.5.1.12 and pointwise over $\Omega \times k^d$ we have

$$(\mathrm{D}f)(\xi_1, \xi_2) = (\mathrm{D}_{\xi_1} f)(\xi_2)$$

as elements of $(E')^*$, and by differentiability both sides are in fact elements of E. It follows that, as $\mathrm{D}f$ is continuous over an open neighborhood of $\gamma([a, b])$ and hence $(\mathrm{D}f)(\gamma(t), \gamma'(t))$ is continuous over $[a, b]$, that, being E quasi-complete, Lemma C.4.3.5 applies and it gives that $(\mathrm{D}f)(\gamma(t), \gamma'(t))$ is integrable over $[a, b]$. We now compute its integral: for the arbitrary $\varphi \in E'$, we have

$$\begin{aligned}
\left\langle \int_{[a,b]} (\mathrm{D}f)(\gamma(t), \gamma'(t))\,\mathrm{d}t, \varphi \right\rangle_{E,E'} &= \int_{[a,b]} \langle (\mathrm{D}f)(\gamma(t), \gamma'(t)) \rangle_{E,E'}\,\mathrm{d}t \\
&= \int_{[a,b]} \mathrm{D}_{\gamma(t)} \langle f(\gamma(t)), \varphi \rangle_{E,E'} (\gamma'(t))\,\mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle f \circ \gamma, \varphi \rangle \right)\,\mathrm{d}t \\
&= \langle f(\gamma(b)), \varphi \rangle_{E,E'} - \langle f(\gamma(a)), \varphi \rangle_{E,E'}
\end{aligned}$$

by the Fundamental Theorem of Calculus. Since the space E' separates the points of E, (C.5.1.37) holds. □

C.5.1.4. *Vector-valued differentiability criterion.* We will now discuss a differentiability criterion, extending Lemma C.5.1.13 to higher order directional derivatives. Such a criterion will hold under the additional assumption on the Hausdorff locally convex topological $\mathbf{C}$-vector space E being quasi-complete. The following Lemma is the key ingredient upon which the differentiability criterion of Theorem C.5.1.24 below rests.

We record here the following consequence of Lemma C.5.1.12, Lemma C.5.1.13, Lemma C.5.1.15 and Lemma C.5.1.16:

THEOREM C.5.1.21 (Strong vector-valued differentiability). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer, k either one of the Archimedean topological fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset, $f : \Omega \to E$ a function. The following conditions are equivalent:*

(1) *f is continuously differentiable over Ω.*
(2) *f is weakly differentiable, the weak differential $\mathrm{D}f : \Omega \times k^d \to (E')^*$ is valued in E and continuous with respect to the product topology on the source and the initial topology on the target E.*
(3) *f is weakly differentiable in the e_i-direction for each $1 \leq i \leq d$, and each of the directional derivatives $\mathrm{D}_i f$ is valued in E and continuous with respect to the Euclidean topology on Ω and the initial topology on E.*
(4) *f is Gateaux-differentiable within $(E')^*$, and the limit*

$$\lim_{h \to 0, h \in W} \frac{f(\xi_1 + h\xi_2) - f(\xi_1)}{h} \tag{C.5.1.38}$$

(with $W \subset k$ the subset $W := \{h \in k : h \neq 0, \xi_1 + h\xi_2 \in \Omega\}$) is, as a function of $(\xi_1, \xi_2) \in \Omega \times k^d$, valued in E and continuous with respect to the product topology on the source and the given topology on E.

PROOF. If (1) holds then f is weakly differentiable by Lemma C.5.1.12((2) $\Rightarrow$ (1)) and the weak differential $\mathrm{D}f$ agrees with the differential in (1), which implies the continuity assertion on $\mathrm{D}f$ as a function $\Omega \times k^d \to E$. Thus, (1) implies (2).

If (2) holds then f is weakly differentiable in the e_i-direction by Lemma C.5.1.15, and by (C.5.1.14) each $\mathrm{D}_i f$ is valued in E and continuous with respect to the Euclidean topology on Ω and the initial topology on E. This implies that (3) holds.

If (3) holds, then since $\mathrm{D}f = \sum_{1 \leq i \leq d} c_i \mathrm{D}_i f$, for $c_i \in (k^d)^*$ the i-th coordinate projection linear form, i.e. for each $(\xi_1, \xi_2) \in \Omega \times k^d$ we have:

$$\mathrm{D}f(\xi_1, \xi_2) = \sum_{1 \leq i \leq d} c_i(\xi_2) \mathrm{D}_i f(\xi_1)$$

and each c_i is continuous, we deduce from (3) that $\mathrm{D}f$ is valued in E and continuous with respect to the product topology on $\Omega \times k^d$ and the given topology on E. We deduce that f is weakly continuously differentiable and by Lemma C.5.1.13(2) is continuously differentiable over Ω, so (1) holds.

If (4) holds then by Lemma C.5.1.16(3) so does (1). If (1) holds then by Lemma C.5.1.13(2) we get (4). □

We will now be discussing an important differentiability criterion in the special case E is quasi-complete, due to Grothendieck (see [**Gro-a**, §2 ,Footnote 4, p. 39]) and whose proof is recorded in more detail by Schwartz in [**Sch-b**, Appendice, Lemma II, p. 146]. We'll give a somewhat different proof below. Before we do so, we include below a result on removable singularities for vector-valued functions, smooth away from an isolated point, adapting an argument from [**Ga-2**, §15.1].

LEMMA C.5.1.22 (Removable singularities in dimension $d = 1$). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k$ be an open subset, and $\xi_0 \in \Omega$ a point. Let $f : \Omega - \{\xi_0\} \to E$ be a function. Assume:*

(1) *E is quasi-complete.*
(2) *For every $\varphi \in E'$, the function*

$$\langle f, \varphi \rangle_{E,E'} : \Omega - \{\xi_0\} \to \mathbf{C} \tag{C.5.1.39}$$

defined as $\xi \mapsto \langle f(\xi), \varphi \rangle_{E,E'}$ is continuously differentiable, and it extends to a continuously differentiable scalar-valued function over Ω.

Then there exists a unique continuous function $F : \Omega \to E$ extending f. Moreover, F is weakly continuously differentiable over Ω.

PROOF. By Lemma C.5.1.13, f is continuous over $\Omega - \{\xi_0\}$ because it is weakly differentiable over $\Omega - \{\xi_0\}$. We let B be a small open ball around ξ_0 whose closure is contained in Ω.

We consider the following function:

$$G : (\Omega - \{\xi_0\}) \times (\Omega - \{\xi_0\}) - \Delta(\Omega - \{\xi_0\}) \to E \tag{C.5.1.40}$$

defined as

$$G(\xi_1, \xi_2) := \frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2} \tag{C.5.1.41}$$

where $\Delta(\Omega - \{\xi_0\})$ is the diagonal in $(\Omega - \{\xi_0\}) \times (\Omega - \{\xi_0\})$. Define the notation

$$U := (\Omega - \{\xi_0\}) \times (\Omega - \{\xi_0\}) - \Delta(\Omega - \{\xi_0\})$$

and

$$C := (B - \{\xi_0\}) \times (B - \{\xi_0\}) - \Delta((\Omega - \{\xi_0\}) \cap B) = U \cap (B \times B) \subset U.$$

Since f is continuous on $\Omega - \{\xi_0\}$, so is G.

By assumption (2), for any $\varphi \in E'$ the function $\langle G, \varphi\rangle_{E,E'} : U \to \mathbf{C}$ is

$$\int_0^1 \langle f, \varphi\rangle'_{E,E'}(\xi_2 + t(\xi_1 - \xi_2))\, \mathrm{d}t,$$

which is a continuous function over $\Omega \times \Omega$ since the function $\langle f, \varphi\rangle_{E,E'}$ has a continuously differentiable extension across ξ_0.

It follows that $\langle G(C), \varphi\rangle_{E,E'}$ is a bounded subset of $\mathbf{C}$. Since $\varphi \in E'$ is arbitrary, $G(C)$ is bounded with respect to the $\sigma(E, E')$-topology on E, so it is bounded with respect to the given topology on E [**Scha**, Ch. IV, 3.3].

We let A be its (bounded!) closed convex circled hull in E. For any $(\xi_1, \xi_2) \in B - \{x ß_0\}$ we therefore have, in light of the design of G from (C.5.1.41) (treating $\xi_1 \neq \xi_2$ separately from $\xi_1 = \xi_2$)

$$f(\xi_1) - f(\xi_2) \in (\xi_1 - \xi_2)A \subset E. \tag{C.5.1.42}$$

For any Cauchy sequence $\{\xi'_k\}_{k\geq 0}$ contained in $\Omega - \{\xi_0\}$ and converging to ξ_0, it follows from (C.5.1.42) that $\{f(\xi'_k)\}_{k\geq 0}$ is a Cauchy sequence in E.

Since E is quasi-complete and Cauchy sequences are bounded, $\{f(\xi'_k)\}$ converges. We define $F(\xi_0)$ to be its limit. The containment in (C.5.1.42) implies that $F(\xi_0)$ does not depend on the choice of convergent Cauchy sequence $\{\xi'_k\}_{k\geq 0}$, and F, thus defined at ξ_0, is sequentially continuous at ξ_0 and hence over Ω upon defining $F := f$ over $\Omega - \{\xi_0\}$. Thus, F is continuous.

We let φ be the arbitrary continuous functional on E. Clearly the continuous function $\langle F, \varphi\rangle_{E,E'}$ is the unique continuous extension of $\langle f, \varphi\rangle_{E,E'} : \Omega - \{\xi_0\} \to \mathbf{C}$ to Ω. That continuous extension is continuously differentiable by hypothesis, so F is weakly differentiable. □

Definition C.5.1.23 (Order-m differentiability). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset, and $f : \Omega \to E$ a function. Let $m \geq 1$ be an integer. If $m = 0$, we say that f is *continuously differentiable* 0 *times* if and only if f is Lipschitz-continuous, i.e. for any $\xi, \xi' \in \Omega$, $\xi \neq \xi'$, such that $|\xi' - \xi| \leq 1$, the ratio:

$$\frac{f(\xi') - f(\xi)}{|\xi' - \xi|} \tag{C.5.1.43}$$

is contained in a bounded subset of E.

If $m \geq 1$, we say that f is *continuously differentiable* m *times* if and only if:

(1) f is weakly differentiable in the e_i-direction for $1 \leq i \leq d$.
(2) For every $1 \leq i \leq d$, the directional derivatives $\mathrm{D}_i f$ is valued in E and continuous with respect to the Euclidean topology on Ω and the initial topology on E.

(3) For every $1 \leq i \leq d$, the directional derivatives $\mathrm{D}_i f$ is continuously differentiable $m-1$ times.

We say f is *infinitely differentiable*, or *smooth*, if and only if for every integer $m \geq 0$, f is continuously differentiable m times.

We denote by $C^m(\Omega, E)$ the $\mathbf{C}$-vector space of E-valued functions continuously differentiable m times, and by $C^\infty(\Omega, E)$, equivalently $\mathcal{E}(\Omega, E)$, the $\mathbf{C}$-vector space of infinitely differentiable E-valued functions.

The following result is due to Grothendieck, and it is recorded in [**Sch-b**, Appendice, Lemme II, p. 146].

THEOREM C.5.1.24. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, k either one of the fields $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology. Let $\Omega \subset k^d$ be an open subset and $f : \Omega \to E$ a function. Assume that E is quasi-complete. Let $m \geq 1$ be an integer. If for every $\varphi \in E'$, the function $\langle f, \varphi \rangle_{E,E'}$ is continuously differentiable m times, then f is differentiable $m-1$ times.*

PROOF. Since $m \geq 1$, from Lemma C.5.1.13(1) we deduce that f is Lipschitz-continuous. This proves the case $m = 1$.

We now assume $m > 1$ and proceed by induction on m. We consider the function, for each $(\xi_1, \xi_2) \in \Omega \times k^d$:

$$F(h) := \frac{f(\xi_1 + h\,\xi_2) - f(\xi_1)}{h} = \int_0^1 (\mathrm{D}f)(\xi_1 + th\xi_2, \xi_2)\,\mathrm{d}t \tag{C.5.1.44}$$

(as a Pettis intgeral with weakly continuously differentiable integrant valued in E, with $F(0) = (\mathrm{D}f)(\xi_1, \xi_2) \in (E')^*$) defined on $h \in k - \{0\}$ such that $\xi_1 + h\,\xi_2 \in \Omega$. This set of elements $W \subset k$ is open since $\Omega \subset k^d$ is open.

For $\varphi \in E'$,

$$\langle F, \varphi \rangle(h) = \int_0^1 \mathrm{D}(\langle f, \varphi \rangle)(\xi_1 + th\xi_2, \xi_2)\,\mathrm{d}t$$

with integrand that is C^1 in (t, h) since $\langle f, \varphi \rangle$ is C^2, so $\langle F, \varphi \rangle$ is C^1 in h near 0. Thus, Lemma C.5.1.22 applies to F, giving a continuous extension across $h = 0$ valued in E.

We claim f satisfies the assumptions of Lemma C.5.1.16(3) with (C.5.1.32) equal to $(\mathrm{D}f)(\xi_1, \xi_2)$. Indeed, formation of the directional derivative $(\partial_{\xi_1} f)(\xi_2)$ is, for each fixed $\xi_2 \in k^d$, continuous as an E-valued function of ξ_1 with respect to the given topology on E by Lemma C.5.1.13(1) applied to $\xi_1 \mapsto (\partial_{\xi_1} f)(\xi_2)$. Then Lemma C.5.1.16(3) applies to give f is C^1 with values in E.

We have shown that for each integer $1 \leq i \leq d$, $\mathrm{D}_i f$ is valued in E and continuous with respect to the Euclidean topology on Ω and the initial topology on E. Note that for each integer $1 \leq i \leq d$, the function $\mathrm{D}_i f$ is E-valued, continuous, and weakly continuously differentiable $m-1$ times. By inductive hypothesis, it is continuously differentiable $m-2$ times. We have therefore shown that f is continuously differentiable $m-1$ times. □

COROLLARY C.5.1.25 (On the agreement of the concepts of E-valued smoothness). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space. Let $d \geq 1$ be an integer, k either one of the two Archimedean topological field $k = \mathbf{R}, \mathbf{C}$, endowed with their Euclidean topology.*

Let $\Omega \subset k^d$ be an open subset, $f : \Omega \to E$ a function. Assume E is quasi-complete. Then, the following conditions are equivalent:

(1) *f is infinitely differentiable.*
(2) *For every $\varphi \in E'$, the $\mathbf{C}$-valued function $\langle f, \varphi \rangle_{E,E'}$ on Ω is infinitely differentiable, i.e. an element of $C^\infty(\Omega)$.*

Note that for $d = 1$ and $k = \mathbf{C}$, this implies that the weakly infinitely differentiable E-valued functions on Ω are exactly the holomorphic E-valued functions (see Definition C.5.2.1(4)).

The concept of vector-valued infinite differentiability defined in [**Tre**, Def. 40.1] is the same as the one in Definition C.5.1.23. The one in force throughout [**Sch-b**] is the "weak" notion in Corollary C.5.1.25(2),

so for E quasi-complete the two notions are equivalent by Corollary C.5.1.25. We may define a concept of infinite differentiability in the sense of Gateaux, building on Lemma C.5.1.16 (requiring directional derivatives to be E-valued and weakly continuous on Ω), and Corollary C.5.1.25 implies that for E quasi-complete it is equivalent to infinite differentiability as defined in Definition C.5.1.23.

For the purposes of this work, E will always be quasi-complete. Thus, there will be no ambiguity as to the meaning of "smooth E-valued function".

C.5.1.5. *Spaces of smooth vector-valued functions.* In this subsection we record a few facts about spaces of smooth vector-valued functions with values in a Hausdorff locally convex quasi-complete topological $\mathbf{C}$-vector space E.

Recall that we write $\mathbf{T}$ for the 1-dimensional compact real torus $\mathbf{R}/\mathbf{Z}$, which is smoothly isomorphic to U(1) under the exponential function $t \mapsto \exp(2\pi i t)$.

DEFINITION C.5.1.26 (Vector-valued smooth function spaces). Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 0$ an integer. Assume E is quasi-complete. We define:

(1) $\mathcal{S}(\mathbf{T}^d, E)$ to be the linear subspace of $C^\infty(\mathbf{R}^d, E)$ of those E-valued smooth functions that are invariant under the continuous left-translation action of the discrete co-compact closed subgroup $\mathbf{Z}^d \subset \mathbf{R}^d$.

We topologize $\mathcal{S}(\mathbf{T}^d, E)$ with the unique Hausdorff topological vector space structure for which a net $\{f_i\}_{i\in I} \subset \mathcal{S}(\mathbf{T}^d, E)$ converges to 0 in $\mathcal{S}(\mathbf{T}^d, E)$ if and only if for every $\varphi \in E'$, the net of smooth complex-valued functions $\{\langle f_i, \varphi\rangle_{E,E'}\}_{i\in I}$ converges to 0 with respect to the topology of uniform convergence over $\mathbf{T}^d$, uniformly as φ varies over each equicontinuous subset of E'.

(2) $\mathcal{S}(\mathbf{R}^d, E)$ to be the linear subspace of those $f \in C^\infty(\mathbf{R}^d, E)$ such that for any complex-valued polynomial function p on $\mathbf{R}^d$, and every integral multi-index $\alpha \in \mathbf{N}^d$, $p \cdot \mathrm{D}^\alpha f$ takes values in a *bounded* subset of E, where $\mathrm{D}^\alpha := \prod_{1\leq i\leq d} \mathrm{D}_i^{\alpha_i}$ (the boundedness can be checked weakly, by [**Scha**, Ch. IV, 3.3]).

We topologize $\mathcal{S}(\mathbf{R}^d, E)$ with the unique Hausdorff topological vector space structure for which a net $\{f_i\}_{i\in I} \subset \mathcal{S}(\mathbf{R}^d, E)$ converges to 0 in $\mathcal{S}(\mathbf{R}^d, E)$ if and only if for every $\varphi \in E'$, the net of smooth complex-valued functions $\{\langle f_i, \varphi\rangle_{E,E'}\}_{i\in I}$ converges to 0 with respect to the topology of uniform convergence over $\mathbf{R}^d$, uniformly as φ varies over each equicontinuous subset of E'.

REMARK C.5.1.27. Definition C.5.1.26 makes sense for an arbitrary Hausdorff locally convex topological $\mathbf{C}$-vector space E (no quasi-completeness hypothesis) by defining smooth E-valued functions as those that are weakly infinitely differentiable and by defining the two respective function spaces $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ as the linear subspaces of $C^\infty(\mathbf{R}^n, E)$ of those smooth E-valued functions $f : \mathbf{R}^d \to E$ such that for any $\varphi \in E'$, the complex-valued smooth function $\langle f, \varphi\rangle_{E,E'}$ is $\mathbf{Z}^d$-invariant over $\mathbf{R}^d$, respectively Schwartz over $\mathbf{R}^d$.

This latter way to define $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ is denoted in [**Sch-b**, §1,p.102] by $\widetilde{\mathrm{H}}^d(E)$ for H either $\mathcal{S}(\mathbf{T}^d)$ or $\mathcal{S}(\mathbf{R}^d)$. By [**Sch-b**, Prop. 8, p. 104], for E quasi-complete this definition and Definition C.5.1.26 are equivalent; see also Corollary C.5.1.25, and [**Sch-b**, Prop. 4, p. 97].

REMARK C.5.1.28 (Fundamental families of seminorms for $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$). In light of Lemma C.2.2.2, the topology thus defined on $\mathcal{S}(\mathbf{T}^d, E)$, respectively $\mathcal{S}(\mathbf{R}^d, E)$, is locally convex: for $\{q_i\}_{i\in I}$ a fundamental family of continuous seminorms for E, a fundamental family of seminorms for $\mathcal{S}(\mathbf{T}^d, E)$ is assigned on each $f \in \mathcal{S}(\mathbf{T}^d, E)$ by

$$\sup_{x\in\mathbf{T}^d} q_i(\mathrm{D}^\alpha f),\ i \in I, \alpha \in \mathbf{N}^d \tag{C.5.1.45}$$

and for $\mathcal{S}(\mathbf{R}^d, E)$ is assigned on each $f \in \mathcal{S}(\mathbf{R}^d, E)$ by

$$\sup_{x\in\mathbf{R}^d} q_i(p \cdot \mathrm{D}^\alpha f),\ i \in I \tag{C.5.1.46}$$

as p varies over all complex-valued polynomial functions on $\mathbf{R}^d$, equivalently over all monomial functions over $\mathbf{R}^d$, and $\alpha \in \mathbf{N}^d$ is an arbitrary multi-index.

REMARK C.5.1.29. Definition C.5.1.26 makes sense for the arbitrary Hausdorff locally convex topological $\mathbf{C}$-vector space E, by defining smooth E-valued functions as being those that are weakly infinitely differentiable, and by defining each of the two function spaces $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ as being the linear subspaces of $C^\infty(\mathbf{R}^n, E)$ of those smooth E-valued functions $f : \mathbf{R}^d \to E$ such that for any $\varphi \in E'$, the complex-valued smooth function $\langle f, \varphi \rangle_{E,E'}$ is $\mathbf{Z}^d$-invariant over $\mathbf{R}^d$, respectively Schwartz over $\mathbf{R}^d$.

This latter way to define $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ is denoted in [**Sch-b**, §1,p.102] by $\widetilde{\mathrm{H}}^d(E)$ for H either $\mathcal{S}(\mathbf{T}^d)$ or $\mathcal{S}(\mathbf{R}^d)$. By [**Sch-b**, Prop.8,p.104], for E quasi-complete this definition and Definition C.5.1.26 are equivalent; see also Corollary C.5.1.25, and [**Sch-b**, Prop.4,p.97].

Most of the following properties of $\mathcal{S}(\mathbf{R}^d, E)$ are due to Schwartz, and we further refer the reader to [**Sch-b**, §1,p.93].

THEOREM C.5.1.30 (Main properties of the function spaces $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$). *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $d \geq 1$ an integer. Assume E is quasi-complete.*

(1) *The spaces $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ are quasi-complete.*
(2) *If E is complete then $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ are complete.*
(3) *If E is a Fréchet space then so are $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$.*
(4) *For $d, d' \geq 1$ two integers, and $X = \mathbf{T}^d, \mathbf{R}^d$, $Y = \mathbf{T}^{d'}, \mathbf{R}^{d'}$ respectively, the injective linear map*

$$C^\infty(X \times Y) \to C^\infty(X, C^\infty(Y)) \tag{C.5.1.47}$$

defined by sending a smooth complex-valued function $f(x, y)$ on $X \times Y$ to the smooth $C^\infty(Y)$-valued function $x \mapsto f(x, \cdot)$ is bijective and it is compatible with the linear subspaces $\mathcal{S}(X \times Y)$ and $\mathcal{S}(X, \mathcal{S}(Y))$, inducing a linear topological isomorphism

$$\mathcal{S}(X \times Y) \to \mathcal{S}(X, \mathcal{S}(Y)). \tag{C.5.1.48}$$

PROOF. (1) and (2) : Let $\{f_i\}_{i \in I} \subset \mathcal{S}(\mathbf{R}^d, E)$ be a bounded Cauchy net. Since $\mathcal{S}(\mathbf{R}^d)$ is complete, it converges weakly; i.e., with respect to the $\sigma(\mathcal{S}(\mathbf{R}^d, E), \mathcal{S}(\mathbf{R}^d, E)')$-topology on $\mathcal{S}(\mathbf{R}^d, E)$.

Since it is bounded by assumption, for each $x \in \mathbf{R}^d$ Remark C.5.1.28 gives that $\{f_i(x)\}_{i \in I}$ is a *bounded* Cauchy net in E, which therefore converges for each $x \in \mathbf{R}^d$ since E is quasi-complete. We denote by $f : \mathbf{R}^d \to E$ the limit function. For any $\varphi \in E'$, $\langle f, \varphi \rangle_{E,E'}$ is the pointwise limit of $\langle f_i, \varphi \rangle$, yet $\langle f_i, \varphi \rangle$ is Cauchy in the complete $\mathcal{S}(\mathbf{R}^d)$ and so has a limit in $\mathcal{S}(\mathbf{R}^d)$. Hence, $\langle f, \varphi \rangle$ is that limit, so it is Schwartz.

By Corollary C.5.1.25, f is smooth as an E-valued function. For any polynomial function p on $\mathbf{R}^d$ and every multi-index $\alpha \in \mathbf{N}^d$, $p \cdot \mathrm{D}^\alpha f$ takes values in a weakly bounded subset of E since each $\langle f, \varphi \rangle$ is Schwartz, so $p \cdot \mathrm{D}^\alpha f$ takes values in a bounded subset of E by [**Scha**, Ch. IV, 3.3]. We deduce that $f \in \mathcal{S}(\mathbf{R}^d, E)$. It remains to show that $f_i \to f$ in $\mathcal{S}(\mathbf{R}^d, E)$. If $\varphi \in E'$ varies over an arbitrary equicontinuous subset A of E' then by equicontinuity we have $\langle f_i, \varphi \rangle_{E,E'} \to \langle f, \varphi \rangle_{E,E'}$ uniformly with respect to $\varphi \in A$. We deduce that $f_i \to f$ in $\mathcal{S}(\mathbf{R}^d, E)$, by definition of the locally convex topology on $\mathcal{S}(\mathbf{R}^d, E)$.

The same argument works if $\{f_i\}_{i \in I}$ is an arbitrary (not necessarily bounded) Cauchy net in $\mathcal{S}(\mathbf{R}^d, E)$ and E is complete rather than just quasi-complete.

The case of $\mathcal{S}(\mathbf{T}^d, E)$ is seen by means of the same argument: we have only used completeness of $\mathcal{S}(\mathbf{R}^d)$ and quasi-completeness, respectively completeness, of E.

(3) : If E is a Fréchet space, then since $\mathcal{S}(\mathbf{T}^d)$ and $\mathcal{S}(\mathbf{R}^d)$ are both metrizable, the description of the topology on $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ by means of the fundamental families of continuous seminorms from (C.5.1.45) and (C.5.1.46) shows that we may choose the index sets I to be countable, which proves that $\mathcal{S}(\mathbf{T}^d, E)$ and $\mathcal{S}(\mathbf{R}^d, E)$ are complete, by (2), and metrizable, hence Fréchet spaces.

(4) : Using the formula

$$\frac{f(x+he_j,y)-f(x,y)}{h}=\int_0^1(\partial_j f)(x+the_j,y)\,\mathrm{d}t$$

one readily checks that (C.5.1.47) makes sense (i.e., $[f]:x\mapsto f(x,\cdot)$ as a map $X\to C^\infty(Y)$ satisfies $\partial_j[f]=[\partial_j f]$ and so is C^∞) and is bijective. The linear map from (C.5.1.48) is continuous, by standard properties of formation of suprema of the absolute value of a continuous complex-valued function on the product and the continuity of the inverse. Surjectivity follows by similar calculations via bijectivity of (C.5.1.47) and the bijection between the seminorm formulas on both sides. □

We will be using Theorem C.5.1.30(4) in Appendix D (see the proof of Lemma D.1.3.7 and the proof of Lemma D.2.2.2(1)), to define the tensor product of tempered distributions on pairs of elementary abelian Lie groups of the form $(\mathbf{R}^d,\mathbf{R}^{d'})$, $(\mathbf{T}^d,\mathbf{T}^{d'})$ and $(\mathbf{Z}^d,\mathbf{Z}^{d'})$. The construction is identical to that of the tensor product of two distributions, discussed in [**Tre**, Def. 40.3], and in this form (i.e. through the use of Theorem C.5.1.30(4)) it is used only in the proof of Lemma D.1.3.7 and Lemma D.2.2.2(1) of Appendix D (we refer the reader to the proof of [**Tre**, Thm. 51.6(51.3)] for a similar argument and a similar use of the binary tensor product of tempered distributions, which is used to make sense of the equicontinuous subsets $A'\otimes B'$ in $\mathcal{S}(\mathbf{R}^{m+n})'$ in the proof of loc.cit.).

C.5.2. Holomorphic vector-valued functions. We start by defining basic concepts of continuity and holomorphy for vector-valued functions on open subsets of $\mathbf{C}$.

C.5.2.1. *Grothendieck's criterion for holomorphy.* For a Hausdorff locally convex topological $\mathbf{C}$-vector space E, we will be identifying E as a linear subspace of $(E')^*$ (although not topologically) under the linear map $\iota_E:E\to(E')^*$ from (C.4.1.1) and studied in Lemma C.4.1.1, when saying that certain linear forms on E' are "elements of E", or are "contained in E". These will ordinarily be integrals in the sense of Pettis, which, recall, are defined for general E as elements of $(E')^*$. The concept of *integrability* was defined in Definition C.4.2.1(3) via such integrals being contained in E.

Recall that we call $\Omega\subset\mathbf{C}$ a *domain*, or a *complex domain*, if $\Omega\subset\mathbf{C}$ is non-empty open and connected.

Definition C.5.2.1. Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, and $\Omega\subset\mathbf{C}$ a domain. We say a function $f:\Omega\to E$ is:

(1) *weakly continuous* if and only if for every continuous linear functional $\varphi\in E'$, the function $\langle f(\xi),\varphi\rangle_{E,E'}$ is continuous as a function of $\xi\in\Omega$ (equivalently, f is continuous when E is given the $\sigma(E,E')$-topology).
(2) *weakly continuously differentiable* k times, for an integer $k\geq 1$, if and only if for every continuous linear functional $\varphi\in E'$, the function $\langle f,\varphi\rangle_{E,E'}$ on Ω is an element of $C^k(\Omega)$.
(3) *Weakly holomorphic* if and only if for every continuous linear functional $\varphi\in E'$, the function $\langle f,\varphi\rangle_{E,E'}$ on Ω is holomorphic in the sense of complex analysis.
(4) *Complex-differentiable*, or *holomorphic*, if and only if for every $\xi_0\in\Omega$

$$\lim_{\xi\to\xi_0}(\xi-\xi_0)^{-1}\left(f(\xi)-f(\xi_0)\right) \tag{C.5.2.1}$$

exists in E as $\xi\to\xi_0$ in the complex sense (in particular, $f:\Omega\to E$ is continuous).

Remark C.5.2.2. Reprising the terminology from Definition C.5.1.8, holomorphy of the E-valued function f on $\Omega\subset\mathbf{C}$ is equivalent to differentiability, upon choosing $k=\mathbf{C}$, $d=1$.

In [**Gro-a**, Thm.1], Grothendieck established the equivalence of the concept of weak holomorphy in Definition C.5.2.1(3) with seemingly stronger notions of holomorphy, showing that analogs of the Cauchy integral formula and familiar results from complex analysis, adapt to the case of E-valued functions with minor changes, as long as vector-valued integrals are contained in E.

REMARK C.5.2.3. In [Gro-a, Thm.1(4)], complex-differentiable functions are termed "strongly differentiable".

We record Grothendieck's Theorem (see [Gro-a, Thm.1]) below. Throughout this work an *arc* $\Gamma \subset \Omega$, for Ω a complex domain, will always refer to a continuous and piecewise-C^1 map $\gamma : [a, b] \to \Omega$, for $a, b \in \mathbf{R}$, $a < b$, such that $\Gamma = \gamma([a, b])$.

The following is immediate from complex analysis:

THEOREM C.5.2.4. *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space, $\Omega \subset \mathbf{C}$ a domain. Let $f : \Omega \to E$ be a function. The following conditions are equivalent:*

(1) *The function f is weakly holomorphic.*
(2) *The function f is weakly continuous in the sense of Definition* C.5.2.1*(1), and for any closed positively oriented simple curve $\Gamma \subset \Omega$ whose interior is contained within Ω we have*

$$\int_\Gamma f(\xi)\,\mathrm{d}\xi = 0 \tag{C.5.2.2}$$

as elements of $(E')^*$.
(3) *f is weakly continuous, and for any closed positively oriented simple curve $\Gamma \subset \Omega$ whose interior is contained within Ω, and for any $\xi_0 \in \Omega$ enclosed in the interior of the contour Γ, we have*

$$f(\xi_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - \xi_0}\,\mathrm{d}\xi \tag{C.5.2.3}$$

as elements of $(E')^*$.

REMARK C.5.2.5. The integrals (C.5.2.2) and (C.5.2.3) are meant in the Pettis sense, computable using a piecewise-C^1 parameterization of Γ as in complex analysis.

If we assume that E is quasi-complete, then the Theorem can be strengthen into the following:

THEOREM C.5.2.6 (Grothendieck's holomorphy criterion). *Let E be a Hausdorff quasi-complete locally convex topological $\mathbf{C}$-vector space, $\Omega \subset \mathbf{C}$ a complex domain of the complex ξ-plane. Let $f : \Omega \to E$ be a function. The following conditions are equivalent:*

(1) *f is weakly holomorphic.*
(2) *f is weakly differentiable k times for every integer $k \geq 1$.*
(3) *f is complex-differentiable k times for every integer $k \geq 1$, in which case for any closed positively oriented simple curve $\Gamma \subset \Omega$ whose interior is contained within Ω, and for any $\xi_0 \in \Omega$ enclosed in the interior of the contour Γ, we have*

$$f^{(k)}(\xi_0) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - \xi_0)^{k+1}}\,\mathrm{d}\xi \tag{C.5.2.4}$$

as elements of $(E')^*$ (the left side belongs to E).
(4) *Around every point $\xi_0 \in \Omega$, we have the following Taylor expansion*

$$f(\xi) = \sum_{k\geq 0} a_k\,(\xi - \xi_0)^k \tag{C.5.2.5}$$

where $a_k = (1/k!)\,f^{(k)}(\xi_0) \in E$ for every $k \geq 0$, where the series is uniformly convergent on every closed disc D around ξ_0 such that $D \subset \Omega$.

In particular, if f is weakly holomorphic then it is holomorphic.

REMARK C.5.2.7. On Theorem C.5.2.6, we make the following observations:

(1) is the first part of [**Gro-a**, Thm.1]. Note that a priori, the derivative $f^{(k)}$ in (1) and (2) is defined as an element of $(E')^*$ by the rule

$$\langle f^{(k)}(\xi), \varphi\rangle_{E,E'} = \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\langle f(\xi), \varphi\rangle_{E,E'} \tag{C.5.2.6}$$

where the right side is defined and continuous in ξ for all $k \geq 1$, since f is weakly holomorphic. The use of quasi-completeness in the proof of Theorem C.5.2.6 from [**Gro-a**, Thm.1] is in that it enables the reduction of the equivalence between parts (1) and (2), on one hand, and (3) on the other, to proving continuity of f (rather than merely weak continuity). Indeed, upon showing that f is continuous, Theorem C.4.3.5 applies since Γ is compact and $\mathrm{d}\xi$ is Radon, so the first two (equivalent) parts and the Cauchy Integral Formula imply that the $(E')^*$-valued integrals in (C.5.2.4) are contained in E by quasi-completeness, so the equality with $f^{(k)}(\xi_0)$ as a weak derivative yields complex-differentiability of f. Continuity of f, however, has to be shown directly and without the aid of quasi-completeness, but rather by means of Lemma C.4.3.1. The argument for continuity is at the bottom of [**Gro-a**, p.38].

(2) The argument at the bottom of [**Gro-a**, p.38] shows that weak holomorphy implies continuity in the presence of quasi-completeness.

The problem of deciding the equivalence of weak holomorphy and the remaining conditions in Theorem C.5.2.4 goes back to Dunford, and we refer the reader to [**Dunford**, Thm. 76] for the special case E is a Banach space.

C.5.3. Meromorphic vector-valued functions. Similarly as for holomorphic functions on domains of the complex plane valued in a quasi-complete Hausdorff locally convex topological $\mathbf{C}$-vector space, we now introduce the concept of meromorphic function.

Definition C.5.3.1. Let $\Omega \subset \mathbf{C}$ be a domain, E a quasi-complete Hausdorff locally convex topological vector space, $D \subset \Omega$ a discrete subset, $f : \Omega - D \to E$ a function. We say the pair (f, D) is a *meromorphic* function on Ω if f is holomorphic on $\Omega - D$, and for every $\xi_0 \in D$ there exists an integer $k \geq 0$ and a sequence

$$\{a_{k'}(f, \xi_0)\}_{k' \geq k} \subset E \tag{C.5.3.1}$$

of elements of E such that the series

$$\sum_{k' \geq k} a_{k'}(f, \xi_0)\,(\xi - \xi_0)^{k'} \tag{C.5.3.2}$$

converges uniformly over the compact subsets of a punctured neighborhood of $\xi_0 \in \Omega$, and its sum is $f(\xi)$ for ξ in any such punctured neighborhood of $\xi_0 \in \Omega$ (equivalently, some $(\xi - \xi_0)^{N_{\xi_0}} f(\xi)$ extends holomorphically across ξ_0 for all $\xi_0 \in D$).

We define:

(1) the *order* of (f, D) at $\xi_0 \in \Omega$, denoted $\mathrm{ord}_{\xi_0}(f)$, to be the minimal integer N such that the conditions on the series (C.5.3.1) are met for the choice $k = N$.

(2) $\xi_0 \in \Omega$ to be a *pole* of f if $\mathrm{ord}_{\xi_0}(f) \leq -1$, in which case we say ξ_0 is a *pole of order* $-\mathrm{ord}_{\xi_0}(f)$.

We define $\mathscr{M}(\Omega, E)$ the complex vector space of meromorphic E-valued functions on Ω.

We check the following, as an application of Theorem C.5.2.4

Lemma C.5.3.2. *Let E be a Hausdorff quasi-complete locally convex topological vector space, $\Omega \subset \mathbf{C}$ a domain. Let $f : \Omega \to E$ be a meromorphic function. For every $\xi_0 \in \Omega$, and every closed positively oriented simple curve $\Gamma \subset \Omega$ enclosing ξ_0 in its interior and with no other possible poles enclosed in its interior and on*

Γ, *we have*

$$a_k(\xi_0, f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - \xi_0)^{k+1}} \, \mathrm{d}\xi \tag{C.5.3.3}$$

for every $k \in \mathbf{Z}$*, the identity being as elements of* E*.*

PROOF. The case $k \geq 0$ in (C.5.2.4), so we focus on $k < 0$. By definition of f being meromorphic at $\xi_0 \in \Omega$ we have by (C.5.3.2)

$$\sum_{k' \geq -N} a_{k'}(\xi_0, f) \, (\xi - \xi_0)^{k'} \tag{C.5.3.4}$$

where the series converges uniformly on every closed disc around ξ_0 entirely contained in Ω, and $(\xi - \xi_0)^N \, f(\xi)$ has a removable singularity at $\xi = \xi_0$ and extends to a holomorphic function on a neighborhood of $\xi_0 \in \Omega$. By writing

$$\frac{f(\xi)}{(\xi - \xi_0)^{k+1}} = \frac{(\xi - \xi_0)^N f(\xi)}{(\xi - \xi_0)^{(N+k)+1}},$$

we're reduced to the settled holomorphic case. □

C.5.3.1. *Vector-valued residues.* We construct the following operators on $\mathscr{M}(\Omega, E)$:

$$\mathrm{Res}_\xi(f) := a_{-1}(\xi, f), \ \xi \in \Omega. \tag{C.5.3.5}$$

LEMMA C.5.3.3. *Let* E *be a Hausdorff quasi-complete locally convex topological* $\mathbf{C}$*-vector space,* $\Omega \subset \mathbf{C}$ *a domain,* f *a meromorphic* E*-valued function on* Ω*. Then:*

(1) *For every* $\xi \in \Omega$*, the map*

$$\mathrm{Res}_\xi : \mathscr{M}(\Omega, E) \to E \tag{C.5.3.6}$$

assigned by the rule $f \mapsto \mathrm{Res}_\xi(f)$ *as in* (C.5.3.5) *is* $\mathbf{C}$*-linear.*

(2) *For every* $\xi_0 \in \Omega$*, we have the identity*

$$\mathrm{Res}_{\xi_0}(f) = \frac{1}{2\pi i} \int_\Gamma f(\xi) \, \mathrm{d}\xi \tag{C.5.3.7}$$

as $\mathbf{C}$*-linear maps* $\mathscr{M}(\Omega, E) \to E$ *for every closed positively oriented simple curve contained within* Ω *and enclosing no further pole other than possibly* ξ_0*, neither in its interior nor on* Γ *itself.*

(3) *If* ξ_0 *is a pole of* f *of order* $k \geq 1$ *then*

$$\mathrm{Res}_{\xi_0}(f) = \lim_{\xi \to \xi_0} \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi^{k-1}} ((\xi - \xi_0)^{k-1} \, f(\xi)) \tag{C.5.3.8}$$

as $\mathbf{C}$*-linear operators* $\mathscr{M}(\Omega, E) \to E$*.*

(4) *Let* E_1, E_2 *be two Hausdorff quasi-complete complex topological vector spaces, and* $T : E_1 \to E_2$ *a continuous* $\mathbf{C}$*-linear operator. Then*

$$\mathrm{Res}_\xi(T \circ (\cdot)) = T \circ \mathrm{Res}_\xi \tag{C.5.3.9}$$

as $\mathbf{C}$*-linear operators* $\mathscr{M}(\Omega, E_2) \to E_2$*, for every* $\xi \in \Omega$*.*

PROOF. Assertions (1) and (3) are clear. For (4), use Lemma C.4.3.8 applied to $X = \Gamma$, $\mu = \mathrm{d}\xi$. □

C.5.4. Vector-valued Residue Theorem. We recall that by a *cycle* Γ on Ω, we mean a formal sum with integer coefficients of positively oriented closed simple curves contained within Ω. The following is therefore the analog of Cauchy's Residue Theorem, for E-valued holomorphic functions on Ω.

THEOREM C.5.4.1 (Residue Theorem for vector-valued meromorphic functions). *Let E be a Hausdorff quasi-complete locally convex complex topological vector space, $\Omega \subset \mathbf{C}$ a domain. Let f be a meromorphic E-valued function on Ω. Let $\Gamma \subset \Omega$ be a cycle which is homologous to zero and does not contain any poles of f. Then*

$$\frac{1}{2\pi i}\int_{\Gamma} f(\xi)\,\mathrm{d}\xi = \sum_{\xi_0 \in D} n(\Gamma, \xi_0)\,\mathrm{Res}_{\xi_0}(f) \tag{C.5.4.1}$$

as $\mathbf{C}$-linear operators $\mathscr{M}(\Omega, E) \to E$, where $n(\Gamma, \xi_0)$ is the winding number of Γ around $\xi_0 \in D$.

PROOF. By pairing both sides of (C.5.4.1) against a general $\varphi \in E'$, we reduce to the classical case with $E = \mathbf{C}$. □

We see from Theorem C.5.4.1 how it is natural to constrain E so as to ensure that f be integrable in the E-valued sense over any closed simple curve Γ. Quasi-completeness of E is assumed for this reason, in light of Theorem C.4.3.5.

The following is an analog of Riemann's Theorem on removable singularities. We refer the reader to [**Kruse**, Thm. 7.4].

LEMMA C.5.4.2 (On removable singularities). *Let E be a Hausdorff quasi-complete locally convex topological $\mathbf{C}$-vector space, $\Omega \subset \mathbf{C}$ a domain, $Z \subset \Omega$ a discrete subset and $f : \Omega - Z \to E$ a holomorphic function. If for every $\xi_0 \in \Omega$ there exists a disc $B \subset \Omega$ containing ξ_0, such that f is bounded on $B - Z$, then f uniquely extends to a holomorphic function $f : \Omega \to E$.*

Here too quasi-completeness is being used to know holomorphic is equivalent to weakly holomorphic. Finer analytic continuation results are available (see [**Jor-b**, Thms.6,8]) again under a local completeness assumption.

We will, throughout this work, always be able to assume quasi-completeness of the Hausdorff locally convex topological $\mathbf{C}$-vector space of values of the holomorphic and meromorphic functions of interest since these spaces will always be the strong dual of a quotient of the Schwartz-Bruhat space of the global, respectively S-local, adeles of a number field F, for F a finite set of places containing all the Archimedean places. Such duals are quasi-complete, and even complete.

C.5.4.1. *Vector-valued Identity Theorem.* We record here below the following analog of the Identity Theorem from classical complex analysis:

LEMMA C.5.4.3. *Let E be a Hausdorff quasi-complete locally convex topological $\mathbf{C}$-vector space, $\Omega \subset \mathbf{C}$ a domain, (f_1, D_1) and (f_2, D_2) two meromorphic E-valued functions on Ω. Let $B \subset \Omega - (D_1 \cup D_2)$ be a non-empty open disc such that*

$$f_1|_B = f_2|_B. \tag{C.5.4.2}$$

Then f_1 and f_2 agree as holomorphic functions on $\Omega - (D_1 \cup D_2)$.

PROOF. For $\varphi \in E'$, the classical Identity Theorem applies to $\langle f_1, \varphi\rangle$ and $\langle f_2, \varphi\rangle$, so $\langle f_1, \varphi\rangle = \langle f_2, \varphi\rangle$ for all φ. Thus, $f_1 = f_2$. □

For a Hausdorff quasi-complete locally convex topological $\mathbf{C}$-vector space E, the complex vector spaces

$$\mathscr{H}(\Omega, E), \quad \mathscr{M}(\Omega, E)$$

can be topologized (the former with the topology of locally uniform convergence, the latter with a certain locally convex inductive limit topology) and are well-behaved. Their respective Hausdorff locally convex

topologies are themselves quasi-complete, and promote the residue operators to continuous linear maps. They have been studied, for example, in [**Jor-a**, **BJM**], in analogy with [**GE**] on the case of $\mathscr{M}(\Omega, E)$ for $E = \mathbf{C}$.

We will not require a study of the complex vector spaces $\mathscr{H}(\Omega, E)$ and $\mathscr{M}(\Omega, E)$ for quasi-complete E: rather, we will be studying, throughout Appendix D and the main text, smaller linear subspaces of holomorphic and meromorphic vector-valued functions satisfying decay conditions uniformly over vertical strips of bounded width, topologized with stronger topologies of uniform convergence over vertical strips of bounded width.

APPENDIX D

Schwartz-Bruhat functions

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D.1. Schwartz-Bruhat space of a Hausdorff locally compact abelian group

In this section we describe, for any Hausdorff locally compact abelian topological group G, the $\mathbf{C}$-vector space of Schwartz functions $\mathcal{S}(G)$ and its locally convex topology. Important examples of such G for our purposes are not only the adele ring $\mathbf{A}_F$ and its factor fields F_v for the places v of F, but also the multiplicative analogues $\mathrm{GL}_1(\mathbf{A}_F)$ and $F_v^\times$ as well as the idele class group C_F and its S-local analogue $C_{F,S}$. (Beware that for non-archimedean v, $\mathcal{S}(F_v^\times)$ is much larger than the more widely-seen space $C_c^\infty(F_v^\times)$ of locally constant functions with compact support away from 0; it includes functions $f(x)$ with sufficiently rapid decay as $|x| \to 0, \infty$.)

We will then recall properties of the Fourier transform on $\mathcal{S}(G)$ and discuss some examples of interest, as well as define the concept of vector-valued tempered distribution on G. The latter is needed to prove "kernel theorems" that express function spaces on a product as suitably completed tensor products of function spaces on the factors; such results are a crucial tool to built operators for a product group from operators associated to the factor groups.

To succinctly state the main result of this Appendix, let us first recall some terminology for Hausdorff locally convex topological vector spaces that is reviewed in much more detail in Appendix C.1. We say such a space E is *bornological* if any seminorm on E that is bounded on bounded subsets is continuous (this is equivalent to another definition given without the language of seminorms in [**Scha**, Ch. II, p. 61], via the relation to seminorms to circled convex subsets in [**Scha**, Ch. II, 1.5, 1.6]). We say E is *barrelled* if every barrel (i.e., closed convex circle absorbing subset) is a neighborhood of 0; see [**Scha**, Ch. II, §7] for more on barrelledness. For instance, Fréchet spaces are barrelled [**Scha**, Ch. II, Cor. to 7.1].

Finally, we say E is *Montel* if it is reflexive (i.e., the natural map to the strong double dual is a topological isomorphism) and has the Heine-Borel property: closed bounded subsets are compact. It is equivalent (by [**Scha**, Ch. IV, 3.1, 5.6]) for E to be barrelled and have the Heine-Borel property. Every barrelled quasi-complete nuclear space is Montel [**Tre**, Cor. 3 to Prop. 50.2], so for example every nuclear Fréchet space is Montel. The latter applies to $\mathcal{S}(\mathbf{R}^N)$, but *not* to inductive limits such as $\mathcal{S}(\mathbf{Q}_p)$ or $\mathcal{S}(\mathbf{A}_F)$ that are complete [**Scha**, Ch. II, 6.6] but non-metrizable (they don't have a countable base of 0-neighborhoods).

The main result we aim to prove in this Appendix is that if G is second countable (i.e., has a countable base for the topology) then $\mathcal{S}(G)$ is a complete bornological Montel space whose strong dual $\mathcal{S}(G)'_\beta$ is nuclear (the strong dual is then also complete and Montel, as holds for the strong dual of any bornological Montel space [**Scha**, Ch. IV, 5.9, 6.1]). Nuclearity is always inherited by the strong dual for nuclear Fréchet spaces [**Tre**, Prop. 50.6], but we need cases such as $G = \mathbf{A}_F$ and $G = C_F$ for which $\mathcal{S}(G)$ is not Fréchet, so the nuclearity of the strong dual of $\mathcal{S}(G)$ is a genuinely additional property for $E = \mathcal{S}(G)$ in general.

The same assertion for $C_c^\infty(G)$ in place of $\mathcal{S}(G)$ is proved as [**Bru**, §8, Cor. 2 to Prop. 10], and [**Bru**, §9, Prop. 11] states that this holds for $\mathcal{S}(G)$ by similar arguments, but no details are provided. (In [**Bru**] the topological hypothesis on G is "separable", which is defined therein to mean "metrizable and covered by countably many compact subsets"; for Hausdorff locally compact abelian groups this is equivalent to being second countable by Remark A.5.2.9, so it matches the topological hypothesis we shall use but we won't need that.). Since $\mathcal{S}(G)$ is much larger than $C_c^\infty(G)$, and studying it entails much greater attention to decay estimates than for $C_c^\infty(G)$, a complete treatment of the matter is provided herein. The arguments presented here follow closely those in the early sections of [**Bru**]. We briefly indicate what happens beyond the second-countable case in Remark D.1.5.8, and primarily focus on second countable G since it is the sole case of relevance for us.

We will be relying on basic duality theory for Hausdorff locally compact abelian groups, for which the reader is referred to [**DE**, Ch. 3]. We will be referring to [**Ru**, Ch. 1, 2] and [**DE**, Ch. 3, 4] for foundational material on the structure of Hausdorff locally compact abelian groups, Pontryagin duality and Fourier Theory, and we will give some references to [**Bru**] and [**Wa**, §2] connected to the definition of the Schwartz space $\mathcal{S}(G)$.

Toward the end of this section we define distributions on G valued in a Hausdorff locally convex topological $\mathbf{C}$-vector space E due to its role in the proof of "kernel theorems" as noted above; this includes studying required properties of differential operators and the Fourier transform.

D.1.1. Fourier Transform. We first briefly recall the definition and basic properties of the Fourier transform on a Hausdorff locally compact abelian group, to be followed by Weil's Formula for the Haar Integral (see Theorem D.1.2.1) and by the Poisson Summation Formula (see Theorem D.1.2.7). They are collected here in the generality of Hausdorff locally compact abelian groups, for later reference.

THEOREM D.1.1.1. *Let G be a Hausdorff locally compact abelian group, and $\widehat{G}$ the set of continuous characters $\chi : G \to U(1)$ endowed with the compact-open topology. Then*

(1) *$\widehat{G}$ is a Hausdorff locally compact abelian group, whose formation is contravariantly functorial with respect to continuous homomorphisms.*
(2) *The duality pairing:*

$$\langle g, \chi \rangle_{G,\widehat{G}} : G \times \widehat{G} \to U(1) \tag{D.1.1.1}$$

is separating and it induces a canonical isomorphism of topological groups

$$\iota_G : G \to \widehat{\widehat{G}} \tag{D.1.1.2}$$

via the assignment:

$$\iota_G(g)(\chi) := \chi(g) \tag{D.1.1.3}$$

(3) *G is compact if and only if $\widehat{G}$ is discrete.*

PROOF. The first assertion in (1) is [**Ru**, Thm. 1.2.6(d)] (topological groups in [**Ru**] are Hausdorff by definition), and the rest of (1) is clear. Assertion (2) is [**Ru**, Thm. 1.7.2], and (3) is [**Ru**, 1.7.3(a)]. □

We recall that for any fixed arbitrary choice of Haar measure μ_G on G, there exists a Fourier transform, defined on $f \in L^1(G, \mu_G)$ on G by the assignment

$$\mathcal{F}_G(f)(\chi) := \widehat{f}(\chi) := \int_G f(g)\,\overline{\chi(g)}\,\mathrm{d}\mu_G(g), \tag{D.1.1.4}$$

noting that $f\overline{\chi}$ is integrable since χ is bounded over G (and continuous).

We further recall that for $f_1, f_2 \in L^1(G, \mu_G)$ there exists a convolution product $f_1 * f_2$ defined by

$$(f_1 * f_2)(x) := \int_G f_1(g)\, f_2(g^{-1}x)\,\mathrm{d}\mu_G(g), \tag{D.1.1.5}$$

with $f_1 * f_2$ a μ_G-measurable function defined μ_G-almost everywhere on G and μ_G-integrable on G with L^1-norm satisfying

$$\|f_1 * f_2\|_1 \leq \|f_1\|_1 \cdot \|f_2\|_1 \tag{D.1.1.6}$$

by [**Ru**, Thm. 1.1.6], and convolution is a commutative algebra structure on $L^1(G, \mu_G)$.

We collect here basic properties of the Fourier transform for later reference, denoting by $\|\cdot\|_u : C_0(G) \to \mathbf{R}_{\geq 0}$ the sup-norm on the space of continuous functions on G vanishing at infinity.

LEMMA D.1.1.2. *Let G be a Hausdorff locally compact abelian group, $f, f_1, f_2 : G \to \mathbf{C}$ three μ_G-measurable functions.*

(1) *If $f \in L^1(G, \mu_G)$ then $\widehat{f} \in C_0(\widehat{G})$.*
(2) *The Fourier transform $\mathcal{F}_G : L^1(G, \mu_G) \to C_0(\widehat{G})$ is an injective linear operator satisfying*

$$\|\widehat{f}\|_u \leq \|f\|_1 \tag{D.1.1.7}$$

(so in particular it is a continuous map).

(3) *If $f_1, f_2 \in L^2(G, \mu_G)$ then $(f_1 * f_2)(x)$ is finite for all $x \in G$ and belongs to $C_0(G)$, satisfying*

$$\|f_1 * f_2\|_u \leq \|f_1\|_2 \cdot \|f_2\|_2. \tag{D.1.1.8}$$

(4) *There exists a unique Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$, called the* **Plancherel measure**, *such that for any $f \in L^1(G, \mu_G) \cap L^2(G, \mu_G)$ we have*

$$\|f\|_2 = \|\widehat{f}\|_2, \tag{D.1.1.9}$$

and the Fourier transform extends uniquely to a unitary isomorphism of Hilbert spaces:

$$\mathcal{F}_G : L^2(G, \mu_G) \to L^2(\widehat{G}, \mu_{\widehat{G}}). \tag{D.1.1.10}$$

(5) *If $f \in L^1(G, \mu_G)$ is such that $\widehat{f} \in L^1(\widehat{G}, \mu_{\widehat{G}})$ then f is continuous on G and the inversion formula holds:*

$$f(x^{-1}) = \mathcal{F}_{\widehat{G}}(\mathcal{F}_G(f))(x) \tag{D.1.1.11}$$

as functions of $x \in G$.

(6) *For $f_1, f_2 \in L^1(G, \mu_G)$ we have*

$$\mathcal{F}_G(f_1 * f_2) = \mathcal{F}_G(f_1) \cdot \mathcal{F}_G(f_2) \tag{D.1.1.12}$$

as functions in $C_0(\widehat{G})$.

Proof. (1) and (2) are [**Ru**, Thm. 1.2.4(a, d)], as well as [**DE**, Cor. 3.3.4] for the injectivity of $\mathcal{F}_G$.
(3) : This is [**Ru**, 1.1.6,(d)] and Hölder's inequality (which establishes the norm-bound in (D.1.1.8)).
(4) : This is [**DE**, Thm.3.4.8 and p. 77].
(5) : This is [**DE**, Thm.3.5.8].
(6) : This is [**Ru**, Thm.1.2.4(b)]. □

D.1.2. Weil's Formula for the Haar Integral. We recall here a fundamental result concerning integrable functions on a Hausdorff locally compact abelian group G with respect to a choice of Haar measure μ: the uniquely determined Haar measure ν on the quotient G/H by a closed subgroup H that has been equipped with a choice of Haar measure μ_H.

Theorem D.1.2.1 (Weil's Quotient Integral Formula). *Let G be a locally compact Hausdorff abelian group, $H \subset G$ a closed subgroup. There exists a Haar measure ν on G/H uniquely determined by choices of a Haar measure μ_G on G and μ_H on H subject to the requirement that for every $f \in C_c(G)$, we have*

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \int_{G/H} \int_H f(xh)\, \mathrm{d}\mu_H(h)\, \mathrm{d}\nu(x). \tag{D.1.2.1}$$

Moreover, for every μ_G-measureable $f : G \to \mathbf{C}$ the assignment $f^H : x \mapsto \int_H f(xh)\, \mathrm{d}\mu_H(h)$ defines μ_G-almost everywhere on G a function that is ν-measurable and if $f \in L^1(G)$ then f^H is ν-integrable on G/H and (D.1.2.1) *holds for f.*

All but the final assertion follow from [**Lang-d**, Ch. XII, Thm. 4.1, Cor. 2.4]. The final L^1 assertion is part of [**DE**, Thm. 1.5.3].

Remark D.1.2.2. Theorem D.1.2.1 has an analogue for non-commutative G but we only ever need it for G commutative, which has the advantage of rendering the group unimodular.

Lemma D.1.2.3. *Let G be a locally compact Hausdorff abelian group, $H \subset G$ a closed subgroup, and μ_G and μ_H Haar measures on G and H respectively. Let ν be the unique invariant Haar measure on the quotient G/H provided by Theorem* D.1.2.1. *Assume that G is σ-finite. Let $f : G \to \mathbf{C}$ be a measurable function. If*

$$\int_{G/H} \int_H |f(xh)|\, \mathrm{d}\mu_H(h)\, \mathrm{d}\nu(x) \tag{D.1.2.2}$$

converges then f is μ_G-integrable over G and (D.1.2.1) *holds.*

PROOF. See [**DE**, Cor. 1.5.4]. □

Here is a basic result on "measureable fundamental domains":

PROPOSITION D.1.2.4. *Let G be a Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup. Let $\Delta \subset G$ a Borel measurable subset. Assume:*

(1) *H is discrete and define μ_H to be the counting measure on H; let $\overline{\mu}$ being the unique Haar measure on G/H induced by the pair μ_G, μ_H.*
(2) *The continuous quotient homomorphism $\pi_H : G \to G/H$ induces a bijection $\Delta \to G/H$.*
(3) *G is second-countable.*

For any $f \in L^1(G, \mu_G)$, the function $S(f) : H \to \mathbf{C}$ defined by

$$S(f)(h) := \int_{\Delta h} f(g)\, \mathrm{d}\mu_G(g) \tag{D.1.2.3}$$

is μ_H-integrable and

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \sum_{h \in H} S(f)(h) \tag{D.1.2.4}$$

with the right side absolutely convergent.

If $f : G/H \to \mathbf{C}$ is a function, then f is $\overline{\mu}$-measurable, respectively $\overline{\mu}$-integrable if and only if $1_\Delta \cdot (f \circ \pi_H)$ is μ_G-measurable, respectively μ_G-integrable. In the latter case,

$$\int_{G/H} f(\overline{g})\, \mathrm{d}\overline{\mu}(\overline{g}) = \int_\Delta (f \circ \pi_H)(g)\, \mathrm{d}\mu_G(g). \tag{D.1.2.5}$$

PROOF. The absolute summability of $S(f)$ in (D.1.2.3) follows from [**Bou-II**, Ch. VII, §2.10, Prop. 15] applied to $X = G$, $\mu = \mu_G$, $F = \Delta$, and to real and imaginary parts of f, using that multiplication by h is continuous and a homeomorphism (so Δh is Borel measurable and the function $n : G \to \mathbf{Z}_{\geq 0}$ is identically 1).

The Borel-measurability, respectively $\overline{\mu}$-integrability criterion, and the identity in (D.1.2.5), is a consequence of [**Bou-II**, §2.10, Thm. 4(c)], where $\varphi_F = 1_\Delta$ in the notation of loc. cit (applied to the real and imaginary parts of f). □

The following application of (D.1.2.5) records that integration on a measurable fundamental domain does not depend on the choice of fundamental domain.

COROLLARY D.1.2.5. *Let G be a Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup. Let $\Delta, \Delta' \subset G$ be two Borel-measurable subsets. Assume:*

(1) *H is discrete and define μ_H to be the counting measure on H.*
(2) *The quotient homomorphism $\pi_H : G \to G/H$ induces bijections $\Delta \to G/H$ and $\Delta' \to G/H$.*
(3) *G is second-countable.*

Let $f : G \to \mathbf{C}$ be a μ_G-measurable function invariant under translation by H (so it is $\overline{f} \circ \pi_H$ for a $\mu_{G/H}$-measureable $\overline{f} : G/H \to \mathbf{C}$). If $1_\Delta f$ is μ_G-integrable then so is $1_{\Delta'} f$ and

$$\int_\Delta f(g)\, \mathrm{d}\mu_G(g) = \int_{\Delta'} f(g)\, \mathrm{d}\mu_G(g). \tag{D.1.2.6}$$

As an application of Weil's Formula for the Haar integral (Theorem D.1.2.1), we next discuss a general form of the Poisson Summation Formula for ease of reference.

SITUATION D.1.2.6. We choose a Haar measure μ_G on G, a Haar measure μ_H on H, and we let $\mu_{G/H}$ be the unique Haar measure on G/H determined by μ_G and μ_H as per Theorem D.1.2.1. We denote by $\mu_{\widehat{G}}$,

$\mu_{\widehat{G/H}}$ the respective Plancherel measures, and choose on $H^\perp$ the unique Haar measure that pulls back to $\mu_{\widehat{G/H}}$ under the isomorphism of Hausdorff locally compact abelian groups $H^\perp \to \widehat{G/H}$ from Lemma D.1.4.6. We denote by $\mu_{H^\perp}$ such choice of Haar measure on $H^\perp$.

By [**DE**, Thm. 3.6.3] we have:

THEOREM D.1.2.7 (Poisson Summation Formula). *Let G be a Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup, as in Situation* D.1.2.6.

Let $f \in L^1(G, \mu_G)$, so the H-invariant function

$$f^H(x) := \int_H f(hx)\,\mathrm{d}\mu_H(h) \tag{D.1.2.7}$$

defined for $\mu_{G/H}$-almost all $x \in G/H$ is $\mu_{G/H}$-measurable and $\mu_{G/H}$-integrable.

(1) *Under the isomorphism $H^\perp \to \widehat{G/H}$, $\widehat{f^H}$ pulls back to $\widehat{f}|_{H^\perp} \in C(H^\perp)$.*
(2) *If $\widehat{f}|_{H^\perp}$ is $\mu_{H^\perp}$-integrable then the Poisson Summation Formula*

$$\int_H f(h^{-1}x)\,\mathrm{d}\mu_H(h) = \int_{H^\perp} \widehat{f}(\chi)\chi(x)\,\mathrm{d}\mu_{H^\perp}(\chi) \tag{D.1.2.8}$$

holds for μ_G-almost all $x \in G$.
(3) *Assume:*
  (i) *f^H is defined for every $x \in G$, and it is continuous.*
  (ii) *$\widehat{f}|_{H^\perp} \in L^1(H^\perp)$ (so the right side of* (D.1.2.8) *converges for all $x \in G$).*

Then (D.1.2.8) *holds for all $x \in G$.*

REMARK D.1.2.8. We note that for (D.1.2.8) to hold for every $x \in G$, it is necessary for f^H to be continuous since the right side of (D.1.2.8) is continuous in x when $\widehat{f}|_{H^\perp} \in L^1(H^\perp)$.

D.1.3. Schwartz space of elementary Lie groups. We fix integers $m, n, r \geq 0$, and a finite abelian group F, with $\mathbf{Z}^r \times F$ endowed with the discrete topology. We denote by $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ the circle group, and define

$$G := \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F. \tag{D.1.3.1}$$

We parametrize G with coordinates $g = (x, z, a, b)$, where $x \in \mathbf{R}^m$, $z \in \mathbf{T}^n$, $a \in \mathbf{Z}^r$ and $b \in F$. Topological groups *isomorphic* to such G for a (unique) choice of integers $m, n, r \geq 0$ and finite abelian group F are called *elementary* abelian Lie groups.

Throughout this section we will recall in what sense elementary abelian Lie groups are the building blocks of general Hausdorff locally compact abelian groups, and how to use them to define the Schwartz space $\mathcal{S}(G)$ as a locally convex topological vector space.

We define

$$\phi : \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{Z}^r \times F \to \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F = G \tag{D.1.3.2}$$

by $\phi(x, t, a, b) := (x, (\exp(2\pi i\, t_j))_{1 \leq j \leq n}, a, b)$ for $t = (t_j) \in \mathbf{R}^n$, so $C^\infty(\mathbf{R}^m \times \mathbf{T}^n)$ is identified with the space of $\{0\} \times \mathbf{Z}^n$-invariant functions in $C^\infty(\mathbf{R}^m \times \mathbf{R}^n)$: let

$$\iota_{m,n} : C^\infty(\mathbf{R}^m \times \mathbf{T}^n) \to C^\infty(\mathbf{R}^m \times \mathbf{R}^n)$$

be the inclusion.

For every $\alpha \in \mathbf{N}^m$ and $\beta \in \mathbf{N}^n$, define

$$D_x^\alpha D_z^\beta := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \circ \frac{\partial^{|\beta|}}{\partial t_1^{\beta_1} \cdots \partial t_n^{\beta_n}} \circ \iota_{m,n} : C^\infty(\mathbf{R}^m \times \mathbf{T}^n) \to C^\infty(\mathbf{R}^m \times \mathbf{R}^n) \tag{D.1.3.3}$$

with $|\alpha| := \sum_{i=1}^m \alpha_i$, $|\beta| := \sum_{i=1}^n \beta_i$. We have

$$D_x^\alpha D_z^\beta (C^\infty(\mathbf{R}^m \times \mathbf{T}^n) \subset \iota_{m,n}(C^\infty(\mathbf{R}^m \times \mathbf{T}^n)) \tag{D.1.3.4}$$

due to invariance of the $\mathbf{C}$-linear operator $\partial^{|\beta|}/(\partial t_1^{\beta_1} \cdots \partial t_n^{\beta_n})$ under additive $\mathbf{Z}^n$-translation on the $\mathbf{R}^n$-factor, so $D_x^\alpha D_z^\beta$ induces a unique $\mathbf{C}$-linear endomorphism of $C^\infty(\mathbf{R}^m \times \mathbf{T}^n)$ that we denote by $\partial_x^\alpha \partial_z^\beta$.

For any $x \in \mathbf{R}^m$ and $\alpha \in \mathbf{N}^m$ we define

$$x^\alpha := \prod_{i=1}^m x_i^{\alpha_i}. \tag{D.1.3.5}$$

D.1.3.1. *Smooth, rapidly decaying functions on elementary abelian Lie groups.* Let's assume not only that G is an elementary abelian Lie group (i.e., isomorphic to one of the form $\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$) but rather that G *is* such a product abelian Lie group. In effect, we fix a choice of "coordinates" on G^0 and on its discrete component group as well as a homomorphic lifting of the discrete component into G. This assumption will be in force until the end of §D.1.3

Definition D.1.3.1. For an elementary abelian Lie group G, we say $f \in C^\infty(G)$ is of *rapid decay* (at infinity) over G if and only if:

(1) for every $\alpha, \beta \in \mathbf{N}^m$ and $\gamma \in \mathbf{N}^n$ and $\delta \in \mathbf{N}^r$, the function

$$\| \cdot \|_{\alpha,\beta,\gamma,\delta} : C^\infty(G) \to [0, +\infty] \tag{D.1.3.6}$$

defined by

$$\|\phi\|_{\alpha,\beta,\gamma,\delta} := \sup_{g \in G} |(a^\delta \, x^\alpha \, \partial_x^\beta \partial_z^\gamma \phi)(x, z, a, b)| \tag{D.1.3.7}$$

is finite on f.

We denote by $\mathcal{S}(G)$ the $\mathbf{C}$-vector space of all smooth functions on G that decay rapidly towards infinity. We topologize $\mathcal{S}(G)$ with the locally convex topology generated by the countable family $\{\| \cdot \|_{\alpha,\beta,\gamma,\delta}\}$ of seminorms. We call $\mathcal{S}(G)$ the *Schwartz-Bruhat space* of G.

Remark D.1.3.2. The locally convex topological vector space $\mathcal{S}(G)$ is Hausdorff, by setting $\alpha, \beta, \gamma, \delta = 0$.

Beware that the specific seminorms on $C^\infty(G)$ defined above depend very much on the choice of coordinates on G. But the topology is *independent* of such a choice thanks to:

Lemma D.1.3.3. *Let $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ for a finite abelian group F and integers $m, n, r \geq 0$. For every continuous automorphisms ϕ of G and $f \in \mathcal{S}(G)$, the pullback $f \circ \phi : G \to \mathbf{C}$ belongs to $\mathcal{S}(G)$ and $f \mapsto f \circ \phi$ is a topological automorphism of $\mathcal{S}(G)$.*

Proof. We will describe a general ϕ as a composition of finitely many types of automorphisms that can be analyzed directly. Since the unique maximal compact subgroup of G is $\mathbf{T}^n \times F$ and its identity component is $\mathbf{T}^n$, Since all automorphisms of $\mathbf{R}^m$ and of $\mathbf{Z}^r$ are linear and of $\mathbf{T}^n$ arise from an element of $\mathrm{GL}_n(\mathbf{Z})$, it is clear that an automorphism arising from ones of the factors $\mathbf{R}^m$, $\mathbf{T}^n$, $\mathbf{Z}^r$, and F has the desired properties. Thus, for a general ϕ we readily reduce to the case when ϕ is the identity on the factor group $\mathbf{T}^n$ and its restriction to each of the four factor groups yields the identity when projected back onto that factor.

By connectedness considerations, the effect of such a ϕ on $\mathbf{R}^m$ is given by $x \mapsto (x, f(x), 0, 0)$ for a continuous homomorphism $f : \mathbf{R}^m \to \mathbf{T}^n$. Automorphisms of the form $(x, z, a, b) \mapsto (x, f(x)z, a, b)$ are readily seen to preserve the Schwartz decay property because $f(x) = (f_1(x), \ldots, f_n(x))$ for $f_j : \mathbf{R}^m \to \mathbf{T}$ given by $x \mapsto e^{2\pi i \langle x, c_j \rangle}$ for some $c_1, \ldots, c_n \in \mathbf{R}^m$. Thus, we can further reduce to the case that ϕ is the identity on the factor group $\mathbf{R}^m$. $\phi(x, z, a, b) = (x + L_1(a), zL_2(a)L_3(b), a, b + L_4(a))$ for homomorphisms

$$L_1 : \mathbf{Z}^r \to \mathbf{R}^m, \;\; L_2 : \mathbf{Z}^r \to \mathbf{T}^n, \;\; L_4 : \mathbf{Z}^r \to F, \;\; L_3 : F \to \mathbf{T}^n.$$

Hence, we can focus on the cases in which all but one of these homomorphisms is trivial. The family of seminorms (D.1.3.10) defining the Schwartz topology allows us to handle the L_1-cases and the L_2-cases (since we can explicitly describe the possibilities for such L_1 and L_2 via "linear" formulas). The L_4-cases are easily handled since in such cases ϕ commutes with every ∂_x^α and ∂_z^β. The L_3-cases are handled by again using the family of seminorms (D.1.3.10). □

D.1.3.2. *Fundamental families of continuous seminorms.* We now discuss different families of seminorms on $\mathcal{S}(G)$ generating the same (Hausdorff) locally convex topology.

LEMMA D.1.3.4. *The following families of seminorms $\mathcal{S}(G) \to [0, +\infty)$ all define the topology on $\mathcal{S}(G)$:*

$$\mathcal{P}_1 : \|f\|_{\mathcal{P}_1,(j,j',\alpha,\beta)} := \sup_{g\in G}\{(1+|x|^j)\,(1+|a|^{j'})\,|\partial_x^\alpha \partial_z^\beta f(x,z,a,b)|\}, \quad j, j' \in \mathbf{N}, \alpha \in \mathbf{N}^m,\ \beta \in \mathbf{N}^n \tag{D.1.3.8}$$

(*where $|x|, |a|$ denote Euclidean norms*),

$$\mathcal{P}_2 : \|f\|_{\mathcal{P}_2,(\alpha,\beta,P)} := \sup_{g\in G}\{|P(x,a)|\,|\partial_x^\alpha \partial_z^\beta f(x,z,a,b)|\}, \quad \alpha \in \mathbf{N}^m,\ \beta \in \mathbf{N}^n,\ P \in \mathbf{C}[X_1,\dots,X_m,Y_1,\dots,Y_r], \tag{D.1.3.9}$$

$$\mathcal{P}_3 : \|f\|_{\mathcal{P}_3,(P,Q)} := \sup_{g\in G}\{|P(x,a)\,Q(\partial_x,\partial_z)f(x,z,a,b)|\} \tag{D.1.3.10}$$

for varying $P \in \mathbf{C}[X_1,\dots,X_m,Y_1,\dots,Y_r], Q \in \mathbf{C}[X_1,\dots,X_m,Z_1,\dots,Z_n]$.

PROOF. We show each collection of seminorms in turn defines the topology on $\mathcal{S}(G)$. First we consider $\mathcal{P}_1$. For every $q \in \mathcal{P}_1$, we have $q = \|\cdot\|_{\mathcal{P}_1,(j,j',\alpha,\beta)}$ for some integers $j, j' \geq 0$ and $\alpha \in \mathbf{N}^m, \beta \in \mathbf{N}^n$. For every $f \in \mathcal{S}(G)$, in terms of the seminorms in (D.1.3.7) we have:

$$q(f) \leq \|f\|_{(0,\alpha,\beta,0)} + C_r^{j'} \sum_{i'=1}^m \|f\|_{(0,\alpha,\beta,\gamma_{i'}(j'))} + C_m^j \sum_{i=1}^m \|f\|_{(\alpha_i(j),\alpha,\beta,0)} + C_r^{j'} C_m^j \sum_{1\leq i,i'\leq m} \|f\|_{(\alpha_i(j),\alpha,\beta,\gamma_{i'}(j'))} \tag{D.1.3.11}$$

where $C_m > 0$ is a constant for which $|x| \leq C_m \max_i |x_i|$ for all $x \in \mathbf{R}^m$ and C_r is likewise for $\mathbf{R}^r$, and $\alpha_i(j) = je_i \in \mathbf{N}^m$, $\gamma_{i'}(j') = j'e'_{i'} \in \mathbf{N}^r$ (so $|x|^j \leq C_m^j \sum_i |x^{\alpha_i(j)}|$, $|a|^{j'} \leq C_r^{j'} \sum_{i'} |a^{\gamma_{i'}(j')}|$.

On the other hand, for every $\eta, \theta \in \mathbf{N}^m$ and $\gamma \in \mathbf{N}^r$, there exist real numbers $C_{\eta,m}, A_{\gamma,r} > 0$ such that $|x^\eta| \leq C_{\eta,m} \cdot (1+|x|^{|\eta|})$ and $|a^\gamma| \leq A_{\gamma,r} \cdot (1+|a|^{|\gamma|})$, showing that we also have

$$\|f\|_{\eta,\theta,\beta,\gamma} \leq C \cdot \|f\|_{\mathcal{P}_1,(|\eta|,|\gamma|,\theta,\beta)} \tag{D.1.3.12}$$

for a real number $C > 0$ depending only on η, γ (and m and r). This settles $\mathcal{P}_1$.

Turning to $\mathcal{P}_2$, every $P \in \mathbf{C}[X_1,\dots,X_m,Y_1,\dots,Y_r]$ is of the form $P = \sum_{\alpha\in\mathbf{N}^m,\delta\in\mathbf{N}^r} c_{\alpha,\delta}\, X^\beta\, Y^\delta$ where all but finitely many of the $c_{\beta,\delta} \in \mathbf{C}$ are nonzero, and $X^\beta := \prod_{i=1}^m X_i^{\beta_i}$ and $Y^\delta := \prod_{i=1}^r Y_i^{\delta_i}$, so for every $f \in \mathcal{S}(G)$ we have

$$\|f\|_{\mathcal{P}_2,(\beta,\gamma,P)} \leq \sum_{\alpha\in\mathbf{N}^m,\delta\in\mathbf{N}^r} |c_{\alpha,\delta}|\,\|f\|_{\alpha,\beta,\gamma,\delta}. \tag{D.1.3.13}$$

Since also $\|\cdot\|_{\alpha,\beta,\gamma,\delta} \in \mathcal{P}_2$ for every $\alpha, \beta \in \mathbf{N}^m$, $\gamma \in \mathbf{N}^n$, $\delta \in \mathbf{N}^r$, $\mathcal{P}_2$ also defines the topology on $\mathcal{S}(G)$.

Finally, since every $Q \in \mathbf{C}[X_1,\dots,X_m]$ is of the form $Q = \sum_{\alpha\in\mathbf{N}^m} c_\alpha X^\alpha$ with all but finitely many of the $c_\alpha \in \mathbf{C}$ being nonzero and $X^\alpha := \prod_{i=1}^m X_i^{\alpha_i}$, every element of $\mathcal{P}_3$ is bounded above by a finite sum of positive constants times elements of $\mathcal{P}_2$. Since likewise every seminorm in $\mathcal{P}_2$ belongs to $\mathcal{P}_3$, the locally convex topologies defined by the respective families $\mathcal{P}_2$ and $\mathcal{P}_3$ of seminorms are the same. □

LEMMA D.1.3.5. *We have:*

(1) *The locally convex topology on* $\mathcal{S}(G)$ *induced by the family of seminorms from* (D.1.3.6) *is equivalent to the locally convex topology on* $\mathcal{S}(G)$ *induced by the family of norms on* $\mathcal{S}(G)$ *defined by*

$$\|f\|_N := \sum_{\alpha,\beta,\gamma,\delta:|\alpha|,|\beta|,|\gamma|,|\delta|\leq N} \|f\|_{\alpha,\beta,\gamma,\delta}, \quad N \in \mathbf{N}. \tag{D.1.3.14}$$

(2) *The locally convex topology on* $\mathcal{S}(G)$ *is Hausdorff and complete, promoting* $\mathcal{S}(G)$ *to a Fréchet space.*

Proof. The proof of (1) is standard, similarly to any locally convex topology defined by a countable family of seminorms.

For (2), we first verify completeness by consider a Cauchy net $\{f_i\}_{i\in I}$ in $\mathcal{S}(G)$ for the locally convex topology induced by the family of seminorms $\|\cdot\|_{(\alpha,\beta,\gamma,\delta)}$ from (D.1.3.6), with I a directed set. The uniform limit of $\{f_i\}_{i\in I}$ in $C_0(G)$ exists, since $C_0(G)$ equipped with the uniform norm is a Banach space. We denote such limit by f. The restriction to every fibre over the projection $G \to \mathbf{Z}^r \times F$ is a uniform limit of smooth functions with uniformity in each of its derivatives too. Formation of the derivatives $\partial_x^\alpha \partial_z^\beta f$, which exist fibre-wise over $\mathbf{Z}^r \times F$, is continuous over G by uniformity of the convergence and so f is smooth over G.

We now show $f_i \to f$ in the topology of $\mathcal{S}(G)$ by using the family of seminorms $\mathcal{P}_1$ in (D.1.3.8) to carry over the classical argument for the completeness of $\mathcal{S}(\mathbf{R}^m)$. We have, for every $\varepsilon > 0$ and uniformly with respect to $(x, z, a, b) \in G$, the net $\{f_i\}_{i\in I}$ being Cauchy in $\mathcal{S}(G)$ and since the locally convex topology on $\mathcal{S}(G)$ is, by Lemma D.1.3.4, equivalently generated by the family of seminorms $\mathcal{P}_1$:

$$|(1+|x|^p)\,(1+|a|^{p'})\,(\partial_x^\beta \partial_z^\gamma f_i)(x,z,a,b) - (1+|x|^p)\,(1+|a|^{p'})\,(\partial_x^\beta \partial_z^\gamma f_j)(x,z,a,b)| < \varepsilon \tag{D.1.3.15}$$

for all large enough $i, j \in I$ with respect to the partial ordering on I (and depending on $\varepsilon > 0$ and on each choice of $p, p' \geq 0$ and $\beta \in \mathbf{N}^m, \gamma \in \mathbf{N}^n$).

The convergence of the net $\{f_i\}_{i\in I}$ to f in $C_0(G)$ being uniform in all derivatives, we may pass to the limit in (D.1.3.15) over $j \in J$ and obtain

$$|(1+|x|^p)\,(1+|a|^{p'})\,(\partial_x^\beta \partial_z^\gamma f_i)(x,z,a,b) - (1+|x|^r)\,(1+|a|^{r'})\,(\partial_x^\alpha \partial_z^\gamma f)(x,z,a,b)| \leq \varepsilon \tag{D.1.3.16}$$

with (x, z, a, b) arbitrary. The exchange in the order of the limit and the derivatives is legitimate by uniformity of the convergence. We may therefore form the supremum in (D.1.3.16) over $(x, z, a, b) \in G$ and obtain that $f \in \mathcal{S}(G)$ and $\|f_i - f\|_{\mathcal{P}_1,(p,p',\beta,\gamma)} \leq \varepsilon$ for every $p, p' \geq 0$ and $\beta \in \mathbf{N}^m, \gamma \in \mathbf{N}^n$.

This shows that $\{f_i\}_{i\in I}$ converges to f in $\mathcal{S}(G)$, proving completeness of $\mathcal{S}(G)$. Since the collection of seminorms $\mathcal{P}_1$ from (D.1.3.8) is countable, $\mathcal{S}(G)$ is metrizeable. As such, $\mathcal{S}(G)$ is a Fréchet space. □

D.1.3.3. *Nuclearity of the Schwartz-Bruhat space of an elementary abelian Lie group.* We claim the Fréchet locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(G)$ is nuclear. In fact, we can identify it with the finite product of π-tensor products

$$\prod_F (\mathcal{S}(\mathbf{R}^m) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(\mathbf{T}^n) \mathbin{\widehat{\otimes}_\pi} \mathbf{s}_r)$$

where $\mathbf{s}_r$ is the (nuclear) space of rapidly decaying sequences $\mathbf{Z}^r \to \mathbf{C}$. This is shown in Lemma D.1.3.7 below.

Remark D.1.3.6. We caution the reader that the nuclear Fréchet space $\mathfrak{s}$ of rapidly decreasing complex $\mathbf{N}$-*indexed* sequences is not $\mathbf{s}_r = \mathcal{S}(\mathbf{Z}^r)$ but there is a linear topological isomorphism $\mathbf{s}_r \to \mathfrak{s}$ for every integer $r \geq 1$ (though we will not use this fact).

We recall the construction of $\mathbf{s}_r$ (see [Tre, Remark 51.1, (51.1), p. 527]). Let $s(\mathbf{Z}^r)$ be the $\mathbf{C}$-vector space of $\mathbf{C}$-valued functions on $\mathbf{Z}^r$; i.e., complex sequences indexed by $\mathbf{Z}^r$. The $\mathbf{C}$-linear subspace $\mathbf{s}_r \subset s(\mathbf{Z}^r)$ consists of those sequences $\{s_a\}_{a\in\mathbf{Z}^r}$ such that

$$p_j(\{s_a\}_{a\in\mathbf{Z}^r}) := \sum_{a\in\mathbf{Z}^r} (1+|a|)^j\,|s_a| < \infty \tag{D.1.3.17}$$

for every integer $j \geq 0$. Endowed with the locally convex topology generated by the family $\{p_j\}_{j\geq 0}$ of seminorms, $\mathbf{s}_r$ is Fréchet since $p_j \leq p_{j+1}$ and finiteness of (D.1.3.17) defines $L^1(\mathbf{Z}^r, \mu_j)$ for the measure μ_j with $\mu_j(\{a\}) = (1+|a|)^j$. Explicitly, $\mathbf{s}_r$ is (topologically) the inverse limit of the inverse system of spaces $L^1(\mathbf{Z}^r, \mu_j)$. Nuclearity of $\mathbf{s}_r$ is [Tre, Thm. 51.5].

LEMMA D.1.3.7 (Nuclearity in the case of elementary abelian Lie groups). *We have:*

(1) *The injective* $\mathbf{C}$*-linear map*

$$\phi : \mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes \mathbf{s}_r \otimes \mathbf{C}^F \to \mathcal{S}(G) \tag{D.1.3.18}$$

defined by

$$\phi(f_1 \otimes f_2 \otimes (\{s_a\}_{a\in\mathbf{Z}^r}) \otimes c) := (f_1 \circ \mathrm{pr}_1) \cdot (f_2 \circ \mathrm{pr}_2) \cdot s \cdot c : G \to \mathbf{C} \tag{D.1.3.19}$$

where $s : G \to \mathbf{C}$ *is* $(x, z, a, b) \mapsto s_a$ *and* $c : F \to \mathbf{C}$ *is a function) induces the projective tensor product topology on the source.*

(2) *The* $\mathbf{C}$*-linear map* ϕ *from* (D.1.3.18) *extends uniquely to a linear topological isomorphism*[1]

$$\phi : \mathcal{S}(\mathbf{R}^m) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}(\mathbf{T}^n) \mathbin{\widehat{\otimes}_\pi} \mathbf{s}_r \mathbin{\widehat{\otimes}_\pi} \mathbf{C}^F \to \mathcal{S}(G) \tag{D.1.3.20}$$

(3) *The locally convex topological* $\mathbf{C}$*-vector space* $\mathcal{S}(G)$ *and its strong dual* $\mathcal{S}(G)'_\beta$ *are complete, nuclear, and Montel (reflexive with all closed bounded subsets compact). In particular, both are barrelled.*

Throughout the proof below, we will be using two ingredients from Appendix C.1 on locally convex spaces: Lemma C.2.2.2 on the topology of uniform convergence on the members of the equicontinuous subsets of the dual, together with Remark C.2.3.8 describing the equicontinuous subsets of the continuous dual of a binary injective tensor product (see also Proposition C.2.4.8 for a description of the latter dual as the space of "integral" continuous bi-linear forms).

The idea is to use nuclearity of the tensor factors (say E, F, for the sake of this introduction) occurring throughout the argument below, to access the equivalence of the π- and ε-tensor topologies, for then comparing either topology to the subspace topology induced from the topology of the Schwartz-Bruhat space of the appropriate fine product.

That one of the two topologies is finer than the subspace topology will be the simpler of the two refinement relations to be shown. It is in the proof of the reverse one that we use the ε-topology, and Lemma C.2.2.2 is used to describpe such topology as the topology of uniform convergence on the subsets of $(E \otimes_\varepsilon F)'$ of the form $(A' \otimes B')^{\circ\circ}$ (bipolar), for $A' \subset E'$ and $B' \subset F'$ equicontinuous.

REMARK D.1.3.8 (On binary tensor products of tempered distributions). Before we can begin the proof of Lemma D.1.3.7, we need to review some facts from Appendix C.1 concerning Schwartz functions valued in certain locally convex spaces since this provides an essential technical tool we will use.

Let $m, m', n, n', r, r' \geq 0$ be fixed integers. Let $T \in \mathcal{S}(\mathbf{R}^m)'_\beta$ and $S \in \mathcal{S}(\mathbf{R}^{m'})'_\beta$ be two tempered distributions. Using Theorem C.5.1.30(4) from Appendix C, we can define the binary tensor product $T \otimes S$ as an element of the strong dual $\mathcal{S}(\mathbf{R}^{m+m'})'_\beta$ (as opposed to only in $\mathcal{S}(\mathbf{R}^m)'_\beta \otimes \mathcal{S}(\mathbf{R}^{m'})'_\beta$) as follows. The linear topological isomorphisms

$$\rho_1 : \mathcal{S}(\mathbf{R}^{m+m'}) \to \mathcal{S}(\mathbf{R}^m, \mathcal{S}(\mathbf{R}^{m'})),\ \rho_2 : \mathcal{S}(\mathbf{R}^{m+m'}) \to \mathcal{S}(\mathbf{R}^{m'}, \mathcal{S}(\mathbf{R}^m)) \tag{D.1.3.21}$$

from (C.5.1.48) lead to defining

$$\langle f, T \otimes S\rangle_{E,E'} = \langle \langle \rho_1(f), S\rangle_{E_2,E'_2}, T\rangle_{E_1,E'_1}. \tag{D.1.3.22}$$

[1]Must expand proof to also yield $\mathcal{S}(\mathbf{R})^{\widehat{\otimes}_\pi m} \simeq \mathcal{S}(\mathbf{R}^m)$ and likewise for $\mathcal{S}(\mathbf{T}^n)$ and $\mathcal{S}(\mathbf{Z}^r)$.

for $f \in E := \mathcal{S}(\mathbf{R}^{m+m'})$, $E_1 := \mathcal{S}(\mathbf{R}^m)$, and $E_2 := \mathcal{S}(\mathbf{R}^{m'})$. The same argument as in [**Tre**, Thm.40.4, p. 416] shows that we have

$$\langle\langle \rho_1(f), S\rangle_{E_2,E_2'}, T\rangle_{E_1,E_1'} = \langle\langle \rho_2(f), T\rangle_{E_1,E_1'}, S\rangle_{E_2,E_2'} \tag{D.1.3.23}$$

for all $f \in E$. The assignment from (D.1.3.22) thereby defines a continuous functional on E, which we denote by $T \otimes S$ following [**Tre**, Def. 40.3, p. 417].

Moreover, the assignment $(T, S) \mapsto T \otimes S$ is a separately continuous bi-linear mapping:

$$\mathcal{S}(\mathbf{R}^m)'_\beta \times \mathcal{S}(\mathbf{R}^{m'})'_\beta \to \mathcal{S}(\mathbf{R}^{m+m'})'_\beta. \tag{D.1.3.24}$$

By [**Tre**, Thm.41.1, p. 421], it is continuous, and therefore induces a continuous linear map:

$$\mathcal{S}(\mathbf{R}^m)'_\beta \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(\mathbf{R}^{m'})'_\beta \to \mathcal{S}(\mathbf{R}^{m+m'})'_\beta. \tag{D.1.3.25}$$

For $A \subset \mathcal{S}(\mathbf{R}^m)'$ and $B \subset \mathcal{S}(\mathbf{R}^{m'})$ equicontinuous subsets, we will be denoting by $A \otimes B$ the subset of $\mathcal{S}(\mathbf{R}^{m+m'})'_\beta$ defined by:

$$A \otimes B := \{T \otimes S : T \in A, S \in B\} \subset \mathcal{S}(\mathbf{R}^{m+m'})' \tag{D.1.3.26}$$

itself an equicontinuous subset.

The same construction applies to $\mathcal{S}(\mathbf{T}^n)$ and $\mathcal{S}(\mathbf{T}^{n'})$, as well as $\mathbf{s}_r$, $\mathbf{s}_{r'}$, replacing $\mathcal{S}(\mathbf{R}^m)$ and $\mathcal{S}(\mathbf{R}^{m'})$. The same notation introduced in (D.1.3.26) will therefore be in force.

Proof of Lemma D.1.3.7. (1) : The map (D.1.3.18) is injective since $\mathrm{Map}(X,k) \otimes \mathrm{Map}(Y,k) \to \mathrm{Map}(X \times Y, k)$ is injective for any sets X, Y and any field k. The proof of the topological aspect of (1) will proceed similarly to the proof of [**Tre**, Thm. 51.6] that treats $\mathbf{R}^m \times \mathbf{R}^n \simeq \mathbf{R}^{n+m}$.

Since boundedness constraints defined by means of the seminorms $q_{\alpha,\beta,\gamma,\delta}$ from (D.1.3.7) do not affect the F-factor in G, $\mathcal{S}(G)$ splits linearly and topologically as $\prod_F \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r)$, as does $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes \mathbf{s}_r \otimes \mathbf{C}^F$ endowed with the projective tensor product topology. We therefore may and do assume that $F = 0$ and rename $\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r$ as G. We now prove that the topology induced on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes \mathbf{s}_r$ by the linear map ϕ from (D.1.3.18) is the π-topology.

The bilinear map:

$$\psi_1 : \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \times \mathbf{s}_r \to \mathcal{S}(G) \tag{D.1.3.27}$$

is separately continuous. Since $\mathbf{s}_r$ and $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)$ are metrizable and $\mathbf{s}_r$ is Baire (even Fréchet), by [**Scha**, Ch. III, Cor̀ to Thm. 5.1] we see that ψ_1 is continuous. As such, by [**Scha**, Ch. III, 6.2] the injection

$$j : \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes \mathbf{s}_r \hookrightarrow \mathcal{S}(G) \tag{D.1.3.28}$$

is continuous for the π-topology on the source, so the subspace topology induced on the source is coarser than the π-topology. Since $\mathbf{s}_r$ is nuclear, so $E \otimes_\pi \mathbf{s}_r \to E \otimes_\varepsilon \mathbf{s}_r$ is a topological isomorphism for any locally convex space E, to check that the topology induced on $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes \mathbf{s}_r$ by j is the π-topology it is enough to check that it is finer than the ε-topology.

We now use the general property for any Hausdorff locally convex topological $\mathbf{C}$-vector space E (see [**Scha**, Ch. IV, Cor. 3 to 1.5]) that the given topology on E is the same as the topology of uniform convergence on the equicontinuous subsets of E', i.e. the $\mathfrak{S}$-topology on E where $\mathfrak{S}$ is the set of the closed convex circled equicontinuous subsets of E'_σ (all weakly compact by [**Scha**, Ch. III, Cor. to 4.3], hence weakly bounded). We further refer the reader to Lemma C.2.2.2, as well as to Remark C.2.3.8, describing the equicontinuous subsets of the continuous dual of a binary injective tensor product of Hausdorff locally convex topological $\mathbf{C}$-vector spaces (as being contained in the polar of a subset of the form $A \otimes B$, for A, B equicontinuous subsets of the respective continuous duals).

Under the continuous bi-linear mapping:

$$B : \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)'_\beta \times (\mathbf{s}_r)'_\beta \to \mathcal{S}(G)'_\beta \tag{D.1.3.29}$$

from Remark D.1.3.8, we claim that subsets of the form

$$U \otimes V := \{\varphi \otimes \theta : \varphi \in U, \theta \in V\} \subset \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)' \otimes (\mathbf{s}_r)'$$

for equicontinuous $U \subset \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)'_\sigma$ and $V \subset (\mathbf{s}_r)'_\sigma$ are sent onto equicontinuous subsets of $\mathcal{S}(G)'$.

Letting $G_1 = \mathbf{R}^m \times \mathbf{T}^n$ and $G_2 = \mathbf{Z}^r$, we have a natural linear topological isomorphism $\mathcal{S}(G) \simeq \mathcal{S}(G_1, \mathcal{S}(G_2))$ and if $f \in \mathcal{S}(G)$ corresponds to a Schwartz-class function $\widetilde{f} : G_1 \to \mathcal{S}(G_2)$ then for $\phi_i \in \mathcal{S}(G_i)'$ the composite function $\phi_2 \circ \widetilde{f} : G_1 \to \mathbf{C}$ lies in $\mathcal{S}(G_1)$ (this is applying functoriality of $\mathcal{S}(G_1, E)$ in the locally convex space E) and

$$(B(\phi_1, \phi_2))(f) = \phi_1(\phi_2 \circ \widetilde{f}).$$

Since $\mathcal{S}(G)$ is a Fréchet space by Lemma D.1.3.5, it is barrelled (as for any Fréchet space [**Scha**, Ch. III, Cor. to 7.1]) Thus, to check equicontinuity of $B(U \otimes V)$ it suffices (by [**Scha**, Ch. III, Thm.4.2]) to check that such subsets of $\mathcal{S}(G)'$ are bounded for the weak topology, which in turn (by the definition of the weak topology) is equivalent to having a bounded set of values on each $f \in \mathcal{S}(G)$. Given an equicontinuous set U of tempered distributions on $\mathbf{R}^m \times \mathbf{T}^n$ and an equicontinuous subset V of $\mathbf{s}'_r$, we need to show that the set of values $(B(u, v))(f) \in \mathbf{C}$ is bounded as u varies through U and v varies through V.

The Schwartz property of $\widetilde{f} : G_1 = \mathbf{R}^m \times \mathbf{T}^n \to \mathcal{S}(G_2)$ expresses that each finite set of partials of $\widetilde{f}$ along $\mathbf{R}^m$ multiplied by any finite collection of polynomials on $\mathbf{R}^m$ lands inside a bounded subset of $\mathcal{S}(G_2)$, so by equicontinuity of $V \subset \mathcal{S}(G_2)'$ the subset $\{v \circ \widetilde{f}\}_{v \in V} \subset \mathcal{S}(G_1)$ is likewise bounded (here we recall that boundedness in a Hausdorff locally convex space can be detected using boundedness under all finite collections of seminorms defining the topology). But then by equicontinuity of U it follows that $\{u(v \circ \widetilde{f})\}_{u \in U, v \in V} \subset \mathbf{C}$ is bounded, establishing the equicontinuity of $B(U \otimes V)$.

This implies, in light of the description of the ε-topology by means of the seminorms $\varepsilon_{U,V}$ from (C.2.3.7), that the identity map on $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes \mathbf{s}_r$ is continuous with respect to the subspace topology via (D.1.3.28) on the source and the ε-topology on the target, because (i) a net $\{T_i\}_{i \in I}$ in $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes_\varepsilon \mathbf{s}_r$ converges in $\mathcal{S}(G)$ if and only if it converges uniformly on the equicontinuous subsets of $\mathcal{S}(G)'$, and (ii) the subsets $U \otimes V \subset \mathcal{S}(G)'$ are examples of such equicontinuous subsets. But by Remark C.2.3.8, this means that $\{T_i\}_{i \in I}$ converges uniformly on the equicontinuous subsets of $(\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes_\varepsilon \mathbf{s}_r)'$, and hence converges in $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes_\varepsilon \mathbf{s}_r$ by Lemma C.2.2.2. This proves that the ε-topology on $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \otimes \mathbf{s}_r$ is at most as fine as the subspace topology arising from (D.1.3.28), and so it completes the proof that the inclusion (D.1.3.28) yields the π-topology as the subspace topology.

To pass from the single tensor product in (D.1.3.28) to the triple tensor product in (D.1.3.18), now let

$$\psi_2 : \mathcal{S}(\mathbf{R}^m) \times \mathcal{S}(\mathbf{T}^n) \to \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n) \tag{D.1.3.30}$$

be the canonical bilinear map. Since $\mathcal{S}(\mathbf{R}^m)$ and $\mathcal{S}(\mathbf{T}^n)$ are Fréchet spaces by Lemma D.1.3.5, the separately continuous bilinear map ψ_2 is continuous by [**Scha**, Ch. III, Cor. to Thm. 5.1]. In particular, the subspace topology arising from the associated linear injection

$$j' : \mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \to \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)$$

is coarser than the π-topology on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n)$.

We will next show that this subspace topology is also finer than the ε-topology on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n)$. Once that is shown, since nuclearity of $\mathcal{S}(\mathbf{R}^m)$ (by [**Tre**, Cor. to Thm. 51.5, p. 530]) identifies that ε-topology with the π-topology, we would have proved the topology induced by j' on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n)$ is exactly the π-topology.

For $V \subset \mathcal{S}(\mathbf{T}^n)$ and $U \subset \mathcal{S}(\mathbf{R}^m)$ varying over neighborhood bases $\mathcal{V}, \mathcal{U}$ at 0 for the locally convex topologies, the convex circled hull of the sets $U \otimes V := \{u \otimes v : u \in U, v \in V\}$ constitute a subbase of 0-neighborhoods in

$\mathcal{S}(\mathbf{R}^m) \otimes_\pi \mathcal{S}(\mathbf{T}^n)$ by the definition of the π-topology. For equicontinuous subsets $A \subset \mathcal{S}(\mathbf{R}^m)'$ and $B \subset \mathcal{S}(\mathbf{T}^n)'$, the subsets $A \otimes B \subset \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n)'$ are equicontinuous by an argument similar to what we have seen above with $\mathcal{S}((\mathbf{R}^m \times \mathbf{T}^n) \times \mathbf{s}_r)$. Thus, exactly as before, it follows that the subspace topology via j' is finer than the ε-topology, as desired.

Since $\phi = (\psi_2 \times \mathrm{id}_{\mathbf{s}_r}) \circ \psi_1$, it follows that the subspace topology induced on $(\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n)) \otimes \mathbf{s}_r$ by ϕ is the projective tensor product of the π-topology on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n)$ and the locally convex topology on $\mathbf{s}_r$. The formation of projective tensor product topologies is associative by inspection of a base of 0-neighborhoods, so the topology induced via (D.1.3.18) on $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes \mathbf{s}_r$ from the topology on $\mathcal{S}(G)$ is the π-topology.

(2) : By (1) and completeness of $\mathcal{S}(G)$ (from Lemma D.1.3.5(2)), our task amounts to showing the denseness of the image of ϕ from (D.1.3.18), for which it is enough to treat the case $F = 0$.

Let $j \geq 0$ be an integer, and write $K'_j \subset \mathbf{Z}^r$ for the finite subset of those vectors $a \in \mathbf{Z}^r$ such that $|a| = \sum_{i=1}^r |a_i| \leq j$. Let $\rho_j : G \to \mathbf{C}$ be the indicator function of $K_j := \mathbf{R}^m \times \mathbf{T}^n \times K'_j \subset \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r = G$. Let $f \in \mathcal{S}(G)$ be the arbitrary element, and denote by $f_j := f \cdot \rho_j : G \to \mathbf{C}$ the product function, for each integer $j \geq 0$, with ρ_j the indicator function of K_j. Since K_j is a subset of G, f_j is an element of $\mathcal{S}(G)$, since it is smooth over G and each of the seminorms $\|\cdot\|_{\alpha,\beta,\gamma,\delta}$ from (D.1.3.7) is finite on f_j, an upper bound on the value of each, being given by $\|f\|_{\alpha,\beta,\gamma,\delta}$. We'll show the sequence $\{f_j\}_{j\geq 0} \subset \mathcal{S}(G)$ converges to f in $\mathcal{S}(G)$. To see this, by Lemma D.1.3.4 it is enough to show that for every pair of integers $N, N' \geq 0$, every $\alpha \in \mathbf{N}^m$, and every $\beta \in \mathbf{N}^n$ we have

$$\lim_{j \to +\infty} \|f - f_j\|_{\mathcal{P}_1,(N,N',\alpha,\beta)} = 0. \tag{D.1.3.31}$$

Keeping in mind that we passed to the case $F = 0$, for every tuple of integers $N_i, N'_i \geq 0$ $(i = 1, 2)$, every $\alpha \in \mathbf{N}^m, \beta \in \mathbf{N}^n$, by a standard compactness argument (similar to classical ones with Schwartz spaces of Euclidean spaces) there is a large enough positive real number C and large enough integers $N_3 \geq N_2$ and $N'_3 \geq N'_2$ such that

(D.1.3.32)

$$|(1+|x|^{N_1})\,(1+|a|^{N'_1})|\,\frac{|(1+|x|^{N_2})\,(1+|a|^{N'_2})\,\partial_x^\alpha \partial_z^\beta h(x,z,a)|}{|(1+|x|^{N_1})\,(1+|a|^{N'_1})|} \leq \frac{C}{|(1+|x|^{N_1})\,(1+|a|^{N'_1})|}\,\|h\|_{\mathcal{P}_1,(N_3,N'_3,\alpha,\beta)}$$

for every $h \in \mathcal{S}(G)$.

To summarize, we obtain the estimate:

$$\|h\|_{\mathcal{P}_1,(N_2,N'_2,\alpha,\beta)} \leq \frac{C}{|(1+|x|^{N_1})\,(1+|a|^{N'_1})|}\,\|h\|_{\mathcal{P}_1,(N_3,N'_3,\alpha,\beta)} \tag{D.1.3.33}$$

for all $(x, a) \in \mathbf{R}^m \times \mathbf{Z}^r$, $\alpha \in \mathbf{N}^m$, all integers $N_1, N'_1, N_2, N'_2 \geq 0$, and for all integers $N_3, N'_3 \geq 0$ large enough. We apply (D.1.3.33) to $h = f - f_j$ to obtain

(D.1.3.34)

$$|(1+|x|^{N_2})\,(1+|a|^{N'_2})\,\partial_x^\alpha \partial_z^\beta (f - f_j)(x,z,a)| \leq \frac{C}{|(1+|x|^{N_1})\,(1+|a|^{N'_1})|}\,(\|f\|_{\mathcal{P}_1,(N_3,N'_3,\alpha,\beta)} + \|f_j\|_{\mathcal{P}_1,(N_3,N'_3,\alpha,\beta)})$$

and upon using $\|f_j\|_{\mathcal{P}_1,(N,N',\alpha,\beta)} \leq \|f\|_{\mathcal{P}_1,(N,N',\alpha,\beta)}$ for all $N, N' \geq 0$ and $\alpha \in \mathbf{N}^m$, $\beta \in \mathbf{N}^n$, we get that for all integers $j \geq 0$:

$$|(1+|x|^{N})\,(1+|a|^{N'})\,\partial_x^\alpha \partial_z^\beta (f - f_j)(x,z,a)| \leq \frac{2C}{1+|a|^{N'_1}}\,\|f\|_{\mathcal{P}_1,(N_3,N'_3,\alpha,\beta)} \tag{D.1.3.35}$$

The left side of (D.1.3.35) is supported away from $K_j \subset G$, and so it is globally bounded above by the supremum over the complement of K_j of the right side of (D.1.3.35), which, upon enlarging $j \geq 0$ (to increase $|a|$ for $a \notin K_j$), we may render as small as we please, uniformly over $\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r$. This proves that (D.1.3.31).

To prove the denseness of the image of ϕ from (D.1.3.18) within $\mathcal{S}(G)$, it is therefore enough to prove that ϕ maps $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes C(K_j')$ onto a dense subset of $\rho_j \cdot \mathcal{S}(G) \subset \mathcal{S}(G)$, since the union of the images of $\mathcal{S}(\mathbf{R}^m) \otimes \mathcal{S}(\mathbf{T}^n) \otimes C(K_j')$ is contained in the image of ϕ, and the union of the $\rho_j \cdot \mathcal{S}(G)$ is dense in $\mathcal{S}(G)$. Since K_j is a finite set, which we enumerate as $K_j = \{x_{j1}, \ldots, x_{jN_j}\}$, for any $f \in \rho_j \cdot \mathcal{S}(G)$ we may write $f = \sum_{i=1}^{N_j} f_i$ with $f_i = \rho_{ij} \cdot f$ where ρ_{ij} is the indicator function of the fibre in $\mathbf{R}^m \times \mathbf{T}^n \times K_j$ over x_{ji}. It is therefore enouh to prove that each f_i can be approximated within $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n \times \{x_{ji}\})$ by elements of the image of ϕ.

We are therefore reduced to the case $G = \mathbf{R}^m \times \mathbf{T}^n$. The same method as above whereby we reduced ourselves to removing a $\mathbf{Z}^r$-factor could be applied if $\mathbf{T}^n$ were replaced with $\mathbf{Z}^n$. We shall access this idea by harness the Fourier transform. More specifically, by [Tre, Thm. 51.3], the Fourier transform for $\mathbf{T}^n$ induces a continuous bijection $\mathcal{F} : \mathcal{S}(\mathbf{T}^n) \to \mathcal{S}(\mathbf{Z}^n)$, which in turn is a topological isomorphism by the Open Mapping Theorem for Fréchet spaces. Thus, we get a linear topological isomorphism

$$\mathcal{S}(G) \simeq \mathcal{S}(\mathbf{R}^m, \mathcal{S}(\mathbf{T}^n)) \overset{\mathcal{F}}{\simeq} \mathcal{S}(\mathbf{R}^m, \mathcal{S}(\mathbf{Z}^n)) \simeq \mathcal{S}(\mathbf{R}^m \times \mathbf{Z}^n)$$

under which ϕ from (D.1.3.18) for $\mathbf{R}^m \times \mathbf{T}^n$ is carried over[2] to the analogue for $\mathbf{R}^m \times \mathbf{Z}^n$. Thus, the denseness assertion is converted into the case of the group $G = \mathbf{R}^m \times \mathbf{Z}^n$, and so then reduces to $G = \mathbf{R}^m$ that is obvious.

(3) : By Lemma D.1.3.5, $\mathcal{S}(G)$ is Hausdorff and complete. By (2), $\mathcal{S}(G)$ is isomorphic to the completed projective tensor product of $\mathcal{S}(\mathbf{R}^m)$, $\mathcal{S}(\mathbf{T}^n)$, $\mathbf{s}_r = \mathcal{S}(\mathbf{Z}^r)$ and $\mathbf{C}^F$. Each is nuclear (see [Tre, Cor. to Thm. 51.5] for $\mathcal{S}(\mathbf{R}^m)$ and [Tre, Thms. 51.3, 51.5] for $\mathcal{S}(\mathbf{T}^n)$ and $\mathbf{s}_r$), so $\mathcal{S}(G)$ is nuclear by [Scha, Ch. III, 7,5]. Since $\mathcal{S}(G)$ is a Fréchet space by Lemma D.1.3.5, its strong dual $\mathcal{S}(G)'_\beta$ is complete (as for any metrizable locally convex space) by [Scha, Ch. II, 8.1; Ch. IV, 6.1]. Likewise nuclearity of the strong dual is inherited from that of the Fréchet $\mathcal{S}(G)$ by [Tre, Prop. 50.6]. The Montel property for $\mathcal{S}(G)$ (in particular, reflexivity) holds as for all nuclear Fréchet spaces [Tre, Cor. 3 to Prop. 50.2], and the strong dual thereby inherits the Montel property (in particular, reflexivity) by [Scha, Ch. IV, 5.9]. Reflexive spaces are barrelled by [Scha, Ch. IV, p. 144], so $\mathcal{S}(G)$ and $\mathcal{S}(G)'_\beta$ are barrelled. □

D.1.4. Structure theory of Hausdorff locally compact abelian groups. In this section, we recall the structure theory of Hausdorff locally compact abelian groups, in preparation for the definition of their Schwartz-Bruhat space of functions.

Definition D.1.4.1. A Hausdorff locally compact abelian group G is called *compactly generated* if and only if there exists a compact neighborhood $U \subset G$ of the identity that generates G as a group.

Theorem D.1.4.2. *Let G be a Hausdorff locally compact abelian group. Assume G is locally Euclidean (as a topological space). There exist integers $m, n \geq 0$ and a discrete abelian group D so that*

$$G \xrightarrow{\simeq} \mathbf{R}^m \times \mathbf{T}^n \times D. \tag{D.1.4.1}$$

This is [DE, Thm. 4.2.4]. We record here the following structural result on Hausdorff locally compact abelian groups, combined with Pontryagin duality:

Theorem D.1.4.3. *Let G be a Hausdorff locally compact abelian group.*

(1) *The group G is compactly generated if and only if $\widehat{G}$ is locally Euclidean, and if G is compactly generated then so is every closed subgroup H and the quotient G/H.*
(2) *The group G is compactly generated and locally Euclidean if and only if there exist integers $m, n, k \geq 0$, a finite abelian group F, and an isomorphism of Hausdorff locally compact abelian groups:*

$$G \xrightarrow{\simeq} \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^k \times F. \tag{D.1.4.2}$$

[2] briefly address calculation for this compatibility of ϕ

The Theorem is [**DE**, Cors. 4.2.5, 4.2.6], apart from the assertions about H and G/H in (1). By the characterization of compact generatedness via the dual being locally Euclidean in (1), the assertions for H and G/H in (1) reduce via Theorem D.1.4.2 to the standard fact that for any G as in (D.1.4.1) all closed subgroups H and quotients G/H are also of the same type (for possibility different m, n, D).

D.1.4.1. *Exhaustion by compactly generated open subgroups, approximation by abelian Lie groups.* Every Hausdorff locally compact abelian group G can be written as a filtered colimit

$$G = \operatorname*{colim}_{H \in \mathrm{CG}(G)} H \tag{D.1.4.3}$$

where $\mathrm{CG}(G)$ is the set of all *open* compactly generated abelian subgroups of G, filtered under open inclusions. As a consequence of Theorem D.1.4.2 and Theorem D.1.4.3, combined with Pontryagin duality and (D.1.4.3) applied to $\widehat{G}$, we get:

LEMMA D.1.4.4. *Let G be a Hausdorff locally compact abelian group, and $\mathrm{CG}(\widehat{G})$ the set of all* open *compactly generated subgroups of $\widehat{G}$, cofiltered under open inclusions. Then the canonical continuous group homomorphism*

$$G \to \lim_{H \in \mathrm{CG}(\widehat{G})} \widehat{H} \tag{D.1.4.4}$$

is an isomorphism of topological groups.

In particular, every Hausdorff locally compact abelian group is canonically isomorphic to a cofiltered inverse limit of locally Euclidean groups. To define the Schwartz space $\mathcal{S}(G)$ of a general Hausdorff locally compact abelian group, we will therefore proceed as in [**Bru**]: first address the case of locally Euclidean groups, and then propagate the definition to general G as a locally convex inductive limit topological $\mathbf{C}$-vector space using (D.1.4.4).

D.1.4.2. *Basic properties of duals and orthogonals.* Recall that Pontryagin duality provides a perfect continuous pairing

$$\langle \cdot, \cdot \rangle_{G,\widehat{G}} : G \times \widehat{G} \to U(1) \subset \mathbf{C}^\times \tag{D.1.4.5}$$

enabling us to construct the following subgroups of G and $\widehat{G}$:

DEFINITION D.1.4.5. Let G be a Hausdorff locally compact abelian group, and $H \subset G$ a subgroup. We define

$$H^\perp := \{\chi \in \widehat{G} : \chi(H) = 1\} \tag{D.1.4.6}$$

and refer to it as the *annihilator* of H in $\widehat{G}$. By Pontryagin duality, for $\Gamma \subset \widehat{G}$ a subgroup we have

$$\Gamma^\perp = \{g \in G : \chi(g) = 1 \text{ for all } \chi \in \Gamma\}. \tag{D.1.4.7}$$

LEMMA D.1.4.6. *Let G be a Hausdorff locally compact abelian group, $H \subset G$ a closed subgroup. Then:*

(1) *$H^{\perp\perp} = H$ as closed subgroups of G.*
(2) *$H^\perp \to \widehat{G/H}$ and $\widehat{G}/H^\perp \to \widehat{H}$ are topological isomorphisms.*
(3) *If H is compact then $H^\perp$ is open in $\widehat{G}$.*
(4) *If H is open in G then $H^\perp$ is compact.*

PROOF. (1) and (2) are [**DE**, Prop.3.6.1]. By compact-discrete duality [**DE**, Prop. 3.1.5], (3) and (4) then follow from (2), noting that every open subgroup of G is closed. □

D.1.4.3. *General structural results.* From Lemma D.1.4.4, if we denote by i a general element of the set $\mathrm{CG}(\widehat{G})$ of open compactly generated abelian subgroups of $\widehat{G}$ and write U_i for Pontryagin dual of the compactly generated open subgroup of G that i labels then

$$\gamma : G \xrightarrow{\simeq} \lim_{i \in \mathrm{CG}(\widehat{G})} U_i. \tag{D.1.4.8}$$

Using Lemma D.1.4.6, the natural continuous map

$$\gamma_i : G \to U_i \tag{D.1.4.9}$$

is a surjective quotient homomorphism with compact kernel

$$K_i := \ker(\gamma_i) \subset G \tag{D.1.4.10}$$

that is the annihilator of the open compactly generated subgroup $\widehat{U_i}$ of $\widehat{G}$. Since (D.1.4.8) is a topological isomorphism, the U_i's become arbitrarily small around 1 in G.

We denote by j the general element of the set $\mathrm{CG}(G)$ of open compactly generated subgroups of G, and write G_j for the open compactly generated subgroup of G that j labels. For any $i \in \mathrm{CG}(\widehat{G})$, since

$$\operatorname*{colim}_{j \in \mathrm{CG}(G)} G_j = G$$

there exists a large enough (with respect to the partial ordering of $\mathrm{CG}(G)$ by upwards inclusion) $j_0 \in \mathrm{CG}(G)$ such that the inclusion homomorphism $K_i \to G$ factors through G_j for all $j \geq j_0$. In other words, K_i is a subgroup of G_j for all $j \geq j_0$ in $\mathrm{CG}(G)$.

The compact K_i is a closed subgroup of the open compactly generated subgroup G_j of G for every $j \geq j_0$. In particular, the quotient group

$$A_{ij} := G_j / K_i \tag{D.1.4.11}$$

is Hausdorff, locally compact, compactly generated and the quotient map $G_j \to A_{ij}$ is clearly open. Since K_i is the kernel of γ_i from (D.1.4.9), A_{ij} is a subgroup of U_i for every $j \geq j_0$ and the natural continuous homomorphism

$$G_j \to \lim_i A_{ij}$$

(carving out open subgroups on both sides of (D.1.4.8)) is an isomorphism. The group A_{ij} is locally Euclidean since it is open in U_i that is locally Euclidean (by Theorem D.1.4.3(1)).

By Lemma D.1.4.3(2), there exist integers $m, n, k \geq 0$ and a finite group F, together with an isomorphism of Hausdorff locally compact abelian groups

$$A_{ij} \xrightarrow{\simeq} \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^k \times F \tag{D.1.4.12}$$

We associate to each $i \in \mathrm{CG}(\widehat{G})$ the set $J_i := \{j \in \mathrm{CG}(G) : K_i \subset G_j\}$. Then J_i inherits a partial order by upwards inclusion of open compactly generated subgroups of G. The collection of open subgroups $A_{ij} \subset U_i$ for all $j \in J_i$ exhausts U_i by open compactly generated subgroups, and each A_{ij} is an abelian Lie group with finite component group.

We therefore have, as Hausdorff locally compact abelian groups

$$\gamma : G \xrightarrow{\simeq} \lim_{i \in \mathrm{CG}(\widehat{G})} \operatorname*{colim}_{j \in J_i} A_{ij} \tag{D.1.4.13}$$

(with open inclusions as transition maps in the colimit) because $U_i = \operatorname*{colim}_{j \in J_i} A_{ij}$. The observation to make is as follows, as in [**Wa**, (1.3)]. For each $j \in \mathrm{CG}(G)$, we may form the set I_j defined as

$$I_j := \{i \in \mathrm{CG}(\widehat{G}) : K_i \subset G_j\} \subset \mathrm{CG}(\widehat{G}) \tag{D.1.4.14}$$

(recall we noted that the K_i's become arbitrarily small in G) which inherits a unique cofiltered set structure from the partial ordering by upwards inclusion on open subgroups of $\widehat{G}$, turned downwards by the Pontryagin duality functor. We thereby obtain:

LEMMA D.1.4.7 (Structure of Hausdorff locally compact abelian groups). *Let G be a Hausdorff locally compact abelian group, $j \in \mathrm{CG}(G)$, and G_j the corresponding open compactly generated subgroup. Then, as Hausdorff locally compact abelian groups, we have a canonical isomorphism:*

$$\delta_j : G_j \xrightarrow{\simeq} \lim_{i \in I_j} A_{ij}. \tag{D.1.4.15}$$

In particular, each G_j is, canonically, a cofiltered limit of abelian Lie groups with finite component group, and the δ_j's induce a unique isomorphism of Hausdorff locally compact abelian groups

$$\delta : G \xrightarrow{\simeq} \operatorname*{colim}_{j \in \mathrm{CG}(G)} \lim_{i \in I_j} A_{ij} \tag{D.1.4.16}$$

(*with open inclusions as transition maps in the colimit*).

The most important takeaways from Lemma D.1.4.7 and the discussion preceding it are that for any Hausdorff locally compact abelian group G, there exist a canonical collection of compact subgroups $k \subset G$ (forming an inverse system under reverse inclusion, and becoming arbitrarily small in G) such that

- $k^\perp$ is open and compactly generated in $\widehat{G}$ (in the discussion above, $K_i^\perp = \widehat{G/K_i} = \widehat{U_i}$ is compactly generated),
- each k is contained in some open compactly generated subgroup Γ of G with such Γ's directed and exhausting G (e.g., the Γ's can be taken to be the G_j's in the preceding discussion),
- the quotient Γ/k is an elementary abelian Lie group.

Moreover, each open compactly generated subgroup Γ of G is isomorphic to the inverse limit of quotients Γ/k', where k' varies through all sufficiently small k as above.

D.1.5. Schwartz-Bruhat space of a Hausdorff locally compact abelian group. The scope of this subsection is to define the Schwartz-Bruhat space of an arbitrary Hausdorff locally compact abelian group G, and fill in the arguments for [**Bru**, §9, Prop. 11] in the special case when G is second countable. We will only briefly indicate how the results extend beyond the second-countable case.

D.1.5.1. *Good subgroups and structural pairs.* Here is terminology and notation following [**Bru**, §9]:

DEFINITION D.1.5.1 (Structural pairs of G). Let G be a Hausdorff locally compact abelian group.

(1) A closed subgroup $k \subset G$ is called *good* if and only if its annihilator $k^\perp \subset \widehat{G}$ is both open (equivalent to compactness of k) and compactly generated.
(2) A pair (Γ, k) of subgroups of G is called *structural* if and only if $\Gamma \subset G$ is an open compactly generated subgroup of G, $k \subset G$ is a good subgroup of G, and $k \subset \Gamma$.

We define $\mathcal{F}$ the set of structural pairs (Γ, k) of G.

Note that if $k_1, k_2 \subset G$ are good subgroups then so is $k_1 \cap k_2$. To see this, it suffices to show that the evident inclusion $k_1^\perp k_2^\perp \subset (k_1 \cap k_2)^\perp$ is an equality. This amounts to surjectivity of $k_1^\perp \times k_2^\perp \to (k_1 \cap k_2)^\perp$, so it suffices to check that the injective dual map $G/(k_1 \cap k_2) \to (G/k_1) \times (G/k_2)$ has closed image. But this image is that of the closed diagonal in $G \times G$ under the quotient map $G \times G \to (G/k_1) \times (G/k_2)$ which is proper since the k_i's are compact.

LEMMA D.1.5.2 (Properties of structural pairs). *Let G be a Hausdorff locally compact abelian group. Then the set $\mathcal{F}$ of structural pairs of G is filtered under the partial ordering:*

$$(k_1, \Gamma_1) \preceq (k_2, \Gamma_2) \tag{D.1.5.1}$$

when $\Gamma_1 \subset \Gamma_2$ and $k_1 \supset k_2$. Moreover, upon denoting by μ_G a fixed choice of Haar measure on G and by $\mu_{\widehat{G}}$ the induced Plancherel measure on $\widehat{G}$, we have:

(1) *For any structural pair (k,Γ) of G, Γ/k is an elementary abelian Lie group.*
(2) *The pair $(\Gamma^\perp, k^\perp)$ is structural for $\widehat{G}$.*
(3) *Let $f : G \to \mathbf{C}$ be an μ_G-integrable function with $\mu_{\widehat{G}}$-integrable Fourier transform, and $H \subset G$ a closed subgroup. Then f is H-invariant if and only if the continuous Fourier transform $\widehat{f} : \widehat{G} \to \mathbf{C}$ of f is supported on $H^\perp$. In particular, for (k,Γ) a structural pair of G and $f : G \to \mathbf{C}$ a μ_G-integrable function with $\mu_{\widehat{G}}$-integrable Fourier transform, then f is supported on Γ and k-invariant if and only if $\widehat{f}$ is $\Gamma^\perp$-invariant and supported on $k^\perp$.*

PROOF. To see that the partial ordering defined by (D.1.5.1) is filtered, it is enough to check that for any two structural pairs (k_1,Γ_1) and (k_2,Γ_2), there exists a third (k,Γ) satisfying $(k_i,\Gamma_i) \preceq (k,\Gamma)$ for $i = 1,2$. Recalling that $k_1 \cap k_2$ is good, we have the relations among structural pairs

$$(k_i,\Gamma_i) \preceq (k_i,\Gamma_1\Gamma_2) \preceq (k_1 \cap k_2, \Gamma_1\Gamma_2).$$

(1) : This is a consequence of Theorem D.1.4.3(2) provided we can show the locally compact Hausdorff group Γ/k is both compactly generated and locally Euclidean. It is compactly generated because it is a quotient of Γ that is compactly generated. For the locally Euclidean property, it is the same as $\widehat{\Gamma/k}$ being compactly generated (by Theorem D.1.4.3(1)). Note that $\widehat{\Gamma/k} = k^\perp \cap \widehat{\Gamma}$ inside $\widehat{G}$, so it is a closed subgroup of the group $k^\perp$ that is compactly generated by the meaning of (k,Γ) being good. Thus, we are done since compact generation is inherited by closed subgroups (Theorem D.1.4.3(1)).

(2) : By definition, $k^\perp$ is open and compactly generated as a subgroup of $\widehat{G}$. It contains $\Gamma^\perp$, and $(\Gamma^\perp)^\perp = \Gamma \subset G$ is open and compactly generated.

(3) : Since f that is μ_G-integrable with Fourier transform $\widehat{f}$ that is $\mu_{\widehat{G}}$-integrable, by Lemma D.1.1.2(5) it follows that f and $\widehat{f}$ are each continuous and Fourier Inversion (D.1.1.11) holds everywhere for both.

Assume f is H-invariant. To show $\widehat{f} : \widehat{G} \to \mathbf{C}$ is supported in $H^\perp$, we pick $\chi \in \widehat{G}$ *not* lying inside $H^\perp$ and aim to show $\widehat{f}(\chi) = 0$. There certainly exists some $h_0 \in H$ such that $\chi(h_0) \neq 1$ (as $\chi \notin H^\perp$), so

$$\widehat{f}(\chi) = \int_G f(g)\overline{\chi}(g)\mathrm{d}g = \int_G f(gh_0)\overline{\chi}(gh_0)\mathrm{d}g = \overline{\chi}(h_0)\widehat{f}(\chi)$$

(the final equality uses the assumed H-invariance of f). This forces $\widehat{f}(\chi) = 0$ as desired. For the converse, if $\widehat{f}$ is supported on $H^\perp$ then to see f is H-invariant we compute for $h \in H$ and $g \in G$ that $f(hg)$ is equal to

$$\mathcal{F}_{\widehat{G}}(\widehat{f})(h^{-1}g^{-1}) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(hg)\,\mathrm{d}\mu_{\widehat{G}}(\chi) = \int_{H^\perp} \widehat{f}(\chi)\chi(hg)\,\mathrm{d}\mu_{\widehat{G}}(\chi) = \int_{H^\perp} \widehat{f}(\chi)\chi(g)\,\mathrm{d}\mu_{\widehat{G}}(\chi),$$

which is equal to $f(g)$ by Fourier inversion (since $\widehat{f}$ is supported on $H^\perp$). This is the H-invariance of f.

Applying the preceding with $H = \Gamma \subset G$ and then with $H = k^\perp \subset \widehat{G}$ (whose orthogonal in G is k), it follows via Fourier Inversion in both directions that f is supported in Γ if and only if $\widehat{f}$ is $\Gamma^\perp$-invariant and f is k-invariant if and only if $\widehat{f}$ is supported in $k^\perp$. □

D.1.5.2. *The Schwartz-Bruhat space $\mathcal{S}(G)$.* For G a Hausdorff locally compact abelian group, and (k,Γ) a structural pair of G, we write:

$$r_{(k,\Gamma)} : \Gamma \to \Gamma/k \tag{D.1.5.2}$$

for the continuous quotient homomorphism. Recall that k is one of the compact groups K_i of (D.1.4.10) and Lemma D.1.4.7, indexed by the set $\mathrm{CG}(\widehat{G})$ of open compactly generated subgroups of $\widehat{G}$ (by associating to k its annihilator), and $r_{(k,\Gamma)}$ is the continuous quotient homomorphism induced by γ_i from (D.1.4.9).

DEFINITION D.1.5.3 (Schwartz-Bruhat functions on Hausdorff locally compact abelian groups). Let G be a Hausdorff locally compact abelian group, $f : G \to \mathbf{C}$ a function.

(1) Let $p \geq 0$ be an integer. We say f is *p-times differentiable* if and only if there exists a good subgroup $k \subset G$ such that f is k-invariant, and for every structural pair (k, Γ) of G the restriction $f|_\Gamma$ is such that the unique function $f_0 : \Gamma/k \to \mathbf{C}$ satisfying $f|_\Gamma = f_0 \circ r_{(k,\Gamma)}$ is C^p on the Lie group Γ/k (for its unique smooth structure respecting the group structure). We denote by $C^p(G)$ the $\mathbf{C}$-vector space of p-times differentiable functions on G. We say f is *smooth* if and only if it is p-times differentiable on G for every integer $p \geq 0$. We denote by $C^\infty(G)$ the $\mathbf{C}$-vector space of smooth functions on G.

(2) We say f is a *Schwartz-Bruhat function* on G if and only if:
   (1) There exists a structural pair (k, Γ) of G such that f is supported in Γ and k-invariant.
   (2) The unique $f_0 \in C^\infty(\Gamma/k)$ such that $f = r_{(k,\Gamma)} \circ f_0$ satisfies the following condition: for a choice of isomorphism

$$\phi_0 : \Gamma/k \to \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F \tag{D.1.5.3}$$

   for unique integers $m, n, r \geq 0$ and a finite abelian group F, the smooth function $f_0 \circ \phi_0^{-1}$ is of rapid decay at infinity on Γ/k in the sense of Definition D.1.3.1(2).

   We denote by $\mathcal{S}(k, \Gamma)$ the $\mathbf{C}$-vector space of smooth functions on G supported in Γ and k-invariant, such that the unique $\mathbf{C}$-valued smooth function induced on the quotient Γ/k has rapid decay at infinity on Γ/k. We topologize $\mathcal{S}(k, \Gamma)$ with the Hausdorff locally convex topology of $\mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F)$ by transport of structure along the isomorphism ϕ_0 of Hausdorff locally compact abelian groups provided by (D.1.5.3); this is well-defined by Lemma D.1.3.3.

We claim that the collection of function spaces $\mathcal{S}(k, \Gamma)$ on G forms a directed system with respect to the partial ordering $(k_1, \Gamma_1) \preceq (k_2, \Gamma_2)$. For this we just have to check:

LEMMA D.1.5.4. *Pullback along the proper map $\Gamma_1/k_2 \twoheadrightarrow \Gamma_1/k_1$ carries $\mathcal{S}(\Gamma_1/k_1)$ to functions on the open subgroup $\Gamma_1/k_2 \subset \Gamma_2/k_2$ whose extension by 0 to Γ_2/k_2 lies in $\mathcal{S}(\Gamma_2/k_2)$ (where each Γ_i/k_i is an elementary abelian Lie group, so the open subgroup $\Gamma_1/k_2 \subset \Gamma_2/k_2$ is also an elementary abelian Lie group). The resulting inclusion $\mathcal{S}(\Gamma_1/k_1) \to \mathcal{S}(\Gamma_2/k_2)$ is a topological embedding (hence closed, by completeness of $\mathcal{S}(\Gamma_1/k_1)$).*

PROOF. We just have to check that for an elementary abelian Lie group A, compact subgroup $k \subset A$, and open subgroup $A' \subset A$:

(i) pullback along $A \twoheadrightarrow A/k$ carries $\mathcal{S}(A/k)$ into $\mathcal{S}(A)$ via a topological embedding onto the subspace of k-invariant elements.

(ii) Extension by zero carries $\mathcal{S}(A')$ into $\mathcal{S}(A)$ via a topological embedding.

First we settle (i). By consideration of the maximal compact subgroup of A and the structure of connected compact abelian Lie groups, we can *choose* an isomorphism

$$A \simeq \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$$

so that k is the product of some of the $\mathbf{T}$-factors and a subgroup $F' \subset F$ *inside* A. Thus, we get compatible descriptions of $\mathcal{S}(A)$ and $\mathcal{S}(k)$, so also $\mathcal{S}(A/k)$, as completed tensor products in (D.1.3.20). This reduces us to the separate cases $A = \mathbf{R}^m, \mathbf{T}^n, \mathbf{Z}^r, F$. The cases of $\mathbf{R}^m$ and $\mathbf{Z}^r$ are obvious ($k = 0$), the case of F is clear. For $A = \mathbf{T}^n$ we know k^0 is a torus that arises as a direct factor of A, so it suffices to separately treat the case $k = k^0$ and k finite. Since $\mathcal{S}(\mathbf{T}^n) = C^\infty(\mathbf{T}^n)$ with the topology of uniform convergence in all derivatives, the case of finite k is clear while for the case of k a torus we have $\mathcal{S}(A) = \mathcal{S}(A')\widehat{\otimes}_\pi \mathcal{S}(k)$, from which the desired properties are immediate.

Turning to (ii), if we fix an isomorphism $A = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ (so F is the torsion subgroup of the component group of A) then the clopen A' contains the identity component $\mathbf{R}^m \times \mathbf{T}^n$ and so is determined by

its component group $\pi_0(A')$ as a subgroup of $\pi_0(A)$. Letting $F' = \pi_0(A')_{\mathrm{tor}}$, we have $A' \subset \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F'$ inside A. The special case $A' = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F'$ is immediate from (D.1.3.20) applied to A and A', so we may now assume $F' = F$. More precisely, now $F \subset A'$ inside A. Thus, A' is determined by the maximal torsion-free quotient M of $\pi_0(A')$ as a subgroup of the corresponding quotient $\mathbf{Z}^r$ for A.

The "elementary divisor" description of $M \subset \mathbf{Z}^r$ allows us to harness the infinite divisibility of $A^0 = \mathbf{R}^m \times \mathbf{T}^n$ to possibly *change* the identification of A with $\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ (adapted to a preliminary choice of analogous decomposition for A') so that we have $A' = \mathbf{R}^m \times \mathbf{T}^n \times M \times F$ inside A with $M \subset \mathbf{Z}^r$ equal to $\oplus_{i=1}^r (n_i \mathbf{Z})$ for some integers $n_i \geq 0$. The family of seminorms (D.1.3.9) applied to A and A' then yields the result. □

We conclude that the directed system of $\mathcal{S}(k,\Gamma)$'s is *strict* in the sense that its transition maps are topological embeddings. Thus, the collection of all Schwartz-Bruhat functions on G is a $\mathbf{C}$-vector space that we call the *Schwartz-Bruhat space* of G and is made into a locally convex topological vector space via the locally convex inductive limit topology induced by the canonical linear isomorphism

$$\mathcal{S}(G) = \operatorname*{colim}_{(k,\Gamma)\in\mathcal{F}} \mathcal{S}(k,\Gamma) \tag{D.1.5.4}$$

whose injective transition maps are topological embeddings.

From the construction it is clear that left translation of G on itself carries $\mathcal{S}(G)$ onto itself, but it is less evident (in such generality) that the action map $G \times \mathcal{S}(G) \to \mathcal{S}(G)$ is continuous. Such continuity is proved in Proposition D.1.9.2.

Remark D.1.5.5. For any Hausdorff locally compact abelian group G, define $C^\infty(G)$ to be the colimit of the spaces $C^\infty(\Gamma/k)$ of smooth functions (similarly to the colimit that defines $\mathcal{S}(G)$). Clearly $\mathcal{S}(G) \subset C^\infty(G)$.

Also, every $\chi \in \widehat{G}$ has smooth restriction to Γ as a function $\chi : G \to \mathbf{C}^\times \subset \mathbf{C}$. Indeed, we can pick an open compactly generated subgroup $U \subset \widehat{G}$ containing χ, so its annihilator k in G is compact by Lemma D.1.4.6(4), hence a good subgroup of G, and by design $\chi(k) = 1$. We may and do replace k with $k \cap \Gamma$ so that $k \subset \Gamma$. Then $\chi|_\Gamma$ descends to a continuous character of the elementary abelian Lie group Γ/k. The latter character is smooth, so $\chi|_\Gamma \in C^\infty(\Gamma)$.

D.1.5.3. *Main Structure Theorems on Schwartz-Bruhat spaces.*

Lemma D.1.5.6 (Strictness of the inductive system $\{\mathcal{S}(k,\Gamma)\}_{(k,\Gamma)}$). *Let G be a second-countable Hausdorff locally compact abelian group, and $B \subset \mathcal{S}(G)$ a bounded subset.*

(1) *The topology on $\mathcal{S}(G)$ is Hausdorff and for any structural pair (k,Γ) of G the locally convex topology induced on $\mathcal{S}(k,\Gamma)$ by $\mathcal{S}(G)$ is the intrinsic locally convex topology of $\mathcal{S}(k,\Gamma) = \mathcal{S}(\Gamma/k)$.*
(2) *There exists a structural pair (k,Γ) of G such that B is the image of a bounded subset of $\mathcal{S}(k,\Gamma)$ into $\mathcal{S}(G)$.*

This result holds without second-countability for G, but the proof is then more technical (e.g., (2) is proved by building a projector via integration over compact subgroups $k \subset G$) and we do not need it.

Proof. Since G is second-countable, there is a *countable* cofinal system of structural pairs (k_n,Γ_n) and so $\mathcal{S}(G)$ is a "strict inductive limit" of a *sequence* of Hausdorff locally convex spaces $\mathcal{S}(k_n,\Gamma_n)$. But such inductive limits $E = \mathrm{colim} E_n$ always satisfy the two asserted properties: E is Hausdorff with $E_n \to E$ a topological embedding [**Scha**, Ch. II, 6.4], and every bounded subset of E arises from a bounded subset of some E_n [**Scha**, Ch. II, 6.5]. □

The following is the main result of this subsection, and a special case of [**Bru**, §9, Prop. 11]:

Theorem D.1.5.7 (Main Structure Theorem on Schwartz-Bruhat spaces). *Let G be a second-countable Hausdorff locally compact abelian group. The Hausdorff locally convex $\mathcal{S}(G)$ is barrelled and bornological,*

Moreover, both $\mathcal{S}(G)$ and its strong dual $\mathcal{S}(G)'_\beta$ are complete, nuclear, and Montel (hence reflexive). In particular, $\mathcal{S}(G)'_\beta$ is barrelled.

PROOF. Since each $\mathcal{S}(k,\Gamma) = \mathcal{S}(\Gamma/k)$ is a Fréchet space by Lemma D.1.3.7, hence each of them is both barrelled and bornological [**Scha**, Ch. II, Cor. to 7.2, Cor. 1 to 8.2], the countable strict inductive limit $\mathcal{S}(G)$ is also barrelled and bornological by [**Scha**, Ch. II, Cor. to 7.1, 8.1].

Each $\mathcal{S}(k,\Gamma) = \mathcal{S}(\Gamma/k)$ is complete, nuclear, and Montel by Lemma D.1.3.7(3). These properties then hold for $\mathcal{S}(G)$ since each is inherited under countable strict inductive limits $E = \mathrm{colim} E_n$ of Hausdorff locally convex topological vector spaces: [**Scha**, Ch. II, 6.6] for completeness, [**Scha**, Ch. III, Cor. to 7.4] for nuclearity, and [**Scha**, Ch. IV, 5.8] for the Montel property (to be precise, the latter reference is for inheriting reflexivity, but for the Heine-Borel property of $E = \mathrm{colim} E_n$ we simply observe that a closed bounded subset $K \subset E$ arises from a bounded subset of some E_n that in turn must be closed in E_n since $E_n \to E$ is a topological embedding, so the Heine-Borel property of E_n makes K compact in E_n, thus also in E).

Turning to the strong dual $\mathcal{S}(G)'_\beta$, it inherits the Montel property from that of $\mathcal{S}(G)$ because the strong dual of a Montel space is always Montel [**Scha**, Ch. IV, 6.1]. For completeness, the strong dual E'_β of any bornological (Hausdorff locally convex) E is complete by [**Scha**, Ch. IV, 5.9]. For nuclearity for $\mathcal{S}(G)'_\beta$, we first note that since $\mathcal{S}(G)$ is nuclear and complete, the strong dual topology on $\mathcal{S}(G)'$ coincides with the Mackey topology $\tau(\mathcal{S}(G)', \mathcal{S}(G))$ (as for any complete nuclear E [**Tre**, top of p. 522]). This is useful because the Mackey topology on the topological dual of a countable strict inductive limit $E = \mathrm{colim} E_n$ coincides with the inverse limit $\lim E'_{n,\tau}$ of the topological duals equipped with their Mackey topologies $\tau'_n = \tau(E'_n, E_n)$ [**Scha**, Ch. IV, Rem. to 4.5]. But with $E_n = \mathcal{S}(\Gamma_n/k_n)$ we likewise have $E'_{n,\tau'_n} = E'_{n,\beta}$, so the known nuclearity of $\mathcal{S}(A)'_\beta$ for elementary abelian Lie groups A gives nuclearity for $\mathcal{S}(G)'_\beta$ since nuclearity is inherited under inverse limits of Hausdorff locally convex topological vector spaces by [**Scha**, Ch. III, Cor. to 7.4]. Barrelledness for $\mathcal{S}(G)'_\beta$ holds as a general property of reflexive spaces (see [**Scha**, Ch. IV p. 144]). □

REMARK D.1.5.8. We note that if G is not second-countable then (see [**Bru**, §9, Prop. 11] $\mathcal{S}(G)$ is still bornological, complete, and Montel but it is *never* nuclear: see the discussion in [**Bru**, top of p. 54], where the absence of second-countability causes the existence of an uncountable topological direct sum of copies of $\mathbf{C}$ to split off from $\mathcal{S}(G)$ linearly and topologically. That obstructs nuclearity because topological direct summands inherit nuclearity by [**Scha**, Ch.III, 7.2] but an uncountable direct sum of copies of $\mathbf{C}$ (with its locally convex inductive limit topology) is never nuclear, by [**Tre**, Thm. 51.2]. The proof of completeness of $\mathcal{S}(G)$ beyond the second-countable case requires some additional work when adapting the proof of completeness of $C_c^\infty(G)$ in [**Bru**, §4, Thm. 1], ultimately resting on Grothendieck's characterization of completeness [**Scha**, Ch. IV, Cor. 2 to Thm. 6.2].

We record below an important property of the Schwartz-Bruhat space $\mathcal{S}(G)$ of a second-countable Hausdorff locally compact abelian group called *acyclicity* (for which we refer the reader to §C.1.5). This is useful when inductive tensor products of nuclear Fréchet spaces carry linear topological embeddings to linear topological embeddings. This will be used throughout the proof of Theorem D.3.3.7(1).

PROPOSITION D.1.5.9 (Acyclicity of $\mathcal{S}(G)$ - I). *Let G be a Hausdorff locally compact abelian group. Assume G is second-countable. Then $\mathcal{S}(G)$ is acyclic.*

PROOF. The space $\mathcal{S}(G)$ is a countable strict locally convex inductive limit of nuclear Fréchet spaces, hence complete by [**Scha**, Ch. II, 6.6]/ But acyclicity and completeness are equivalent for such inductive limits by [**Wngr-2**, §3, Thm. 3.3, p. 253], which applies because nuclear Fréchet space are always Montel (by [**Scha**, Ch. IV, 5.6] and [**Tre**, Cor. 1 to Prop. 50.2]). □

The following Theorem is a consequence of Proposition D.1.5.9:

THEOREM D.1.5.10 (Acyclicity of $\mathcal{S}(G)$ - II). *Let G be a Hausdorff locally compact abelian group. Assume G is second-countable, and choose a sequence of structural pairs (k_m, Γ_m) for $m \geq 0$ such that the short algebraically exact sequence*

$$0 \to \bigoplus_{m\geq 0} \mathcal{S}(k_m, \Gamma_m) \xrightarrow{\sigma} \bigoplus_{m\geq 0} \mathcal{S}(k_m, \Gamma_m) \xrightarrow{q} \mathcal{S}(G) \to 0, \tag{D.1.5.5}$$

where $\sigma((f_m)_{m\geq 0}) := (f_m - f_{m-1})_{m\geq 0}$ (upon defining $f_{-1} := 0$) and q is the quotient map, is topologically exact; i.e. σ is a linear topological embedding with closed range, and q is a quotient map.

PROOF. The quotient map q is, explicitly, assigned by $q((f_m)_{m\geq 0}) = \sum_{m\geq 0} f_m$, noting that for all but finitely many integers $m \geq 0$, we have $f_m = 0$, and hence the sum converges in $\mathcal{S}(G)$. The continuity of q and σ follows from the universal property of locally convex direct sums, so $\ker(q)$ is closed due to the Hausdorff property of $\mathcal{S}(G)$.

The linear inverse of σ is, explicitly:

$$\sigma^{-1}((f_m)_{m\geq 0}) = (\sum_{0\leq j\leq m} f_j)_{m\geq 0}. \tag{D.1.5.6}$$

Indeed, we have $\sigma^{-1}(\sigma((f_m)_{m\geq 0})) = (\sum_{0\leq j\leq m}(f_j - f_{j-1}))_{m\geq 0} = (f_m)_{m\geq 0}$, following the convention $f_{-1} := 0$. Note that σ^{-1} from (D.1.5.6) only makes sense on $\ker(q)$, which ensures that the sums $\sum_{0\leq j\leq m} f_j$ vanish for all large enough integers $m \geq 0$. The main difficulty is to show $\sigma^{-1}|_{\ker(q)}$ is continuous.

By Proposition D.1.5.9 the strict inductive limit $\mathcal{S}(G)$ is acyclic. Since $\{\mathcal{S}(k_m, \Gamma_m)\}_{m\geq 0}$ is a defining inductive sequence of the topology on $\mathcal{S}(G)$, by [**Vogt-1**, §1, Cor. 1.3, p. 60], the map σ is a linear topological isomorphism onto its range. □

We refer the reader to §C.1.5, for a discussion of acyclicity for locally convex inductive limits of Hausdorff locally convex topological $\mathbf{C}$-vector spaces, with applications to characterizing when a closed subspace $F \subset E = \lim E_n$ is a "limit subspace": when is the natural continuous linear bijection $\lim(F \cap E_n) \to F$ a homeomorphism?

D.1.6. Differential operators on Hausdorff locally compact abelian groups. For G a (real or complex) Lie group, we have the notion of its Lie algebra $\mathfrak{g}$, which, as a linear space is the tangent space at the neutral element of G, the Lie group being a differentiable manifold. The Lie algebra can be defined purely topologically. For an arbitrary Hausdorff locally compact abelian group, a Lie "algebra" can be defined as follows:

DEFINITION D.1.6.1. Let G be a topological group. We define the *topological Lie algebra* of G to be the set

$$\mathcal{L}(G) := \mathrm{Hom}_{\mathrm{cont}}(\mathbf{R}, G) \tag{D.1.6.1}$$

of one-parameter subgroups of G, i.e. of all continuous homomorphisms $\mathbf{R} \to G$, topologized with the compact open topology.

REMARK D.1.6.2. The space $\mathcal{L}(G)$ makes sense for non-commutative G too, and is defined in Definition D.1.6.1 for arbitrary topological groups. In special cases, it carries finer structures that approximate the defining elements of the Lie algebra of a Lie group. In general, we may write

$$\exp_G : \mathcal{L}(G) \to G \tag{D.1.6.2}$$

for the continuous map

$$\exp_G(a) := a(1),\ a \in \mathcal{L}(G) \tag{D.1.6.3}$$

and refer to it as the *exponential map* of G. In favorable situations, it is possible to identify on $\mathcal{L}(G)$ the structure of a Lie algebra and construct on G the structure of a Lie group. For instance, if G admits a continuous faithful finite dimensional complex representation, then G is a Lie group.

REMARK D.1.6.3. For G a Hausdorff locally compact abelian group, $\mathcal{L}(G)$ is a Hausdorff locally convex topological $\mathbf{R}$-vector space. We will never need this, but let us briefly indicate where this topology comes from. A canonical $\mathbf{R}$-vector space structure is defined by pointwise addition and pre-composition with $\mathbf{R}$-scaling on itself. By Pontryagin duality, G is isomorphic as a Hausdorff locally compact abelian group to its own Pontryagin bidual, and so by Lemma D.1.4.7 it admits a neighborhood base at the neutral element comprised of *quasi-convex* subsets: subsets $U \subset G$ such that for every $g \in G$ outside U there exists $\chi \in \widehat{G}$ with $\mathrm{Re}(\chi(u)) \geq 0$ for every $u \in U$ and $\mathrm{Re}(\chi(g)) < 0$ (such χ is an element of the analog of the polar set of U as defined for subsets of locally convex topological $\mathbf{C}$-vector spaces). Thus, by [**Au**, Prop. 1.2(ii), (iii)], $\mathcal{L}(G)$ with its $\mathbf{R}$-linear structure is a Hausdorff locally convex $\mathbf{R}$-vector space.

The following description of $\mathcal{L}(G)$ for G a Hausdorff locally compact abelian group is a consequence of Pontryagin duality. We endow

$$\mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \widehat{\mathbf{R}}) \tag{D.1.6.4}$$

with the compact-open topology. The topological space $\mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \widehat{\mathbf{R}})$ from (D.1.6.4) is an abelian group upon defining the group operation pointwise.

LEMMA D.1.6.4. *Let G be a Hausdorff locally compact abelian group. Then $\mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \widehat{\mathbf{R}})$ is a Hausdorff locally compact abelian group. The assignment:*

$$\Psi : \mathcal{L}(G) \to \mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \widehat{\mathbf{R}}) \tag{D.1.6.5}$$

defined by $a \mapsto \widehat{a}$ is an isomorphism of Hausdorff locally compact abelian groups.

PROOF. Pontryagin duality ensures Ψ is bijective, so we just need to check that Ψ and its inverse are continuous. More broadly, for any locally compact abelian G_1 and G_2, the map $\mathrm{Hom}(G_1, G_2) \to \mathrm{Hom}(\widehat{G_1}, \widehat{G_2})$ is checked to be continuous for the compact-open topologies. □

D.1.6.1. *Degree-1 differential operators and differential characterization of smoothness.* We now characterize the notion of smooth complex valued function on a Hausdorff locally compact abelian group G. Using the topological Lie algebra $\mathcal{L}(G)$ of a Hausdorff locally compact abelian group, we can define differential operators on G. We will then use them to provide a characterization of smoothness, and check that the definition recovers the familiar one when G is of the form $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$, for integers $m, n, r \geq 0$ and a finite group F, the definition of which was initiated in Definition 2.1.1.4.

DEFINITION D.1.6.5. Let G be a Hausdorff locally compact abelian group, and $a \in \mathcal{L}(G)$ a one-parameter subgroup. We define the following $\mathbf{C}$-linear map:

$$\partial_a : C^1(G) \to C(G) \tag{D.1.6.6}$$

by the assignment:

$$(\partial_a f)(g) := \lim_{t \to 0} \frac{f(ga(t)) - f(g)}{t} \tag{D.1.6.7}$$

the limit being formed within $\mathbf{C}$ (we'll soon see this limit always exists). We refer to ∂_a as the *invariant differential operator* of a.

We verify below that $g \mapsto (\partial_a f)(g)$, for $f \in C^1(G)$, is defined for all g and continuous over G.

LEMMA D.1.6.6. *Let G be a Hausdorff locally compact abelian group, and $f \in C^1(G)$. Then $\partial_a f$ is a continuous function on G. Moreover, ∂_a preserves the $\mathbf{C}$-linear subspace $C^\infty(G) \subset C^1(G)$ and satisfies*

$$\partial_a(f_1 f_2) = f_1 \partial_a(f_2) + f_2 \partial_a(f_1).$$

We call the operators $\partial_a : C^\infty(G) \to C^\infty(G)$ first-order differential operators.

PROOF. Fix $f \in C^1(G)$ and a structural pair (k, Γ) of G such that f is supported on Γ and is k-invariant. Then the unique function $f_0 \in C(\Gamma/k)$ such that $f_0 \circ r_{(k,\Gamma)} = f$, for $r_{(k,\Gamma)} : \Gamma \to \Gamma/k$ the continuous quotient homomorphism from (D.1.5.2), is C^1 on the elementary abelian Lie group Γ/k in the sense of Definition D.1.5.3(1). The one-parameter group $a : \mathbf{R} \to G$ has homomorphic image contained in a path-connected subgroup of G. In particular, it is contained in every open compactly generated subgroup, hence it lands in Γ.

Thus, we have a one-parameter subgroup $a_0 : \mathbf{R} \to \Gamma/k$ by composition with the continuous quotient homomorphism $r_{(k,\Gamma)}$, and by k-invariance of f we have $f(ga(t)) = f_0(\overline{g} a_0(t))$ where $\overline{g} := r_{(k,\Gamma)}(g)$. Since $f_0 \in C^1(\Gamma/k)$, we can rewrite (D.1.6.7) as

$$\lim_{t \to 0} \frac{f_0(\overline{g} a_0(t)) - f_0(\overline{g})}{t}.$$

This reduces the claims in the lemma to the case when G is an elementary abelian Lie group, say say isomorphic to $\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ for integers $m, n, r \geq 0$ and a finite abelian group F. The image of a is contained within $\mathbf{R}^m \times \mathbf{T}^n \times \{0\} \times \{0\}$. Since $\mathbf{R}$ and $\mathbf{R}^m \times \mathbf{T}^n$ are Lie groups, a is smooth and $\partial_a f$ is the directional derivative of f along the smooth curve a. Now all of the assertions can be seen by inspection. □

LEMMA D.1.6.7 (Invariance and commutation of degree-one differential operators). *Let $a, b : \mathbf{R} \rightrightarrows G$ be one-parameter subgroups; i.e., $a, b \in \mathcal{L}(G)$. Then:*

(1) *∂_a is a G-invariant $\mathbf{C}$-linear operator on $C^\infty(G)$ that depends $\mathbf{R}$-linearly on a, and it restricts to a continuous endomorphism of $\mathcal{S}(G)$.*
(2) *As $\mathbf{C}$-linear operators on $C^\infty(G)$, we have $[\partial_a, \partial_b] = 0$; i.e., $\partial_a \partial_b = \partial_b \partial_a$.*

PROOF. Pick $f \in C^\infty(G)$, so $f \in C^\infty(\Gamma/k)$ ffor some compact subgroup $k \subset G$ with open compactly generated orthogonal $k^\perp$ and an open compactly generated subgroup $\Gamma \subset G$ containing k. In particular, f is k-invariant and Γ contains the support of f.

Since $U := a(\mathbf{R}) \cdot \Gamma \cup b(\mathbf{R}) \cdot \Gamma$ is an open subset of G that is a symmetric neighborhood of the neutral element, it generates an open subgroup of G that is also compactly generated (since both $\mathbf{R}$ and Γ are so). We may replace Γ with this open subgroup so that a, b are valued in Γ. The direct limit definitions of $C^\infty(G)$ and $\mathcal{S}(G)$ as well as of the topology on the latter thereby allows us to rename Γ as G to arrange that G is compactly generated. Since a and b take values in the identity component of G, we can also assume without loss of generality that G is connected.

It follows that G/k is an abelian Lie group, and by design f arises by pullback from some $f_0 \in C^\infty(G/k)$. Thus, $\partial_a f$ is the pullback of the output of the differential operator $\partial_{\overline{a}}$ on $f_0 \in C^\infty(G/k)$ where $\overline{a}$ is the composition of a with the quotient homomorphism $G \to G/k$. The same holds for $\partial_b f$ using the composition $\overline{b}$ of b with the quotient homomorphism $G \to G/k$. To check (1) and (2) it now suffices to do so with G replaced by G/k' for each compact subgroup $k' \subset G$ with compactly generated orthogonal ${k'}^\perp$. In other words, we may rename G/k as G to assume that G is a connected abelian Lie group. In this case the assertions are standard. □

D.1.6.2. *General differential operators.* We now discuss general differential operators on Hausdorff locally compact abelian groups.

DEFINITION D.1.6.8 (Differential operators on Hausdorff locally compact abelian groups). Let G be a Hausdorff locally compact abelian group.

(1) Let $\mathrm{Diff}(G)$ be the commutative $\mathbf{C}$-subalgebra with identity inside $\mathrm{End}_{\mathbf{C}}(C^\infty(G))$ generated by the invariant differential operators $\{\partial_a\}_{a\in\mathcal{L}(G)}$. We refer to $\mathrm{Diff}(G)$ as the $\mathbf{C}$-algebra of *invariant differential operators* of G. We denote by $\mathrm{Diff}^1(G)$ the $\mathbf{C}$-linear subspace of $\mathrm{Diff}(G)$ generated by the set $\{\partial_a\}_{a\in\mathcal{L}(G)}$. We denote by $\mathrm{Diff}^1_r(G)$ the $\mathbf{R}$-linear subspace $\mathbf{R}$-linearly generated by the same set $\{\partial_a\}_{a\in\mathcal{L}(G)}$.

(2) Let $\mathcal{L}(G)^*$ be the set of (finite) words on the set $\mathcal{L}(G)$. For any $w\in\mathcal{L}(G)^*$, let $D_w\in\mathrm{Diff}(G)$ be the corresponding differential operator (understood to be the identity when w is empty). For any $D\in\mathrm{Diff}(G)$, an identity of the form

$$D=\sum_{w\in\mathcal{L}(G)^*} c_w\, D_w \tag{D.1.6.8}$$

with w varying over all the words on the elements $\mathcal{L}(G)$, $c_w\in\mathbf{C}$ with $c_w=0$ for all but finitely many w, as a *presentation* for D.

We define by $\mathcal{P}_D$ the set of all presentations of D in $\mathrm{Diff}(G)$, and we define the *degree* of D to be

$$\deg(D):=\inf_{\mathcal{P}_D}\max_{c_w\neq 0}\ell(w) \tag{D.1.6.9}$$

where $\ell(w)$ is the length of the finite word w on $\mathcal{L}(G)$, the maximum is formed over all the finitely many words w in a given presentation of D whose complex coefficient c_w is nonzero, and the infimum is formed over the set of presentations of D in $\mathrm{Diff}(G)$.

REMARK D.1.6.9. The constants in $\mathrm{Diff}(G)$ have degree 0, and for every nonzero $a\in\mathcal{L}(G)$ we have $\deg(\partial_a)=1$, justifying the notation for $\mathrm{Diff}^1(G)$. Note that the differential operator $D=0$ has degree 0 but the presentation $\partial_a\partial_b-\partial_b\partial_a$ for it leads to the value $\max_{c_w\neq0}\ell(w)=\max\{\ell(\partial_a\partial_b),\ell(\partial_b\partial_a)\}=2$.

The following is [**Wa**, §2,Def.2.4]:

DEFINITION D.1.6.10. Let G be a Hausdorff locally compact abelian group, $D\in\mathrm{Diff}(G)$ an invariant differential operator. We define the function

$$p(D):\widehat{G}\to\mathbf{C} \tag{D.1.6.10}$$

by the assignment

$$p(D)(\chi):=(D\chi)(e) \tag{D.1.6.11}$$

where $e\in G$ is the neutral element. We refer to $p(D)$ as the *polynomial of* D, or *symbol* of D.

REMARK D.1.6.11. Definition D.1.6.10 is meaningful since continuous characters of a Hausdorff locally compact abelian group are smooth on any open compactly generated subgroup Γ by Remark D.1.5.5 and there is an evident restriction map $\mathrm{Diff}(G)\to\mathrm{Diff}(\Gamma)$ via the equality $\mathcal{L}(\Gamma)=\mathcal{L}(G)$ (since $\mathbf{R}$ has no proper open subgroup).

EXAMPLE D.1.6.12. For $G=\mathbf{R}$ we have $\widehat{G}=\{\chi_t=e^{2\pi it(\cdot)}\}_{t\in\mathbf{R}}$, anjd for $D=c\,d/dx$ with $c\in\mathbf{C}$ we have $p(D)(\chi_t)=2\pi ict$. For $G=\mathbf{T}=\mathbf{R}/\mathbf{Z}$ we have $\widehat{G}=\{\chi_m:x\mapsto\mapsto e^{2\pi imx}\}_{m\in\mathbf{Z}}$, and for $D=c\,d/dx$ with $c\in\mathbf{C}$ we have $p(D)(\chi_m)=2\pi icm$.

More broadly, consider $G=\mathbf{R}^m\times\mathbf{T}^n\times\mathbf{Z}^r\times F$ with elements denoted (x,z,a,b) and its dual $\widehat{G}=\mathbf{R}^m\times\mathbf{Z}^r\times\mathbf{T}^r\times\widehat{F}$ with elements denoted $(\widetilde{x},\widetilde{a},\widetilde{z},\widetilde{b})$. A general 1-parameter subgroup $\lambda:\mathbf{R}\to\widehat{G}$ has the form $t\mapsto(t\widetilde{c},0,t\widetilde{c}' \bmod \mathbf{Z}^r,0)$ for unique $\widetilde{c}\in\mathbf{R}^m$, $\widetilde{c}'\in\mathbf{R}^r$, and when G is viewed as the dual of $\widehat{G}$ the character $\chi_{(x,z,a,b)}:\widehat{G}\to\mathrm{U}(1)$ associated to $(x,z,a,b)\in G$ is

$$(\widetilde{x},\widetilde{a},\widetilde{z},\widetilde{b})\mapsto e^{2\pi x\cdot\widetilde{x}}e^{2\pi iz\cdot\widetilde{a}}\,\widetilde{z}^{\,a}\,\widetilde{b}(b)$$

(using the standard dot products on $\mathbf{R}^m$ and $\mathbf{R}^r$).

The operator ∂_λ is $\widetilde{c}\cdot\partial_{\widetilde{x}} + \widetilde{c}'\cdot\partial_{\widetilde{z}}$, so it is readily computed that

$$p(\partial_\lambda)(\chi_{(x,z,a,b)}) = 2\pi i(x\cdot\widetilde{c} + a\cdot\widetilde{c}'). \tag{D.1.6.12}$$

We will see in Lemma D.1.6.19(1) with the aid of Fourier transform that for every open compactly generated subgroup $\Gamma' \subset \widehat{G}$, the restriction of $p(D) : \widehat{G} \to \mathbf{C}$ to Γ' belongs to $C^\infty(\Gamma')$. In particular, $p(D)$ is continuous.

LEMMA D.1.6.13. *Let G be a Hausdorff locally compact abelian group. For $\chi \in \widehat{G}$ and $a_1, \ldots, a_n \in \mathcal{L}(G)$,*

$$\partial_{a_n}\cdots\partial_{a_1}(\chi) = \left(\prod p(\partial_{a_i})(\chi)\right)\chi$$

as functions $G \to \mathbf{C}$ (equivalently, $(\partial_{a_n}\cdots\partial_{a_1}(\chi))\chi^{-1}$ is a constant function on G). In particular,

$$p(\partial_{a_n}\cdots\partial_{a_1})(\chi) = \prod p(\partial_{a_i})(\chi). \tag{D.1.6.13}$$

Moreover, for every compact subgroup $k' \subset \widehat{G}$ and $D \in \mathrm{Diff}(G)$, the function $p(D) : \widehat{G} \to \mathbf{C}$ is k'-invariant.

PROOF. The identity (D.1.6.13) follows from the first asserted identity by evaluating at $e \in G$. To prove the first assertion, by induction it suffices to treat the case $n = 1$: $\partial_a(\chi) = c_a\chi$ as functions for some $c_a \in \mathbf{C}$ (and then $c_a = p(\partial_a)(\chi)$ by evaluating at the identity). From the definition of ∂_a,

$$(\partial_a(\chi))(g) = \lim_{t\to 0}\frac{\chi(ga(t)) - \chi(g)}{t} = \left(\lim_{t\to 0}\frac{\chi(a(t)) - 1}{t}\right)\chi(g).$$

Thus, setting c_a to be the final limit expression does the job.

For k'-invariance of $p(D)$, by $\mathbf{C}$-linearity in D (and the fact that invariance is obvious for constant operators), it suffices to treat the case $D = \partial_{a_n}\cdots\partial_{a_1}$. for some $n \geq 1$. By (D.1.6.13) it further suffices to treat the case $n = 1$: to show $p(\partial_a)(\chi\psi) = p(\partial_a)(\chi)$ for ψ in a compact subgroup of $\widehat{G}$. The Leibniz rule gives $p(\partial_a)(\chi\psi) = p(\partial_a)(\chi) + p(\partial_a)(\psi)$, so we just want $p(\partial_a)$ to vanish on all compact subgroups $k' \subset \widehat{G}$. Such subgroups are precisely $\widehat{G/U}$ for an open subgroup $U \subset G$. But the characters $\chi \in \widehat{G/U}$ are exactly U-invariant characters on G, and any character of the latter type is constant near $e \in G$ (namely equal to 1), so applying D kills it near e. □

LEMMA D.1.6.14. *Let G be a Hausdorff locally compact abelian group. Define the assignment:*

$$\sigma : \mathrm{Diff}^1_r(G) = \{\partial_a\}_{a\in\mathcal{L}(G)} \to \mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \mathbf{R}) \tag{D.1.6.14}$$

by setting $\sigma(\partial_a) : \widehat{G} \to \mathbf{R}$ to be the continuous homomorphism satisfying

$$\exp(2\pi i(\sigma(\partial_a)(\chi))t) = \langle a(t), \chi\rangle_{G,\widehat{G}}$$

as functions of $t \in \mathbf{R}$.

This map σ is an $\mathbf{R}$-linear isomorphism, upon endowing the target with the pointwise $\mathbf{R}$-linear structure. The following diagram:

$$\begin{array}{ccc} \mathcal{L}(G) & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \widehat{\mathbf{R}}) \\ {\scriptstyle a\mapsto\partial_a}\downarrow & & \uparrow{\scriptstyle \simeq} \\ \mathrm{Diff}^1_r(G) & \xrightarrow[\sigma]{} & \mathrm{Hom}_{\mathrm{cont}}(\widehat{G}, \mathbf{R}) \end{array} \tag{D.1.6.15}$$

commutes, where the rightmost vertical arrow is induced by the isomorphism $\mathbf{R} \simeq \widehat{\mathbf{R}}$ defined by $t \mapsto e^{2\pi i t(\cdot)}$.

PROOF. The top side of the diagram is a bijective homomorphism by Lemma D.1.6.4, and it is $\mathbf{R}$-linear by design. The right side is also an $\mathbf{R}$-linear isomorphism by design, and the left side is an $\mathbf{R}$-linear isomorphism by Lemma D.1.6.7(1). Thus, the bottom is an $\mathbf{R}$-linear isomorphism once it is shown that the diagram commutes. We now focus on the commutativity.

The map $t \mapsto \langle a(t), \chi\rangle_{G,\widehat{G}} = \chi(a(t))$ is a continuous character of $\mathbf{R}$. There therefore exists a unique $\lambda_{a,\chi} \in \mathbf{R}$ such that $\langle a(t), \chi\rangle_{G,\widehat{G}} = \exp(2\pi i \lambda_{a,\chi}\, t)$ as continuous functions of $t \in \mathbf{R}$. Formation of $\lambda_{a,\chi}$ is readily checked to be continuous in (a, χ) with respect to the compact-open topology on $\mathcal{L}(G)$ and the compact-open topology on $\widehat{G}$. The assignment $a \mapsto (\chi \mapsto \lambda_{a,\chi})$ is the composition of the top horizontal and (inverse of the) rightmost vertical arrows in (D.1.6.15).

The equation

$$\langle a(t), \chi\rangle_{G,\widehat{G}} = \exp(2\pi i(\sigma(\partial_a)(\chi))t) \tag{D.1.6.16}$$

determines the one-parameter subgroup $a : \mathbf{R} \to G$ uniquely by Theorem D.1.1.1(2). It follows that we have

$$\lambda_{a,\chi} = \sigma(\partial_a)(\chi)$$

for all $\chi \in \widehat{G}$, and hence $a \mapsto (\chi \mapsto \lambda_{a,\chi})$ (which, recall, equals the composition of going across the top and right side of the diagram (D.1.6.15)) agrees with $a \mapsto (\chi \mapsto \sigma(\partial_a)(\chi))$ that is the composition along the left and bottom sides of (D.1.6.15). Thus, the diagram commutes. □

COROLLARY D.1.6.15 (Values of the polynomials of degree-1 differential operators). *Let G be a Hausdorff locally compact abelian group, $a \in \mathcal{L}(G)$ a 1-parameter subgroup. Then:*

$$p(\partial_a) = 2\pi i \sigma(\partial_a) \tag{D.1.6.17}$$

as functions $\widehat{G} \to \mathbf{C}$. In particular, $p(\partial_a)$ takes values within $\mathbf{R}(1)$.

PROOF. In the proof of Lemma D.1.6.13 we saw

$$p(\partial_a)(\chi) = \lim_{t\to 0} \frac{\chi(a(t)) - 1}{t},$$

which is the derivative at $t = 0$ of $\chi(a(t)) = \exp(2\pi i(\sigma(\partial_a))(\chi)t)$ (equality by (D.1.6.16)). This derivative value is clearly $2\pi i(\sigma(\partial_a))(\chi)$. □

We conclude with a "coordinate-free" characterization of smooth functions on an elementary abelian Lie group G that are of rapid decay at infinity.

LEMMA D.1.6.16. *Let G be an elementary abelian Lie group, $f : G \to \mathbf{C}$ a smooth function on G. The following conditions on f are equivalent:*

(1) *f is a Schwartz-Bruhat function.*
(2) *For every pair $(D_1, D_2) \in \mathrm{Diff}(G) \times \mathrm{Diff}(\widehat{G})$, the product function $p(D_2) \cdot D_1(f)$ on G is continuous and globally bounded (with the bound depending on D_1 and D_2).*

REMARK D.1.6.17. Since $D_2 \in \mathrm{Diff}(\widehat{G})$, we use the natural isomorphism $G \simeq \widehat{\widehat{G}}$ to view $p(D_2)$ as a function on G.

PROOF. Consider $f \in C^\infty(G)$. There exists a structural pair (k, Γ) of G such that f is supported on Γ and the unique $f_0 \in C^\infty(\Gamma/k)$ such that $f_0 \circ r_{(k,\Gamma)} = f$ (with $r_{(k,\Gamma)} : \Gamma \to \Gamma/k$ the continuous quotient homomorphism of (D.1.5.2)), with Γ/k an elementary abelian Lie group. (as Γ/k is an elementary abelian Lie group; see also Lemma D.1.3.3). Clearly $D_1(f)$ is supported on Γ, and it is k-invariant since D_1 is built from invariant first-order differential operators. Thus, $D_1(f)$ arises from a function on Γ/k. The same then holds for $p(D_2) \cdot D_1(f)$ since this product is certainly supported in Γ and $p(D_2)$ is k-invariant by the last assertion in Lemma D.1.6.13. Hence, we can rename Γ/k as G and assume $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ for integers

$m, n, r \geq 0$ and a finite abelian group F. Any one-parameter subgroup $a : \mathbf{R} \to G$ has image contained in $\mathbf{R}^m \times \mathbf{T}^n \times \{0\} \times \{0\}$ by connectedness.

By $\mathbf{C}$-bilinearity in D_1 and D_2, we may focus on D_1 and D_2 that are each products of finitely many ∂_a's for 1-parameter subgroups a of G and $\widehat{G}$ respectively. As a warm-up, let's consider the case $D_1 = \partial_{a_1}$ and $D_2 = \partial_{a_2}$, with $a_1 \in \mathcal{L}(G)$ and $a_2 \in \mathcal{L}(\widehat{G})$. Viewing $\mathbf{T}$ as $\mathbf{R}/\mathbf{Z}$, we have $a_1(t) = (tc, tc' \bmod \mathbf{Z}, 0, 0)$ for some unique $c \in \mathbf{R}^m$ and $c' \in \mathbf{R}^n$. Using the analogous description $\widehat{G} = \mathbf{R}^m \times \mathbf{Z}^n \times \mathbf{T}^r \times \widehat{F}$ (with elements denoted $(\widetilde{x}, \widetilde{a}, \widetilde{z}, \widetilde{b})$), we likewise have $a_2(t) = (te, 0, td, 0)$ for unique $e \in \mathbf{R}^m$ and $d \in \mathbf{R}^r$. Then

$$(p(\partial_{a_2}) \cdot \partial_{a_1} f)(x, z, a, b) = p(\partial_{a_2})(x, z, a, b) \cdot \left(\sum_{j=1}^{m} c_j \, \partial_{x_j} f(x, z, a, b) + \sum_{\ell=1}^{n} c'_\ell \, \partial_{z_\ell} f(x, z, a, b) \right) \tag{D.1.6.18}$$

where $p(\partial_{a_2})(x, z, a, b) = 2\pi i(x \cdot e + a \cdot d)$ by (D.1.6.12). Thus, the right side of (D.1.6.18) equals

$$2\pi i \sum_{j=1}^{m} (x \cdot e + a \cdot d) c_j \, \partial_{x_j} f(x, z, a, b) + 2\pi i \sum_{\ell=1}^{n} (x \cdot e + a \cdot d) c'_\ell \, \partial_{z_\ell} f(x, z, a, b) \tag{D.1.6.19}$$

for $e \in \mathbf{R}^m$, $d \in \mathbf{R}^r$ encoding a_2 and $c \in \mathbf{R}^m$, $c' \in \mathbf{R}^n$ encoding a_1.

In general, as we let D_2 vary through all non-empty products of finitely many ∂_a's with a varying through the 1-parameter subgroups of $\widehat{G}$ we see that $p(D_2) : G \to \mathbf{C}$ varies through all products of linear forms in (x, a); the empty product (corresponding to the constant operator $D_2 = 1$) has $p(D_2) = 1$. Likewise, as we let D_1 vary through all such non-empty products with 1-parameter subgroups of G clearly $D_1(f)$ varies through the output of f when hit by all products of finitely many first-order linear differential operators in the ∂_{x_j}'s and ∂_{z_ℓ}'s (and $D_1(f) = f$ for the empty product corresponding to $D_1 = 1$). The boundedness condition in (2) thereby expresses *exactly* the finiteness on f for all seminorms in (D.1.3.9), and that condition defines $\mathcal{S}(G)$ as a subspace of $C^\infty(G)$ by Lemma D.1.3.4. □

D.1.6.3. *Differential operators, symbols and the Fourier transform.* We conclude this review on the Schwartz-Bruhat space of a Hausdorff locally compact abelian group G, with a fundamental property of the Fourier transform $\mathcal{F}_G : L^1(G, \mu_G) \to C_0(G)$, where μ_G is a fixed choice of Haar measure on G, uniquely determining a dual Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ (see Lemma D.1.1.2).

For a structural pair (k, Γ) of G, we choose the volume-1 Haar measure on k and denote it μ_k, and together with the restriction μ_G to the open subgroup $\Gamma \subset G$ this uniquely determines a Haar measure $\mu_{\Gamma/k}$ on the elementary abelian Lie group Γ/k by Theorem D.1.2.1. Likewise, the measures μ_G and μ_k determine a Haar measure $\mu_{G/k}$ on G/k. By Lemma D.1.1.2 we likewise get the Plancherel measures $\mu_{\widehat{G}}$ on $\widehat{G}$, $\mu_{\widehat{\Gamma/k}}$ on $\widehat{\Gamma/k} = k^\perp/\Gamma^\perp$, and $\mu_{k^\perp}$ on $k^\perp = \widehat{G/k}$. Then $\mu_{\widehat{\Gamma/k}}$ and $\mu_{k^\perp}$ determine a Haar measure $\mu_{\Gamma^\perp}$ on the compact subgroup $\Gamma^\perp \subset \widehat{G}$.[3]

We use μ_k to define the continuous linear map $\pi_{(k,\Gamma)} : \mathcal{S}(G) \to \mathcal{S}(k, \Gamma) = \mathcal{S}(\Gamma/k)$ assigned by

$$\pi_{(k,\Gamma)}(f)(x) := \int_k f(gx) \, \mathrm{d}\mu_k(g), \ x \in \Gamma \tag{D.1.6.20}$$

on each $f \in \mathcal{S}(G)$. To check that this output really lies in $\mathcal{S}(\Gamma/k)$ and that the linear map is continuous, the inductive limit process defining the topology of $\mathcal{S}(G)$ reduces us to checking:

[3]There are a variety of things to clarify that I am sure are implicit later: is $\mu_{k^\perp}$ the restriction of $\mu_{\widehat{G}}$ to the open subgroup $k^\perp$? Is $\mu_{k^\perp}$ the one determined just from μ_G and μ_k without mention of Γ at all? Is $\mu_{\Gamma^\perp}$ the volume-1 measure? Can any of these be seen directly without much headache?

LEMMA D.1.6.18. *If A is an elementary abelian group and k is a compact subgroup equipped with its volume-1 measure μ_k then for $f \in \mathcal{S}(A)$ the k-invariant function $x \mapsto \int_k f(xg)\,\mathrm{d}\mu_k(g)$ lies in $\mathcal{S}(A/k)$. Moreover, this map $\pi : \mathcal{S}(A) \to \mathcal{S}(A/k)$ is continuous and surjective.*

PROOF. Averaging over k (say on continuous functions) is the composite of the analogous averaging over k^0 followed by averaging over k/k^0, so it suffices to separately treat the cases when k is connected and when k is finite.

First we treat finite k. In this case the k-averaging involves only finite sums and so the output of such averaging is seen to be valued in $\mathcal{S}(A)^k$ and continuous as an $\mathcal{S}(A)$-valued operation. We then just need to show the inclusion $\mathcal{S}(A/k) \to \mathcal{S}(A)^k$ is an equality and the resulting map $\pi : \mathcal{S}(A) \to \mathcal{S}(A/k)$ is continuous. Fix an identification

$$A = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F \tag{D.1.6.21}$$

for finite F. Letting $F' \subset F$ be the image of k under $A \to (A/A^0)_{\mathrm{tor}} = F$, it suffices to separately treat the cases $F' = 0$ and $k \simeq F'$. In the latter case the containment $k \subset \mathbf{T}^n \times F$ and the infinite divisibility of $\mathbf{T}$ allows us to use compatible decompositions of $k \simeq F'$ and F as products of cyclic groups to possibly *change* the identification (D.1.6.21) to arrange that $k \subset F$ *as subgroups of* A. Then we have compatible linear topological descriptions

$$\mathcal{S}(A) = \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r) \otimes \mathbf{C}^F, \;\; \mathcal{S}(A/k) = \mathcal{S}(\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r) \otimes \mathbf{C}^{F/k}$$

from which all is clear.

Now assume $F' = 0$, so $k \subset \mathbf{T}^n$ inside A and hence (D.1.3.20) provides compatible identifications

$$\mathcal{S}(A) = \mathcal{S}(\mathbf{R}^m \times \mathbf{Z}^r)\widehat{\otimes}_\pi \mathcal{S}(\mathbf{T}^n) \otimes \mathbf{C}^F, \;\; \mathcal{S}(A/k) = \mathcal{S}(\mathbf{R}^m \times \mathbf{Z}^r)\widehat{\otimes}_\pi \mathcal{S}(\mathbf{T}^n/k) \otimes \mathbf{C}^F$$

(with $\mathbf{T}^n/k$ also a torus). We thereby reduce to the case $F = 0$. Since the finite k is contained in $\mathbf{T}^n$, via any of the families of seminorms in Lemma D.1.3.4 the k-action on $\mathcal{S}(A)$ is seen to be continuous and so the completed tensor product descriptions of $\mathcal{S}(A)$ and $\mathcal{S}(A/k)$ reduce checking $\mathcal{S}(A/k) = \mathcal{S}(A)^k$ to the special case $A = \mathbf{T}^n$ that is clear (since $\mathcal{S}(T) = C^\infty(T)$ for any torus T). The continuity of $\mathcal{S}(A) \to \mathcal{S}(A/k)$ is seen by using the family of seminorms (D.1.3.10) applied to A and A/k (via the compatible product decompositions of these groups as above and any choice of isomorphism $\mathbf{T}^n/k \simeq \mathbf{T}^n$).

It remains to treat k connected, so k is a torus and therefore contained in the unique maximal torus $\mathbf{T}^n \subset A$. Since any containment of tori splits as a topological group factor, we can choose (D.1.6.21) so that k is a product of some of the $\mathbf{T}$-factors. In particular, $A = A' \times k$ for an elementary abelian Lie group A'. For the special case $A = k$, this k-averaging construction $\int_k : \mathcal{S}(k) \to \mathbf{C}$ is readily seen to be continuous. In general, by (D.1.3.20) we have naturally $\mathcal{S}(A')\widehat{\otimes}_\pi \mathcal{S}(k) \simeq \mathcal{S}(A)$. By Grothendieck's explicit description of elements of completed projective tensor products of Fréchet spaces [**Scha**, Ch. III, 6.4], everything in this completed tensor product can be written in the form

$$\sum_{n \geq 1} \lambda_n (f_n \otimes F_n)$$

for an absolutely summable sequence (λ_n) in $\mathbf{C}$ and sequences $(f_n) \subset \mathcal{S}(A')$ and $(F_n) \subset \mathcal{S}(k)$ that converge to 0. In particular, a general $f \in \mathcal{S}(A)$ can be written pointwise as an absolutely summable series

$$f(a', g) = \sum_{n \geq 1} \lambda_n f_n(a') F_n(g)$$

for $(a', g) \in A' \times k = A$; note that the f_n's and F_n's are respectively uniformly bounded on A' and k respectively.

Thus, for $x_0 = (a'_0, g_0) \in A$ we have

$$\int_k f(x_0 g)\,\mathrm{d}\mu_k(g) = \sum_{n\geq 1} \lambda_n f_n(a'_0) \int_k F_n(g_0 g)\,\mathrm{d}\mu_k(g) = \sum_{n\geq 1} \lambda_n f_n(a'_0) \int_k F_n\,\mathrm{d}\mu_k.$$

In other words,

$$\int_k f((a', g')g)\,\mathrm{d}\mu_k(g) = \sum_{n\geq 1} \lambda_n f_n(a') \int_k F_n\,\mathrm{d}\mu_k$$

for all $(a', g') \in A' \times k = A$. Since the F_n's are uniformly bounded and $f_n \to 0$ in $\mathcal{S}(A')$, the series

$$\sum_{n\geq 1} \lambda_n f_n \int_k F_n\,\mathrm{d}\mu_k$$

has Cauchy partial sums in $\mathcal{S}(A') = \mathcal{S}(A/k)$ and so converges in $\mathcal{S}(A/k)$ by completeness. This proves k-averaging is valued in $\mathcal{S}(A/k)$ and agrees with the visibly *continuous* linear $\mathrm{id}\widehat{\otimes}_\pi \int_k : \mathcal{S}(A')\widehat{\otimes}_\pi \mathcal{S}(k) \to \mathcal{S}(A')$. □

Since $\mu_k(k) = 1$, it is clear that $\pi_{(k,\Gamma)} : \mathcal{S}(G) \to \mathcal{S}(k, \Gamma) = \mathcal{S}(\Gamma/k)$ is also a projector. We likewise use $\mu_{\Gamma^\perp}$ to define the continuous linear map $\pi_{(\Gamma^\perp, k^\perp)} : \mathcal{S}(\widehat{G}) \to \mathcal{S}(\Gamma^\perp, k^\perp) = \mathcal{S}(k^\perp/\Gamma^\perp)$ by

$$\pi_{(\Gamma^\perp,k^\perp)}(f)(x) := \int_{\Gamma^\perp} f(gx)\,\mathrm{d}\mu_{\Gamma^\perp}(g),\ x \in k^\perp \tag{D.1.6.22}$$

on each $f \in \mathcal{S}(\widehat{G})$. That $\pi_{(\Gamma^\perp,k^\perp)}$ is also a projector amounts to checking $\mu_{\Gamma^\perp}(\Gamma^\perp) = 1$, but rather than carry out such a direct verification (which seems a bit complicated)[4] we get it immediately from the projector property for $\pi_{(k,\Gamma)}$ via Lemma D.1.6.19(2, 3) below, with that part (3) a consequence of the Poisson Summation Formula (see Theorem D.1.2.7 or [**DE**, Thm.3.6.3]).

Lemma D.1.6.19 (Fourier transform and differential operators on Schwartz-Bruhat spacees). *Let G be a Hausdorff locally compact abelian group, and (k, Γ) a structural pair of G. Let μ_G be a fixed choice of Haar measure on G, and $\mu_{\widehat{G}}$ the Plancherel measure on $\widehat{G}$ determined by μ_G.*

(1) *For any invariant differential operator D on G,*

$$\mathcal{F}_G(Df) = p(D) \cdot \widehat{f} \tag{D.1.6.23}$$

for every $f \in \mathcal{S}(G)$; i.e., as functions of $\chi \in \widehat{G}$, we have

$$\widehat{Df}(\chi) = p(D)(\chi) \cdot \widehat{f}(\chi). \tag{D.1.6.24}$$

(2) *The linear map $\mathcal{F}_G : \mathcal{S}(G) \to C_0(\widehat{G})$ is valued in $\mathcal{S}(\widehat{G})$ and continuous with respect to the locally convex topologies on $\mathcal{S}(G)$ and $\mathcal{S}(\widehat{G})$. As such, it is a linear topological isomorphism.*

(3) *The Fourier transform $\mathcal{F}$ interacts with the projectors $\pi_{(k,\Gamma)}$ and $\pi_{(\Gamma^\perp,k^\perp)}$ according to the following identity:*

$$\mathcal{F}_G \circ \pi_{(k,\Gamma)}(f) = \int_{k^\perp} \widehat{f}(\chi)\,\chi\,\mathrm{d}\mu_{k^\perp}(\chi) \tag{D.1.6.25}$$

where $\mu_{k^\perp}$ is the Plancherel measure on $k^\perp$ uniquely determined by $\mu_{\Gamma/k}$ and μ_G. In particular, we have

$$\mathcal{F}_G \circ \pi_{(k,\Gamma)}(f) = \pi_{(\Gamma^\perp,k^\perp)} \circ \mathcal{F}_{\widehat{G}} \tag{D.1.6.26}$$

intertwining $\mathcal{S}(k, \Gamma)$ and $\mathcal{S}(\Gamma^\perp, k^\perp)$.

Proof. (1) : This is a consequence of (1) in the proof of [**Wa**, Thm. 3.2] together with the calculations for (2) and (3) on [**Wa**, p. 313]. We record the calculation, because our convention in (D.1.1.4) to define the Fourier transform incorporate conjugation on the character, contrary to the definition in loc.cit.

[4] I am skeptical the proof of the Lemma below invoked here really does not require knowing this volume-1 property in the details, in which case volume-1 should be proved directly.

First we treat the special case $D = \partial_a$ for a 1-parameter subgroup $a : \mathbf{R} \to G$:

$$\begin{aligned}
\mathcal{F}_G(\partial_a f)(\chi) &= \int_G \lim_{t\to 0} \frac{f(ga(t)) - f(g)}{t}\overline{\chi(g)}\,\mathrm{d}\mu_G(g) \\
&= \lim_{t\to 0}\left\{\int_G \frac{f(ga(t)^{-1}a(t))}{t}\overline{\chi(ga(t)^{-1})}\,\mathrm{d}\mu_G(ga(t)^{-1}) - \int_G \frac{f(g)}{t}\,\overline{\chi}(g)\,\mathrm{d}\mu_G(g)\right\} \\
&= \lim_{t\to 0}\left\{\frac{\chi(a(t))}{t}\int_G f(g)\overline{\chi}(g)\,\mathrm{d}\mu_G(g) - \frac{1}{t}\int_G f(g)\overline{\chi}(g)\,\mathrm{d}\mu_G(g)\right\} \\
&= \lim_{t\to 0}\left\{\frac{\chi(a(t)) - 1}{t}\right\}\int_G f(g)\overline{\chi(g)}\,\mathrm{d}\mu_G(g) \\
&= p_{\partial_a}(\chi)\,\widehat{f}(\chi)
\end{aligned}$$

where the exchange of the order of integration and formation of the limit is legitimate by Lebesgue's Dominated Convergence, because χ^{-1} is bounded over G and the family of Schwatz functions $g \mapsto \frac{f(ga(t))-f(g)}{t}$ on G for $t \neq 0$ that converges pointwise to $\partial_a f$ on G as $t \to 0$ also converges to it in $L^1(G, \mu_G)$ as $t \to 0$.

To see the L^1-convergence, we shall use the group-theoretic analogue of Hadamard's Lemma:

$$\frac{f(ga(t)) - f(g)}{t} = \int_0^1 (\partial_a f)(ga(tu))\,\mathrm{d}u.$$

The right side as a function of g for each $t \neq 0$ converges to $\partial_a f$ in $L^1(G, \mu_G)$ as $t \to 0$, due to Fubini's Theorem and the Dominated Convergence Theorem on $G \times [0, 1]$ (as $(\partial_a f)(ga(v)) - (\partial_a f)(g) \in L^1(G \times [0, 1])$ by translation-invariance of μ_G). As usual, really one works with arbitrary sequences $t_n \to 0$ to infer the conclusion for general $t \to 0$.

For general D, linearity reduces us to the special case that D is a product of ∂_{a_i}'s for finitely many $a_1, \ldots, a_n \in \mathcal{L}(G)$. The multiplicativity in (D.1.6.13) allows us to carry out induction on n (the base case $n = 1$ being settled above).

(2) : To check $\mathcal{F}_G(f)$ is Schwartz on $\widehat{G}$ for $f \in \mathcal{S}(G)$, the key point is to reduce to elementary abelian Lie groups. Pick a structural pair (k, Γ) so that $f \in \mathcal{S}(k, \Gamma)$. The continuous $\widehat{f}$ is supported on $k^\perp$ and $\Gamma^\perp$-invariant by Lemma D.1.5.2(3). Strictly speaking, that result assumes $\widehat{f} \in L^1(\widehat{G})$ which we don't yet know, but the required parts of the proof never use that hypothesis: $\widehat{f}$ is supported in $k^\perp$ because in the proof of Lemma D.1.5.2(3) the argument to show $\widehat{f}$ is supported in $H^\perp$ when f is H-invariant for a closed subgroup $H \subset G$ only requires an L^1-condition on f (in particular, pointwise Fourier inversion is not used). Likewise the $\Gamma^\perp$-invariance for $\widehat{f}$ can be shown by adapting a calculation from that earlier proof: if $\chi_0 \in \Gamma^\perp$ then since f is supported on the open subgroup Γ we have

$$\widehat{f}(\chi_0\chi) = \int_G f(g)\overline{\chi_0\chi}(g)\,\mathrm{d}\mu_G(g) = \int_\Gamma f(\gamma)\overline{\chi_0\chi}(g)\,\mathrm{d}\mu_G(\gamma) = \int_\Gamma f(\gamma)\overline{\chi}(g)\,\mathrm{d}\mu_G(\gamma)$$

since $\chi_0(\gamma) = 1$ (as $\chi_0 \in \Gamma^\perp$). The calculation then runs in reverse to recover $\widehat{f}(\chi)$ as desired.

Thus, $\widehat{f}$ arises from a continuous function on $k^\perp/\Gamma^\perp$. As such, we claim $\widehat{f}$ viewed on $k^\perp/\Gamma^\perp = \widehat{\Gamma/k}$ is exactly $\mathcal{F}_{\Gamma/k}(f_0)$ for the continuous L^1-function f_0 on Γ/k arising from f.[5] Thus, to prove $\widehat{f} \in \mathcal{S}(\widehat{G})$ it suffices to show $\widehat{f_0} \in \mathcal{S}(\widehat{\Gamma/k})$. Once this is proved, the resulting linear map $\mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\widehat{G})$ is the direct limit of Fourier transforms at the (k, Γ)-levels,[6] so the topological isomorphism property for $\mathcal{F}_G$ between Schwartz spaces reduces to the case of elementary abelian Lie groups.

So now $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$. Differentiation under the integral sign, combined with the assumed Schwartz-Bruhat property of f and the explicit description $\widehat{G} = \mathbf{R}^m \times \mathbf{Z}^n \times \mathbf{T}^r \times \widehat{F}$, shows $\widehat{f}$ is smooth. The

[5] I am sure proving this requires clearing up the questions on measures posed in a prior footnote. Being off by a scalar will be harmless for this.

[6] Up to some nonzero scalar depending on the level? Harmless if so.

calculation even shows that all $\partial_{x_j}(\widehat{f})$'s and $\partial_{z_\ell}(\widehat{f})$'s are themselves Fourier transforms of Schwartz-Bruhat functions, and we can keep iterating. Thus, to verify the characterization in Lemma D.1.6.16 of the Schwartz-Bruhat property for smooth functions in the case of the smooth $\widehat{f}$ for general $f \in \mathcal{S}(G)$ it suffices to check that $p(D)\widehat{f}$ is bounded on $\widehat{G}$ for all $D \in \mathrm{Diff}(G)$. But this is immediate from (1) since $\mathcal{F}_G(Df) \in C_0(\widehat{G})$.

It remains to show the resulting linear map $\mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\widehat{G})$ is a topological isomorphism. But since f and $\widehat{f}$ are now known to always be L^1 for $f \in \mathcal{S}(G)$, Fourier inversion holds pointwise in both directlons by Lemma D.1.1.2(5). Hence, as long as $\mathcal{F}_G$ is shown to be continuous for general elementary abelian Lie groups G we can apply that to $\widehat{G}$ to get continuity in both directions (or use the Open Mapping Theorem for Fréchet spaces). We are going to harness (D.1.3.20) to reduce to the study of the factor groups $\mathbf{R}^m$, $\mathbf{T}^n$, $\mathbf{Z}^r$, and F, but we have to be careful since the completed tensor product there has no reason to interact well with an operation not yet known to be continuous.

To get around that problem, we will use Grothendieck's explicit description of elements of completed projective tensor products $E\widehat{\otimes}_\pi F$ of Fréchet spaces [**Scha**, Ch. III, 6.4]: everything in the completed tensor product can be written in the form

$$\sum_{j\geq 1} \lambda_j (e_j \otimes f_j)$$

for an absolutely summable sequence (λ_j) in $\mathbf{C}$ and sequences $(e_j) \subset E$ and $(f_j) \subset F$ that converge to 0. This will be applied to $\mathcal{S}(\mathbf{R}^m)\widehat{\otimes}_\pi \mathcal{S}(\mathbf{T}^n \times \mathbf{Z}^r \times F)$.

Before we cary that out, note that the Cauchy-Schwartz argument (as in [**KG**, Part I, Thm. 29, p. 72]) using multiplication and division by $1 + |x|^N + |a|^N$ for $N \geq 2m, 2r$ that shows the topology on $\mathcal{S}(G)$ can be defined using L^1-norms rather than sup-norms tells us that any sequence $\{\phi_j\}$ converging to 0 in $\mathcal{S}(G)$ has a uniform upper bound on all $\|\phi_j\|_1$'s, so in particular the Fourier transforms $\widehat{\phi_j}$ have uniformly bounded sup-norms over G. For $G_1 = \mathbf{R}^m$ and $G_2 = \mathbf{T}^n \times \mathbf{Z}^r \times F$, write a general $f \in \mathcal{S}(G) = \mathcal{S}(G_1)\widehat{\otimes}_\pi \mathcal{S}(G_2)$ in the form provided by Grothendieck:

$$f = \sum_{j\geq 1} \lambda_j f_j \otimes F_j$$

for an absolutely summable sequence (λ_j) in $\mathbf{C}$ and sequences $f_j \in \mathcal{S}(G_1)$ tending to 0 and $F_j \in \mathcal{S}(G_2)$ tending to 0.

By Fubini's Theorem, the Dominated Convergence Theorem, and the uniform bounds on the L^1-norms of all $\widehat{f_j}$'s and all $\widehat{F_j}$'s, we conclude that pointwise

$$\widehat{f}(\chi_1, \chi_2) = \sum_{j\geq 1} \widehat{f_j}(\chi_1)\widehat{F_j}(\chi_2)$$

for characters χ_i of G_i, and this is a uniformly convergent sum over $\widehat{G}$. It follows that *if* we know continuity of each $\mathcal{F}_{G_i}$ between Schwartz spaces then $\mathcal{F}_{G_1}\widehat{\otimes}_\pi \mathcal{F}_{G_2}$ makes sense on $\mathcal{S}(G)$ and coincides with $\mathcal{F}_G$, providing continuity of the latter. Thus, it suffices to separately treat G_1 and G_2.

We have peeled off $\mathbf{R}^m$ on its own, and in a similar way we can peel off $\mathbf{T}^n$ and then $\mathbf{Z}^r$ from G_2, so finally our task comes down to the separate cases of $\mathbf{R}^m$, $\mathbf{T}^n$, $\mathbf{Z}^r$, and F. The case of F is trivial and the case of $\mathbf{R}^m$ (for which we only need to treat $m = 1$) follows from the classical interaction of derivative and multiplication operators through the Euclidean Fourier transform. The remaining cases of $\mathbf{T}^n$ and $\mathbf{Z}^r$ are treated by [**Tre**, Thm. 51.3].

(3) : The identity (D.1.6.25) is a consequence[7] of the Poisson Summation Formula in its form of [**DE**, Thm.3.6.3].[8] □

[7] Really? Need to say something about the calculation to make clear it isn't hiding some real work with the measures to check it works on the nose.

[8] Why is an external reference given here rather than citing (D.1.2.8) already stated in this document?

REMARK D.1.6.20. We note that by Lemma D.1.6.19(1), symbols of differential operators on $\widehat{G}$ are multipliers of the Schwartz-Bruhat space of G; i.e., for any $D \in \mathrm{Diff}(\widehat{G})$ and $f \in \mathcal{S}(G)$, $p(D) \cdot f \in \mathcal{S}(G)$.

For $G = \mathbf{R}$, the Lemma recovers the familiar compatibilities between Fourier transform and derivatives. Indeed, by Example D.1.6.12, $\sigma(\partial_x) = x$ as a function on $\mathbf{R}$ viewed as the character group of $\mathbf{R}$ itself, and (D.1.6.23) reads $\widehat{\partial_x f} = 2\pi i\, x \cdot \widehat{f}$ by Corollary D.1.6.15, which is [**Fol**, Ch.8,Thm.8.22(e)].

D.1.7. Distributional Poisson Summation. We record here the following version of the Poisson Summation Formula (see Theorem D.1.2.7), with the aim of using it throughout Chapter **??** of this work. It is used in combination with Theorem 4.6.2.1, Theorem 4.6.1.5 and Lemma 4.4.2.6, to construct absolute Schauder bases in Theorem 7.3.1.6.

THEOREM D.1.7.1 (Distributional Poisson Summation). *Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G. Let $k \subset G$ be a closed subgroup, and μ_k a choice of Haar measure on k. Let $\mu_{G/k}$ be the unique Haar measure on G/k determined by the choices μ_G and μ_k as per Theorem D.1.2.1.*

The linear map:

$$\Phi_k : \mathcal{S}(G) \to C(G/k) \tag{D.1.7.1}$$

defined in (D.1.2.7) *by*

$$\Phi_k(\phi)(x) := \int_k \phi(hx)\,\mathrm{d}\mu_k(h) \tag{D.1.7.2}$$

for all $x \in G$ is valued in $\mathcal{S}(G/k)$. The induced linear operator $\Phi_k : \mathcal{S}(G) \to \mathcal{S}(G/k)$ satisfies the following properties:

(1) *The following identity holds for all $\phi \in \mathcal{S}(G)$, with $\widehat{\phi}$ the Fourier transform $\widehat{\phi}(\chi) := \int_G \phi(x)\overline{\chi}(x)\,\mathrm{d}\mu_G(x)$, and for all $x \in G$:*

$$\Phi_k(\phi)(x) = \int_{k^\perp} \widehat{\phi}(\chi)\chi(x)\,\mathrm{d}\mu_{k^\perp}(\chi) \tag{D.1.7.3}$$

where $\mu_{k^\perp}$ is the Haar measure on $k^\perp = \widehat{G/k} \subset \widehat{G}$ dual to $\mu_{G/k}$.

(2) *The map Φ_k is a continuous and surjective. If G is second-countable then it is a topological quotient map.*

PROOF. When G is an elementary abelian Lie group, everything apart from the identity in (1) is handled by Lemma D.1.6.18. We shall reduce all but (1) for general G to such cases. The space $\mathcal{S}(G)$ is the locally convex inductive limit of spaces $\mathcal{S}(k', \Gamma')$ for structural pairs (k', Γ') for G. Since $k\Gamma'/k$ is an open and compactly generated subgroup of G/k in which $k'k/k'$ is a compact subgroup whose annihilator $k^\perp \cap {k'}^\perp \subset k^\perp = \widehat{G/k}$ is open in $\widehat{G/k}$ (as ${k'}^\perp$ is open in $\widehat{G}$) and compactly generated (as it is an open subgroup in ${k'}^\perp$ due to openness of $k^\perp$ in $\widehat{G}$, hence a closed subgroup of the compactly generated ${k'}^\perp$, so Theorem D.1.4.3(1) applies), we see that $(k'k/k', k\Gamma'/k)$ is a structural pair for G/k. The space $\mathcal{S}(G/k)$ is the locally convex inductive limit of spaces $\mathcal{S}(k'k/k, k\Gamma'/k)$ (since $G \to G/k$ is proper, so pairs of that special form are cofinal among all structural pairs for G/k).

Consider Φ_k applied to $f \in \mathcal{S}(k', \Gamma') = \mathcal{S}(\Gamma'/k')$. The resulting k-invariant function $\Phi_k(f)$ viewed on G/k is $k'k/k$-invariant due to the k'-invariance of f, so averaging $\Phi_k(f)$ over $k'k/k$ has the effect of multiplying it by a volume for $k'k/k$ having nothing to do with f. But that further average is $\Phi_{k'k}(f)$, and $\Phi_{k'k}$ on $\mathcal{S}(k', \Gamma')$ is a nonzero scalar multiple of $\Phi_{k'k/k'}$ on $\mathcal{S}(\Gamma'/k')$. The settled case of elementary abelian Lie groups tells us that

$$\Phi_{k'k/k'} : \mathcal{S}(\Gamma'/k') \to C(k\Gamma'/kk')$$

lands inside $\mathcal{S}(k\Gamma'/kk')$ and the resulting map $\mathcal{S}(\Gamma'/k') \to \mathcal{S}(k\Gamma'/kk')$ is a continuous surjection. Thus, Φ_k in general lands inside $\mathcal{S}(G/k)$ with $\Phi_k : \mathcal{S}(G) \to \mathcal{S}(G/k)$ a continuous surjection. When G is second-countable, both $\mathcal{S}(G)$ and $\mathcal{S}(G/k)$ are strict LF-spaces. The Open Mapping Theorem for LF-spaces [**Gro-f**, Ch. 4, §5, Thm. 2(a), p. 148] then ensures Φ_k is a quotient map for such G.

It remains to prove (D.1.7.3). This is an application of Fourier inversion. Since $\Phi_k(\phi)$ and its Fourier transform are respectively Schwartz-Bruhat on G/k and $\widehat{G/k} = k^\perp$, it satisfies Fourier inversion pointwise everywhere on G/k by Lemma D.1.1.2(5). Since $\mu_{k^\perp}$ is the dual measure to $\mu_{G/k}$, for all $x \in G/k$ we have

$$\Phi_k(\phi)(x) = \int_{k^\perp} \mathcal{F}_{G/k}(\Phi_k(\phi))(\chi)\chi(x)\,\mathrm{d}\mu_{k^\perp}(x)$$

where $\mathcal{F}_{G/k}$ is defined using integration against $\mu_{G/k}$. Thus, it suffices to show

$$\mathcal{F}_{G/k}(\Phi_k(\phi))(\chi) = \widehat{\phi}(\chi)$$

for $\chi \in \widehat{G/k} = k^\perp$, where $\widehat{\phi}$ is defined using integration against μ_G.

By definition,

$$\widehat{\phi}(\chi) = \int_G \phi(g)\overline{\chi}(g)\,\mathrm{d}\mu_G(g) = \int_{G/k}\left(\int_k \phi(hg)\overline{\chi}(hg)\,\mathrm{d}\mu_k(h)\right)\,\mathrm{d}\mu_{G/k}(\overline{g}).$$

In the integrand, $\chi(hg) = \chi(g)$ since $\chi \in k^\perp$, so the inner integral becomes

$$\int_k \phi(hg)\overline{\chi}(g)\,\mathrm{d}\mu_k(h) = \Phi_k(\phi)(g)\overline{\chi}(g).$$

This is what we needed. □

D.1.8. Support of tempered distributions on Hausdorff locally compact abelian groups. Let G be a Hausdorff locally compact abelian group. Recall that $\mathcal{S}(G)$ is endowed with a natural Hausdorff locally convex topology, propagated from the special case G is an elementary abelian Lie group (see §D.1.5, and [**DE**, Ch. 3, 4]) using the structure theory of Hausdorff locally compact abelian groups. In Theorem D.1.5.7 (which is [**Bru**, §10, Prop. 11]) the main properties of the Hausdorff locally convex topology on $\mathcal{S}(G)$ and its strong dual $\mathcal{S}(G)'_\beta$ were recorded when G is second-countable.

We refer to $\mathcal{S}(G)'_\beta$ (the continuous dual of $\mathcal{S}(G)$, equipped with the Hausdorff locally convex "strong dual" topology of uniform convergence on the bounded subsets of $\mathcal{S}(G)$) as the space of *tempered distributions* on G. When G is an elementary abelian Lie group, say $G \simeq \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$, the description of $\mathcal{S}(G)$ as a completed tensor product in (D.1.3.20) implies density of the image of the continuous injection $\mathcal{D}(G) := C_c^\infty(G) \to \mathcal{S}(G)$ by reduction to the classical case of $\mathbf{R}^m$ [**KG**, Thm. 30, p. 74] (where the proof for $m = 1$ easily adapts to general m) and the straightforward case of $\mathbf{Z}^r$, so passage to direct limit with subspaces $\mathcal{S}(k, \Gamma)$ then gives the same density result for general G. In particular, the induced map on topological linear duals $\mathcal{S}(G)' \to \mathcal{D}(G)'$ is *injective* for general G. In this way we view the space of tempered distributions as a special subspace of the space of all distributions (continuous linear functionals on compactly supported smooth functions), equipped with a finer strong dual topology.

D.1.8.1. *Kernel Theorems.* We let $(m, k, r) \in \mathbf{N}^3$ be a triple of non-negative integers, F a finite abelian group, and consider the elementary abelian Lie group:

$$G := \mathbf{R}^m \times \mathbf{T}^k \times \mathbf{Z}^r \times F.$$

Describing tempered distributions on G amounts to describing tempered distributions on $\mathbf{R}^m$, $\mathbf{T}^k$, $\mathbf{Z}^r$ and F, respectively, in light of the following:

LEMMA D.1.8.1 (Kernel Theorem for Schwartz-Bruhat spaces). *Let G_1, G_2 be Hausdorff locally compact abelian groups. Assume G_1 and G_2 are second-countable. Then:*

(1) The canonical linear map from $\mathcal{S}(G_1)\otimes\mathcal{S}(G_2)$ to $\mathcal{S}(G_1\times G_2)$ is continuous with respect to the inductive tensor product topology on the source, and it extends to a continuous linear map

$$\mathcal{S}(G_1)\,\overline{\otimes}\,\mathcal{S}(G_2)\to\mathcal{S}(G_1\times G_2) \tag{D.1.8.1}$$

which is a linear topological isomorphism.

(2) The canonical linear map $\mathcal{S}(G_1)'\otimes\mathcal{S}(G_2)'\to(\mathcal{S}(G_1)\otimes_\iota\mathcal{S}(G_2))'=(\mathcal{S}(G_1)\overline{\otimes}\mathcal{S}(G_2))'\simeq\mathcal{S}(G_1\times G_2)'$ extends to a continuous linear map

$$\mathcal{S}(G_1)'_\beta\,\widehat{\otimes}_\pi\mathcal{S}(G_2)'_\beta\to\mathcal{S}(G_1\times G_2)'_\beta \tag{D.1.8.2}$$

which is a topological isomorphism.

In (D.1.8.2) it is equivalent by Corollary C.2.3.12 to use $\widehat{\otimes}_\varepsilon$ since the spaces $\mathcal{S}(G_j)'_\beta$ are nuclear by Theorem D.1.5.7. For the proof of (2), we further refer the reader to [**Al-B**, Lemma 3, p. 803].

PROOF. (1) : Assume first that G_1 and G_2 are elementary abelian Lie groups. In this case $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are Fréchet spaces, so the inductive tensot product topology on $\mathcal{S}(G_1)\otimes\mathcal{S}(G_2)$ coincides with the projective tensor product topology by [**Scha**, Ch. III, 5.1]. Hence, the desired result in these cases is immediate from Lemma D.1.3.7(2).

In the general case, we note that for any second-countable Hausdorff locally compact abelian group G, the Hausdorff locally convex topological vector space $\mathcal{S}(G)$ is a strict LF-space since it is the inductive limit of Fréchet spaces $\mathcal{S}(k,\Gamma)\simeq\mathcal{S}(\Gamma/k)$ as (k,Γ) varies over a *countable* cofinal system of structural pairs for G (see Definition D.1.5.1); i.e., $\Gamma\subset G$ is an open and compactly generated subgroup, and $k\subset G$ is a compact subgroup contained in Γ whose open orthogonal is compactly generated (noting that Γ/k is therefore an abelian Lie group). That such a countable collection of pairs exists uses that G is second-countable. The desired assertion then follows from Corollary C.2.3.12 and (the end of) Proposition C.2.3.13(2); the latter result rests on the countability aspect noted above.

(2) : If G_1 and G_2 are elementary abelian Lie groups then $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are nuclear Fréchet spaces, so by (1) and Corollary C.2.3.12 the assertion is that the formation of the strong duals of these Schwartz spaces "commutes" with completed projective tensor products. This is a general property of nuclear Fréchet spaces [**Tre**, Prop. 50.7] (bearing in mind also that the strong dual of a nuclear Fréchet space is always nuclear [**Tre**, Prop. 50.6]).

In general, by expressing each $\mathcal{S}(G_j)$ as a countable strict inductive limit $\varinjlim\mathcal{S}(\Gamma_{m,j}/k_{m,j})$ (whose topological embedding transition maps are closed embeddings by completeness of each $E_m:=\mathcal{S}(\Gamma_{m,1}/k_{m,1})$ and $F_m:=\mathcal{S}(\Gamma_{m,2}/k_{m,2})$), the argument at the end of the proof of Lemma C.1.6.1(2) gives the topological identifications

$$\mathcal{S}(G_1)'_\beta=\lim E'_{m,\beta},\ \ \mathcal{S}(G_2)'_\beta=\lim F'_{m,\beta}.$$

(with all transition maps surjective). Since $\mathcal{S}(G_1\times G_2)'_\beta$ is compatibly identified as a topological vector space with the inverse limit of the spaces $(\mathcal{S}(\Gamma_{m,1}/k_{m,1})\widehat{\otimes}_\pi\mathcal{S}(\Gamma_{m,2}/k_{m,2}))'_\beta$ by (1), the desired result thereby reduces to checking that for the countable inverse systems $\{E'_{m,\beta}\}$ and $\{F'_{m,\beta}\}$, the natural continuous linear map

$$\lim E'_{m,\beta}\widehat{\otimes}_\pi\lim F'_{m,\beta}\to\lim(E_m\widehat{\otimes}_\pi F_m)'_\beta=\lim(E'_{m,\beta}\widehat{\otimes}_\pi F'_{m,\beta}) \tag{D.1.8.3}$$

is a topological isomorphism (the passage of strong dual through the completed projective tensor product is an instance of (1)).

By [**Jar**, Section 16.3, Cor. 2], the formation of completed *injective* tensor products $\widehat{\otimes}_\varepsilon$ commutes with inverse limits of Hausdorff topological vector spaces in the natural manner. Thus, it suffices to show that the completed projective tensor products on each side of (D.1.8.3) are identified with the corresponding

completed injective tensor products (via the natural map from $\widehat{\otimes}_\pi$ to $\widehat{\otimes}_\varepsilon$). For this it suffices by Corollary C.2.3.12 to check that one of the factor spaces in each such completed tensor product is nuclear. By [**Tre**, Prop. 50.6] each $E'_{m,\beta}$ and $F'_{m,\beta}$ is nuclear due to the E_m's and F_m's being nuclear Fréchet, so the right side of (D.1.8.3) is settled. For the left side of (D.1.8.3), we use that an inverse limit of nuclear spaces is nuclear [**Scha**, Ch. III, Cor. to 7.4]. □

D.1.8.2. *Supports.* We recall that a tempered distribution T on $G = \mathbf{T}^n$ is simply a distribution, since $\mathcal{S}(\mathbf{T}^n) = \mathcal{D}(\mathbf{T}^n)$ $(:= C_c^\infty(\mathbf{T}^n))$. In particular, the *support* of T is (by definition) the smallest closed subset K of $\mathbf{T}^n$ such that $\langle T, f\rangle_{\mathcal{S}(\mathbf{T}^n)',\mathcal{S}(\mathbf{T}^n)} = 0$ for any $f \in \mathcal{S}(\mathbf{T}^k)$ supported in $\mathbf{T}^n - K$. Since the support of any $f \in \mathcal{S}(\mathbf{T}^n)$ is compact, using a smooth partition of unity it is equivalent to say that the support of T is the closed subset $K \subset \mathbf{T}^n$ whose complement is the union of those open $U \subset \mathbf{T}^n$ such that $\langle T, f\rangle_{\mathcal{S}(\mathbf{T}^n)',\mathcal{S}(\mathbf{T}^n)} = 0$ for every Schwartz function f on $\mathbf{T}^n$ supported in U.

The notion of *support* of a tempered distribution on $\mathbf{R}^m$ is defined by adapting the notion of support of distributions on $\mathbf{R}^m$ as in [**Horm**, Ch. II, §2.2, Def. 2.2.2]: it is the closed subset $K \subset \mathbf{R}^m$ whose complement is the union of those open $U \subset \mathbf{R}^m$ such that $\langle T, f\rangle_{\mathcal{S}(\mathbf{R}^m)',\mathcal{S}(\mathbf{R}^m)} = 0$ for every Schwartz function f on $\mathbf{R}^m$ supported over U.

DEFINITION D.1.8.2. Let G be a Hausdorff locally compact abelian group. Let $T \in \mathcal{S}(G)'$ be a tempered distribution on G.

(1) The *vanishing locus* of T is defined to be the union of all open subsets $U \subset G$ such that $\langle T, f\rangle_{\mathcal{S}(G)',\mathcal{S}(G)} = 0$ for every $f \in \mathcal{S}(G)$ supported in U.
(2) The *support* of T, denoted $\mathrm{supp}(T) \subset G$, is defined to be the complement of $V(T)$ in G.

PROPOSITION D.1.8.3. *Let $G = \mathbf{R}^m \times \mathbf{T}^k \times \mathbf{Z}^r \times F$ be an elementary abelian Lie group. Let $T \in \mathcal{S}(G)'$ be a tempered distribution of the form $T_1 \otimes T_2 \otimes T_3 \otimes T_4$, for $T_1 \in \mathcal{S}(\mathbf{R}^m)'$, $T_2 \in \mathcal{S}(\mathbf{T}^n)'$, $T_3 \in \mathcal{S}(\mathbf{Z}^r)'$, $T_4 \in \mathcal{S}(F)'$. Then we have:*

$$\mathrm{supp}(T) = \mathrm{supp}(T_1) \times \mathrm{supp}(T_2) \times \mathrm{supp}(T_3) \times \mathrm{supp}(T_4). \tag{D.1.8.4}$$

PROOF. Define $G_1 = \mathbf{R}^m$, $G_2 = \mathbf{T}^n$, $G_3 = \mathbf{Z}^r$, $G_4 = F$. For the containment "$\supset$", we prove the reverse for the complements: if $\xi \in G - \mathrm{supp}(T)$ then we want $\xi \notin \mathrm{supp}(T_1) \times \mathrm{supp}(T_2) \times \mathrm{supp}(T_3) \times \mathrm{supp}(T_4)$. By the definition of $\mathrm{supp}(T)$, there are open sets $U_i \subset G_i$ such that $\xi \in U := U_1 \times U_2 \times U_3 \times U_4$ and T kills any $f \in \mathcal{S}(G)$ supported inside U. It suffices to show $U_j \subset G_j - \mathrm{supp}(T_j)$ for some j. If not, for each $1 \le j \le 4$ there exists $f_j \in \mathcal{S}(G_j)$ supported inside U_j such that $\langle T_j, f_j\rangle_{\mathcal{S}(G_j)',\mathcal{S}(G_j)} \neq 0$. Then for $f = f_1 \otimes \cdots \otimes f_4 \in \mathcal{S}(G)$ we have $\langle T, f\rangle_{\mathcal{S}(G)',\mathcal{S}(G)} = \prod \langle T_j, f_j\rangle_{\mathcal{S}(G_j)',\mathcal{S}(G_j)} \neq 0$. But $\mathrm{supp}(f) \subset \prod \mathrm{supp}(f_j) \subset U$, so this contradicts that T kills all Schwartz-Bruhat functions on G supported inside U.

For the reverse containment, if $\xi \in G$ has jth component $\xi_j \in G_j - \mathrm{supp}(T_j)$ for some $1 \le j \le 4$ then we want to show $\xi \notin \mathrm{supp}(T)$. By definition of $\mathrm{supp}(T_j)$, there exists an open $U_j \subset G_j$ containing ξ_j so that every $f_j \in \mathcal{S}(G_j)$ supported inside U_j is killed by T_j. Shrinking U_j allows us to assume it has compact closure. It suffices to show any $f \in \mathcal{S}(G)$ supported inside $(\prod_{i \neq j} G_i) \times U_j$ is killed by T. We will do this by first passing to the compactly supported case.

The construction in the proof of density of $C_c^\infty(\mathbf{R}^m)$ inside $\mathcal{S}(\mathbf{R}^m)$ in [**KG**, p. 76] via $\chi_N \varphi \to \varphi$ for $\chi_N(x) = \chi(x/N)$ for a smooth function χ equal to 1 on $[-1,1]^m$ and supported inside $[-2,2]^m$ does not increase support, so we can apply it at the level of the completed tensor product description of $\mathcal{S}(G)$ in (D.1.3.20). We can make a similar density proof for $C_c(\mathbf{Z}^r)$ inside $\mathcal{S}(\mathbf{Z}^r)$ by using $\chi_N : \mathbf{Z}^r \to \mathbf{C}$ that is the characteristic function of $[-N, N]^r$. In this way, we can express any $f \in \mathcal{S}(G)$ supported inside $(\prod_{i \neq j} G_i) \times U_j$ as a limit of a sequence of functions $f_k \in C_c^\infty(G)$ supported inside $(\prod_{i \neq j} G_i) \times U_j$. By continuity of T it

suffices to show $\langle T, f_k\rangle_{\mathcal{S}(G)',\mathcal{S}(G)} = 0$ for all k. In other words, it is enough to treat the case of compactly supported f.

The compact support $K \subset G$ of f has compact image $K_j \subset U_j$. We can pick a smooth function ψ on U_j equal to 1 on K_j and having compact support $K_j' \subset U_j$. Multiplication by ψ on $\mathcal{S}(G_j)$ has image inside the closed subspace $C^\infty_{K_j'}(G_j)$ of smooth functions on G_j supported inside $K_j' \subset U_j$, so

$$f = \psi f \in C^\infty_{K_j'}(G_j)\widehat{\otimes}_\pi \mathcal{S}(\prod_{i\neq j} G_i).$$

Since T_j kills $C^\infty_{K_j'}(G_j)$, we conclude that T kills f. □

LEMMA D.1.8.4. *Let G be a Hausdorff locally compact abelian group, $T \in \mathcal{S}(G)'$ a tempered distribution. For each structural pair (k,Γ) for G (i.e., $\Gamma \subset G$ an open compactly generated subgroup, $k \subset G$ a compact subgroup with open compactly generated orthogonal $k^\perp$, and $k \subset \Gamma$; see Definition* D.1.5.1*), let $T_{(k,\Gamma)} \in \mathcal{S}(\Gamma/k)'$ denote composition of T with the continuous map $\mathcal{S}(\Gamma/k) \simeq \mathcal{S}(k,\Gamma) \hookrightarrow \mathcal{S}(G)$ (see Definition* D.1.5.3*).*

The open subset $V(T) = G - \mathrm{supp}(T) \subset G$ is contained in the union of the open preimages under $\Gamma \to \Gamma/k$ of $V(T_{(k,\Gamma)}) \subset \Gamma/k$ as (k,Γ) varies over the directed set of all structural pairs of G.

PROOF. A point $\overline{g} \in \Gamma/k$ belongs to $V(T_{(k,\Gamma)})$ if and only if there exists an open neighborhood $\overline{U} \subset \Gamma/k$ of $\overline{g}$ such that $\langle T_{(k,\Gamma)}, f\rangle = 0$ for every $f \in \mathcal{S}(\Gamma/k)$ supported in $\overline{U}$. This vanishing condition says

$$\langle T, \widetilde{f}\rangle_{\mathcal{S}(G)',\mathcal{S}(G)} = 0$$

for $\widetilde{f}$ the extension by zero from Γ to G of the composition of f with the quotient homomorphism $\Gamma \to \Gamma/k$. Such f's are exactly the k-invariant Schwartz-Bruhat functions supported in the open preimage U of $\overline{U}$ in Γ.

Now consider a point $g \in V(T)$. There exists an open and compactly generated subgroup $\Gamma \subset G$ containing g and a compact subgroup $k \subset G$ contained in Γ and with compactly generated orthogonal $k^\perp \subset \widehat{G}$. Note that Γ is necessarily also closed in G. By definition of $V(T)$, there exists an open neighborhood $U \subset G$ of g such that $\langle T, f\rangle_{\mathcal{S}(G)',\mathcal{S}(G)} = 0$ for every $f \in \mathcal{S}(G)$ whose support is contained in U. We can shrink U around g so that $U \subset \Gamma$. By Lemma D.1.4.7, we can shrink k so that $kg \subset U \subset \Gamma$, and then find an open $V \subset U$ around g so that $kV \subset U$. Thus, we can replace U with kV to arrange that U is the preimage in Γ of an open subset $\overline{U} \subset \Gamma/k$ containing $\overline{g} := g \bmod k$. This implies $\overline{U} \subset V(T_{(k,\Gamma)})$, so since the initial arbitrary choice of $g \in V(T)$ belongs to U, we have shown $V(T)$ is contained in the desired union. □

D.1.9. Continuity of translation on Schwartz-Bruhat spaces. With the aim to make Lemma 4.3.3.1 and its corollaries applicable, we discuss continuity properties of the left-translation action of a Hausdorff locally compact abelian group G on its Schwartz-Bruhat space $\mathcal{S}(G)$. Since both G and $\mathcal{S}(G)$ are generally studied via direct limits from the case of elementary abelian Lie groups, we begin with a basic lemma on inheritance of continuity through locally convex inductive limits.

LEMMA D.1.9.1. *Let H be a Hausdorff locally compact topological group that is the union of an increasing sequence of open subgroups $\{H_n\}_{n\geq 1}$. Let $\{E_n\}$ be a strict inductive system of Hausdorff locally convex topological vector spaces such that every E_n is barrelled, and let $E = \varinjlim E_n$.*

If an action of H on E makes H_n act continuously on E_n for every n then theaction map $H \times E \to E$ is continuous.

PROOF. The space E is barrelled by [**Scha**, Ch. II, Cor. 1 to 7.2]. Each $h \in H$ lies in some H_{n_0}, so the h-action on E carries E_n into itself for all $n \geq n_0$ with $h : E_n \to E_n$ a topological isomorphism for such n (since H_n acts continuously on E_n). Thus, $h : E \to E$ is a topological isomorphism, so the H-action on E is encoded in a group homomorphism $\rho : H \to \mathcal{L}(E)$ into the vector space of continuous linear endomorphisms of E. Since H is locally compact, continuity for its action on E is equivalent to the conjunction of two properties:

(i) equicontinuity at $0 \in E$ for the subset $\rho(K) \subset \mathcal{L}(E)$ for every compact subset $K \subset H$, (ii) for each $x \in E$ and 0-neighborhood $U \subset E$ the existence of an e-neighborhood $\Omega \subset H$ such that $\Omega.x \subset x + U$.

First we verify (i). The barrelledness of E ensures (by [**Scha**, Ch. III, 4.2]) that equicontinuity holds if the subset $K.x \subset E$ is weakly bounded for every $x \in E$: that is, $\{\ell(k.x)\}_{k\in K}$ is bounded in $\mathbf{C}$ for each continuous linear functional $\ell \in E'$. Thus, it suffices that $K.x$ is compact for each $x \in E$. The open cover $\{H_n \cap K\}$ of K has a finite subcover, so $K \subset H_n$ for sufficiently large n. By taking n big enough, also $x \in E_n$. Since H_n acts continuously on E_n, we see that $K.x$ is compact inside E_n and so the boundedness of $\ell(K.x)$ follows.

To verify (ii), pick n for which $x \in E_n$. Since H_n acts continuously on E_n, there is an e-neighborhood $\Omega \subset H_n$ such that $\Omega.x \subset x + (U \cap E_n) \subset x + U$. Since H_n is open in H, we are done. □

PROPOSITION D.1.9.2. *Let G be a second-countable Hausdorff locally compact abelian group. The left-translation action $\rho : G \times \mathcal{S}(G) \to \mathcal{S}(G)$ is continuous. In particular, for any compact subset $K \subset G$ the map $K \times \mathcal{S}(G) \to \mathcal{S}(G)$ is uniformly continuous over K in the sense that for any 0-neighborhood $U \subset \mathcal{S}(G)$ there is a 0-neighborhood $V \subset \mathcal{S}(G)$ such that $K.V \subset U$.*

PROOF. If the G-action is continuous then the uniform continuity property for compact subsets $K \subset G$ is immediate from the fact that $k.0 = 0$ for all $k \in K$. To prove the continuity of the G-action we first reduce to the case that G is an elementary abelian Lie group. Since G is second-countable, there is a countable directed system of structural pairs (k_n, Γ_n) (see Definition D.1.5.1) such that $\operatorname{colim} \mathcal{S}(k_n, \Gamma_n) \to \mathcal{S}(G)$ is a linear topological isomorphism with each injection $\mathcal{S}(\Gamma_n/k_n) = \mathcal{S}(k_n, \Gamma_n) \to \mathcal{S}(G)$ a topological embedding. We apply Lemma D.1.9.1 to $H = G$ with $H_n = \Gamma_n$ and $E_n = \mathcal{S}(k_n, \Gamma_n) \simeq \mathcal{S}(\Gamma_n/k_n)$ (so $E = \mathcal{S}(G)$). Note that the barrelledness of such E_n's follows from $\mathcal{S}(A)$ being a Fréchet space for elementary abelian Lie groups A, since every Fréchet space is barrelled [**Scha**, Ch. II, Cor. to 7.1]. This reduces the continuity task to the case of the groups Γ_n/k_n that are elementary abelian Lie groups.

We can write $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ for integers $m, n, r \geq 0$ and a finite abelian group F. By (D.1.3.20), $\mathcal{S}(G)$ is the completed projective tensor product of $\mathcal{S}(\mathbf{R}^m)$, $\mathcal{S}(\mathbf{T}^n)$, $\mathcal{S}(\mathbf{Z}^r)$, and $\mathcal{S}(F) = \mathbf{C}^F$. The left G-action on $\mathcal{S}(G)$ preserves the subspace given by the "uncompleted" projective tensor product. To reduce the continuity verification to the action of G on the uncompleted projective tensor product, we apply:

LEMMA D.1.9.3. *Let E be a Hausdorff topological vector space and H a topological group equipped with a left action on the completion $\widehat{E}$ preserving the subspace E. If each $h \in H$ acts continuously on $\widehat{E}$ and the action map $H \times E \to E$ is continuous then the action map $H \times \widehat{E} \to \widehat{E}$ is continuous.*

PROOF. Since H is a topological group whose elements individually act continuously on $\widehat{E}$ (by hypothesis), it suffices to prove the action map on $\widehat{E}$ is continuous at (e, x) for each $x \in \widehat{E}$. A base of x-neighborhoods in $\widehat{E}$ is given by $x + \overline{U}$ for 0-neighborhoods $U \subset E$. Thus, for each such U we seek a 0-neighborhood $V \subset E$ and an e-neighborhood $W \subset H$ such that $W.(x + \overline{V}) \subset x + \overline{U}$. Since each $h \in H$ acts continuously on $\widehat{E}$, it is equivalent to show $W.(x + V) \subset x + \overline{U}$ for some W and V. This holds for $x = 0$: the assumed continuity of the H-action on E provides a 0-neighborhood $V \subset E$ and an e-neighborhood $W \subset H$ such that $W.V \subset U \subset \overline{U}$.

To handle general $x \in \widehat{E}$, pick a 0-neighborhood $U' \subset E$ so that $U' + U' \subset U$ and then a 0-neighborhood U'' so that $U'' + U'' - U'' \subset U'$ By the verified continuity at $(e, 0)$, there exists an e-neighborhood $W' \subset H$ and 0-neighborhood $U_0 \subset E$ such that $W'.\overline{U}_0 \subset \overline{U''}$. Thus, for $x_0 \in E$ with $x - x_0 \in \overline{U''}$ we have

$$W'.x \subset W'.(\overline{U}_0) + W'.x_0 \subset \overline{U''} + W'.x_0.$$

By the assumed continuity of the H-action on E, we can find an e-neighborhood $W'' \subset W'$ so that $W''.x_0 \subset x_0 + U''$. Thus,

$$W''.x \subset \overline{U''} + W''.x_0 \subset \overline{U''} + U'' + x_0 \subset \overline{U''} + U'' - \overline{U''} + x \subset x + \overline{U'}.$$

The continuity of the H-action on E also provides an e-neighborhood $W \subset W''$ and a 0-neighborhood $V \subset E$ such that $W.V \subset U'$. Hence,

$$W.(x+V) \subset W.x + W.V \subset W''.x + U' \subset x + \overline{U'} + U' \subset x + \overline{U}$$

as desired. □

We are reduced to proving the continuity of the G-action on the projective tensor product $\mathcal{S}(\mathbf{R}^m) \otimes_\pi \mathcal{S}(\mathbf{T}^n) \otimes_\pi \mathcal{S}(\mathbf{Z}^r) \otimes_\pi \mathcal{S}(F)$. To further reduce to the separate treatment of the cases $G = \mathbf{R}^m, \mathbf{T}^n, \mathbf{Z}^r, F$, we next apply:

LEMMA D.1.9.4. *If H is a topological group acting continuously on Hausdorff locally convex topological vector spaces E_1, E_2 then the natural action of H on $E_1 \otimes_\pi E_2$ is continuous.*

PROOF. Since H is a topological group whose elements individually act continuously on $\widehat{E}$ (by hypothesis), it suffices to prove continuity at (e,x) for each $x \in E_1 \otimes E_2$. By writing x as a finite sum of elementary tensors, we reduce to the case that $x = x_1 \otimes x_2$ is an elementary tensor. A neighborhood base of x in $E_1 \otimes_\pi E_2$ is given by the x-translates of the convex circled hulls of the sets $U_1 \otimes U_2$ of elementary tensors $u_1 \otimes u_2$ with $u_j \in U_j$ for a base of convex circled 0-neighborhoods $U_j \subset E_j$. For a choice of U_1 and U_2, pick convex circled 0-neighborhoods $U_j'' \subset U_j' \subset E_j$ such that

$$\mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') \subset \mathrm{cch}(U_1 \otimes U_2)$$

and

$$x_1 \otimes U_2'' \subset \mathrm{cch}(U_1' \otimes U_2'), \;\; U_1'' \otimes x_2 \subset \mathrm{cch}(U_1' \otimes U_2')$$

(we can use $U_j'' = \lambda U_j'$ for a nonzero small $\lambda \in \mathbf{C}$ such that $\lambda x_i \in U_i'$ for $i = 1, 2$).

Continuity of the H-action on each E_j provides a convex circled 0-neighborhood $V_j \subset E_j$ and an e-neighborhood $W_j \subset H$ so that $W_j.V_j \subset U_j'$ and $W_j.x_j \subset x_j + U_j''$. Thus, $W := W_1 \cap W_2$ carries the convex circled hull of $V_1 \otimes V_2$ into that of $U_1' \otimes U_2'$ and carries $x = x_1 \otimes x_2$ into $x + x_1 \otimes U_2'' + U_1'' \otimes x_2 + U_1' \otimes U_2'$ (recall $U_j'' \subset U_j'$). This implies

$$\begin{aligned} W.(x + \mathrm{cch}(V_1 \otimes V_2)) &\subset x + x_1 \otimes U_2'' + U_1'' \otimes x_2 + U_1' \otimes U_2' + \mathrm{cch}(U_1' \otimes U_2') \\ &\subset x + \mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') + \mathrm{cch}(U_1' \otimes U_2') \\ &\subset x + \mathrm{cch}(U_1 \otimes U_2). \end{aligned} \tag{D.1.9.1}$$

The verifies the continuity of the action at (e,x) for general $x \in \widehat{E}$. □

We are reduced to treating the separate cases $G = \mathbf{R}^m, \mathbf{T}^n, \mathbf{Z}^r, F$. The case $G = F$ a finite abelian group holds because the natural F-action on $\mathbf{C}^F$ is clearly continuous. Next suppose $G = \mathbf{R}^m$. It suffices to prove two properties: the continuity of $G \times \mathcal{S}(G) \to \mathcal{S}(G)$ at (e,f) for each $f \in \mathcal{S}(G)$ and the equicontinuity at $0 \in \mathcal{S}(G)$ for each compact subset $K \subset G$. For each $f \in \mathcal{S}(\mathbf{R}^m)$, 0-neighborhood $U \subset \mathcal{S}(\mathbf{R}^m)$, and compact subset $K \subset \mathbf{R}^m$, we seek a 0-neighborhood $\Omega \subset \mathbf{R}^m$ and a 0-neighborhood $V \subset \mathcal{S}(\mathbf{R}^m)$ such that $\Omega.f \subset f + U$ and $K.V \subset U$. This says (i) if Ω is sufficiently small then $\Omega.f$ is sufficiently near f, and (ii) K is equicontinuous at 0.

For (i), we use the family of seminorms (D.1.3.8) defining the topology on $\mathcal{S}(\mathbf{R}^m)$. Given $\varepsilon > 0$, $j \in \mathbf{N}$, and $\alpha \in \mathbf{N}^m$ it suffices to show that for all $c \in \mathbf{R}^m$ sufficiently near 0 (depending on ε, j, α and f)

$$\sup_{x \in \mathbf{R}^m} |(1+|x|^j)\partial_x^\alpha f(x+c) - (1+|x|^j)\partial_x^\alpha f(x)| \leq \varepsilon.$$

Since

$$(1+|x+c|^j) - (1+|x|^j) = \left(1 - \frac{1+|x|^j}{1+|x+c|^j}\right)(1+|x+c|^j)$$

with $|1-(1+|x|^j)/(1+|x+c|^j)| \leq \varepsilon$ for *all* $x \in \mathbf{R}^m$ when c is sufficiently near 0, for such c we have

$$|(1+|x+c|^j)\partial_x^\alpha f(x+c) - (1+|x|^j)\partial_x^\alpha f(x+c)| \leq \varepsilon \sup_{y\in\mathbf{R}^m} |(1+|y|^j)\partial_y^\alpha f(y)|$$

where the supremum on the right side is *finite*. Thus, it suffices to instead show that for all $c \in \mathbf{R}^m$ sufficiently near 0 (depending on ε, j, α and f) that

$$\sup_{x\in\mathbf{R}^m} |(1+|x+c|^j)\partial_x^\alpha f(x+c) - (1+|x|^j)\partial_x^\alpha f(x)| \leq \varepsilon.$$

Since f is Schwartz, there exists $R > 0$ so that if $|x| > R$ then $|(1+|x|^j)\partial_x^\alpha f(x)| \leq \varepsilon/2$. If $|c| \leq R/2$ and $|x| > 3R/2$ then $|x+c| > R$, so for such c and x we have

$$|(1+|x+c|^j\partial_x^\alpha f(x+c) - (1+|x|^j)\partial_x^\alpha f(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It instead $|x| \leq 3R/2$ then continuity in (x, c) for

$$(1+|x+c|^j\partial_x^\alpha f(x+c) - (1+|x|^j)\partial_x^\alpha f(x)$$

ensures that this difference has absolute value at most ε whenever $|x| \leq 3R/2$ provided that c is sufficiently near 0. This settles (i).

For (ii), we pick $\varepsilon > 0$, $j \in \mathbf{N}$, and $\alpha \in \mathbf{N}^m$ and will show that for all $c \in K$ we have

$$\sup_{x\in\mathbf{R}^m} |(1+|x|^j)\partial_x^\alpha f(x+c)| \leq \varepsilon \tag{D.1.9.2}$$

provided that $\sup_{x\in\mathbf{R}^m} |(1+|x|^i)\partial_x^\alpha f(x)|$ is small enough (in terms of ε, j, and K) for all $0 \leq i \leq j$. Since $|c| \leq B_K$ for some constant B_K, the supremum in (D.1.9.2) can be rewritten as bounded from above:

$$\sup_{y\in\mathbf{R}^m} |(1+|y-c|^j)\partial_y^\alpha f(y)| \leq \sup_{y\in\mathbf{R}^m} |(1+(|y|+B_K)^j)\partial_y^\alpha f(y)| \leq \sum_{i=0}^{j} \binom{j}{i} B_K^{j-i} \sup_{y\in\mathbf{R}^m} |(1+|y|^i)\partial_y^\alpha f(y)|.$$

This upper bound is at most ε when each of the terms in the sum is small enough (in terms of ε, j, and B_K). This settles (ii), and so completes the treatment of the case $G = \mathbf{R}^m$.

The case $G = \mathbf{T}^n$, so $\mathcal{S}(G) = C^\infty(\mathbf{T}^n)$, is an easier version of the argument for $\mathbf{R}^m$ with no polynomial multiplier thanks to compactness of the group $\mathbf{T}^n$ (and translation-invariance of the differential operators ∂_{z_j} on $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$). The case $G = \mathbf{Z}^r$ is also an easier version of the argument for $\mathbf{R}^m$ with $\alpha = 0$ (i.e., no differential operator). □

D.1.10. Examples: global and S-local idele classes. As per the general definition of the $\mathbf{C}$-linear space of Schwartz-Bruhat functions on a Hausdorff locally compact abelian group G, upon calling G either C_F or $C_{F,S}$, then $\mathcal{S}(G)$ is the locally convex inductive limit of a system of Fréchet spaces $\mathcal{S}(\Gamma/k)$ where $\Gamma \subset G$ is open and compactly generated, $k \subset \Gamma$ is compact with (open and) compactly generated annihilator $k^\perp$ in $\widehat{G}$. In particular, if G is second-countable then $\mathcal{S}(G)$ is a strict LF-space.

Let F be a number field, S a finite set of places of F containing all Archimedean places. In this section we describe explicitly the Hausdorff locally convex topological $\mathbf{C}$-vector spaces $\mathcal{S}(C_{F,S})$ and $\mathcal{S}(C_F)$.

We call $S_f := S \cap \Sigma_{F,f}$. Recall that $\mathbf{A}_{F,S}$ is defined as $\prod_{v\in S} F_v \times \prod_{v\notin S} \mathcal{O}_{F_v}$, and is a topological subring of the global adele ring $\mathbf{A}_F$. For any finite subset S of places of F containing all Archimedean places, define the canonical insertion homomorphism

$$i_{F,S} : \mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{\Sigma_F - S}) = \mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^\times_{\Sigma_F - S} \to \mathrm{GL}_1(\mathbf{A}_F) \tag{D.1.10.1}$$

with $\widehat{\mathcal{O}}_{\Sigma_F - S} := \prod_{v\notin S} \mathcal{O}_{F_v}$. The *$S$-idele class group* is defined as

$$C_{F,S} := (\mathrm{GL}_1(\mathbf{A}_{F,S}) \times \widehat{\mathcal{O}}^\times_{\Sigma_F - S})/\mathcal{O}^\times_{F,S}.$$

The following result is straightforward.

LEMMA D.1.10.1. *Let F be a number field, S a finite set of places of F containing all Archimedean places.*

(1) *The homomorphism $i_{F,S}$ from* (D.1.10.1) *is continuous injective and open, and is equivariant with respect to the diagonal continuous left-translation action of $\mathcal{O}_{F,S}^\times$ on source and target.*
(2) *The Hausdorff locally compact abelian group $\mathrm{GL}_1(\mathbf{A}_F)$ is the direct limit of the Hausdorff locally compact abelian groups $\mathrm{GL}_1(\mathbf{A}_{F,S})$ with transition homomorphisms induced by the $i_{F,S}$.*
(3) *The continuous homomorphism $i_{F,S}$ induces a unique continuous injective open homomorphism:*

$$i_{F,S} : C_{F,S} \to C_F \tag{D.1.10.2}$$

with finite cokernel $\mathrm{Pic}(\mathcal{O}_{F,S})$ (so it is an isomorphism between identity components $C_{F,S}^0 \simeq C_F^0$ for all S, and an isomorphism for all sufficiently large S).

D.1.10.1. *Complements on the locally convex topology on Schwartz-Bruhat spaces.* For convenient reference, we wish to record several families of seminorms that define the topology on $\mathcal{S}(G)$ for some classes of Hausdorff locally compact abelian groups G. To get started, the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{R})$ (whose construction is reviewed in Definition 2.1.1.4 for $\mathbf{R}^m$ with any $m \geq 1$) is usually defined using sup-norms but it can also be defined by a family of seminorms involving the Fourier transform and L^2-norms:

PROPOSITION D.1.10.2. *Let $\mathcal{F}_{\mathbf{R}} : \mathcal{S}(\mathbf{R}) \simeq \mathcal{S}(\mathbf{R})$ be the Fourier transform $\mathcal{F}_{\mathbf{R}}(f) : x \mapsto \int_{\mathbf{R}} f(y)e^{-2\pi ixy}\,\mathrm{d}y$, and for integers $m, n \geq 0$ let $\nu_{m,n} : \mathcal{S}(\mathbf{R}) \to \mathbf{R}_{\geq 0}$ be the seminorm*

$$\nu_{m,n}(\phi) := \|\partial_x^m \mathcal{F}_{\mathbf{R}}(\partial_t^n \phi(t))(x)\|_{L^2(\mathbf{R},\mathrm{d}x)}\,. \tag{D.1.10.3}$$

The family of seminorms $\{\nu_{m,n}\}_{m,n\geq 0}$ defines the usual locally convex topology on $\mathcal{S}(\mathbf{R})$.

In particular, for a subset $B \subset \mathcal{S}(\mathbf{R})$ to be bounded in the topological vector space $\mathcal{S}(\mathbf{R})$ it is necessary and sufficient that for each pair of integers $m, n \geq 0$ the subset

$$\nu_{m,n}(B) \subset \mathbf{R}_{\geq 0}$$

is bounded.

PROOF. The assertion on boundedness within $\mathcal{S}(\mathbf{R})$ follows from the rest because by [**Scha**, Ch. IV, 3.3] boundedness for a subset of a Hausdorff locally convex topological vector space is the same as weak boundedness (i.e., boundedness in the weak topology defined by continuous linear functionals).

By the usual identity for pulling a derivative operator out of a Fourier transform [**Fol**, Thm.8.22(d)], for every $\phi \in \mathcal{S}(\mathbf{R})$ we have

$$\nu_{m,n}(\phi) = (2\pi)^m \|x^m \partial_x^n \phi(x)\|_{L^2(\mathbf{R},\mathrm{d}x)}.$$

Thus, for every integer $N \geq 1$ we have

$$\nu_{m,n}(\phi) \leq (2\pi)^m \|(1+|x|)^m \partial_x^n \phi(x)\|_{L^2(\mathbf{R},\mathrm{d}x)} \leq (2\pi)^m \left(\int_{\mathbf{R}} \frac{1}{(1+|x|)^{2N}}\,\mathrm{d}x\right)^{1/2} \sup_{x\in\mathbf{R}} (1+|x|)^{m+N} |\partial_x^n \phi(x)|.$$

In particular, since the seminorms $\nu'_{m,n}(\phi) := \sup_{x\in\mathbf{R}} |(1+|x|)^m \partial_x \phi(x)|$ generate the topology of $\mathcal{S}(\mathbf{R})$, each seminorm $\nu_{m,n}$ is continuous on $\mathcal{S}(\mathbf{R})$ for each integer $m, n \geq 0$. Thus, the family of seminorms $\{\nu_{m,n}\}_{m,n\geq 0}$ induces on $\mathcal{S}(\mathbf{R})$ a topology that is coarser than the usual topology on $\mathcal{S}(\mathbf{R})$ (defined in Definition 2.1.1.4).

Since $\mathcal{F}_{\mathbf{R}} : \mathcal{S}(\mathbf{R}) \simeq \mathcal{S}(\mathbf{R})$ is a topological automorphism, the identity for passing a derivative inside a Fourier transform implies that the topology on $\mathcal{S}(\mathbf{R})$ is also defined by the family of seminorms

$$\begin{aligned}\sup_{x\in\mathbf{R}} |\partial_x^m \mathcal{F}_{\mathbf{R}}(\partial_t^n \phi(t))(x)| &= \sup_{x\in\mathbf{R}} |\mathcal{F}_{\mathbf{R}}((-2\pi i t)^m \partial_t^n \phi(t))(x)| \\ &\leq (2\pi)^m \int_{\mathbf{R}} |t^m \partial_t^n \phi(t)|\,\mathrm{d}t \\ &\leq (2\pi)^m \int_{\mathbf{R}} |(1+|t|)^m \partial_t^n \phi(t)|\,\mathrm{d}t.\end{aligned} \tag{D.1.10.4}$$

The final integral is equal to

$$\begin{aligned}\int_{\mathbf{R}} \left|\frac{(1+|t|)^{m+N}\partial_t^n\phi(t)}{(1+|t|)^N}\right| \mathrm{d}t &\leq \left(\int_{\mathbf{R}} \frac{1}{(1+|t|)^{2N}}\,\mathrm{d}t\right)^{1/2} \|(1+|t|)^{m+N}\partial_t^n\phi\|_{L^2(\mathbf{R},\mathrm{d}x)} \\ &\leq C_N \sum_{1\leq k\leq m+N} (2\pi)^{-k}\binom{m+N}{k}\nu_{k,n}(\phi)\end{aligned} \tag{D.1.10.5}$$

for

$$C_N := \left(\int_{\mathbf{R}} \frac{1}{(1+|t|)^{2N}}\,\mathrm{d}t\right)^{1/2}.$$

Thus, by using $N = 1$ (or any fixed choice of $N \geq 1$) we see that the topology defined by the $\nu_{m,n}$'s is also finer than the usual topology on $\mathcal{S}(\mathbf{R})$, so the two topologies coincide. □

Next, we recall some families of seminorms using L^p for $1 \leq p < \infty$ in place of L^∞ to define the topology on $\mathcal{S}(\mathbf{R}^m)$ for $m \geq 1$, as well as an analogue for $\mathcal{S}(\mathbf{T}^n)$ for $n \geq 1$. This will in turn lead to a version for $\mathcal{S}(G)$ for "sufficiently large" elementary abelian Lie group quotients G of the idele class group C_F and the S-idele class group $C_{F,S}$ for a number field F.

Example D.1.10.3. The Hausdorff locally convex topology on $\mathcal{S}(\mathbf{R}^m)$ in Definition D.1.3.1 (with $r, n = 0$ there) is defined by the seminorms

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbf{R}^m} |x^\alpha \partial_x^\beta f(x)| \tag{D.1.10.6}$$

for $\alpha, \beta \in \mathbf{N}^m$. It is clearly equivalent to use the slightly modified family of seminorms $\{\|\cdot\|_{\alpha,\beta}\}$ from (2.1.1.3) for $\alpha, \beta \in \mathbf{N}^m$ (this is the special case of (D.1.3.8) with $r, n = 0$). The topology is also defined by each of the following families based on integrals (for a proof see [**KG**, Part I, Ch. IV, Thm. 29, pp. 72-4]:

(1) for $\alpha, \beta \in \mathbf{N}^m$,

$$p'_{\alpha,\beta}(f) := \|x^\alpha \partial_x^\beta f(x)\|_{L^1(\mathbf{R}^m,\mathrm{d}x)}; \tag{D.1.10.7}$$

(2) for $\alpha, \beta \in \mathbf{N}^m$ and fixed $1 < p < \infty$,

$$p''_{\alpha,\beta,p}(f) := \|x^\alpha \partial_x^\beta f(x)\|_{L^p(\mathbf{R}^m,\mathrm{d}x)} \tag{D.1.10.8}$$

(the proof of equality of topologies in [**KG**, Part I, Ch. IV, Thm. 29, pp. 72-4] treats $p = 2$, but the same method adapts readily to any $1 < p < \infty$).

Example D.1.10.4. For $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, the Hausdorff locally convex topology on $\mathcal{S}(\mathbf{T}^n) = C^\infty(\mathbf{T}^n)$ in Definition D.1.3.1 (with $m, r = 0$ there) is defined by the seminorms

$$p_\beta(f) := \sup_{z\in\mathbf{T}^n} |\partial_z^\beta f(z)| \tag{D.1.10.9}$$

for $\beta \in \mathbf{N}^n$. The Sobolev Inequality allows this to be replaced with an L^p-counterpart for any $1 \leq p < \infty$. More precisely, for each such fixed p we claim the family of seminorms

$$p''_{\beta,p}(f) := \|\partial_z^\beta f(z)\|_{L^p(\mathbf{T}^n,\mathrm{d}z)} \tag{D.1.10.10}$$

for $\beta \in \mathbf{N}^n$ defines the same topology on $C^\infty(\mathbf{T}^n)$.

For the convenience of the reader, we review the standard argument. Since the volume-1 measure is being used on $\mathbf{T}^n$, clearly $p''_{\beta,p} \leq p_\beta$. To go in reverse and bound each p_β from above in terms of finitely many $p_{\alpha,p}$'s for a fixed $1 \leq p < \infty$, it suffices to treat $p = 1$ since Hölder's inequality on the compact $\mathbf{T}^n$ gives

$$\|\phi\|_{L^1(\mathbf{T}^n,\mathrm{d}z)} \leq \|1\|_{L^q(\mathbf{T}^n,\mathrm{d}z)}\|f\|_{L^p(\mathbf{T}^n,\mathrm{d}z)} = \|f\|_{L^p(\mathbf{T}^n,\mathrm{d}z)}$$

where $1/p + 1/q = 1$.

By the Sobolov Inequality (more precisely, [**Wrn**, Cor. (b) to 6.22]), for a fixed integer $s > n/2$ there is a constant $C > 0$ such that for all $\phi \in C^\infty(\mathbf{T}^n)$ we have

$$\sup_{z\in\mathbf{T}^n} |\phi(z)| \leq C\|\phi\|_s \tag{D.1.10.11}$$

where the Sobolev norm $\|\cdot\|_s$ is defined by

$$\|\phi\|_s^2 := \sum_{\xi\in\mathbf{Z}^n} (1 + |\xi|^2)^s |\widehat{\phi}(\xi)|^2.$$

Clearly $|\widehat{\phi}(0)| \leq \|\phi\|_{L^1(\mathbf{T}^n,\mathrm{d}z)}$. For each nonzero $\xi \in \mathbf{Z}^n$ and a fixed integer $N \geq 1$ (we will soon require $N \geq s + (n+1)/2$) and an index j (depending on ξ) such that $\|\xi\|_{\sup} = |\xi_j|$ we have via integration by parts along the jth factor $\mathbf{T}$ of $\mathbf{T}^n$ that

$$\widehat{\phi}(\xi) = \frac{1}{(2\pi i\xi_j)^N}\widehat{\partial_{z_j}^N\phi}(\xi).$$

Since the Fourier coefficients are each bounded above by the L^1-norm, for a constant C_n satisfying $|x| \leq C_n|x|_{\sup}$ for all $x \in \mathbf{R}^n$ we have

$$|\widehat{\phi}(\xi)| \leq \left(\frac{C_n}{2\pi|\xi|}\right)^N \max_{|\beta|=N} \|\partial_z^\beta\phi\|_{L^1(\mathbf{T}^n,\mathrm{d}z)}$$

(with the multi-index $\beta \in \mathbf{N}^n$ varying with degree N).

Thus, requiring $N \geq s + (n+1)/2$, we have

$$\|\phi\|_s^2 \leq \|\phi\|^2_{L^1(\mathbf{T}^n,\mathrm{d}z)} + \left(\frac{C_n}{2\pi}\right)^{2N} \left(\sum_{\xi\in\mathbf{Z}^n-\{0\}} \frac{(1+|\xi|^2)^s}{|\xi|^{2N}}\right) \max_{|\beta|=N} \|\partial_z^\beta\phi\|^2_{L^1(\mathbf{T}^n,\mathrm{d}z)}$$

where the sum over $\xi \in \mathbf{Z}^n - \{0\}$ is finite since $2N - 2s \geq n + 1$. Feeding this into (D.1.10.11), controlling the L^1-norms of all degree-N partials of $\phi \in C^\infty(\mathbf{T}^n)$ controls ϕ uniformly pointwise over $\mathbf{T}^n$ as desired.

EXAMPLE D.1.10.5. The Hausdorff locally convex topology on $\mathcal{S}(\mathbf{Z}^r)$ in Definition D.1.3.1 (with $m, n = 0$ there) is defined by the family of seminorms

$$p_\gamma(f) := \sup_{a\in\mathbf{Z}^r} |a^\gamma f(a)| \tag{D.1.10.12}$$

for $\gamma \in \mathbf{N}^r$. For a fixed integer $1 \leq p < \infty$, we claim that the topology is also defined by the family of seminorms

$$p_{\gamma,p}(f) = \|a^\gamma f(a)\|_{L^p(\mathbf{Z}^r)} := \left(\sum_{a\in\mathbf{Z}^r} |a^\gamma f(a)|^p\right)^{1/p}.$$

Clearly $p_\gamma \leq p_{\gamma,p}$ for each γ (termwise domination), so it remains to show that each $p_{\gamma,p}$ is controlled by the p_α's for finitely many $\alpha \in \mathbf{N}^r$ (depending on γ).

Adapting the argument used for $\mathbf{R}^m$ in Example D.1.10.3 via Hölder's inequality, we multiply and divide by $(1 + |a|^2)^r$ (whose reciprocal has finite L^p-norm since $2r \geq r + 1$) to get

$$p_{\gamma,p}(f) \leq \|(1 + |a|^2)^{-r}\|_{L^p(\mathbf{Z}^r)} \sup_{a\in\mathbf{Z}^r} |(1 + |a|^2)^r a^\gamma f(a)|$$

with finite L^p-norm multiplier having nothing to do with f. Via binomial expansion, the supremum on the right side is controlled by the $p_\alpha(f)$'s for finitely many α's depending only on γ (and r). This completes the proof.

The three preceding examples are the key ingredients in the proof of a coordinate-free variant of Lemma D.1.3.4 incorporating a version with L^q-norms for $1 \leq .q < \infty$, inspired by [**Wa**, §3, p. 312].

PROPOSITION D.1.10.6. *Let G be an elementary abelian Lie group. For each invariant differential operator $D' \in \mathrm{Diff}(\widehat{G})$ on the dual group $\widehat{G}$, let $p(D') \in C^\infty(G)$ be the associated symbol (smoothness via Lemma* D.1.6.19(1)*). The locally convex topology on $\mathcal{S}(G)$ is defined by each of the following families of seminorms:*

(1) $\|p(D') \cdot D(f)\|_{\sup}$ *for varying $D \in \mathrm{Diff}(G)$ and $D' \in \mathrm{Diff}(\widehat{G})$,*
(2) *for a fixed integer $1 \leq q < \infty$, $\|p(D') \cdot D(f)\|_{L^q(G)}$ for varying $D \in \mathrm{Diff}(G)$ and $D' \in \mathrm{Diff}(\widehat{G})$,*
(3) *for a basis $\{\partial_j\}_{1 \leq j \leq d}$ of $\mathcal{L}(G) = \mathcal{L}(G^0)$ and a basis $\{\partial'_i\}_{1 \leq i \leq d'}$ of $\mathcal{L}(\widehat{G})$, $\|p(\partial'^\beta) \cdot \partial^\alpha(f)\|_{\sup}$ for $\alpha \in \mathbf{N}^d$, $\beta \in \mathbf{N}^{d'}$.*
(4) *for a fixed integer $1 \leq q < \infty$, a basis $\{\partial_j\}_{1 \leq j \leq d}$ of $\mathcal{L}(G) = \mathcal{L}(G^0)$, and a basis $\{\partial'_i\}_{1 \leq i \leq d'}$ of $\mathcal{L}(\widehat{G})$, $\|p(\partial'^\beta) \cdot \partial^\alpha(f)\|_{L^q(G)}$ for $\alpha \in \mathbf{N}^d$, $\beta \in \mathbf{N}^{d'}$.*

By Lemma D.1.6.16, for any $f \in \mathcal{S}(G)$ the sup-norms in (1) and (3) are finite. Likewise, by Remark D.1.6.20 we see that $p(D') \cdot D(f) \in \mathcal{S}(G)$ for all pairs (D, D'), so the L^q-norms in (2) and (4) are finite.

PROOF. Since the formation of $p(D')$ is linear in D', it is clear that (1) and (3) define the same locally convex topology on $\mathcal{S}(G)$, as do (2) and (4). In particular, (3) and (4) are independent of the choice of such bases and it suffices to show (3) and (4) define the usual topology on $\mathcal{S}(G)$ for some common choice of such bases. Fixing an isomorphism

$$G \simeq \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$$

for integers $m, n, r \geq 0$ and a finite abelian group F, we have compatibly

$$\widehat{G} \simeq \mathbf{R}^m \times \mathbf{Z}^n \times \mathbf{T}^r \times \widehat{F}.$$

For (3) we use the basis of $\mathcal{L}(G) = \mathcal{L}(\mathbf{R}^m \times \mathbf{T}^n)$ corresponding to the standard partial derivatives along the factors $\mathbf{R}$ of $\mathbf{R}^m$ and $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ of $\mathbf{T}^n$, and for (4) we use the basis of $\mathcal{L}(\widehat{G}) = \mathcal{L}(\mathbf{R}^m \times \mathbf{T}^r)$ corresponding to the standard partial derivatives along the factors $\mathbf{R}$ of $\mathbf{R}^m$ and $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ of $\mathbf{T}^r$.

By Example D.1.6.12 and Lemma D.1.6.13, in terms of the "coordinates" $(x, z, a, b) \in \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ we see that the operators $p(\partial'^\beta) \cdot \partial^\alpha$ for varying α and β are exactly $\mathbf{C}^\times$-multiples of the operators

$$x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu : \mathcal{S}(G) \to \mathcal{S}(G).$$

That (3) defines the usual topology on $\mathcal{S}(G)$ now follows from (D.1.3.9). It remains to prove that for each $1 \leq q < \infty$, the seminorms

$$f \mapsto \|x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu(f)\|_{L^q(G)}$$

on $\mathcal{S}(G)$ define the usual topology on $\mathcal{S}(G)$.

We shall do this by using each of Examples D.1.10.3–D.1.10.5. First we show that each sup-norm

$$\|x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu(f)\|_{\sup}$$

over G is bounded above by a sum of $\mathbf{R}_{>0}$-multiples of finitely many L^q-seminorms over G (with the constant multipliers and L^q-seminorms depending only on γ, δ, λ, and μ, not on f). Since

$$\sup_{(x,z,a,b)} g(x, z, a, b) = \sup_{(z,a,b)} (\sup_x g(x, z, a, b)),$$

for each (z,a,b) we view $x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu f(x,z,a,b)$ as an element of $\mathcal{S}(\mathbf{R}^m)$ and apply Example D.1.10.3 to get a finite set of pairs of multi-indices (γ',λ') and positive constants $C_{\gamma',\delta'}$ depending only on (γ,λ) so that

$$\sup_{(x,z,a,b)} |x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu f(x,z,a,b)| \leq \sup_{(z,a,b)} \sum_{(\gamma',\lambda')} C_{\gamma',\lambda'} \left(\int_{\mathbf{R}^m} |x^{\gamma'} \partial_x^{\lambda'} a^\delta \partial_z^\mu f(x,z,a,b)|^q \, \mathrm{d}x \right)^{1/q}$$
$$\leq \sum_{(\gamma',\lambda')} C_{\gamma',\lambda'} \left(\int_{\mathbf{R}^m} \sup_{(z,a,b)} |\partial_z^\mu a^\delta x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}x \right)^{1/q}, \tag{D.1.10.13}$$

where the sup inside the integral is a continuous function of $x \in \mathbf{R}^m$ since z varies over the compact $\mathbf{T}^n$.

Writing $\sup_{(z,a,b)} = \sup_{(a,b)} \sup_z$, we next apply Example D.1.10.4 to get a finite set of multi-indices μ' and positive constants $C_{\mu'}$ depending only on μ so that the integrand at the end of (D.1.10.13) satisfies

$$\sup_{(z,a,b)} |\partial_z^\mu a^\delta x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \leq \sup_{(a,b)} \sum_{\mu'} C_{\mu'} \int_{\mathbf{T}^n} |\partial_z^{\mu'} a^\delta x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}z$$
$$\leq \sum_{\mu'} C_{\mu'} \int_{\mathbf{T}^n} \sup_{(a,b)} |a^\delta \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}z. \tag{D.1.10.14}$$

Thus,

$$\sup_{(x,z,a,b)} |x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu f(x,z,a,b)| \leq \sum_{(\gamma',\lambda')} C_{\gamma',\lambda'} \left(\sum_{\mu'} C_{\mu'} \int_{\mathbf{R}^m \times \mathbf{T}^n} \sup_{(a,b)} |a^\delta \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}z \, \mathrm{d}x \right)^{1/q}$$

(the inner supremum over the countably many pairs (a,b) is measureable).

Now by Example D.1.10.5, there is a finite set of multi-indices δ' and positive constants $C_{\delta'}$ depending only on δ so that

$$\sup_{(a,b)} |a^\delta \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \leq \sup_b \sum_{\delta'} C_{\delta'} \int_{\mathbf{Z}^r} |a^{\delta'} \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}a$$

(counting measure on $\mathbf{Z}^r$). Hence, the left side of (D.1.10.13) is bounded above by

$$\sum_{(\gamma',\lambda')} C_{\gamma',\lambda'} \left(\sum_{(\mu',\delta')} C_{\mu'} C_{\delta'} \int_{\mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r} \sup_{b \in F} |a^{\delta'} \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'} f(x,z,a,b)|^q \, \mathrm{d}a \, \mathrm{d}z \, \mathrm{d}x \right)^{1/q}.$$

For any non-negative function h on F clearly $\sup_{b \in F} h(b) \leq \sum_{b \in F} h(b)$, so we finally obtain

$$\|x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu(f)\|_{\sup} \leq \sum_{(\gamma',\lambda')} C_{\gamma',\lambda'} \left(\sum_{(\mu',\delta')} C_{\mu'} C_{\delta'} \|a^{\delta'} \partial_z^{\mu'} x^{\gamma'} \partial_x^{\lambda'}(f)\|_{L^q(G)}^q \right)^{1/q}$$

where the finitely many multi-indices over which we sum and the positive constant multipliers on the right side only depend on the multi-indices on the left side. This shows that each of the sup-seminorms on G is controlled by finitely many of the L^q-seminar's over G.

To go in reverse, we want to control each choice of L^q-seminorm on $\mathcal{S}(G)$ in terms of finitely many sup-seminorms. Fix one such L^q-seminorm ν. We will use the isomorphism

$$\mathcal{S}(G) \simeq \mathcal{S}(\mathbf{R}^m) \widehat{\otimes}_\pi \mathcal{S}(\mathbf{T}^n) \widehat{\otimes}_\pi \mathcal{S}(\mathbf{Z}^r) \widehat{\otimes}_\pi \mathcal{S}(F)$$

from Lemma D.1.3.7(2). By Examples D.1.10.3–D.1.10.5 (along with the easy case of the finite-dimensional $\mathcal{S}(F) = \mathbf{C}^F$), on each of the four factor spaces in this completed tensor product the topology is defined by the corresponding family of L^q-norms. Thus, a base of 0-neighborhoods in the uncompleted tensor product

consists of *convex combinations* of elementary tensors whose constituent vectors have any specified finite collection of L^q-seminorms below any desired positive bound.

Hence, by the L^q-triangle inequality and Fubini's Theorem, in the topology of $\mathcal{S}(G)$ each $f \in \mathcal{S}(G)$ is an infinite series $\sum_{k\geq 0} f_k$ where every f_k is a finite sum of elementary tensors and for every $k \geq 1$ we have $\nu(f_k) \leq 1/2^k$. It follows that the infinite series is also ν-convergent to f. Successive use of the explicit description of tensor product seminorms [**Scha**, Ch. III, 6.3] shows that on *finite sums* of elementary tensors we have

$$\|x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu(\cdot)\|_{L^q(G)} \leq \|x^\gamma \partial_x^\lambda(\cdot)\|_{L^q(\mathbf{R}^m)} \otimes_\pi \|\partial_z^\mu(\cdot)\|_{L^q(\mathbf{T}^n)} \otimes_\pi \|a^\delta(\cdot)\|_{L^q(\mathbf{Z}^r)} \otimes_\pi \|\cdot\|_{L^q(F)}.$$

But $\nu(f)$ is the limit of $\nu(\sum_{k\leq N} f_k)$ as $N \to \infty$ (with each partial sum $\sum_{k\leq N} f_k$ a finite sum of elementary tensors), so we can pass to the limit and get

$$\|x^\gamma a^\delta \partial_x^\lambda \partial_z^\mu(\cdot)\|_{L^q(G)} \leq \|x^\gamma \partial_x^\lambda(\cdot)\|_{L^q(\mathbf{R}^m)} \widehat{\otimes}_\pi \|\partial_z^\mu(\cdot)\|_{L^q(\mathbf{T}^n)} \widehat{\otimes}_\pi \|a^\delta(\cdot)\|_{L^q(\mathbf{Z}^r)} \widehat{\otimes}_\pi \|\cdot\|_{L^q(F)}.$$

In particular, the seminorms on the left side are all *continuous* on $\mathcal{S}(G)$. This proves that the L^q-seminorms on $\mathcal{S}(G)$ are each controlled by finitely many of the usual sup-norms on $\mathcal{S}(G)$, as desired. □

D.1.10.2. *Families of continuous seminorms on $\mathcal{S}(C_F)$.* Let F be a number field, S a finite set of places of F containing all the Archimedean places. We denote by C_F^1, respectively $C_{F,S}^1$, the closed subgroups of idele classes of idelic norm 1 of the Hausdorff locally compact abelian groups C_F, respectively $C_{F,S}$ of idele classes. C_F^1 and $C_{F,S}^1$ are compact Hausdorff (see Theorem A.1.4.3).

The following are families of seminorms that generate the Hausdorff locally convex topology on $\mathcal{S}(C_{F,S}^0)$ and $\mathcal{S}(C_F^0)$, where G^0 denotes the identity component of a topological group G.

Proposition D.1.10.7 (Fundamental family of continuous seminorms on $\mathcal{S}(C_F^0)$). *Let F be a number field. Let τ be a continuous homomorphic section to the idelic norm. Let $k \subset C_F^0$ be a compact subgroup (so $k \subset (C_F^0)^1 = (C_F^1)^0$) with compactly generated orthogonal $k^\perp$ in the dual group of C_F^0.*

If k is sufficiently small then there exist continuous 1-parameter subgroups $\gamma_j : \mathbf{R} \to C_F^1$, $1 \leq j \leq r_1+2r_2-1$ such that the set of seminorms

$$p_{r,m,\alpha}(f) := \sup_{g\in C_F^0/k} |\partial_\tau^m(\partial_{\gamma_1}^{\alpha_1} \cdots \partial_{\gamma_{r_1+2r_2-1}}^{\alpha_{r_1+2r_2-1}})(f)(g)(1+|\log|g|_F|^r)| \tag{D.1.10.15}$$

for $r \in \mathbf{N}$, $m \in \mathbf{N}$, $\alpha \in \mathbf{N}^{r_1+2r_2-1}$ generate the Schwartz-Bruhat topology on $\mathcal{S}(k, C_F^0)$.

Remark D.1.10.8. We note that the image of each γ_j from Proposition D.1.10.7 is contained in $(C_F^1)^0$ since it is a connected subgroup of C_F^1.

Proof. Upon choosing a continuous homomorphic section τ to the idelic norm, we have an induced isomorphism of Hausdorff locally compact abelian groups:

$$\overline{\theta}_\tau : \mathbf{R}_{>0} \times C_F^1 \to C_F \tag{D.1.10.16}$$

defined as $\overline{\theta}_\tau(t, g) := \tau(t)g$.

By [**AT**, Ch. IX, §1. Thm. 3], the compact norm-1 subgroup $G \subset C_F^0$ is $(\mathbf{R}/\mathbf{Z})^{r_2} \times \mathbf{S}^{r_1+r_2-1}$ where $\mathbf{S} = \lim_{x\mapsto Nx} \mathbf{R}/\mathbf{Z} \simeq \mathbf{A}_\mathbf{Q}/\mathbf{Q}$. Since G is compact, so all elementary abelian Lie group quotients of G are compact, $\mathcal{S}(G) = C^\infty(G)$. Thus, Lemma D.1.8.1(1) identifies the Schwartz-Bruhat space $\mathcal{S}(G) = C^\infty(G)$ with

$$C^\infty((\mathbf{R}/\mathbf{Z})^{r_2}) \overline{\otimes} C^\infty(\mathbf{S}^{r_1+r_2-1})$$

where $C^\infty((\mathbf{R}/\mathbf{Z})^{r_2})$ is equipped with the Fréchet topology of uniform convergence of functions and their derivatives or arbitrary order.

Via the compatibility of $\overline{\otimes}$ with countable strict inductive limits in Proposition C.2.3.13(2), we claim that that for every integer $r \geq 0$ (such as $r = r_1 + r_2 - 1$) the space $C^\infty(\mathbf{S}^r)$ is naturally identified with the countable *strict* locally convex inductive limit of copies of $C^\infty((\mathbf{R}/\mathbf{Z})^r)$ under the *topological embedding* transition operators induced by the positive integer-multiplication maps on $(\mathbf{R}/\mathbf{Z})^r$. To see this, we may pass to the case $r = 1$ to reduce to checking that every smooth function on $\mathbf{S}$ factors through some finite layer $\mathbf{R}/\mathbf{Z}$ in the inverse limit defining $\mathbf{S}$. By definition, smooth functions on $\mathbf{S}$ are K-invariant for a proper compact subgroup $K \subset \mathbf{S}$ such that the nonzero orthogonal $K^\perp \subset \widehat{\mathbf{S}} = \mathbf{Q}$ is finitely generated, hence contained inside $(1/N)\mathbf{Z}$ for some $N \geq 1$. The latter provides a finite layer through which the K-invariant smooth functions factor, as desired.

Passing to a sufficiently large finitely generated subgroup $k^\perp \subset \widehat{G} = \mathbf{Z}^{r_2} \times \mathbf{Q}^{r_1+r_2-1}$ (a cofinal system of which is given by $\mathbf{Z}^{r_2} \times (1/N)\mathbf{Z}^{r_1+r_2-1}$) corresponds to passing to a sufficiently small compact subgroup $k \subset G$ with finitely generated orthogonal, so we may assume k is the closed subgroup of $\mathbf{S}^{r_1+r_2-1}$ corresponding to kernel of projection to a finite layer $(\mathbf{R}/\mathbf{Z})^{r_1+r_2-1}$.

The completed inductive tensor product $\mathcal{S}((\mathbf{R}/\mathbf{Z})^{r_2}) \,\overline{\otimes}\, \mathcal{S}(k, \mathbf{S}^{r_1+r_2-1})$, which by Corollary C.2.3.12 is topologically isomorphic to the completed projective tensor product $\mathcal{S}((\mathbf{R}/\mathbf{Z})^{r_2}) \,\widehat{\otimes}_\pi \mathcal{S}(k, \mathbf{S}^{r_1+r_2-1})$, is therefore linearly topologically isomorphic to $\mathcal{S}(k, G)$ equipped with the topology of uniform convergence of functions and their directional derivatives of arbitrary order along the coordinate continuous 1-parameter subgroups induced by the continuous homomorphism

$$\mathbf{R}^{r_2} \times (\mathbf{R}^{r_1+r_2-1} \times \{0\}) \to (\mathbf{R}/\mathbf{Z})^{r_2} \times (\mathbf{R}^{r_1+r_2-1} \times \widehat{\mathbf{Z}}) \to (\mathbf{R}/\mathbf{Z})^{r_2} \times \mathbf{S}^{r_1+r_2-1}$$

further composed with the continuous quotient homomorphism $(\mathbf{R}/\mathbf{Z})^{r_2} \times \mathbf{S}^{r_1+r_2-1} \to (\mathbf{R}/\mathbf{Z})^{r_2} \times (\mathbf{S}^{r_1+r_2-1}/k)$ with $\mathbf{S}^{r_1+r_2-1}/k = (\mathbf{R}/\mathbf{Z})^{r_1+r_2-1}$. These induce continuous 1-parameter subgroups of C_F^1.

Since $\mathcal{S}(C_F^0/k)$ is linearly and topologically isomorphic to $\mathcal{S}(\mathbf{R}_{>0}) \,\overline{\otimes}\, \mathcal{S}((C_F^1)^0/k)$ (equal to the completed projective tensor product since $(C_F^1)^0/k$ is an elementary abelian Lie group), we are reduced to showing that the Hausorff locally convex topology on $\mathcal{S}(\mathbf{R}_{>0})$ is induced by the set of seminorms

$$p_{r,m}(f) := \sup_{t \in \mathbf{R}_{>0}} |((t\partial_t)^m f(t))(1 + |\log(t)|^r)|.$$

Upon performing the change of variables $t = e^x$ ($x \in \mathbf{R}$), the equality with the topology on $\mathcal{S}(\mathbf{R}_{>0})$ follows from Example D.1.10.3. □

PROPOSITION D.1.10.9 (Fundamental family of continuous seminorms on $\mathcal{S}(k, C_F)$). *Let F be a number field. Let $\gamma_1, \ldots, \gamma_{r_1+2r_2-1}$ be continuous 1-parameter subgroups of C_F^1 as in Proposition* D.1.10.7. *Let τ be a choice of continuous homomorphic section to the idelic norm.*

For $k \subset \Gamma$ a sufficiently small compact subgroup whose annihilator is open and compactly generated in $\widehat{C_F}$, the following sets of seminorms on $\mathcal{S}(k, C_F)$ generate the subspace topology:

(1) *for varying $r \in \mathbf{N}$, $m \in \mathbf{N}$, $\alpha \in \mathbf{N}^{r_1+2r_2-1}$, and a fixed real number $p \geq 1$, $\{\nu_{r,m,\alpha}^p\}$ defined on $\mathcal{S}(k, C_F)$ by*

$$\nu_{r,m,\alpha}^p(f) := \left(\int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} |(t\partial_t)^m (\partial_{\gamma_1}^{\alpha_1} \cdots \partial_{\gamma_{r_1+2r_2-1}}^{\alpha_{r_1+2r_2-1}})(f)(\tau(t)g)(1 + |\log(t)|^r)|^p \, \mathrm{d}\overline{\mu}_1(g) \right) \frac{\mathrm{d}t}{t} \right)^{1/p}. \tag{D.1.10.17}$$

(2) *For the above sets of seminorms, the seminorms obtained upon replacing $1 + |\log(\cdot)|^r$ by $|\log(\cdot)|^r$.*
(3) *For the above sets of seminorms, the seminorms obtained upon replacing $\{1 + |\log(\cdot)|^r\}_{r \in \mathbf{N}}$ by the absolute value of all polynomials in $|\log(t)|$ with $\mathbf{C}$-coefficients and of arbitrary degree.*

PROOF. We denote by

$$D_j := \partial_{\gamma_j} : \mathcal{S}(C_F) \to \mathcal{S}(C_F) \tag{D.1.10.18}$$

for $1 \leq j \leq r_1 + 2r_2 - 1$ the linear differential operators induced by the continuous 1-parameter subgroups $\{\gamma_j\}_{1 \leq j \leq r_1+2r_2-1}$ of $C_F^1 \subset C_F$. Since each $a \in \mathcal{L}(C_F)$ takes value in the identity component C_F^0, we have $\mathcal{L}(C_F) = \mathcal{L}(C_F^0)$. By [**AT**, Ch. IX, §1, Thm. 3] we have an isomorphism of topological groups

$$C_F^0 \simeq (\mathbf{R}/\mathbf{Z})^{r_2} \times \mathbf{R}_{>0} \times \mathbf{S}^{r_1+r_2-1}$$

for $\mathbf{S} = \mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$, so as $\mathbf{R}$-vector spaces we have

$$\mathcal{L}(C_F^0) \simeq \mathcal{L}((\mathbf{R}/\mathbf{Z})^{r_2}) \oplus \mathcal{L}(\mathbf{R}_{>0}) \oplus \mathcal{L}(\mathbf{S}^{r_1+r_2-1})$$

where $\mathcal{L}(\mathbf{S}^{r_1+r_2-1})$ is $\mathbf{R}$-linearly isomorphic to $\mathcal{L}((\mathbf{R}/\mathbf{Z})^{r_1+r_2-1})$ via projection $\mathrm{pr} : \mathbf{S} \to \mathbf{R}/\mathbf{Z}$ to the initial layer since the transition maps among copies of $(\mathbf{R}/\mathbf{Z})^{r_1+r_2-1}$ realizing $\mathbf{S}^{r_1+r_2-1}$ as an inverse limit, are finite-degree covering maps (so inducing $\mathbf{R}$-linear isomorphisms on Lie algebras). In particular, $\dim_{\mathbf{R}} \mathcal{L}(C_F) = 1 + (r_1 + 2r_2 - 1) = [F : \mathbf{Q}]$ and so the $\mathbf{R}$-linearly independent $\{\partial_\tau, D_j\}_{1 \leq j \leq r_1+2r_2-1}$ are a basis of $\mathcal{L}(C_F)$.

In light of the isomorphism of Hausdorff locally compact abelian groups

$$\overline{\theta}_\tau : \mathbf{R}_{>0} \times C_F^1 \to C_F \tag{D.1.10.19}$$

induced by the choice τ of continuous homomorphic section to the idelic norm, to compute the $\mathbf{R}$-linear space of symbols of invariant differential operators on the dual group C_F it is enough to compute the symbols of the linear differential operators on the dual group of $\mathbf{R}_{>0}$, since the dual of the compact C_F^1 is discrete (and so has no nonzero invariant differential operator).

We can k small enough so that C_F^0/k has the same dimension as $\mathcal{L}(C_F^0)$ (as we may clearly do). Integrating k-invariant functions on C_F and over C_F/k differs only in a constant multiplier by a volume, so via Proposition D.1.10.6 we see that for each real number $p \geq 1$, as we vary $P \in \mathbf{R}[X]$, $m \in \mathbf{N}$, and $\alpha \in \mathbf{N}^{r_1+2r_2-1}$ the associated seminorms

$$\nu_{P,m,\alpha}^p(f) := \left(\int_{\mathbf{R}_{>0}} \left(\int_{C_F^1} |(t\partial_t)^m (\partial_{\gamma_1}^{\alpha_1} \cdots \partial_{\gamma_{r_1+2r_2-1}}^{\alpha_{r_1+2r_2-1}})(f)(\tau(t)g)| P(\log(t))|^p \, \mathrm{d}\overline{\mu}_1(g) \right) \frac{\mathrm{d}t}{t} \right)^{1/p}$$

generate the Hausdorff locally convex topology on $\mathcal{S}(C_F)$. That the Schwartz-Bruhat topology on $\mathcal{S}(C_F)$ remains unaltered upon changing the seminorms $\nu_{r,m,\alpha}^p$ into the seminorms from (D.1.10.17), or those obtained upon replacing $(1 + |\log(t)|^r)$ from (D.1.10.17) by $|\log(t)|^r$, is a consequence of the triangle inequality for L^p-norms. □

D.1.10.3. *Families of continuous seminorms on $\mathcal{S}(C_{F,S})$.* The following is the analog of Proposition D.1.10.9 in the case of $C_{F,S}$, for S a finite set of places of F containing the Archimedean places.

Proposition D.1.10.10 (Fundamental family of continuous seminorms on $\mathcal{S}(k, C_{F,S})$). *Let F be a number field, S a finite set of places containing the Archimedean places. Let $\gamma_1, \ldots, \gamma_{r_1+2r_2-1}$ be the continuous 1-parameter subgroups of the open subgroup $C_{F,S}^1 \subset C_F^1$ of finite index provided by Proposition* D.1.10.7. *Let τ be a continuous homomorphic section to the S-idelic norm.*

For $k \subset C_{F,S}$ a sufficiently small compact subgroup whose open annihilator is compactly generated in $\widehat{C_{F,S}}$, the following sets of seminorms on $\mathcal{S}(k, C_{F,S})$ generate the usual topology:

(1) *for varying $r \in \mathbf{N}$, $m \in \mathbf{N}$, $\alpha \in \mathbf{N}^{r_1+2r_2-1}$, and a fixed real number $p \geq 1$, $\{\nu_{r,m,\alpha}^p\}$ on $\mathcal{S}(k, C_{F,S})$ defined by*

(D.1.10.20)

$$\nu_{r,m,\alpha}^p(f) := \left(\int_{\mathbf{R}_{>0}} \left(\int_{C_{F,S}^1} |(t\partial_t)^m (\partial_{\gamma_1}^{\alpha_1} \cdots \partial_{\gamma_{r_1+2r_2-1}}^{\alpha_{r_1+2r_2-1}})(f)(\tau(t)g)(1 + |\log(t)|^r)|^p \, \mathrm{d}\overline{\mu}_{S,1}(g) \right) \frac{\mathrm{d}t}{t} \right)^{1/p}.$$

(2) *For the above sets of seminorms, the seminorms obtained upon replacing $1 + |\log(\cdot)|^r$ by $|\log(\cdot)|^r$.*

(3) *For the above sets of seminorms, the seminorms obtained upon replacing* $\{1+|\log(\cdot)|^r\}_{r\in\mathbf{N}}$ *by the absolute value of the polynomials in* $|\log(t)|$ *of arbitrary degree.*

The proof of Proposition D.1.10.10 proceeds as for Proposition D.1.10.9, upon noting that we have $C^0_{F,S} = C^0_F$ for all S by Lemma D.1.10.1(3). We further note that the seminorms in (D.1.10.20) agree with the seminorms in (D.1.10.17) for all large enough S, since $C_{F,S} = C_F$ for all sufficiently large S by Lemma D.1.10.1(3).

D.2. Schwartz-Bruhat functions on modulated groups

In this section we introduce modulated groups, a special case of Hausdorff locally compact abelian groups G, unifying examples of interest to this work, such as global and S-local idele class groups. This section will enable systematic use of linear and topological properties of the Mellin transform for G equipped with a proper continuous surjective homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$; such a G is called $\mathbf{R}_{>0}$-*modulated*.

D.2.1. Preparations on modulated groups. The *compact* Hausdorff abelian group C^1_F acts on $\mathcal{S}(C_F)$ by continuous left-translation: $(g.f)(x) = f(g^{-1}x)$ (for the continuity property more broadly. See Proposition D.1.9.2). The same holds for the compact Hausdorff abelian groups $C^1_{F,S}$ acting on $\mathcal{S}(C_{F,S})$, and $J^1_{F,S}$ acting on $\mathcal{S}(J_{F,S})$. The Hausdorff locally compact abelian groups C_F, $C_{F,S}$ and $J_{F,S}$ are three examples of "modulated" groups. This concept was discussed by Weil in [**Weil-e**, §3, p. 6] with the aim of treating certain representation-theoretic preparations towards the explicit formulas for Artin L-functions in a unified manner, encompassing Weil groups.

D.2.1.1. *Modulated groups: definition and canonical measure.* Let G be a second-countable Hausdorff locally compact abelian group equipped with a continuous proper homomorphism

$$|\cdot|_G : G \to \mathbf{R}_{>0}$$

(so $G^1 := \ker(|\cdot|_G)$ is compact). For a choice of continuous character $\omega \in \widehat{G^1}$, let $\mathrm{H}^0(\omega, \mathcal{S}(G)) \subset \mathcal{S}(G)$ denote the subspace of Schwartz-Bruhat functions f on G that are ω-homogenous in the sense that $g.f = \omega(g)f$ for all $g \in G^1$; this is a closed subspace since the G-action on $\mathcal{S}(G)$ is continuous.

Recall from Theorem D.1.5.7 that the Hausdorff locally convex topological vector space $\mathcal{S}(G)$ is complete, and by Proposition D.1.9.2 every compact subset $K \subset G$ acts uniformly continuously on $\mathcal{S}(G)$; in particular, the G^1-action on $\mathcal{S}(G)$ is uniformly continuous. This allows us to build averaging operators over G^1 valued in $\mathcal{S}(G)$ via Pettis integration, as follows.

Define the $\mathbf{C}$-linear map

$$\Phi_{G,\omega} : \mathcal{S}(G) \to \mathrm{H}^0(\omega, \mathcal{S}(G)) \tag{D.2.1.1}$$

by

$$\Phi_{G,\omega}(f)(h) := \frac{1}{\mu_1(G^1)} \int_{G^1} \overline{\omega}(g) f(g^{-1}h)\, \mathrm{d}\mu_1(g) \tag{D.2.1.2}$$

with μ_1 a fixed choice of Haar measure on G^1. It is clear from the definition that the function $\Phi_{G,\omega}(f) : G \to \mathbf{C}$ is ω-homogenous, but it is not so obvious that this function on G belongs to $\mathcal{S}(G)$; this is where Pettis integration enters. We use Lemma 4.3.3.1(1) applied to the compact group G^1 acting uniformly continuously on the complete locally convex topological vector space $E = \mathcal{S}(G)$ via $f \mapsto \overline{\omega}(g)\, g.f$ and the collection of continuous linear functionals $\varphi \in E'$ given by evaluation at points $h \in G$. That is, we have an $\mathcal{S}(G)$-*valued* Pettis integral

$$\Phi_{G,\omega}(f) = \frac{1}{\mu_1(G^1)} \int_{G^1} \langle \overline{\omega}(g) g.f, \cdot \rangle_{E,E'}\, \mathrm{d}\mu_{G^1}(g) \in E \subset (E')^*$$

whose value at any $h \in G$ is the ordinary integral in (D.2.1.2). The averaging operator $\Phi_{G,\omega} : \mathcal{S}(G) \to \mathcal{S}(G)$ over G^1 is a continuous projector onto $\mathrm{H}^0(\omega, \mathcal{S}(G))$ by Lemma 4.3.3.1(1,2).

Definition D.2.1.1 (Modulated groups). Let G be a second-countable Hausdorff locally compact abelian group. We say G is *modulated* if there is given a non-trivial continuous proper homomorphism

$$|\cdot|_G : G \to \mathbf{R}_{>0}. \tag{D.2.1.3}$$

We call $|\cdot|_G$ the *module* of G, and denote by Γ the closed image of $|\cdot|_G$ in $\mathbf{R}_{>0}$. We define:

(1) $G^1 := \ker(|\cdot|_G)$, a the compact subgroup of G;
(2) the *canonical* Haar measure μ_{can} on G to be the unique Haar measure on G such that for every real number $\Lambda > 1$, upon denoting by $G^{[1/\Lambda,\Lambda]}$ the compact subset of G defined as the pre-image of the compact interval $[1/\Lambda, \Lambda] \subset \mathbf{R}_{>0}$ under the proper homomorphism $|\cdot|_G$, we have

$$\mu_{\mathrm{can}}(G^{[1/\Lambda,\Lambda]}) = 2\log(\Lambda) + O(1) \tag{D.2.1.4}$$

as $\Lambda \to +\infty$ (such a measure exists because $|\cdot|_G$ is non-trivial).

The condition (D.2.1.4) uniquely determines μ_{can}, but to see the existence of μ_{can} it is useful first to show (as one sees directly in all cases of interest) that $|\cdot|_G$ admits continuous homomorphic sections:

Remark D.2.1.2. The closed subgroup $\Gamma \subset \mathbf{R}_{>0}$ is either of the form $t^{\mathbf{Z}}$ for a unique $t \in \mathbf{R}_{>1}$ or is $\mathbf{R}_{>0}$ (the case $|\cdot|_G$ is surjective), and in all cases we claim $|\cdot|_G$ admits continuous homomorphic sections $\tau : \Gamma \to G$ (each thereby inducing an isomorphism $G \simeq G^1 \times \Gamma$ depending on the choice of τ). This is clear when Γ is infinite cyclic. When $|\cdot|_G$ is surjective, the existence of τ follows from the proper $|\cdot_G : G \to \mathbf{R}_{>0}$ being a quotient map and G necessarily having an open subgroup of the form $K \times \mathbf{R}^N$ for some compact group K and integer $N \geq 1$ [**Ru**, Thm. 1.2.4].

Modulated groups are discussed by Weil in his paper on the explicit formulas for Artin L-functions of number fields; we refer the reader to [**Weil-e**, §3, pp. 5-6]. Many non-compact Hausdorff locally compact groups of arithmetic integer, abelian and non-abelian, are modulated. The examples throughout this work will be $(C_F, |\cdot|_F)$, $(C_{F,S}, |\cdot|_F \circ i_{F,S})$, and $(J_{F,S}, |\cdot|_{F,S})$, with $i_{F,S} : C_{F,S} \to C_F$ the continuous open injective homomorphism induced by the inclusion $\mathrm{GL}_1(\mathbf{A}_{F,S} \times \widehat{\mathcal{O}}_{\Sigma_F - S}) \to \mathrm{GL}_1(\mathbf{A}_F)$.

Remark D.2.1.3 (Canonical measure on modulated groups). For G a Hausdorff locally compact modulated abelian group and $\tau : \Gamma \to G$ a continuous homomorphic section to the module $|\cdot|_G$ (which defines an isomorphism $G^1 \times \Gamma \simeq G$), to build μ_{can} we use

$$\mu_{\mathrm{can}} = \mu_{1,\mathrm{can}} \times \mu_\Gamma$$

where $\mu_{1,\mathrm{can}}$ is the volume-1 Haar measure on G^1 and μ_Γ is a Haar measure on $\Gamma \subset \mathbf{R}_{>0}$ satisfying

$$\mu_\Gamma([1/\Lambda, \Lambda] \cap \Gamma) = 2\log(\Lambda) + O(1) \tag{D.2.1.5}$$

as $\Lambda \to +\infty$.

To see such a μ_Γ exists, we separately treat the cases $\Gamma = \mathbf{R}_{>0}$ and $\Gamma = t^{\mathbf{Z}}$ for some $t > 1$. In the case $\Gamma = \mathbf{R}_{>0}$ we use $\mu_\Gamma = \mathrm{d}t/t$ (in this case the $O(1)$-term in (D.2.1.5) vanishes). In the case $\Gamma = t^{\mathbf{Z}}$ with $t > 1$, take μ_Γ to be counting measure multiplied by $\log(t)$; this satisfies (D.2.1.5) because the difference between $2\log(\Lambda)$ and

$$\#\{m \in \mathbf{Z} : 1/\Lambda \leq t^m \leq \Lambda\}\log(t)$$

is bounded (by $\log(t)$) as $\Lambda \to +\infty$.

Remark D.2.1.4. In light of Remark D.2.1.3, we make the following observations on our choice of Haar measures $\overline{\mu}$ on C_F, $\overline{\mu}_1$ on C_F^1, $\overline{\mu}_S$ on $J_{F,S}$, and $\overline{\mu}_{S,1}$ on $J_{F,S}^1$. The Haar measures $\overline{\mu}_1$ on C_F^1 and its open

subgroup $C^1_{F,S}$, and $\overline{\mu}_{S,1}$ on $J^1_{F,S}$, generally do *not* give C^1_F, $C^1_{F,S}$, and $J^1_{F,S}$ volume 1 and are therefore $\overline{\mu}$ and $\overline{\mu}_S$ not the "canonical" choices on C_F and $J_{F,S}$ in the sense of μ_{can} on a modulated group as discussed above.

Since the pair of measures $\overline{\mu}$ on C_F and $\overline{\mu}_1$ on C^1_F, as well as the pair $\overline{\mu}_S$ on $J_{F,S}$ and $\overline{\mu}_{S,1}$ on $J^1_{F,S}$, induces $\mathrm{d}t/t$ on the idelic-norm image $\mathbf{R}_{>0}$, it follows that the canonical measure μ_{can} on C_F is

$$\mu_{\mathrm{can}} = \frac{1}{\overline{\mu}_1(C^1_F)}\,\overline{\mu} \tag{D.2.1.6}$$

and similarly the canonical measure on $J_{F,S}$ is

$$\mu_{S,\mathrm{can}} = \frac{1}{\overline{\mu}_{S,1}(J^1_{F,S})}\,\overline{\mu}_S. \tag{D.2.1.7}$$

We will be adhering to our choice of normalizations $\overline{\mu}$ and $\overline{\mu}_S$ due to their general familiarity from Tate's dissertation [**CF**, Ch. XV, §2.3] and their role in the proof of the Class Number Formula contained therein (see [**CF**, Ch. XV, §4, Thms. 4.3.2, 4.4.1]).

D.2.2. Complements on the G^1-isotypes of $\mathcal{S}(G)$ for modulated groups G. By Lemma 4.3.3.1(2) (applied to the G^1-action on $E = \mathcal{S}(G)$ defined by $f \mapsto \overline{\omega}(g)\, g.f$ for $g \in G^1$), $\Phi_{G,\omega}$ induces a linear topological isomorphism

$$\Phi_{G,\omega} : \mathcal{S}(G)(\omega) \to \mathrm{H}^0(\omega, \mathcal{S}(G)) \tag{D.2.2.1}$$

where $\mathcal{S}(G)(\omega)$ is the linear quotient $\mathcal{S}(G)/\ker(\Phi_{G,\omega})$ equipped with the quotient topology. The kernel of $\Phi_{G,\omega}$ is the closure in $\mathcal{S}(G)$ of the subspace of differences $\{f - \overline{\omega}(g)\, g.f \,:\, f \in \mathcal{S}(G), g \in G^1\}$ by Lemma 4.3.3.1(3), so $\mathcal{S}(G)(\omega)$ is the maximal Hausdorff quotient of the space of G^1-coinvariants of $\mathcal{S}(G)$ under the action $f \mapsto \overline{\omega}(g)\, g.f$ $(g \in G^1)$. For this reason, $\mathcal{S}(G)(\omega)$ is also denoted $\mathrm{H}_{0,\mathrm{cont}}(\omega, \mathcal{S}(G))$.

For a Hausdorff locally convex topological $\mathbf{C}$-vector space E equipped with a left action ρ_G of G by linear topological automorphisms, we say ρ_G is *uniformly continuous* on a subset $\Sigma \subset G$ if and only if for every 0-neighorhood $U \subset E$ there exists 0-neighborhood $V \subset E$ so that $\rho_G(\sigma, x) \in U$ for all $\sigma \in \Sigma$. Equivalently, this says that for every net $\{x_i\}_{i\in I} \subset E$ converging to $x \in E$ and every 0-neighborhood $U \subset E$ there exists i_0 so that $\rho_G(\sigma, x_i) - \rho_G(\sigma, x) = \rho_G(g, x_i - x) \in U$ for all $i \geq i_0$ and all $\sigma \in \Sigma$. For example, Proposition D.1.9.2 the G-action on $\mathcal{S}(G)$ is uniformly continuous on any compact subset $K \subset G$.

Lemma D.2.2.1 (G^1-isotypes under coarsening of the topology - I). *Let G be a Hausdorff locally compact modulated abelian group equipped with a module*

$$|\cdot|_G : G \to \mathbf{R}_{>0}. \tag{D.2.2.2}$$

Let μ_1 be a choice of Haar measure on the compact Hausdorff abelian group G^1. Let

$$i : \mathcal{S}(G) \hookrightarrow E$$

be a continuous injective linear map to a Hausdorff locally convex topological $\mathbf{C}$-vector space.

Assume E is complete and is equipped with a uniformly continuous left action ρ_G of G making i be G-equivariant (using the continuous left-translation action on the source). Choose a continuous character $\omega \in \widehat{G^1}$ and denote by

$$\Phi^E_{G,\omega} : E \to \mathrm{H}^0(\omega, E) := \{u \in E : \rho_G(g, u) = \omega(g)u \text{ for all } g \in G^1\} \tag{D.2.2.3}$$

the continuous linear averaging map

$$\Phi^E_{G,\omega}(u) := \frac{1}{\mu_1(G^1)} \int_{G^1} \overline{\omega}(g)\rho_G(g, u)\, \mathrm{d}\mu_1(g) \tag{D.2.2.4}$$

provided by Lemma 4.3.3.1 (applied to the G^1-action on E defined by $u \mapsto \overline{\omega}(g)\, \rho_G(g, u)$ for $g \in G^1$).

The following hold:

(1) $i^{-1}(\ker(\Phi^E_{G,\omega})) = \ker(\Phi_{G,\omega})$ *as closed subspaces of* $\mathcal{S}(G)$*, and* i *induces continuous injective linear maps (each also denoted as* i*)*

$$\mathcal{S}(G)(\omega) \to E(\omega) := E/\ker(\Phi^E_{G,\omega}),\ \ \mathrm{H}^0(\omega, \mathcal{S}(G)) \to \mathrm{H}^0(\omega, E) \tag{D.2.2.5}$$

(2) *We have* $i^{-1}(\mathrm{H}^0(\omega, E)) = \mathrm{H}^0(\omega, \mathcal{S}(G))$ *as closed subspaces of* $\mathcal{S}(G)$*.*

(3) *If* i *has dense range then so do the continuous linear maps in* (D.2.2.5)*.*

(4) *For any* $T \in \mathscr{L}(\mathcal{S}(G))$ *continuous with respect to the Hausdorff locally convex topology induced on* $\mathcal{S}(G)$ *by* i*, if* i *has dense range then* T *extends by continuity to a unique* $\overline{T} \in \mathscr{L}(E)$*. If, furthermore, there exists a topological automorphism* α *of* G *leaving* $|\cdot|_G$ *unaffected such that* T *satisfies*

$$T \circ \rho_G(g, \cdot) = \rho_G(\alpha(g), T(\cdot)) \tag{D.2.2.6}$$

for every $g \in G^1$ *then the map* $\overline{T}$ *satisfies*

$$\overline{T} \circ \rho_G(g, \cdot) = \rho_G(\alpha(g), \overline{T}(\cdot)) \tag{D.2.2.7}$$

and for $\alpha^*\omega := \omega \circ \alpha^{-1}$[9] *there exist unique continuous linear maps* T_ω*,* $\overline{T}_\omega$ *making the diagrams*

$$\begin{array}{ccccc} \mathcal{S}(G) & \longrightarrow & \mathcal{S}(G)(\omega) & \xrightarrow[\simeq]{\Phi_{G,\omega}} & \mathrm{H}^0(\omega, \mathcal{S}(G)) \\ \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T_\omega} & & \Big\downarrow{\scriptstyle T} \\ \mathcal{S}(G) & \longrightarrow & \mathcal{S}(G)(\alpha^*\omega) & \xrightarrow[\Phi_{G,\alpha^*\omega}]{\simeq} & \mathrm{H}^0(\alpha^*\omega, \mathcal{S}(G)) \end{array} \tag{D.2.2.8}$$

and

$$\begin{array}{ccccc} E & \longrightarrow & E(\omega) & \xrightarrow[\simeq]{\Phi^E_{G,\omega}} & \mathrm{H}^0(\omega, E) \\ \Big\downarrow{\scriptstyle \overline{T}} & & \Big\downarrow{\scriptstyle \overline{T}_\omega} & & \Big\downarrow{\scriptstyle \overline{T}} \\ E & \longrightarrow & E(\alpha^*\omega) & \xrightarrow[\Phi^E_{G,\alpha^*\omega}]{\simeq} & \mathrm{H}^0(\alpha^*\omega, E) \end{array} \tag{D.2.2.9}$$

commute. The operator $\overline{T}_\omega$ *is the unique extension by continuity of* T_ω*.*

In both diagrams, the composites across the rows are the actual averaging operators $\Phi_{G,\omega}$, $\Phi^E_{G,\omega}$, etc., with the induced isomorphisms as the right arrows in the rows (denoted as $\Phi_{G,\omega}$, etc. via abuse of notation) due to the isomorphism in (4.3.3.2) (where the group denoted G there is G^1 in the present setting).

Proof. The formation of Pettis integrals (over the compact G^1) with values in the complete E (equipped with a uniformly continuous G^1-action) is functorial with respect to G^1-equivariant continuous linear maps between such spaces E: see Lemma C.4.3.8. Thus, (1) and (2) are a consequence of the injectivity of i and the definitions of $\Phi_{G,\omega}$ and $\Phi^E_{G,\omega}$.

Assertion (3) follows from Lemma 4.3.3.1(2) since the continuous linear maps $\Phi_{G,\omega}$ and $\Phi^E_{G,\omega}$ are both open onto the respective spaces of ω-isotypic vectors. For (4), the existence and uniqueness of $\overline{T}$ arises from the universal mapping property of E as the completion of $\mathcal{S}(G)$ with respect to the Hausdorff subspace topology on it arising from the complete E. The uniqueness of T_ω and $\overline{T}_\omega$ is due to the surjectivity of the left horizontal maps in the top row of the two diagrams, and the existence follows from T and $\overline{T}$ taking ω-isotypic vectors to $\alpha^*(\omega)$-isotypic vectors. □

In light of Lemma D.2.2.1, for $i : \mathcal{S}(G) \to E$ continuous linear with dense range and satisfying the assumptions of the Lemma, we will identify $E(\omega)$ and $\mathrm{H}^0(\omega, E)$ as the need arises, each equipped with their

[9]Now composing with α^{-1}, was composing with α before but seems need α^{-1}: T carries ω-isotypic to $\omega \circ \alpha^{-1}$-isotypic.

induced continuous action of G, without mentioning the linear topological isomorphism induced by $\Phi^E_{G,\omega}$. We will refer to such Hausdorff locally convex complete topological $\mathbf{C}$-vector spaces as the *ω-isotype* of $\mathcal{S}(G)$, respectively E, for each continuous character $\omega \in \widehat{G^1}$.

Similarly, for $T \in \mathscr{L}(\mathcal{S}(G))$ compatible with the continuous left-translation action of G^1 on $\mathcal{S}(G)$ up to continuous automorphisms of G (i.e. satisfying the identity in (D.2.2.6) for every $g \in G^1$) we will refer to T_ω and $\overline{T}_\omega$ as the operators induced by T "on isotypes". Constructions on isotypes of $\mathcal{S}(G)$ are inherited by the isotypes of E under the assumptions of the Lemma on E and on continuous endomorphisms of $\mathcal{S}(G)$ extending to E compatibly with the continuous actions of G on source and target.

Next we record a description of the isotypes of $\mathcal{S}(G)$ for G a modulated Hausdorff locally compact abelian group. We refer the reader to §C.2.3.4, particularly Corollary C.2.3.12 and Proposition C.2.3.13, for the main properties of the inductive tensor product topology $\otimes_\iota$ and its completion $\overline{\otimes}$ that are used below.

LEMMA D.2.2.2 (G^1-isotypes under coarsening of the topology - II). *Let G be a second-countable Hausdorff locally compact abelian modulated group, with module $|\cdot|_G$, and let τ be a continuous homomorphic section to the module $|\cdot|_G$.*

(1) *The section τ induces a linear topological isomorphism*

$$\mathcal{S}(G^1)\,\overline{\otimes}\,\mathcal{S}(\Gamma) \to \mathcal{S}(G) \tag{D.2.2.10}$$

compatible with the continuous left-translation action of $G^1 \times \Gamma$ on the source through the tensor factors, and of G on the target.

(2) *Let E be a Hausdorff locally convex topological $\mathbf{C}$-vector space that is complete and bornological and equipped with a continuous left action ρ^E_G of G. Assume:*
   - *(i) The action ρ^E_G of G on E is sequentially uniformly continuous (so the E-valued ω-isotypic averaging operator $\Phi^E_{G,\omega}$ over G^1 via Lemma 4.3.3.1(b) makes sense, allowing us to define the quotient space $E(\omega) = \mathrm{H}_{0,\mathrm{cont}}(\omega, E) := E/\ker(\Phi^E_{G,\omega})$ of "continuous ω-isotypic coinvariants").*
   - *(ii) There exists a continuous injective linear map $i : \mathcal{S}(G) \to E$ with dense range that is equivariant with respect to the continuous left-translation action of G on $\mathcal{S}(G)$ and the action ρ^E_G on E.*
   - *(iii) For any continuous character $\chi \in \widehat{G}$, the continuous linear operator $M_\chi : \mathcal{S}(G) \to \mathcal{S}(G)$ defined as multiplication by the smooth function $\chi : G \to \mathbf{C}$ is also continuous with respect to the subspace topology on $\mathcal{S}(G)$ induced by i from E.*

   Letting $\chi : G = G^1 \times \Gamma \to \mathbf{C}^\times$ be defined by $\chi(g^1\tau(\gamma)) = \overline{\omega}(g^1)$, the composite injective continuous linear map $\mathcal{S}(\Gamma) \xrightarrow{\tau} \mathcal{S}(G) \xrightarrow{i} E$ gives $\mathcal{S}(\Gamma)$ a (necessarily coarser Hausdorff locally convex) subspace topology whose completion maps isomorphically via M_χ onto the Hausdorff quotient $E(\omega)$ of E.

The meaning of the final assertion is that under the linear topological automorphism M_χ of $\mathcal{S}(G)$ equipped with the subspace topology from E, the natural map $M_\chi(\mathcal{S}(\Gamma)) \to E(\omega)$ identifies the target with the completion of the source.

PROOF. Assertion (1) is a special case of Lemma D.1.8.1(1). For assertion (2), we first treat the case $E = \mathcal{S}(G)$ with i the identity. We have $\mathcal{S}(\Gamma) = \mathrm{H}^0(G^1, \mathcal{S}(G))$ (functions independent of the second component in $G = \Gamma \times G^1$), and $M_\chi(g.f) = \chi(g)(g.M_\chi(f))$ for $g \in G$, so M_χ carries the space of G^1-invariants onto the space of ω-isotypic vectors. In other words, $M_\chi(\mathcal{S}(\Gamma)) = \mathrm{H}^0(\omega, \mathcal{S}(G))$, Thus, the composite map $M_\chi(\mathcal{S}(\Gamma)) \to \mathcal{S}(G) \hookrightarrow \mathcal{S}(G)(\omega)$ is a topological isomorphism due to (4.3.3.2) applied to $E = \mathcal{S}(G)$ and the group G^1 acting by $f \mapsto \overline{\omega}(g)\, g.f$ for $g \in G^1$. This settles the special case $E = \mathcal{S}(G)$ with i the identity.

Now we turn to the general case of (2). The assumed continuity of M_χ with respect to the topology on $\mathcal{S}(G)$ by E under i allows us to conclude by using Lemma D.2.2.1(4) with $T = M_\chi$[10] *provided* that the Hausdorff quotient $E(\omega)$ of E inherits completness from E and the continuous linear map $M_\chi(\mathcal{S}(\Gamma)) \to E(\omega)$ is a topological embedding with dense image. Let us now verify these two properties.

The completeness of $E(\omega)$ holds because the natural map $\mathrm{H}^0(\omega, E) \to E(\omega)$ is a topological isomorphism (see (4.3.3.2)) and $\mathrm{H}^0(\omega, E)$ inherits completeness due to its closedness inside E. The map $M_\chi(\mathcal{S}(\Gamma)) \to E(\omega)$ (where $\mathcal{S}(\Gamma)$) is given the subspace topology from E) is topologically identified with the map $\mathrm{H}^0(\omega, \mathcal{S}(G)) \to \mathrm{H}^0(\omega, E) \simeq E(\omega)$ that is a topological embedding when $\mathcal{S}(G)$ is given the subspace topology from E. The image of this map is also dense because the map is linearly identified with $\mathcal{S}(G)(\omega) \to E(\omega)$ whose image is dense since $\mathcal{S}(G)$ is dense in E. □

D.2.2.1. *A kernel Theorem for $\mathbf{R}_{>0}$-modulated groups.* Lemma D.2.2.2 takes the following form for G any modulated group such that the module $|\cdot|_G$ is surjective; i.e., for which $\Gamma = \mathbf{R}_{>0}$. (Examples of interest include $G = C_F, C_{F,S}, J_{F,S}$.) We refer to these modulated groups as *$\mathbf{R}_{>0}$-modulated*.

Corollary D.2.2.3. [11] *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group; i.e., $\Gamma = \mathbf{R}_{>0}$. Fix a choice of $\omega \in \widehat{G^1}$ and a continuous homomorphic section τ to $|\cdot|_G$, and let $\chi : G = G^1 \times \Gamma \to \mathbf{C}^\times$ be the character $\chi(g^1\tau(\gamma)) = \overline{\omega}(g^1)$. Let $M_\chi : \mathcal{S}(G) \simeq \mathcal{S}(G)$ be multiplication by χ.*

(1) *The natural continuous linear map $\mathcal{S}(\mathbf{R}_{>0}) \simeq M_\chi(\mathcal{S}(\Gamma)) \to \mathcal{S}(G)(\omega)$ is a topological isomorphism;*
(2) *Let E be a complete Hausdorff locally convex topological vector space equipped with a continuous left G-action and a G-equivariant continuous linear injection $i : \mathcal{S}(G) \to E$ such that $M_\chi : \mathcal{S}(G) \to \mathcal{S}(G)$ is a topological isomorphism when $\mathcal{S}(G)$ is equipped with the subspace topology from E.*

   When $\mathcal{S}(\Gamma) \xhookrightarrow{\tau} \mathcal{S}(G)$ is equipped with the subspace topology from E, the natural map $M_\chi(\mathcal{S}(\Gamma)) \to E(\omega)$ is a topological embedding that identifies the target with the completion of the source.

Proof. This is Lemma D.2.2.2 in the special case $\Gamma = \mathbf{R}_{>0}$. □

D.3. Schwartz-Bruhat functions with I-exponential decay

We now relate spaces of Schwartz-Bruhat functions on the positive real line with exponential or semi-exponential decay to spaces of holomorphic functions on the complex plane or half-planes, with rapid decay uniformly over vertical strips of bounded width.

D.3.1. Schwartz-Bergman spaces. We begin with the following application of the Asymptotic Residue Theorem (see Theorem B.3.2.1).

Lemma D.3.1.1. *Let $I \subset \mathbf{R}$ be a non-empty open bounded interval, and $N \geq 0$ an integer.*

Let ϕ be a holomorphic function on the open vertical strip $\mathbf{C}_I$ of bounded width. Assume that for every non-empty open interval $J \subset \mathbf{R}$ strictly contained within I (i.e., $\overline{J} \subset I$), $s^N\phi(s)$ is bounded over $\mathbf{C}_J$. Then for every integer $k \geq 0$, $\partial_s^k\phi$ satisfies the same condition.

Proof. We write $I := (a, b)$ for real numbers $a < b$. For any non-empty open interval $J \subset \mathbf{R}$ strictly contained within I, there exists $\varepsilon > 0$ small enough such that $a + 2\varepsilon < b - 2\varepsilon$ and $J \subset [a + 2\varepsilon, b - 2\varepsilon]$. By hypothesis, $s^N\phi$ is bounded over $\mathbf{C}_{[a+\varepsilon, b-\varepsilon]}$. We fix a point $z \in \mathbf{C}_{(a+2\varepsilon, b-2\varepsilon)}$.

[10] No, that cited Lemma is *not* what we need since automorphisms α of G are not relevant here; we are using a variant of the Lemma where T intertwines the isotypic subspaces for two characters of G^1 without any automorphism of G being relevant. For χ as just defined, $T(g.f) = \chi(g)\, g.(Tf)$, so T carries G^1-invariants exactly onto $\mathrm{H}^0(\omega, \mathcal{S}(G))$.

[11] This result is never cited anywhere, so why is it stated? The preceding result entirely subsumes it anyway.

For $k \geq 0$ a fixed arbitrary integer, the function:

$$f_z(s) := \frac{s^N \phi(s)}{(s-z)^{k+1}} \tag{D.3.1.1}$$

is meromorphic over $\mathbf{C}_I$, with a pole of order at most $k+1$ at $s = z$. For $s = \sigma + iT$, the function of T given by

$$u_\sigma(T) := \frac{(\sigma+iT)^N \phi(\sigma+iT)}{(\sigma+iT-z)^{k+1}} = f_z(\sigma+iT)$$

satisfies $\lim_{|T|\to+\infty} \sup_{\sigma\in[a+\varepsilon,b-\varepsilon]} |u_\sigma(T)| = 0$ since

$$|u_\sigma(T)| \leq \frac{\sup_{s\in\mathbf{C}_{[a+\varepsilon,b-\varepsilon]}} |s^N\phi(s)|}{|\sigma+iT-z|^{k+1}} = \frac{\sup_{s\in\mathbf{C}_{[a+\varepsilon,b-\varepsilon]}} |s^N\phi(s)|}{(|\sigma+\mathrm{Re}(z)|^2+|T-\mathrm{Im}(z)|^2|^{(k+1)/2}} \leq \frac{\sup_{s\in\mathbf{C}_{[a+\varepsilon,b-\varepsilon]}} |s^N\phi(s)|}{|T-\mathrm{Im}(z)|^{k+1}} \tag{D.3.1.2}$$

where the right side of (D.3.1.2) is finite for $T \neq \mathrm{Im}(z)$.

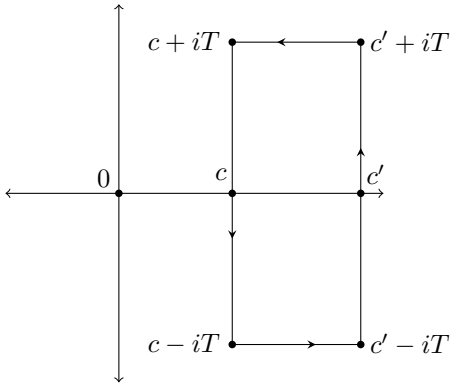


FIGURE 1. A counterclockwise rectangular contour, with $c = a+\varepsilon$, $c' = b-\varepsilon$

By the Cauchy Integral Formula applied to the contour in Figure 1, we have

$$\partial_s^k(s^N\phi)|_{s=z} = -\frac{k!}{2\pi i}\int_{a+\varepsilon-iT}^{a+\varepsilon+iT} f_z(s)\,\mathrm{d}s + \frac{k!}{2\pi i}\int_{b-\varepsilon-iT}^{b-\varepsilon+iT} f_z(s)\,\mathrm{d}s + \frac{k!}{2\pi i}\int_{a+\varepsilon-iT}^{b-\varepsilon-iT} f_z(s)\,\mathrm{d}s - \frac{k!}{2\pi i}\int_{a+\varepsilon+iT}^{b-\varepsilon+iT} f_z(s)\,\mathrm{d}s. \tag{D.3.1.3}$$

In light of (D.3.1.2), we have:

$$\frac{k!}{2\pi}\int_{a+\varepsilon-iT}^{b-\varepsilon-iT} |f_z(s)|\,|\mathrm{d}s| + \frac{k!}{2\pi}\int_{a+\varepsilon+iT}^{b-\varepsilon+iT} |f_z(s)|\,|\mathrm{d}s| \leq A\left\{\frac{1}{|T-\mathrm{Im}(z)|^{k+1}} + \frac{1}{|-T-\mathrm{Im}(z)|^{k+1}}\right\} \tag{D.3.1.4}$$

where

$$A := \frac{k!(b-a-2\varepsilon)\sup_{s\in\mathbf{C}_{[a+\varepsilon,b-\varepsilon]}} |s^N\phi(s)|}{2\pi} > 0.$$

We deduce that for $k \geq 1$ we have

$$|\partial_s^k(s^N\phi)||_{s=z} \leq \frac{k!}{2\pi}\int_{-\infty}^{+\infty} \frac{|a+\varepsilon+it|^N|\phi(a+\varepsilon+it)|}{(|a+\varepsilon-\mathrm{Re}(z)|^2+|t-\mathrm{Im}(z)|^2)^{(k+1)/2}}\,\mathrm{d}t \tag{D.3.1.5}$$

$$+ \frac{k!}{2\pi}\int_{-\infty}^{+\infty} \frac{|b-\varepsilon+it|^N|\phi(b-\varepsilon+it)|}{(|b-\varepsilon-\mathrm{Re}(z)|^2+|t-\mathrm{Im}(z)|^2)^{(k+1)/2}}\,\mathrm{d}t \tag{D.3.1.6}$$

where the integrals converge since $s^N\phi(s)$ is bounded on vertical lines within $\mathbf{C}_I$ and $k+1 > 1$.

Since $z \in \mathbf{C}_{(a+2\varepsilon,b-2\varepsilon)}$, then $|a+\varepsilon-\mathrm{Re}(z)| \geq \varepsilon$, and $|b-\varepsilon-\mathrm{Re}(z)| \geq \varepsilon$, and so the two integrals in (D.3.1.5), which are bounded independently of $\mathrm{Im}(z)$ (because $s^N\phi$ is bounded on $\mathbf{C}_{[a+\varepsilon,b-\varepsilon]}$), are bounded above by a positive constant independent of $\mathrm{Re}(z)$ too. In particular, $\partial_s^k(s^N\phi)$ is bounded over $\mathbf{C}_{(a+2\varepsilon,b-2\varepsilon)}$, and hence $\partial_s^k(s^N\phi)$ is bounded over $\mathbf{C}_J$ for every integer $k \geq 1$; this also holds for $k=0$ by hypothesis.

We want to show $s^N\partial_s^k\phi$ is bounded over $\mathbf{C}_J$ for every non-empty open interval J with $\overline{J} \subset I$. For $k=0$ this is given in the hypotheses, and $N=0$ was settled above (for all k), so we now assume $k, N \geq 1$. We have

$$\partial_s^k(s^N\phi) = \sum_{0\leq m\leq \min(k,N)} \binom{k}{m} \frac{N!}{(N-m)!} s^{N-m}\, \partial_s^{k-m}\phi(s)$$

over $\mathbf{C}_I$, and by separating off the term for $m=0$ we see that if the desired boundedness assertion holds for $\partial_s^j\phi$ for every $0 \leq j \leq k-1$ (i.e., for each such j, $s^N\partial_s^j\phi$ is bounded over $\mathbf{C}_J$ for any non-empty open interval J with $\overline{J} \subset I$) then it also holds for $\partial_s^k\phi$. We conclude by induction on the integer $k \geq 0$. □

The following Lemma is reminiscent of properties of Bergman Spaces and their completeness (see [**HKZ**, Prop.1.2]), replacing the role of the open unit disc with that of an open vertical strip:

LEMMA D.3.1.2 (Uniform Schwartz topology and L^p-convergence on vertical lines). *Let $I \subset \mathbf{R}$ be a non-empty open interval, and $\mathcal{S}(\mathbf{C}_I)$ the $\mathbf{C}$-vector space of holomorphic functions on $\mathbf{C}_I$ that are Schwartz (in terms of vertical decay properties) uniformly on the vertical strips inside $\mathbf{C}_I$ of bounded width (so all functions $s^N\phi$ for $N \geq 0$ and $\phi \in \mathcal{S}(\mathbf{C}_I)$ are bounded on $\mathbf{C}_J$ for all open intervals J with $\overline{J} \subset I$).*

Let $p > 1$ be a real number, and $a_\bullet := \{a_m\}_{m\geq 0} \subset I$ a sequence of real numbers such that $\inf(I)$ and $\sup(I)$ are accumulation points of $a_\bullet$ (for instance, if $\inf(I) = 0$ then 0 is an accumulation point and if $\sup(I) = +\infty$ then $\limsup_{m\geq 0} a_m = +\infty$).

When $\mathcal{S}(\mathbf{C}_I)$ is endowed with the topology of L^p-convergence on all vertical lines $a_m + i\mathbf{R}$ for all $\{s^N\phi\}_{N\geq 0}$ for $\phi \in \mathcal{S}(\mathbf{C}_I)$ then it is a separable Fréchet space. This topology is equal to the Hausdorff locally convex topology of Schwartz convergence on every vertical line of $\mathbf{C}_I$. In particular, this topology does not depend on the choice of $p > 1$ nor on the choice of $a_\bullet$.

PROOF. We will treat the case $I = (0, +\infty)$ for the sake of clarity, the general case being established with the same method, with only notational differences. Let $a_\bullet := \{a_m\}_{m\geq 0} \subset \mathbf{Q}_{>0}$ be a sequence of positive rational numbers such that 0 is an accumulation point, and $\limsup_{m\geq 0} a_m = +\infty$. Consider the linear map

$$A_{a_\bullet} : \mathcal{S}(\mathbf{C}_{>0}) \to \prod_{m\geq 0}\prod_{N\geq 0} L^p(\mathbf{R}, \mathrm{d}x) \tag{D.3.1.7}$$

defined as $A_{a_\bullet}(\phi) := (s^N\phi(s)|_{\mathrm{Re}(s)=a_m})_{m,N\geq 0}$.

The map $A_{a_\bullet}$ is injective since elements of $\mathcal{S}(\mathbf{C}_{>0})$ are holomorphic by definition . Let $\{\phi_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{C}_{>0})$ be a sequence, and assume that the sequence $\{A_{a_\bullet}(\phi_k)\}_{k\geq 0}$ converges, so $\{\phi_k|_{a_m+i\mathbf{R}}\}_{m\geq 0}$ converges to some $(\alpha_m)_{m\geq 0} \in \prod_{m\geq 0} L^p(\mathbf{R}, \mathrm{d}x)$.

Let $0 < A < B$ be arbitrary real numbers, and consider the vertical strip $\mathbf{C}_{(A,B)}$ of bounded width within $\mathbf{C}_{>0}$ for $\mathrm{Re}(s) \in (A, B)$. There exist integers m_1, m_2 such that $a_{m_1} < A < B < a_{m_2}$. Since each ϕ_k is holomorphic over the vertical strip $\mathbf{C}_{(a_{m_1},a_{m_2})}$ and is uniformly Schwartz, by Theorem B.2.2.4 we obtain

$$\phi_k(s) = -\frac{1}{2\pi i}\int_{a_{m_1}-i\infty}^{a_{m_1}+i\infty} \frac{\phi_k(z)}{z-s}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{a_{m_2}-i\infty}^{a_{m_2}+i\infty} \frac{\phi_k(z)}{z-s}\,\mathrm{d}z \tag{D.3.1.8}$$

for any $s \in \mathbf{C}_{(A,B)}$.

Since $a_{m_1} < A$ and $a_{m_2} > B$, the function $(z-s)^{-1}$ for $z \in (a_{m_1} + i\mathbf{R}) \cup (a_{m_2} + i\mathbf{R})$ and $s \in \mathbf{C}_{(A,B)}$ is uniformly bounded. Since $\{\phi_k|_{a_{m_j}+i\mathbf{R}}\}_{k\geq 0}$ converges in $L^p(\mathbf{R}, \mathrm{d}x)$ for $j = 1, 2$, both integrals in (D.3.1.8) as

sequences of functions of $s \in \mathbf{C}_{(A,B)}$ satisfy

(D.3.1.9)

$$\int_{a_{m_j}-i\infty}^{a_{m_j}+i\infty} \left|\frac{\phi_k(z)-\alpha_{m_j}(\mathrm{Im}(z))}{z-s}\right| |\mathrm{d}z| \leq \left(\int_{\mathbf{R}} \left|\phi_k(a_{m_j}+ix)-\alpha_{m_j}(x)\right|^p \,\mathrm{d}x\right)^{1/p} \left(\int_{\mathbf{R}} \left|\frac{1}{a_{m_j}+ix-s}\right|^q \,\mathrm{d}x\right)^{1/q}$$

for $j = 1, 2$ by Hölder's Inequality, with $1/p + 1/q = 1$ (we have $q = p/(p-1) > 1$).

The first of the two factors on the right side of (D.3.1.9) is arbitrarily small for $j = 1, 2$ as $k \to +\infty$ since $\phi_k \to \alpha_{m_j}$ on $a_{m_j} + i\mathbf{R}$ in the L^p-sense. The second factor on the right side, namely

$$\left(\int_{\mathbf{R}} \left|\frac{1}{a_{m_j}+ix-s}\right|^q \,\mathrm{d}x\right)^{1/q}, \tag{D.3.1.10}$$

is bounded above (upon writing $s = \sigma + i\tau$) by

$$\left(\int_{\mathbf{R}} \left(\frac{1}{|a_{m_j}-\sigma|^2+|x-\tau|^2}\right)^{q/2} \mathrm{d}x\right)^{1/q} \leq \left(\int_{\mathbf{R}} \frac{1}{(|a_{m_j}-A|^2+x^2)^{q/2}} \,\mathrm{d}x\right)^{1/q} < \infty. \tag{D.3.1.11}$$

The final upper bound is a positive constant depending only on a_{m_j}, A, B but not on $s \in \mathbf{C}_{(A,B)}$.

In particular, $\{\phi_k\}_{k\geq 0}$ as a sequence of holomorphic functions on the vertical strip $\mathbf{C}_{(A,B)}$ converges pointwise to some function $\phi(s)$, namely on the strip $a_{m_1} < \sigma < a_{m_2}$ we have

$$\phi(s) = -\frac{1}{2\pi i}\int_{a_{m_1}+i\mathbf{R}} \frac{\alpha_{m_1}(\mathrm{Im}(z))}{s-z}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{a_{m_2}+i\mathbf{R}} \frac{\alpha_{m_2}(\mathrm{Im}(z))}{s-z}\,\mathrm{d}z$$

where both integrals are oriented upward and are absolutely convergent by Hölder's Inequality: the corresponding L^1-norms for $j = 1, 2$ are bounded above by

$$\frac{1}{2\pi}\left(\int_{\mathbf{R}} |\alpha_{m_j}(y)|^p \,\mathrm{d}y\right)^{1/p} \left(\int_{a_{m_j}+i\mathbf{R}} \frac{|\mathrm{d}z|}{|z-s|^q}\right)^{1/q} < \infty.$$

The convergence to ϕ is uniform on $\mathbf{C}_{(A,B)}$, so ϕ holomorphic over the vertical strip $\mathbf{C}_{(A,B)}$.

Since $(A, B) \subset \mathbf{R}_{>0}$ is arbitrary (here we use that 0 is an accumulation point of $a_\bullet$ and $\limsup_{m\geq 0} a_m = +\infty$, so given any $(A, B) \subset \mathbf{R}_{>0}$ we can always find $a_{m_1} < A$ and $a_{m_2} > B$ belonging to $a_\bullet$), we see that the limit in $\prod_{m\geq 0} L^p(\mathbf{R}, \mathrm{d}x)$ of the image under $A_{a_\bullet}$ from (D.3.1.7) of the sequence $\{\phi_k\}_{k\geq 0} \subset \mathcal{S}(\mathbf{C}_{>0})$, remains in $\mathscr{O}(\mathbf{C}_{>0})$; i.e., it is holomorphic over $\mathbf{C}_{>0}$.

Using (D.3.1.8) and adapting the preceding considerations with $s^N\phi_k$ in place of ϕ_k for each $N \geq 1$, the uniform Schwartz condition on the limit holomorphic function ϕ over the vertical strips within $\mathbf{C}_{>0}$ of bounded width is verified using the convergence of the sequence of integrals (for each integer $N \geq 1$) in

$$|s^N\phi_k(s)| \leq \frac{1}{2\pi}\int_{\mathbf{R}} \left|\frac{(a_{m_1}+ix)^N\phi_k(a_{m_1}+ix)}{a_{m_1}+ix-s}\right| \mathrm{d}x + \frac{1}{2\pi}\int_{\mathbf{R}} \left|\frac{(a_{m_2}+ix)^N\phi_k(a_{m_2}+ix)}{a_{m_2}+ix-s}\right| \mathrm{d}x. \tag{D.3.1.12}$$

We note that since each $s^N\phi$ is bounded over the vertical strips of bounded width in $\mathbf{C}_{>0}$ (staying away from the left edge), so is each $s^N(\partial_s^k\phi)$ for each integer $k \geq 0$, as a consequence of Lemma D.3.1.1. It follows that we have $\phi \in \mathcal{S}(\mathbf{C}_{>0})$. It follows that $A_{a_\bullet}$ has (sequentially) closed range in $\prod_{m\geq 0}\prod_{N\geq 0} L^p(\mathbf{R}, \mathrm{d}x)$. Thus, $\mathcal{S}(\mathbf{C}_{>0})$ is a Fréchet space.

The preceding work (especially on the strip $a_{m_1} < \sigma < a_{m_2}$ with uniformity) shows that the topology on $\mathcal{S}(\mathbf{C}_I)$ agrees with L^p-convergence on every vertical line of $\mathbf{C}_I$. This in turn agrees with convergence in $\mathcal{S}(\mathbf{R})$ on every such vertical line due to Example D.1.10.3, so it agrees with the topology defined via $A_{a_\bullet}$. □

Definition D.3.1.3 (Schwartz-Bergman spaces). Fix $1 < p < \infty$ and $I \subset \mathbf{R}$ a non-empty open interval. We define the *Schwartz-Bergman* space $A\mathcal{S}(\mathbf{C}_I)$ to be the $\mathbf{C}$-linear space $\mathcal{S}(\mathbf{C}_I)$ of holomorphic functions on $\mathbf{C}_I$ that are Schwartz uniformly on the vertical strips of bounded width within $\mathbf{C}_I$, topologized with the

Hausdorff locally convex topology generated by the countable family of seminorms $\{p_{m,N}\}_{m,N\geq 0}$ defined as:

$$p_{m,N}(\phi) := \|(a_m + ix)^N \phi(a_m + ix)\|_{L^p(\mathbf{R},\mathrm{d}x)},\ m, N \geq 0$$

with $\{a_m\}_{m\geq 0} \subset I$ a choice of sequence of positive rational numbers as in Lemma D.3.1.2.

REMARK D.3.1.4. Lemma D.3.1.2 shows that $A\mathcal{S}(\mathbf{C}_I)$ is a separable Fréchet space, and its topology does not depend on the choice of $\{a_m\}_{m\geq 0}$, nor on the choice of $p > 1$.

D.3.2. Fourier-Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$. We now relate the spaces of Schwartz-Bruhat functions $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, $\mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$ and $\check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$ with the Schwartz-Bergman spaces $A\mathcal{S}(\mathbf{C}_I)$ for $I = \mathbf{R}$, $I = \mathbf{R}_{>0}$ and $I = (-\infty, 0)$ by means of the Mellin transform. We begin with a definition.

DEFINITION D.3.2.1 (Schwartz-Bruhat functions on $\mathbf{R}_{>0}$ with I-exponential decay). Let $I \subset \mathbf{R}$ be a non-empty interval. We define $\mathcal{S}_I(\mathbf{R}_{>0}) \subset C^\infty(\mathbf{R}_{>0})$ to be the linear subspace of those smooth functions ϕ on $\mathbf{R}_{>0}$ satisfying the following condition: $t^a\phi(t) \in \mathcal{S}(\mathbf{R}_{>0})$ for all $a \in I$.

We call $\mathcal{S}_I(\mathbf{R}_{>0})$ the $\mathbf{C}$-linear space of *Schwartz-Bruhat functions of I-exponential decay*. We topologize $\mathcal{S}_I(\mathbf{R}_{>0})$ with the (visibly Hausdorff) projective topology [**Scha**, Ch. II, §5] induced by the t^a-multipliiers

$$T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}) \tag{D.3.2.1}$$

for every $a \in I$ (i.e., $T_a(\phi) = t^a\phi$). In other words, the topology on $\mathcal{S}_I(\mathbf{R}_{>0})$ is the subspace topology arising from the linear injection $T = \prod_{a\in I} T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})$, Equivalently, a net $\{\phi_\alpha\}$ in $\mathcal{S}_I(\mathbf{R}_{>0})$ converges to $\phi \in \mathcal{S}_I(\mathbf{R}_{>0})$ if and only if $T_a(\phi_\alpha) \to T_a(\phi)$ in $\mathcal{S}(\mathbf{R}_{>0})$ for all $a \in I$.

REMARK D.3.2.2. Each operator T_a intertwines with the left-translation action by $\mathbf{R}_{>0}$ up to a controlled scalar multiplication: $T_a(c.\phi) = c^a(c.(T_a(\phi)))$ for $c \in \mathbf{R}_{>0}$. Thus, $\mathcal{S}_I(\mathbf{R}_{>0})$ has a natural $\mathbf{R}_{>0}$-action via $(c.f)(t) = f(c^{-1}t)$ that is moreover *continuous*, and uniformly so for the action of compact subsets of $\mathbf{R}_{>0}$ (thanks to the known case of $\mathcal{S}(\mathbf{R}_{>0}) \simeq \mathcal{S}(\mathbf{R})$).

D.3.2.1. *Basic properties of $\mathcal{S}_I(\mathbf{R}_{>0})$.*

LEMMA D.3.2.3. *Let $I \subset \mathbf{R}$ be a non-empty interval. The following hold:*

(1) *The Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}_I(\mathbf{R}_{>0})$ is Fréchet and nuclear (hence Montel).*
(2) *The Hausdorff strong dual $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$ is complete, nuclear, and Montel.*
(3) *If $0 \in I$, then $\mathcal{S}_I(\mathbf{R}_{>0})$ is naturally a linear subspace of $\mathcal{S}(\mathbf{R}_{>0})$ and the resulting injective linear inclusion*

$$\mathcal{S}_I(\mathbf{R}_{>0}) \subset \mathcal{S}(\mathbf{R}_{>0})$$

is continuous.
(4) *If $I = [0, +\infty)$ then $\mathcal{S}_I(\mathbf{R}_{>0}) = \mathcal{S}_{\mathrm{SE}}(\mathbf{R}_{>0})$, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{R}_{>0})$.*
(5) *If $I = (-\infty, 0]$ then $\mathcal{S}_I(\mathbf{R}_{>0}) = \check{\mathcal{S}}_{\mathrm{SE}}(\mathbf{R}_{>0})$, as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{R}_{>0})$.*
(6) *If $I = \mathbf{R}$ then $\mathcal{S}_I(\mathbf{R}_{>0}) = \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ as $\mathbf{C}$-linear subspaces of $\mathcal{S}(\mathbf{R}_{>0})$.*

The implication "nuclear Fréchet ⇒ Montel" noted in (1) is a special case of [**Tre**, Cor. 3 to Prop. 50.2] (since all Fréchet spaces are barrelled [**Scha**, Ch. II, Cor. to 7.1]).

PROOF. (1) : Since the topology on $\mathcal{S}_I(\mathbf{R}_{>0})$ is the subspace topology inherited from the product topology on $\prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})$ via a linear injection $T = \prod_{a\in I} T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})$, $\mathcal{S}_I(\mathbf{R}_{>0})$ is nuclear because arbitrary products and subspaces of nuclear spaces are nuclear [**Scha**, Ch. III, Thm. 7.4].

Since $\{a\} \subset I$ for every $a \in I$, we have a linear inclusion $S_a : \mathcal{S}_I(\mathbf{R}_{>0}) \subset \mathcal{S}_{\{a\}}(\mathbf{R}_{>0})$, and S_a is clearly continuous. Letting $R_a : \mathcal{S}_{\{b\}}(\mathbf{R}_{>0}) \simeq \mathcal{S}_{\{b+a\}}(\mathbf{R}_{>0})$ be the linear homeomorphism given by t^{-a}-multiplication for all $a, b \in \mathbf{R}$, the linear image of $\mathcal{S}_I(\mathbf{R}_{>0})$ within the product $\prod_{a\in I} \mathcal{S}_{\{a\}}(\mathbf{R}_{>0})$ under the continuous linear

map $\prod_{a\in I} S_a$ is the closed linear subspace of those elements $\{\phi_a : a \in I\}$ satisfying $(R_b^{-1} \circ R_a)(\phi_a) = \phi_b$ for all $a, b \in I$. Hence, $\mathcal{S}_I(\mathbf{R}_{>0})$ is complete since $\prod_{a\in I} S_a$ is a linear topological embedding and each $\mathcal{S}_{\{a\}}(\mathbf{R}_{>0})$ is complete (being isomorphic to $\mathcal{S}_{\{0\}}(\mathbf{R}_{>0}) = \mathcal{S}(\mathbf{R}_{>0})$).

We may write I as the countable union of closed intervals $[a, b] \subset \mathbf{R}$, and by a similar argument $\mathcal{S}_I(\mathbf{R}_{>0})$ is therefore realized as a topological subspace (hence as a locally convex topological $\mathbf{C}$-linear subspace) of the countable product of the $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$'s, the linear topological embedding being induced by the natural continuous linear maps $\mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$.

We further claim that $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$ is carried onto a closed subspace of the product $\mathcal{S}_{\{a\}}(\mathbf{R}_{>0}) \times \mathcal{S}_{\{b\}}(\mathbf{R}_{>0})$ via the "diagonal" map $f \mapsto (f, f)$. To see this, via the change of variables $e^x = t^{b-a}$ we reduce to showing (i) if $f \in \mathcal{S}(\mathbf{R})$ satisfies $e^x f \in \mathcal{S}(\mathbf{R})$ then $e^{cx} f \in \mathcal{S}(\mathbf{R})$ for all $0 < c < 1$, (ii) the space of pairs $(f_1, f_2) \in \mathcal{S}(\mathbf{R})$ with $f_2 = e^x f_1$ is closed. Since $e^{cx} f = e^{(c-1)x}(e^x f)$ with $c - 1 < 0$, by separately treating $x < 0$ and $x > 0$ we get (i). The closedness in (ii) follows from consideration of convergent sequences (as $\mathcal{S}(\mathbf{R})$ is metrizable). Finally, the continuous linear bijection $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0}) \to \mathcal{S}_{\{a\}}(\mathbf{R}_{>0}) \times \mathcal{S}_{\{b\}}(\mathbf{R}_{>0})$ onto a closed subspace is a topological embedding because the same calculations show that convergence in $\mathcal{S}_{\{c\}}(\mathbf{R}_{>0})$ for $a < c < b$ is controlled by simultaneous convergence in both $\mathcal{S}_{\{a\}}(\mathbf{R}_{>0})$ and $\mathcal{S}_{\{b\}}(\mathbf{R}_{>0})$.

It follows that $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$, and hence $\mathcal{S}_I(\mathbf{R}_{>0})$, is a Fréchet space (as a countable product of complete metric spaces is a complete metric space). Being nuclear and Fréchet, we are done.

(2) : By [**Scha**, Ch. II, 8.1; Ch. IV, 6.1], $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$ is complete. By [**Tre**, Prop. 50.6], $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$ is nuclear. Finally, by [**Scha**, Ch. IV, 5.9], $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$ is Montel.

(3) : If $0 \in I$ then the inclusion of intervals $\{0\} \subset I$ induces an injective continuous linear operator $\mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}_{\{0\}}(\mathbf{R}_{>0})$. We conclude by observing that we have $\mathcal{S}_{\{0\}}(\mathbf{R}_{>0}) = \mathcal{S}(\mathbf{R}_{>0})$ by definition.

Assertions (4), (5) and (6) are a consequence of the definitions (see Definition B.2.1.1).[12] □

REMARK D.3.2.4 (On nuclearity of $\mathcal{S}_I(\mathbf{R}_{>0})$). Let $I \subset \mathbf{R}$ be a non-empty interval. The definition of $\mathcal{S}_I(\mathbf{R}_{>0})$ realizes it, through the proof of Lemma D.3.2.3(1), as a subspace of $\mathfrak{s}^{\mathbf{N}}$ where $\mathfrak{s} = \mathcal{S}(\mathbf{Z})$ is the nuclear Fréchet space of rapidly decreasing $\mathbf{Z}$-indexed sequences. The relation of such Schwartz functions to $\mathfrak{s}$ is explained in the proof of [**Tre**, Cor. to Thm. 51.5] via Fourier series, and the inheritance of nuclearity under passage to subspaces and arbitrary products is [**Tre**, (50.3), (50.5) in Prop. 50.1] (or see [**Scha**, Ch. III, 7.4]).

D.3.2.2. *Fourier-Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$.*

PROPOSITION D.3.2.5 (Fourier-Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$). *Let $I \subset \mathbf{R}$ be an open non-empty interval. The assignment $\phi \mapsto M(\phi)(s) := \int_{\mathbf{R}_{>0}} \phi(t) t^s \frac{\mathrm{d}t}{t}$ is defined on $\mathcal{S}_I(\mathbf{R}_{>0})$ and valued in $A\mathcal{S}(\mathbf{C}_I)$. It induces a linear topological isomorphism*

$$M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I). \tag{D.3.2.2}$$

PROOF. By Proposition B.2.2.3(2), the visibly holomorphic $M(\phi)$ on $\mathbf{C}_I$ is Schwartz uniformly on every vertical strip $\mathbf{C}_J$ for every open bounded interval J with $\overline{J} \subset I$. It follows that $M(\phi)$ is an element of $A\mathcal{S}(\mathbf{C}_I)$.

The resulting linear operator $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \to A\mathcal{S}(\mathbf{C}_I)$, M_I as in (D.3.2.2) is injective by Fourier Inversion. To show it is surjective. pick $u \in A\mathcal{S}(\mathbf{C}_I)$. By injectivity of M_I, without loss of generality we may assume that I is bounded. Proposition B.2.1.8 implies that there exists a unique $\phi \in C_0^\infty(\mathbf{R}_{>0})$ such that $M(\phi) = u$ over $\mathbf{C}_I$. By Lemma D.3.1.1 combined with differentiation under the integral sign and the Identity Theorem,[13] $(\log(t))^m t^\sigma \phi \in C_0^\infty(\mathbf{R}_{>0})$ for every $\sigma \in I$ and for every integer $m \geq 0$. This forces $\phi \in \mathcal{S}_I(\mathbf{R}_{>0})$.

The map M_I is sequentially continuous by continuity of Fourier transform on $\mathcal{S}(\mathbf{R})$ since $M_I(\phi_m)|_{\sigma+i\mathbf{R}} \to M_I(\phi)|_{\sigma+i\mathbf{R}}$ as Schwartz functions on $\mathbf{R}$ for any sequence $\{\phi_m\}_{m\geq 0} \subset \mathcal{S}_I(\mathbf{R}_{>0})$ converging to ϕ and any

[12] There should really be a lemma after that earlier Definition which explains such consequences (4), (5), (6) using terminology and notation that makes sense back there.

[13] clarify how the Identity Theorem is being used, via "injectvity" of $\phi \mapsto M(\phi)$ and $\log(t) = o(t^\varepsilon)$ as $t \to +\infty$ for all $\varepsilon > 0$.

$\sigma \in (a,b)$, which implies convergence in $A\mathcal{S}(\mathbf{C}_I)$ by design of the Hausdorff locally convex topology on $A\mathcal{S}(\mathbf{C}_I)$ from Definition D.3.1.3. Since $\mathcal{S}(\mathbf{C}_I)$ is a Fréchet space, as is $\mathcal{S}_I(\mathbf{R}_{>0})$ by Lemma D.3.2.3(1), sequential continuity of M_I implies it is continuous. Thus, by the Open Mapping Theorem for Fréchet spaces, the continuous bijection M_I is a linear topological isomorphism. □

DEFINITION D.3.2.6 (Fourier-Mellin transform on $\mathcal{S}_I(\mathbf{R}_{>0})$). Let $I \subset \mathbf{R}$ be a non-empty open interval. We define the *Fourier-Mellin transform* on $\mathcal{S}_I(\mathbf{R}_{>0})$ to be the linear topological isomorphism M_I from (D.3.2.2).

D.3.2.3. *Fundamental families of continuous seminorms on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$.* Next we provide a countable family of seminorms on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ that generate its Hausdorff locally convex topology from Lemma D.3.2.3(6) for $I = \mathbf{R}$ (as $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0}) = \mathcal{S}_I(\mathbf{R}_{>0})$ for $I = \mathbf{R}$).

LEMMA D.3.2.7 (Countable fundamental families of seminorms on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$). *For $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}_{\geq 0}$, and $m \geq 0$ an integer, define $\|\cdot\|_{a,m}$ and $\|\cdot\|_{a,b,m}$ on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ by*

$$\|\phi\|_{a,m} := \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, \max\{t, t^{-1}\}^{2a} \, \frac{\mathrm{d}t}{t}\right)^{1/2} \tag{D.3.2.3}$$

and

$$\|\phi\|_{a,b,m} := \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, \max\{t, t^{-1}\}^{2a} \, |\log(t)|^{2b} \, \frac{\mathrm{d}t}{t}\right)^{1/2} \tag{D.3.2.4}$$

(note that $\|\cdot\|_{a,0,m} = \|\cdot\|_{a,m}$). Then

(1) *$\|\cdot\|_{a,m}$ and $\|\cdot\|_{a,b,m}$ are finite on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ (hence seminorms) and continuous.*
(2) *The collection $\{\|\cdot\|_{a,m}\}$ as a varies over a countable subset $\{a_k\}_{k\geq 0}$ of positive real numbers such that $\limsup_{k\geq 0} a_k = +\infty$ and $m \geq 0$ varies over all the nonnegative integers generates on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ its Hausdorff locally convex topology.*
(3) *The collection $\{\|\cdot\|_{a,b,m}\}$ as a varies over a countable subset $\{a_k\}_{k\geq 0}$ of positive real numbers such that $\limsup_{k\geq 0} a_k = +\infty$, $b \geq 0$ varies over the nonnegative integers, and $m \geq 0$ varies over the nonnegative integers generates on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$ its Hausdorff locally convex topology.*

PROOF. (1) : We have the inequalities:

$$\begin{aligned}\|\phi\|_{a,m} &\leq \left(\int_{[1,+\infty)} |(t\partial_t)^m \phi(t)|^2 \, t^{2a} \, \frac{\mathrm{d}t}{t}\right)^{1/2} + \left(\int_{(0,1)} |(t\partial_t)^m \phi(t)|^2 \, t^{-2a} \, \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leq \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, t^{2a} \, \frac{\mathrm{d}t}{t}\right)^{1/2} + \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, t^{-2a} \, \frac{\mathrm{d}t}{t}\right)^{1/2}\end{aligned} \tag{D.3.2.5}$$

and

$$\begin{aligned}\|\phi\|_{a,b,m} &\leq \left(\int_{[1,+\infty)} |(t\partial_t)^m \phi(t)|^2 \, t^{2a} \, |\log(t)|^{2b} \, \frac{\mathrm{d}t}{t}\right)^{1/2} + \left(\int_{(0,1)} |(t\partial_t)^m \phi(t)|^2 \, t^{-2a} \, |\log(t)|^{2b} \, \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leq \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, t^{2a} \, |\log(t)|^{2b} \, \frac{\mathrm{d}t}{t}\right)^{1/2} + \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \, t^{-2a} \, |\log(t)|^{2b} \, \frac{\mathrm{d}t}{t}\right)^{1/2}.\end{aligned} \tag{D.3.2.6}$$

This proves that the integrals defining $\|\cdot\|_{a,m}$ and $\|\cdot\|_{a,b,m}$ converge on any $\phi \in \mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$, and that the resulting seminorms are continuous. In particular, the topology defined in (2) is coarser than the topology on $\mathcal{S}_{\mathrm{E}})\mathbf{R}_{>0})$. This completes the proof of (1).

Since $\|\cdot\|_{a,0,m} = \|\cdot\|_{a,m}$ and clearly (2) implies (3), it remains to prove (2). For each real number c and each integer $m \geq 0$ we have

$$\left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 t^{2c} \frac{\mathrm{d}t}{t}\right)^{1/2} \leq \left(\int_{\mathbf{R}_{>0}} |(t\partial_t)^m \phi(t)|^2 \max\{t, t^{-1}\}^{2|c|} \frac{\mathrm{d}t}{t}\right)^{1/2}. \tag{D.3.2.7}$$

By Lemma D.3.1.2 (with $p = 2$), Remark D.3.1.4 and Proposition D.3.2.5, the topology generated by the $\|\cdot\|_{a,m}$'s as $a \in \mathbf{R}$ varies over a sequence of nonnegative real numbers $\{a_k\}_{k\geq 0}$ such that $\limsup_{k\geq 0} a_k = +\infty$ and m varies over the nonnegative integers is finer than the topology on $\mathcal{S}_{\mathrm{E}}(\mathbf{R}_{>0})$. □

D.3.3. Schwartz-Bruhat functions with I-exponential decay on $\mathbf{R}_{>0}$-modulated groups. We let $(G, |\cdot|_G)$ be a modulated group and throughout this subsection we assume the module $|\cdot|_G : G \to \mathbf{R}_{>0}$ is surjective. We refer to such $(G, |\cdot|_G)$ as an *$\mathbf{R}_{>0}$-modulated group*. All examples of modulated groups arising throughout this work fulfill this assumption.

Note that upon choosing a continuous homomorphic section τ to the module $|\cdot|_G$ of G we get an isomorphism of topological groups $\alpha_\tau : G \simeq \mathbf{R}_{>0} \times G^1$. In particular, G is compactly generated (even if G isn't assumed to be second-countable) since G^1 is compact Hausdorff. Any compact subgroup $k \subset G$ is contained inside G^1 since $\mathbf{R}_{>0}$ has no non-trivial compact subgroup, and the orthogonal of k in $\widehat{G}$ is in turn identified with the direct product of $\widehat{\mathbf{R}_{>0}} \simeq \mathbf{R}$ and the orthogonal of k in the discrete $\widehat{G^1}$ since the formation of dual groups respects direct products and $G = \mathbf{R}_{>0} \times G^1$ via τ.

Thus, the condition that the subgroup $k^\perp \subset \widehat{G}$ is compactly generated is the same as the condition that the orthogonal of k in $\widehat{G^1}$ is compactly generated. Since G^1 is compact, so $\widehat{G^1}$ is discrete, the orthogonal of k in $\widehat{G^1}$ is discrete. It follows that the subgroup $k^\perp \subset \widehat{G}$ is compactly generated if and only if the orthogonal of k in $\widehat{G^1}$ is finitely generated.

Definition D.3.3.1 (Schwartz-Bruhat functions with I-exponential decay on $\mathbf{R}_{>0}$-modulated groups). Let $(G, |\cdot|_G)$ be an $\mathbf{R}_{>0}$-modulated group. Let $I \subset \mathbf{R}$ be a non-empty interval. We define $\mathcal{S}_I(G)$ to be the $\mathbf{C}$-vector space of those continuous functions ϕ on G satisfying $|\cdot|_G^a \phi \in \mathcal{S}(G)$ for all $a \in I$. We refer to $\mathcal{S}_I(G)$ as the $\mathbf{C}$-vector space of *Schwartz-Bruhat functions on G with I-exponential decay*. Similarly to Example D.3.2.2, $\mathcal{S}_I(G)$ has a left G-action via $(g.\phi)(x) = \phi(g^{-1}x)$.

(1) For $I = [0, +\infty)$, we write $\mathcal{S}_{\mathrm{SE}}(G) := \mathcal{S}_I(G)$ and refer to its elements as Schwartz-Bruhat functions on G with *forward semi-exponential decay*.
(2) For $I = (-\infty, 0]$, we write $\check{\mathcal{S}}_{\mathrm{SE}}(G) := \mathcal{S}_I(G)$ and refer to its elements as Schwartz-Bruhat functions on G with *backward semi-exponential decay*.
(3) For $I = \mathbf{R}$, we write $\mathcal{S}_{\mathrm{E}}(G) := \mathcal{S}_I(G)$ and refer to its elements as Schwartz-Bruhat functions on G with *exponential decay*.

For $G = \mathbf{R}_{>0}$ this definition of $\mathcal{S}_I(G)$ coincides with Definition D.3.2.1. In that special case, we saw in the proof of the Fréchet property for $\mathcal{S}_I(\mathbf{R}_{>0})$ in Lemma D.3.2.3(1) that membership in $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0})$ amounts to membership in $\mathcal{S}_{\{a\}}(\mathbf{R}_{>0})$ and $\mathcal{S}_{\{b\}}(\mathbf{R}_{>0})$. The same holds in general:

Lemma D.3.3.2. *For an $\mathbf{R}_{>0}$-modulated group G and real numbers $a < c < b$, if $\phi : G \to \mathbf{C}$ satisfies $|\cdot|_G^a \phi, |\cdot|_G^b \phi \in \mathcal{S}(G)$ then $|\cdot|_G^c \phi \in \mathcal{S}(G)$.*

Proof. By renaming $|\cdot|_G^a \phi$ as ϕ and replacing the modulus with $|\cdot|_G^{b-a}$, we have to show that if $\phi, |\cdot|_G \phi \in \mathcal{S}(G)$ then $|\cdot|_G^c \phi \in \mathcal{S}(G)$ for all $0 < c < 1$. By definition of $\mathcal{S}(G)$, there exists a compact subgroup $k \subset G$ with $k^\perp$ compactly generated such that ϕ is k-invariant. Since $|\cdot|_G$ is trivial on k, we can pass to $G/k = \mathbf{R}_{>0} \times (G^1/k)$ so that G^1 is an elementary abelian Lie group.

Using $\mathbf{R} \simeq \mathbf{R}_{>0}$ via e^x, we can arrange that $G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^r \times F$ for integers $m \geq 1$, $n \geq 0$, $r \geq 0$, and a finite abelian group F with the modulus given by $|(x, z, a, b)| = e^{x_1}$. We are given $\phi(x, z, a, b) \in \mathcal{S}(G)$ such

that $e^{x_1}\phi(x,z,a,b) \in \mathcal{S}(G)$ and want to show that if $0 < c < 1$ then $e^{cx_1}f(x,z,a,b) \in \mathcal{S}(G)$. The calculation used to treat the case $G = \mathbf{R}$ in the proof of Lemma D.3.2.3(1) carries over essentially without change to the present more general setting using the seminorms (D.1.3.8)). One difference from the case $G = \mathbf{R}$ is that when $m > 1$ or $r > 0$ we can "go out to infinity" while keeping x_1 bounded or even equal to 0 but the required decay estimates still carry through. □

We fix an $\mathbf{R}_{>0}$-modulated group G and now topologize $\mathcal{S}_I(G)$ by following the same approach used to topologize the Schwartz-Bruhat space of G.

Definition D.3.3.3 (Topology on $\mathcal{S}_I(G)$). Let G be an $\mathbf{R}_{>0}$-modulated group, $k \subset G$ a compact subgroup with compactly generated orthogonal $k^\perp \subset \widehat{G}$. We define

$$\mathcal{S}_I(k,G) \subset \mathcal{S}_I(G) \tag{D.3.3.1}$$

to be the linear subspace of k-invariant Schwartz-Bruhat functions with I-exponential decay. We topologize $\mathcal{S}_I(k,G)$ with the projective topology induced by the linear operators

$$T_a : \mathcal{S}_I(k,G) \to \mathcal{S}(k,G) = \mathcal{S}(G/k) \tag{D.3.3.2}$$

assigned by $T_a(\phi) := |\cdot|_G^a\phi$.

We topologize $\mathcal{S}_I(G)$ as the locally convex inductive limit of the family $\{\mathcal{S}_I(k,G)\}_k$ of linear subspaces of $\mathcal{S}_I(G)$ as $k \subset G$ varies over the compact subgroups with finitely generated orthogonal, partially ordered by downward inclusion. (The inductive limit is $\mathcal{S}_I(G)$ as a vector space since whether or not a given function $f : G \to \mathbf{C}$ is k-invariant is the same as for $|\cdot|_G^a f$ for $a \in \mathbf{R}$ because $|\cdot|_G$ is trivial on k.) We refer to this topology on $\mathcal{S}_I(G)$ as the *inductive topology* on $\mathcal{S}_I(G)$.

Example D.3.3.4. When the preceding topology on $\mathcal{S}_I(G)$ is applied to the modulated group $G = \mathbf{R}_{>0}$ with $|\cdot|_G : G \to \mathbf{R}_{>0}$ the identity map, it recovers the topology in Definition D.3.2.1 since $\mathbf{R}_{>0}$ has no non-trivial compact subgroups.

Example D.3.3.5. For any $a \in \mathbf{R}$, multiplication by $|\cdot|_G^a$ carries $\mathcal{S}_{\{a\}}(G)$ linearly and topologically isomorphically onto $\mathcal{S}(G)$ (as functions on G, $\mathcal{S}_{\{a\}}(G) = |\cdot|_G^{-a}\mathcal{S}(G)$). For $a, b \in \mathbf{R}$, correspondingly multiplication by $|\cdot|_G^{a-b}$ carries $\mathcal{S}_{\{a\}}(G)$ linearly and topologically isomorphically onto $\mathcal{S}_{\{b\}}(G)$.

For $I = [a,b]$, the space $\mathcal{S}_I(k,G)$ is identified via $f \mapsto (f,f)$ linearly and topologically with a closed subspace of $\mathcal{S}_{\{a\}}(k,G) \times \mathcal{S}_{\{b\}}(k,G)$. To prove this, by passing to $G/k = \mathbf{R}_{>0} \times (G^1/k)$ as in the proof of Lemma D.3.3.2 we can carry over the same considerations with e^{cx} for $0 \le c \le 1$ as in the proof of the closed embedding property for the case $G = \mathbf{R}_{>0}$ that arose in the proof of the Fréchet condition in Lemma D.3.2.3(1).

A general non-empty interval $I \subset \mathbf{R}$ is exhausted by countably many compact intervals, so we can similarly conclude that $\mathcal{S}_I(k,G)$ is a closed subspace of a *countable* direct product of copies of the space $\mathcal{S}(k,G) = \mathcal{S}(G/k)$ that is nuclear Fréchet, so $\mathcal{S}_I(k,G)$ is nuclear [**Scha**, Ch. III, 7.4] and Fréchet. It likewise follows that $\mathcal{S}_I(k,G)$ is separable (i.e., has a countable dense subset) since every subset of a separable metric space is separable and each metric space $\mathcal{S}(G/k)$ is separable by (D.1.3.20) (ultimately because the spaces $\mathcal{S}(\mathbf{Z}) \simeq \mathcal{S}(\mathbf{T})$ and $\mathcal{S}(\mathbf{R})$ are separable).[14]

Since the transition maps $\mathcal{S}(k,G) \to \mathcal{S}(k',G)$ for $k' \subset k$ are topological embeddings, the same holds for $\mathcal{S}_I(k,G) \to \mathcal{S}_I(k',G)$. Thus, when G is second-countable we see that the topology on $\mathcal{S}_I(G)$ is a countable strict inductive limit. Continuing to assume G is second-countable, since each $\mathcal{S}_I(k,G)$ is topologized as a projective limit of copies of the Hausdorff locally convex space $\mathcal{S}(k,G) = \mathcal{S}(G/k)$ that is nuclear and complete (see Lemma D.1.3.7(3) applied to the elementary abelian Lie group G/k) it follows that each of the spaces $\mathcal{S}_I(k,G)$

[14]Need to give reference for $\mathcal{S}(\mathbf{R})$ being separable.

and then their countable strict inductive limit $\mathcal{S}_I(G)$, is both nuclear and complete [**Scha**, Ch. III, Cor. to 7.4; Ch. II, 5.3, 6.6].

REMARK D.3.3.6. Let G be an $\mathbf{R}_{>0}$-modulated group and $I \subset \mathbf{R}$ a non-empty interval. For every $a \in \mathbf{R}$, the linear operation $\phi \mapsto |\cdot|_G^a \cdot \phi$ carries each $\phi \in \mathcal{S}_I(G)$ to a smooth function on G and so this operation takes value within $\mathcal{S}_{-a+I}(G)$. By design of the topology on $\mathcal{S}_I(G)$ and $\mathcal{S}_{-a+I}(G)$ from Definition D.3.3.1, this operator is a linear topological isomorphism $M_a : \mathcal{S}_I(G) \to \mathcal{S}_{-a+I}(G)$. In particular, $\mathcal{S}_{\mathrm{E}}(G)$ is stable under multiplication by continuous quasi-characters of arbitrary exponent, and multiplication by any of them is a linear topological automorphism.

D.3.3.1. *The main Structure Theorem on Schwartz-Bruhat functions with I-exponential decay.* The topological space of $\Omega(G)$ of continuous quasi-characters of any modulated group $(G, |\cdot|_G)$ is endowed with a natural structure of Riemann surface, namely the open subsets $\{|\cdot|_G^s \chi_0\}$ for $\chi_0 \in \widehat{G^1}$ (these are the connected components) have the topology of either infinite cylinders (when the image of $|\cdot|_G$ is discrete) or $\mathbf{C}$ (in the $\mathbf{R}_{>0}$-modulated case). In the latter case, the connected components are the fibers of the continuous surjective restriction map $\Omega(G) \to \Omega(G^1) = \widehat{G^1}$.

In the $\mathbf{R}_{>0}$-modulated case, using unitary base points χ_0 yields a well-defined function $\mathrm{Re} : \Omega(G) \to \mathbf{R}$ via $|\cdot|_G^s \chi_0 \mapsto \mathrm{Re}(s)$ (this is the usual "real part" on copies of $\mathbf{C}$ whose base point has an ambiguity corresponding to vertical translation). For any non-empty interval $I \subset \mathbf{R}$ we then denote by $\Omega(G)_I$ the subset of points in $\Omega(G)$ whose "real part" belongs to I (this is a copy of $\mathbf{C}_I \subset \mathbf{C}$ for each connected component of $\Omega(G)$). We endow the interior of $\Omega(G)_I$ inside $\Omega(G)$, if non-empty, with the structure of Riemann surface inherited from the ambient $\Omega(G)$.

THEOREM D.3.3.7 (Structure of $\mathcal{S}_I(G)$ for $\mathbf{R}_{>0}$-modulated groups). *Let $I \subset \mathbf{R}$ be a non-empty interval. Let $(G, |\cdot|_G)$ be a second-countable $\mathbf{R}_{>0}$-modulated group and τ a continuous homomorphic section to the module homomorphism $|\cdot|_G : G \to \mathbf{R}_{>0}$. Then:*

(1) *the choice of section τ induces a linear topological isomorphism:*

$$\mathcal{S}(G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) \xrightarrow{\simeq} \mathcal{S}_I(G) \tag{D.3.3.3}$$

compatible with the left-translation action of $G^1 \times \mathbf{R}_{>0}$ on the source through the first and second projection and on the tensor factors, and of G on the target,[15] *and promoting the following diagram:*

$$\begin{array}{ccc} \mathcal{S}(k, G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) & \longrightarrow & \mathcal{S}_I(k, G) \\ \downarrow & & \downarrow \\ \mathcal{S}(G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) & \longrightarrow & \mathcal{S}_I(G) \end{array} \tag{D.3.3.4}$$

to a commutative diagram of continuous linear maps whose horizontal arrows are linear topological isomorphisms.

Each bounded subset in $\mathcal{S}_I(G)$ is contained in some $\mathcal{S}_I(k, G)$, which in turn is the image of the space

$$\mathcal{S}(k, G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) \simeq \mathcal{S}(k, G^1) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}_I(\mathbf{R}_{>0}) \tag{D.3.3.5}$$

under (D.3.3.3) *for $k \subset G^1$ a compact subgroup such that $k^\perp$ is compactly generated (this is a topological isomorphism by Corollary* C.2.3.12*).*

(2) *The Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}_I(G)$ is complete, nuclear, and Montel and a strict LF-space. In particular, it is barrelled and bornological. If G^1 has compactly generated dual group then $\mathcal{S}_I(G)$ is a nuclear Fréchet space.*

[15] Somewhere perhaps the continuity of G-action on $\mathcal{S}_I(G)$ for the inductive topology should be addressed.

(3) *The Hausdorff strong dual $\mathcal{S}_I(G)'_\beta$ is complete, nuclear and Montel.*
(4) *If $0 \in I$ then $\mathcal{S}_I(G)$ is naturally a linear subspace of $\mathcal{S}(G)$ and the injective linear inclusion $\mathcal{S}_I(G) \subset \mathcal{S}(G)$ is continuous. If I is compact then the inductive topology on $\mathcal{S}_I(G)$ is equal to the projective topology induced by the linear maps $T_a : \mathcal{S}_I(G) \to \mathcal{S}(G)$ assigned by $T_a(\phi) := |\cdot|_G^a \phi$, for every $a \in I$.*

REMARK D.3.3.8. The usefulness of the second-countability assumption has been seen in the preceding discussion, and it also provides a cofinal *sequence* of compact subgroups $k_m \subset G^1$ with compactly generated (equivalently finitely generated) orthogonal in $\widehat{G^1}$. This allows reductions to the case that G is an elementary abelian Lie group via Corollary C.2.3.12 and Proposition C.2.3.13.

REMARK D.3.3.9 (On the role of acyclicity of Schwartz-Bruhat spacess). In §C.1.5, we discussed inductive limits of Hausdorff locally convex topological $\mathbf{C}$-vector spaces, and the concept of acyclicity, one of whose applications is to provide a criterion to detect limit subspaces of LF-spaces (i.e. closed subspaces whose subspace topology is itself the LF-topology induced from the ambient): see Corollary C.1.5.13.

In Proposition D.1.5.9, we proved that for any Hausdorff locally compact second countable abelian group, the Schwartz-Bruhat space $\mathcal{S}(G)$ is acyclic, because it is a countable strict inductive limit of Fréchet spaces (strictness is the key point).

Acyclicity of the Schwartz-Bruhat space of G^1 will be used in proving part (4) of Theorem D.3.3.7 by realizing $\mathcal{S}_I(G)$ as a limit subspace of a strict LF-space, hence itself an LF-space. This will enable us to apply the Open Mapping Theorem (see [**Gro-f**, Ch. 4, §5, Thm. 2(1)]) to the continuous linear bijection from (D.3.3.3) with $\mathcal{S}_I(G)$ re-topologized as in part (4).

Before we prove Theorem D.3.3.7, we discuss the following aspect on $\mathcal{S}_I(G)$. We defined on $\mathcal{S}_I(G)$ the inductive topology. There is another natural topology on $\mathcal{S}_I(G)$, whose nature is projective. One part of the following result is that (at least for second-countable G) both topologies have the same bounded subsets.[16]

PROPOSITION D.3.3.10 (On the bounded subsets of $\mathcal{S}_I(G)$). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let τ be a choice of continuous homomorphic section to the module $|\cdot|_G$. The canonical continuous linear map*

$$\mathcal{S}_I(\mathbf{R}_{>0}) \,\overline{\otimes}\, \mathcal{S}(G^1) \to \mathcal{S}_I(\mathbf{R}_{>0}) \,\widehat{\otimes}_\pi \mathcal{S}(G^1) \tag{D.3.3.6}$$

is a bijection identifying bounded subsets in both directions.

For $k \subset G$ a compact subgroup with compactly generated orthogonal, the canonical continuous linear map

$$\mathcal{S}_I(\mathbf{R}_{>0}) \,\overline{\otimes}\, \mathcal{S}(k, G^1) \to \mathcal{S}_I(k, G) \tag{D.3.3.7}$$

is a linear topological isomorphism onto the separable nuclear Fréchet space $\mathcal{S}_I(k, G)$.

(1) *The bounded subsets of $\mathcal{S}_I(G)$ are separable and metrizable.*
(2) *The bounded subsets of $\mathcal{S}_I(G)$ are the same with respect to the inductive topology and with respect to the coarser projective topology $\mathscr{T}$ on $\mathcal{S}_I(G)$ induced by the continuous linear maps $T_a : \mathcal{S}_I(G) \to \mathcal{S}(G)$ assigned by $T_a(\phi) := |\cdot|_G^a \phi$ for every $a \in I$.*
(3) *$B \subset \mathcal{S}_I(G)$ is bounded if and only if $T_a(B)$ is bounded in $\mathcal{S}(G)$ for every $a \in I$, equivalently for every $a \in J$ with $J \subset I$ a countable subset such that the endpoints of I are accumulation points for J.*

That each $\mathcal{S}_I(k, G)$ is separable nuclear Fréchet was addressed in Example D.3.3.5.

PROOF. (1) : Since G is second-countable, there exists a cofinal sequence $\{k_m\}_{m\geq 0}$ of compact subgroups satisfying $k_{m+1} \subset k_m$ for every integer $m \geq 0$ with each $k_m^\perp$ is compactly generated. The natural map $\varinjlim_{m\geq 0} \mathcal{S}_I(k_m, G) \to \mathcal{S}_I(G)$ is then a linear topological isomorphism with each $\mathcal{S}_I(k_m, G) \to \mathcal{S}_I(G)$ a topological

[16] Somewhere should continuity of G-action on $\mathcal{S}_I(G)$ for the projective topology be addressed?

embedding [**Scha**, Ch. II, 6.4]. By [**Scha**, Ch. II, 6.5], a subset $B \subset \mathcal{S}_I(G)$ is bounded if and only if there exists an integer $m_0 \geq 0$ and a bounded subset $B_0 \subset \mathcal{S}_I(k_{m_0}, G)$ such that B is the image of B_0 under the linear topological embedding $\mathcal{S}_I(k_{m_0}, G) \to \mathcal{S}_I(G)$. Since $\mathcal{S}_I(k_{m_0}, G)$ is a separable Fréchet space, its topological subspace B_0 is then also separable and metrizable. This proves (1).

Before proving (2), we show that the canonical continuous linear map in (D.3.3.6) is a bijection. The argument we give uses the same idea as the proof of [**Gro-b**, Ch. I, §3, No. 1, Prop. 14.II, p. 76]. As a first step, we rewrite the source and target spaces each in a more convenient form. For the source, since $\mathcal{S}_I(\mathbf{R}_{>0})$ and each $\mathcal{S}(k, G^1)$ are nuclear Fréchet with $\{\mathcal{S}(k_m, G^1)\}$ a countable strict inductive system for $\{k_m\}$ as in the proof of (1), the end of Proposition C.2.3.13(2) gives that the natural linear map

$$\varinjlim \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(k, G^1) \to \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$$

is a topological isomorphism. By Corollary C.2.3.12, the completed inductive tensor products in the inductive limit on the left can each be replaced with the corresponding completed projective tensor product.

Now we harness properties of $\mathcal{S}_I(\mathbf{R}_{>0})$ and its strong dual to rewrite those completed tensor products, as well as the one in the target of (D.3.3.6). Recall from Lemma D.3.2.3(1,2) that $\mathcal{S}_I(\mathbf{R}_{>0})$ is Montel (hence reflexive), nuclear and complete, and its strong dual $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$ (whose own dual is $\mathcal{S}_I(\mathbf{R}_{>0})$ by reflexivity of the latter) is complete and barrelled (as it is reflexive, hence barrelled [**Scha**, Ch. IV, 5.6]). Thus, for any complete Hausdorff locally convex space F (such as $\mathcal{S}(G^1)$ or $\mathcal{S}(k, G^1)$) it follows from [**Tre**, Prop. 50.5, p. 522] that the natural map

$$\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi F \to \mathscr{L}_b(\mathcal{S}_I(\mathbf{R}_{>0})'_\beta, F)$$

to the space of continuous linear maps $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta \to F$ (topologized by uniform convergence on bounded subsets of $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta$) is a topological isomorphism.

The map we want to show is a bijection is thereby linearly and topologically identified with the natural map

$$\varinjlim \mathscr{L}_b(\mathcal{S}_I(\mathbf{R}_{>0})'_\beta, \mathcal{S}(k, G^1)) \to \mathscr{L}_b(\mathcal{S}_I(\mathbf{R}_{>0})'_\beta, \mathcal{S}(k, G^1)).$$

The injectivity is now clear, and it remains to show that every continuous linear map $\mathcal{S}_I(\mathbf{R}_{>0})'_\beta \to \mathcal{S}(G^1)$ lands inside one of the subspaces $\mathcal{S}(k, G^1)$.

Define $E := \mathcal{S}_I(\mathbf{R}_{>0})$. Let p be a continuous *norm* (not just seminorm) on E; e.g., pullback along $\mathcal{S}_I(\mathbf{R}_{>0}) \hookrightarrow \mathcal{S}_{\{a\}}(\mathbf{R}_{>0}) \simeq \mathcal{S}(\mathbf{R}_{>0})$ of the sup-norm on the latter. For the closed unit ball $U \subset E$ with respect to p, we claim that the linear span W of the polar $U^\circ := \{\ell \in E' : |\ell(U^\circ)| \leq 1\}$ in the dual E' is dense for the strong-dual topology. (Although references may differ on whether the "polar" is taken in a real or complex sense [**NB**, Def. 8.3.1, Rem. 8.3.2], since U is circled the meanings of U° and $U^{\circ\circ}$ are the same for both notions.) By the Hahn-Banach Theorem, it is the same to show that a continuous linear functional on E'_β vanishing on W (equivalently on U°) must be 0. By semi-reflexivity of E, the continuous linear functionals on E'_β are the same as on E'_σ [**Scha**, Ch. IV, 5.5]. Thus, it is the same to show W is dense in E' for the weak topology $\sigma(E', E)$. The continuous dual of $E'_{\sigma(E',E)}$ is E [**Scha**, Ch. IV, 1.2], so we want to show that if $x \in E$ annihilates W (or equivalently U°) then $x = 0$.

For any such x, the line $\mathbf{R}x$ lies inside the bipolar $U^{\circ\circ}$. Since U is convex and closed for the given topology of E, it is also closed for the weak topology $\sigma(E, E')$ [**Scha**, Ch. IV, 3.1]. Thus, by the Bipolar Theorem (see [**Scha**, Ch. IV, 1.5] for the real case, and [**NB**, Thm. 8.3.8] for the complex case) we have $U^{\circ\circ} = U$. The condition $\mathbf{R}x \subset U$ then forces $x = 0$ since U is the closed unit ball for a norm on E. This completes the proof that U° has dense span in E'_β. Each topologically embedded subspace $\mathcal{S}(k, G^1) \subset \mathcal{S}(G^1)$ is complete and hence closed, so to show that a continuous $T : E'_\beta \to \mathcal{S}(G^1)$ has image contained in some $\mathcal{S}(k, G^1)$ it suffices by our density conclusion to show $T(U^\circ)$ is contained in some $\mathcal{S}(k, G^1)$.

Since T is continuous, it carries bounded sets onto bounded sets. Every bounded subset of $\mathcal{S}(G^1)$ is contained in some $\mathcal{S}(k, G^1)$ by [**Scha**, Ch. ,II, 6.5], so it suffices to show that the polar U° of the closed unit

ball U for the continuous norm p on E is bounded in the topology of E'_β. This strong dual has topological dual E since E is reflexive, and by [**Scha**, Ch. IV, 3.3] boundedness in a Hausdorff locally convex space is equivalent to weak-boundedness, which is to say boundedness under all continuous linear functionals. This says that for all $x \in E$, as ℓ varies in U° then the values $\ell(x)$ in $\mathbf{C}$ are bounded. We can scale x to lie in U since it is a 0-neighborhood for the topology of E, so then by definition of U° we have $|\ell(x)| \leq 1$ for all such ℓ. This completes the proof that (D.3.3.6) is bijective.

Next we show that under the bijection (D.3.3.6), bounded sets match on both sides. The preceding discussion found two topologies on the linear space underlying $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$: the strict-LF topology induced by the continuous linear bijection (D.3.3.6), and the projective tensor topology. That strict LF-topology is finer than the projective tensor product topology since (D.3.3.6) is continuous. Since the strict-LF topology on $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is ultrabornological,[17] it is finer than the topology topology $\mathscr{T}_\pi^{\mathrm{ubor}}$ that is coarsest among the ultrabornological topologies on $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ that are finer than the projective tensor topology; the topology $\mathscr{T}_\pi^{\mathrm{ubor}}$ is constructed at the start of [**Jar**, §13.3] as a locally convex inductive limit of Banach spaces, so it is an LF-space topology.[18]

By the Open Mapping Theorem for LF-spaces (see [**Gro-f**, Ch. 4, §5, Thm. 2(1), p. 148]), the continuous linear bijection

$$\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1) \to (\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1), \mathscr{T}_\pi^{\mathrm{ubor}})$$

between LF-spaces is a linear topological isomorphism. By [**Jar**, §13.3, Prop. 1(b)], $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ with its given topology and with the topology $\mathscr{T}_\pi^{\mathrm{ubor}}$ have the same Banach disks,[19] so it follows that $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ and $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ have the same Banach disks via the continuous bijection between them.

Let $B \subset \mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ be a bounded subset. We want to show its preimage under the bijection (D.3.3.6) is also bounded. It is harmless to replace B with a larger bounded subset, so we may replace B with its convex circled hull to arrange that B is also convex and circled (a "disk" in the terminology of [**NB**, Def. 4.2.7]). Its closure is also bounded [**Scha**, Ch. I, 5.1(ii)], so we may further assume B is closed. But then B is complete since $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is complete, so B is sequentially complete. As such it is a Banach disk in $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ by [**NB**, Thm.13.1.2(b)], so the corresponding pre-image in $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ is also a Banach disk by the foregoing. Since by definition Banach disks in a locally convex space are bounded, we have shown that $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ and $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ have the same bounded subsets. In particular, the coarsest bornological topology $\mathscr{T}^{\mathrm{bor}}$ that is finer than the projective tensor topology on $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ (this is constructed in [**Jar**, §13.3, Prop. 1(a)]) is already ultrabornological.[20]

(2) : Since the identity map on $\mathcal{S}_I(G)$ is continuous when going from the inductive topology to the projective topology, bounded sets for the former are bounded for the latter. The main issue is therefore to show that a bounded set B for the projective topology arises from a bounded set in the inductive topology, and more specifically that any such B arises from $\mathcal{S}_I(k, G)$ for some compact subgroup $k \subset G$ with compactly generated orthogonal. By design, the bounded sets for the projective topology are the preimages of bounded sets under the linear injection

$$\mathcal{S}_I(G) \hookrightarrow \prod_{a \in I} \mathcal{S}(G).$$

Bounded subsets of arbitrary products of topological vector spaces are precisely subsets of products of bounded subsets of the factors [**Scha**, Ch. I, 5.5], so we are led to study boundedness inside $\mathcal{S}(G)$ more closely.

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[17] Is this the same as what Grothendieck calls "strictly bornological"? Does it require a countability condition? Reference for various definitions?

[18] Why are the members of the direct limit construction actually complete? And why is the colimit process "countable" as required in the definition of an LF-space?

[19] Is the reader going to be told what "Banach disks" means?

[20] Why? And why is this last sentence being mentioned? Omit if not actually used: if the reader hasn't been told what "ultrabornological" means and some relation with being bornological then this cannot be understood.

The same argument as given above upon replacing for the study of (D.3.3.6), upon replacing $\mathcal{S}_I$ with $\mathcal{S}$, proves that the continuous linear map $\mathcal{S}(G) \simeq \mathcal{S}(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1) \to \mathcal{S}(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ is bijective and that the two sides have the *same* bounded subsets. It follows that so do $\prod_{a\in I} \mathcal{S}(G)$ and $\prod_{a\in I}(\mathcal{S}(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1))$.

By [**Jar**, §15.4, Thm. 1], completed projective tensor products pass through arbitrary topological products of Hausdorff locally convex spaces, so the natural continuous linear map

$$(\prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1) \to \prod_{a\in I} (\mathcal{S}(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1))$$

is a topological isomorphism. The bounded subsets in the latter thereby give rise to the bounded subsets in the projective topology on $\mathcal{S}_I(G)$. Since $\mathcal{S}(G^1)$ is nuclear, the functor $(\cdot)\widehat{\otimes}_\pi \mathcal{S}(G^1)$ carries closed embeddings to closed embeddings by [**Tre**, Prop. 43.7, Thm. 50.1]. Hence, $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1) \to (\prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$ induced by the linear topological embedding $\prod_{a\in I} T_a : \mathcal{S}_I(\mathbf{R}_{>0}) \to \prod_{a\in I} \mathcal{S}(\mathbf{R}_{>0})$ is a linear topological embedding. This shows that the inductive topology and the topology $\mathscr{T}$ on $\mathcal{S}_I(G)$ have the same bounded subsets.[21]

(3) : The topology $\mathscr{T}$ on $\mathcal{S}_I(G)$ is equal to the projective topology induced by the linear operators T_a for a varying over J only, as opposed to all of I. Thus, we may conclude by using that a subset of a product $\prod E_i$ of topological vector spaces is bounded exactly when its projection into each E_i is bounded [**Scha**, Ch. I, 5.5] □

Proof of Theorem D.3.3.7. (1) : Since G is $\mathbf{R}_{>0}$-modulated, it is compactly generated and hence $\mathcal{S}(G)$ is the locally convex inductive limit of the $\mathcal{S}(k, G)$ as k varies over the collection of compact subgroups of G^1 with compatctly generated $k^\perp$. The analogue holds for $\mathcal{S}_I(G)$ in relation to the $\mathcal{S}_I(k, G)$'s, with each $\mathcal{S}_I(k, G)$ linearly and topologically isomorphic to $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(k, G^1)$ by (D.3.3.7), and that in turn is identified with $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(k, G^1)$ by Corollary C.2.3.12 since $\mathcal{S}_I(\mathbf{R}_{>0})$ and $\mathcal{S}(k, G^1)$ are Fréchet spaces. Passing to the inductive limit over a countable cofinal system of such k's (as exists since G is second-countable) then yields (D.3.3.3) and the rest of (1), since a bounded subset of a countable strict inductive limit is the same as a bounded subset of one of the layers of the inductive limit [**Scha**, Ch. II, 6.5].

(2) : In Example D.3.3.5 we saw that each $\mathcal{S}_I(k, G)$ is nuclear Fréchet, so each of these is also barrelled and bornological [**Scha**, Ch. II, Cor. to 7.1, 8.1] as well as Montel [**Tre**, Cor. 3 to Prop. 50.2]. The countable strict inductive limit $\mathcal{S}_I(G)$ of such $\mathcal{S}_I(k, G)$'s (so a strict LF-space) is then also complete, nuclear, Montel, barrelled, and bornological, as all of these properties are inherited under countable strict inductive limits (see [**Scha**, Ch. II, 6.6] for completeness).

Assume now that G^1 has compactly generated dual group. Since the dual $\widehat{G^1}$ is discrete, compact generation amounts to being finitely generated, so the dual G^1 of $\widehat{G^1}$ has the form $\mathbf{T}^n \times F$ for some $n \geq 0$ and a finite abelian group F. In this case $G = \mathbf{R}_{>0} \times G^1$ is an elementary abelian Lie group, so we can take $k = 0$; i.e., $\mathcal{S}_I(G) = \mathcal{S}_I(0, G)$ is metrizable.

(3) : Since $\mathcal{S}_I(G)$ is bornological, its strong dual is complete by [**Scha**, Ch. IV, 6.1], and since $\mathcal{S}_I(G)$ is Montel its strong dual is also Montel [**Scha**, Ch. IV, 5.9]. To deduce nuclearity of $\mathcal{S}_I(G)'_\beta$, we can argue exactly as in the proof that $\mathcal{S}(G)'_\beta$ is nuclear (Theorem D.1.5.7) to reduce it to the case G is an elementary abelian Lie group. In such cases we have seen $\mathcal{S}_I(G$ is nuclear Fréchet, so its strong dual is nuclear by [**Tre**, Prop. 50.6].

(4) : Since $0 \in I$, we have a continuous linear injective map $\mathcal{S}_I(G) \to \mathcal{S}_{\{0\}}(G)$, and also the linear topological identification $\mathcal{S}_{\{0\}}(G) = \mathcal{S}(G)$ by design of $\mathcal{S}_{\{0\}}(G)$.

Finally, assuming $I = [a, b]$ with $a < b$, we want to show the continuous diagonal inclusion

$$\mathcal{S}_I(G) \hookrightarrow \mathcal{S}_{\{a\}}(G) \times \mathcal{S}_{\{b\}}(G) \simeq \mathcal{S}(G) \times \mathcal{S}(G)$$

[21] No, this does not prove it since we have made *no contact* with $\mathcal{S}_I(k, G)$'s. Moreover, there has not been clear contact made between $\mathcal{S}_I(G)$ and $\mathcal{S}_I(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(G^1)$, which seems to be invoked here yet is what comes up in the *next* proof, risking an appearance of circularity. The "$\mathbin{\overline{\otimes}}$" version (D.3.3.7) has not been proved either. Too many loose ends.

is a topological embedding. (The analogue for $a = b$ using just one factor on the right side is immediate from the definitions.) The content here is that the source has its topology defined by an inductive limit process, and such colimits do not generally interact well with closed embeddings.

By (D.3.3.3) and its analogous for usual Schwartz-Bruhat spaces in (D.1.8.1), the diagonal inclusion is identified with $(\cdot)\,\overline{\otimes}\,\mathcal{S}(G^1)$ applied to the analogous map

$$\mathcal{S}_I(\mathbf{R}_{>0}) \hookrightarrow \mathcal{S}_{\{a\}}(\mathbf{R}_{>0}) \times \mathcal{S}_{\{b\}}(\mathbf{R}_{>0}) \simeq \mathcal{S}(\mathbf{R}_{>0}) \times \mathcal{S}(\mathbf{R}_{>0})$$

that was seen to be a closed embedding in the proof of Lemma D.3.2.3(1).

We call $E_1 := \mathcal{S}_I(\mathbf{R}_{>0})$, $E_2 := \mathcal{S}_{\{a\}}(\mathbf{R}_{>0}) \times \mathcal{S}_{\{b\}}(\mathbf{R}_{>0})$, and $F := \mathcal{S}(G^1)$. Then E_1, E_2 are both nuclear Fréchet spaces and F is a strict locally convex inductive limit of a sequence of nuclear Fréchet spaces since G^1 is second-countable (as G second-countable). By Proposition C.2.3.14, the continuous injective linear map $E_1\,\overline{\otimes}\,F \to E_2\,\overline{\otimes}\,F$ is a linear topological embedding. Thus, we are done. □

Remark D.3.3.11 (Bornologicity of $\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(G^1)$). Assume G is second-countable. In (D.3.3.6) it was seen that the canonical continuous linear map

$$\mathcal{S}_I(G) = \mathcal{S}_I(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(G^1) \to \mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(G^1) \tag{D.3.3.8}$$

is bijective and that the bounded subsets are the same on both sides. Since the bounded sets on both sides of (D.3.3.8) coincide and the source is bornological (as it is the countable strict inductive limit of the spaces $\mathcal{S}_I(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(k, G^1) \simeq \mathcal{S}_I(k, G)$ that are Fréchet), the bornological topology associated to $\mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(G^1)$ (in the sense of [**Scha**, p. 63]) is the topology of $\mathcal{S}_I(\mathbf{R}_{>0})\,\overline{\otimes}\,\mathcal{S}(G^1)$.

When G^1 has finitely generated dual group, so $G^1 \simeq \mathbf{T}^n \times F$ for a finite abelian F and hence G^1 is an elementary abelian Lie group, the canonical continuous linear map $\mathcal{S}_I(G) \to \mathcal{S}_I(\mathbf{R}_{>0})\,\widehat{\otimes}_\pi\mathcal{S}(G^1)$ is a topological isomorphism by Corollary C.2.3.12.

But if the second-countable G^1 does *not* have finitely generated (equivalently not compactly generated) dual group then we claim this continuous linear bijection is *not* an isomorphism. In particular, this provides a class of examples for which formation of projective tensor products does not commute with locally convex inductive limits, even when they are strict and all member spaces of the colimit are nuclear Fréchet and the other tensor factor is nuclear Fréchet (but not finite-dimensional).

Indeed, we can pick a strictly decreasing cofinal sequence $\{k_m\}_{m\geq 1}$ of compact subgroups of G with compactly generated orthogonal in $\widehat{G}$, so then $\mathcal{S}(G^1)$ is a strictly increasing sequence of closed subspaces $\mathcal{S}(k_m, G^1)$ that are compatibly topologically complemented via integration averaging over compact subgroups. Writing E_{m+1}) for the resulting topological complement of $\mathcal{S}(k_m, G^1)$ inside $\mathcal{S}(k_{m+1}, G^1)$, the locally convex inductive limit $\mathcal{S}(G^1)$ of the $\mathcal{S}(k_m, G^1)$'s is thereby also a locally convex direct sum of $E_1 := \mathcal{S}(k_1, G^1)$ and the E_{m+1})'s. The space $\mathcal{S}_I(\mathbf{R}_{>0})$ is nuclear and infinite-dimensional, so not normable [**Tre**, Cor. 2 to Prop. 50.2]. Since $E_{m+1} \neq 0$ because $k_m/k_{m+1} \neq 0$, [**Gro-b**, Ch. I, §1, No. 3, Rem. 1, p. 47] applies to the locally convex space $\oplus E_m = \mathcal{S}(G^1)$ to yield that the linear bijection (D.3.3.8) is not a topological isomorphism.

This is likewise a source of examples for which the subspace topology induced by the injective linear map $\mathcal{S}(G_1) \otimes \mathcal{S}(G_2) \to \mathcal{S}(G_1 \times G_2)$ is strictly finer than the projective tensor product topology (compare with Lemma D.1.8.1(1)).

Proposition D.3.3.12 (G^1-isotypes of $\mathcal{S}_I(G)$). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I \subset \mathbf{R}$ be a non-empty interval. The operation $(g.f)(x) = f(g^{-1}x)$ on the space of functions $f : G \to \mathbf{C}$ preserves $\mathcal{S}_I(G)$ and as such defines a continuous action of G on $\mathcal{S}_I(G)$.*

For any continuous character $\omega \in \widehat{G^1}$, there exists a unique linear map

$$\Phi^I_{G,\omega} : \mathcal{S}_I(G) \to \mathrm{H}^0(\omega, \mathcal{S}_I(G)) \tag{D.3.3.9}$$

to the space of ω-isotypic vectors in $\mathcal{S}_I(G)$ making the following diagram of continuous linear maps commute

$$\begin{array}{ccc} \mathcal{S}_I(G) & \xrightarrow{\prod_{a\in I} T_a} & \prod_{a\in I}\mathcal{S}(G) \\ {\scriptstyle \Phi^I_{G,\omega}}\downarrow & & \downarrow{\scriptstyle \prod_{a\in I}\Phi_{G,\omega}} \\ \mathrm{H}^0(\omega, \mathcal{S}_I(G)) & \xrightarrow[\prod_{a\in I} T_a]{} & \prod_{a\in I}\mathrm{H}^0(\omega, \mathcal{S}(G)) \end{array} \tag{D.3.3.10}$$

where $\Phi_{G,\omega} : \mathcal{S}(G) \to \mathrm{H}^0(\omega, \mathcal{S}(G))$ is the projector from (D.2.1.1)*. Moreover $\Phi^I_{G,\omega}$ is a continuous projector (so surjective) and open.*

Terminology D.3.3.1 (Passage to G^1-isotypes). We will refer to the linear map $\Phi^I_{G,\omega} : \mathcal{S}_I(G) \to \mathrm{H}^0(\omega, \mathcal{S}_I(G))$ as the *passage to the ω-isotype* operator.

Proof of Proposition D.3.3.12. Let ρ_G denoted the continuous left-translation action of G on $\mathcal{S}(G)$: $(\rho_G(g))(f) : x \mapsto f(g^{-1}x)$ for $x \in G$. Upon denoting by $\mathcal{S}_a$ the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{S}(G)$ equipped with the continuous action $|\cdot|^a_G \cdot \rho_G$, the induced continuous action of G on the product $\prod_{a\in I}\mathcal{S}_a$ preserves the range of the linear injection $\prod_{a\in I} T_a$ that is a topological (even closed) embedding by design of the Hausdorff locally convex topology on $\mathcal{S}_I(G)$ because

$$|g|^a_G(\rho_G(g))(|x|^a f(x)) = |g|^a_G|g^{-1}x|^a_G f(g^{-1}x) = |x|^a_G f(g^{-1}x).$$

Thus, $(g.f)(x) = f(g^{-1}x)$ preserves $\mathcal{S}_I(G) \subset \prod_{a\in I}\mathcal{S}_a$ and establishes the desired continuity for that G-action on $\mathcal{S}_I(G)$.

Since $|\cdot|_G$ is trivial on G^1, a direct calculation shows $\Phi_{G,\omega}(|\cdot|^a_G f) = |\cdot|^a_G \Phi_{G,\omega}(f)$, so if $f \in \mathcal{S}_I(G)$ then $\Phi_{G,\omega}(f) : G \to \mathbf{C}$ belongs to $\mathcal{S}_I(G)$ and is certainly ω-isotypic. The injectivity of the horizontal maps in (D.3.3.10) shows that this is the unique $\Phi^I_{G,\omega}$ and that it is a continuous projector. Any continuous projector of a topological space onto a subspace is a quotient mapping (since the inclusion of the subspace is a continuous section), so in the topological group setting it is open. □

D.3.3.2. *Fourier-Melllin transform on $\mathcal{S}_I(G)$.* We conclude this subsection on the structure of $\mathcal{S}_I(G)$ for second-countable $\mathbf{R}_{>0}$-modulated groups G with the following result on the Fourier-Mellin transform in analogy with, and generalizing, Proposition D.3.2.5 that treats the case $G = \mathbf{R}_{>0}$.

Theorem D.3.3.13 (Fourier-Mellin transform on $\mathcal{S}_I(G)$). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated group. Let $I \subset \mathbf{R}$ be a non-empty open interval, and let μ be a choice of Haar measure on G.*

For $\phi \in \mathcal{S}_I(G)$, its Fourier-Mellin transform $M(\phi) : c \mapsto \int_G \phi(g)c(g)\,\mathrm{d}\mu(g)$ for $c \in \Omega(G)$ is absolutely convergent on $\Omega(G)_I$ and is holomorphic in c, defining a linear map

$$M_I : \mathcal{S}_I(G) \to \mathscr{H}(\Omega(G)_I) \tag{D.3.3.11}$$

that is G-equivariant where we use the left-translation action of G on source and the G-action $(g.f)(c) = c(g)f(c)$ on the target.

Under the linear topological isomorphism from (D.3.3.3)*, M_I matches the linear topological isomorphism*

$$\check{\mathcal{F}}_{G^1} \mathbin{\overline{\otimes}} M_I : \mathcal{S}(G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) \to \mathcal{S}(\widehat{G^1}) \mathbin{\overline{\otimes}} A\mathcal{S}(\mathbf{C}_I) \tag{D.3.3.12}$$

where $\mathcal{F}_{G^1}$ is the Fourier transform on $\mathcal{S}(G^1)$ and $\check{\mathcal{F}}_{G^1}$ is the composition of $\mathcal{F}_{G^1}$ with the inversion automorphism on G^1.

Proof. Pick a continuous section τ to $|\cdot|_G$ and use it to define a linear topological isomorphism

$$C^\infty(G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) = \mathcal{S}(G^1) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) \simeq \mathcal{S}_I(G)$$

from (D.3.3.3). We shall use this to show that $M(\phi)$ is absolutely convergent pointwise on $\Omega(G)_I$ and holomorphic on this open domain inside $\Omega(G)$. Scale the Haar measure on $G = G^1 \times \mathbf{R}_{>0}$ to be $\mu_1 \times (\mathrm{d}t/t)$ where μ_1 is the volume-1 Haar measure on G^1.

We may and do pick a compact subgroup $k \subset G^1$ such that $\phi \in \mathcal{S}_I(k, G)$, so ϕ arises from

$$C^\infty(G^1/k) \mathbin{\overline{\otimes}} \mathcal{S}_I(\mathbf{R}_{>0}) = C^\infty(G^1/k) \mathbin{\widehat{\otimes}_\pi} \mathcal{S}_I(\mathbf{R}_{>0})$$

(topological isomorphism by (D.3.3.5)). By Grothendieck's description of elements of completed projective tensor products as infinite series [**Scha**, Ch. III, 6.4], we have

$$\phi = \sum_{n \geq 1} \lambda_n h_n \otimes f_n$$

for an absolutely summable sequence $\{\lambda_n\}$ in $\mathbf{C}$ and sequences $\{h_n\} \subset C^\infty(G^1/k)$ and $\{f_n\} \subset \mathcal{S}_I(\mathbf{R}_{>0})$ that converge to 0 in the respective locally convex topologies (all such infinite series converge in the completed projective tensor product). Pointwise on G we have $\phi(\tau(t)g^1) = \sum_{n \geq 1} \lambda_n h_n(g^1) f_n(t)$, an absolutely convergent sum since h_n and f_n certainly converge to 0 pointwise. Since $h_n \to 0$ uniformly on G^1 (as the sup-norm on $C^\infty(G^1/k)$ is among the seminorms defining the topology), so $|h_n| \leq B$ for all n and some $B > 0$, for $c \in \Omega(G)_I$ with $\mathrm{Re}(c) = a \in I$ we have

$$|\phi(\tau(t)g^1)c(\tau(t)g^1)| \leq \sum_{n \geq 1} B|\lambda_n||f_n(t)|t^a.$$

We have arranged $\mu = \mu_1 \times \mathrm{d}t/t$ with $\mu_1(G^1) = 1$, so for $a \in I$ we have

$$\sum_{n \geq 1} B|\lambda_n| \int_G |f_n(t)|t^a \mathrm{d}\mu(g) = \sum_{n \geq 1} B|\lambda_n| \int_{\mathbf{R}_{>0}} |f_n(t)t^a| \, \mathrm{d}t/t$$

with $f_n(t)t^a \in \mathcal{S}(\mathbf{R}_{>0})$ a sequence of functions tending to 0 in $\mathcal{S}(\mathbf{R}_{>0})$ because $f_n \to 0$ in $\mathcal{S}_I(\mathbf{R}_{>0})$ and $a \in I$. Provided the integrals $\int_{\mathbf{R}_{>0}} |f_n(t)t^a| \, \mathrm{d}t/t$ are uniformly bounded, the convergence of $\sum_{n \geq 1} |\lambda_n|$ would then allow us to conclude by the Dominated Convergence Theorem that the integral defining $M(\phi)$ on $\Omega(G)_I$ is pointwise absolutely convergent. Thus, it suffices to show that if φ_n is a sequence in $\mathcal{S}(\mathbf{R}_{>0})$ converging to 0 then $\varphi_n \to 0$ in $L^1(\mathbf{R}_{>0})$. But this is immediate from the fact that the topology on $\mathcal{S}(\mathbf{R}_{>0})$ can be described using L^1-convergence in smooth functions and their higher derivatives (see (D.1.10.8) for $\mathbf{R} \simeq \mathbf{R}_{>0}$).

The preceding method of proof of absolute convergence pointwise on $\Omega(G)_I$ also gives the pointwise equality

$$M(\phi)(c) = \sum_{n \geq 1} \lambda_n \int_G h_n(g^1) f_n(\tau(t)) c(\tau(t)g^1) \mathrm{d}\mu(\tau(t)g^1) \tag{D.3.3.13}$$

with uniform control on decay of the tail as c varies in vertical strips $\Omega(G)_J$ for compact intervals $J \subset I$ since (i) the decay $t^a f_n \to 0$ in $\mathcal{S}(\mathbf{R}_{>0})$ for $a \in J$ is uniformly controlled as such a varies (we only need to consider a an endpoint of J), (ii) $\sum_{n \geq N} |\lambda_n| \to 0$ as $N \to \infty$. Thus, (D.3.3.13) converges uniformly on $\Omega(G)_J$ for compact $J \subset I$ (so uniformly on compact subsets of $\Omega(G)_I$), reducing the verification of holomorphicity to the special case $\phi = h \otimes f$ for $h \in C^\infty(G^1/k)$ and $f \in \mathcal{S}_I(\mathbf{R}_{>0})$. In such cases, with c varying in the connected component of $\Omega(G)_I$ labelled by some $\omega \in \widehat{G^1}$ (i.e., $c|_{G^1} = \omega$) and using the unitary base point $\chi(\tau(t)g^1) = \omega(g^1)$ (so $c = \chi| \cdot |_G^s$) we have

$$M(h \otimes f)(c) = \int_{G^1} h\omega \, \mathrm{d}\mu_1 \cdot \int_{\mathbf{R}_{>0}} f(t)t^s \, \mathrm{d}t/t = (\mathcal{F}_{G^1}(h))(1/\omega)(M_I(f))(s) \tag{D.3.3.14}$$

where $M_I : \mathcal{S}_I(\mathbf{R}) \simeq A\mathcal{S}(\mathbf{C}_I) \subset \mathscr{H}(\mathbf{C}_I)$ is as in (D.3.2.2) (the Fourier transform $\mathcal{F}_{G^1}(h)$ is evaluated on $1/\omega = \overline{\omega}$ rather than ω due to the appearance of complex conjugation in Definition D.1.1.4 for the Fourier transform on locally compact abelian groups).

With the convergence and holomorphicity aspects of $M(\phi)$ established, if ϕ is k-invariant for a compact subgroup $k \subset G^1$ (with finitely generated orthogonal in $\widehat{G^1}$) then we claim the restriction of $M(\phi)$ to the connected component $\Omega(G)_{I,\omega}$ over $\omega \in \widehat{G^1}$ vanishes except possibly when $\omega(k) = 1$ (i.e., $\omega \in k^\perp$). In other words, under the product decomposition

$$\mathscr{H}(\Omega(G)_I) = \prod_{\omega \in \widehat{G}} \mathscr{H}(\Omega(G)_{I,\omega})$$

we claim the holomorphic $M(\phi)$ is supported in the factors labeled by $k^\perp = \widehat{G^1/k}$, which is to say it arises from the subspace $\mathscr{H}(\Omega(G/k)_I) = \prod_{\omega \in \widehat{G^1/k}} \Omega(G/k)_{I,\omega}$ embedded using 0 along all factors not labeled by $\widehat{G^1/k}$ (note that $\Omega(G/k)_{I,\omega} = \Omega(G)_{I,\omega}$ when $\omega(k) = 1$). This is a direct calculation: if $\omega|_k$ is non-trivial then for $c \in \Omega(G)_{I,\omega}$ the assumed k-invariance of ϕ implies

$$M(\phi)(c) = \int_G \phi(g)c(g)\,\mathrm{d}\mu(g) = \int_{G/k} \phi(\overline{g})c(g) \left(\int_k c(x)\,\mathrm{d}\mu_k(x) \right) \mathrm{d}\overline{\mu}(\overline{g})$$

(with g a representative of $\overline{g} \in G/k$), and the inner integral over k is $\int_k \omega\,\mathrm{d}\mu_k = 0$ since $\omega|_k$ is non-trivial.

Thus, M_I for G is obtained as the direct limit of the corresponding Fourier-Mellin transforms for G/k as k varies through a countable cofinal decreasing sequence $\{k_m\}$ of compact subgroups $k \subset G^1$ with finitely generated orthogonal inside $\widehat{G^1}$. Topologically $\mathcal{S}_I(G)$ is the locally convex inductive limit of the spaces $\mathcal{S}_I(k_m, G) = \mathcal{S}_I(G/k_m)$, so by the good behavior of $\overline{\otimes}$ with respect to countable strict inductive limits (Proposition C.2.3.13(2)) we may and do now focus on each G/k_m in place of G; i.e., we consider G that is an elementary abelian Lie group (equivalently, the compact G^1 has compactly generated dual group).

The choice of τ identifies $\Omega(G)_I$ with $\widehat{G^1} \times \mathbf{C}_I$ for $\widehat{G^1}$ discrete, so $\mathscr{H}(\Omega(G)_I) = \prod_{\omega \in \widehat{G^1}} \mathscr{H}(\mathbf{C}_I)$. Equipping the ω-component with the unitary base point $\chi(\tau(t)g^1) = \omega(g^1)$, (D.3.3.14) says that the restriction of the Fourier-Mellon transform to an elementary tensor $h \otimes f \in C^\infty(G^1) \otimes \mathcal{S}_I(\mathbf{R}_{>0}) \subset \mathcal{S}_I(G)$ is the tuple

$$\prod_{\omega \in \widehat{G^1}} (\check{\mathcal{F}}_{G^1}(h))(\omega) M_I(f) \in \prod_{\omega \in \widehat{G^1}} A\mathcal{S}(\mathbf{C}_I).$$

For any (discrete) set Σ and $\mathbf{C}$-vector space V, the natural map

$$\mathrm{Map}(\Sigma, \mathbf{C}) \otimes V \to \prod_{\sigma \in \Sigma} V$$

defined by $h \otimes v \mapsto (h(\sigma)v)$ is injective (via passage to the direct limit from the case of finite-dimensional V), so in particular we have

$$\mathcal{S}(\widehat{G^1}) \otimes A\mathcal{S}(\mathbf{C}_I) \hookrightarrow \prod_{\omega \in \widehat{G^1}} A\mathcal{S}(\mathbf{C}_I)$$

and $\check{\mathcal{F}}_{G^1}(h) \in \mathcal{S}(\widehat{G^1})$.

We have shown that the restriction of M to the subspace $\mathcal{S}(G^1) \otimes \mathcal{S}(\mathbf{R}_{>0}) \subset \mathcal{S}_I(G)$ is the composition of the algebraic isomorphism

$$\check{\mathcal{F}}_{G^1} \otimes M_I : \mathcal{S}(G^1) \otimes \mathcal{S}(\mathbf{R}_{>0}) \simeq \mathcal{S}(\widehat{G^1}) \otimes A\mathcal{S}(\mathbf{C}_I)$$

followed by the injection of the latter into $\prod_{\omega \in \widehat{G^1}} A\mathcal{S}(\mathbf{C}_I) \subset \prod_{\omega \in \widehat{G^1}} \mathscr{H}(\mathbf{C}_I) = \mathscr{H}(\Omega(G)_I)$. Since the Fourier transform $\check{\mathcal{F}}_{G^1} : \mathcal{S}(G^1) \to \mathcal{S}(\widehat{G^1})$ is a linear topological isomorphism (Lemma D.1.6.19(2)) and likewise for $M_I : \mathcal{S}_I(\mathbf{R}_{>0}) \simeq A\mathcal{S}(\mathbf{C}_I)$ by (D.3.2.2) It follows that M restricts to a linear *topological* isomorphism

$$\mathcal{S}(G^1) \otimes \mathcal{S}_I(\mathbf{R}_{>0}) \simeq \mathcal{S}(\widehat{G^1}) \otimes A\mathcal{S}(\mathbf{C}_I)$$

where the two sides are given either the inductive or projective tensor product topologies, both of which coincide due to Corollary C.2.3.12 since the tensor factors are Fréchet (recall G^1 is now an elementary abelian Lie group).

We have shown via Grothendieck's infinite series description of $\mathcal{S}(G^1)\widehat{\otimes}_\pi \mathcal{S}_I(\mathbf{R}_{>0}) \simeq \mathcal{S}_I(G)$ that M carries $\sum \lambda_n h_n \otimes f_n$ (for absolutely summable $\{\lambda_n\}$ and null sequences h_n, f_n) to the holomorphic function whose restriction to $\Omega(G)_{I,\omega} = \mathbf{C}_I$ is

$$\sum_{n\geq 1} \lambda_n(\check{\mathcal{F}}_{G^1}(h))(\omega) M_I(f_n).$$

Thus, it remains to show that for the complete Hausdorff locally convex topological vector space $V = A\mathcal{S}(\mathbf{C}_I)$ and the discrete finitely generated abelian group $\Gamma = \widehat{G^1}$, the unique continuous map

$$\theta : \mathcal{S}(\Gamma)\widehat{\otimes}_\pi V \to \prod_{\gamma\in\Gamma} V$$

arising via completeness of the target [**Scha**, Ch. II, 5.3] and continuity of the evaluation maps $\mathcal{S}(\Gamma)\otimes_\pi V \to V$ at every $\gamma \in \Gamma$ satisfies $\theta(\sum \lambda_n(h_n \otimes v_n)) = (\sum \lambda_n h_n(\gamma)v_n)_{\gamma\in\Gamma}$ (the infinite series in V is convergent by completeness and local convexity of the topology) and is *injective*.

The explicit description of θ on Grothendieck's infinite series is immediate from continuity considerations, so the composite map

$$\mathcal{S}_I(G) \simeq \mathcal{S}(G^1)\,\widehat{\otimes}_\pi \mathcal{S}_I(\mathbf{R}_{>0}) \overset{\check{\mathcal{F}}_{G^1}\widehat{\otimes} M_I}{\simeq} \mathcal{S}(\widehat{G^1})\,\widehat{\otimes}_\pi A\mathcal{S}(\mathbf{C}_I) \overset{\theta}{\to} \prod_{\omega\in\widehat{G^1}} \mathscr{H}(\mathbf{C}_I) = \mathscr{H}(\Omega(G)_I)$$

coincides with the Fourier-Mellin transform M. Hence, injectivity of θ is reduced to injectivity of M, and that in turn follows from combining injectivity of restriction of holomorphic functions to vertical lines in $\Omega(G)_I$ and injectivity of the Fourier transform on Schwartz functions. □

D.3.4. Uniform Riemann-Lebesgue Lemma on $\mathbf{R}_{>0}$-modulated groups. We include here an analog for second-countable $\mathbf{R}_{>0}$-modulated groups of the Uniform Riemann-Lebesgue Lemma from Proposition B.2.2.3. We refer the reader to §D.1.6 for differential operators on Hausdorff locally compact abelian groups and their interaction with the Fourier transform on Schwartz-Bruhat functions (see Lemma D.1.6.19).

We recall for ease of access that for G an $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group, $\Omega(G)$ denotes the Riemann surface of continuous quasi-characters on G, and for any non-empty interval $I \subset \mathbf{R}$ we write $\Omega(G)_I$ for the domain

$$\Omega(G)_I := \{c \in \Omega(G) : \sigma(c) \in I\}$$

where $\sigma : \Omega(G) \to \mathbf{R}$ is the exponent function, associating to a continuous quasi-character $c \in \Omega(G)$ the unique real number $\sigma(c)$ such that $c = \chi|\cdot|_G^{\sigma(c)+it}$ for a, equivalently any, choice $\chi \in \widehat{G}$ of continuous (unitary) character such that $\chi|_{G^1} = c|_{G^1}$, and $t \in \mathbf{R}$ the real number uniquely determined by χ such that $c = \chi|\cdot|_G^{\sigma(c)+it}$.

Upon changing the choice of χ, the real number $t \in \mathbf{R}$ changes accordingly, but $\sigma(c)$ does not change, and it is therefore uniquely associated to c. The function $\sigma : \Omega(G) \to \mathbf{R}$ is continuous and surjective. The next result is a generalization of Proposition B.2.2.3(2), but first we need to record a useful observation:

Remark D.3.4.1. Let I be a non-empty interval. We claim any invariant differential operator D on $C^\infty(G)$ preserves the subspace $\mathcal{S}_I(G) \subset C^\infty(G)$. It suffices to treat the case $D = \partial_b$ for a 1-parameter subgroup $b : \mathbf{R} \to G$. For $a \in I$ and $\phi \in \mathcal{S}_I(G)$ we want to show that $|\cdot|_G^a D(\phi) \in \mathcal{S}(G)$. By hypothesis $|\cdot|_G^a \phi \in \mathcal{S}(G)$, so $D(|\cdot|_G^a \phi) \in \mathcal{S}(G)$. The product rule for $D = \partial_b$ gives

$$D(|\cdot|_G^a \phi) = r|\,|_G^a \phi + |\cdot|_G^a D(\phi)$$

where $|b(t)|_G = e^{rt}$ for some $r \in \mathbf{R}$. Since $r|\cdot|_G^a \phi \in \mathcal{S}(G)$ by the definition of $\mathcal{S}_I(G)$, and we have noted that $D(|\cdot|_G^a \phi) \in \mathcal{S}(G)$, it follows that $|\cdot|_G^a D(\phi) \in \mathcal{S}(G)$ as desired.

THEOREM D.3.4.2 (Uniform Riemann-Lebesgue - II). *Let G be a second-countable $\mathbf{R}_{>0}$-modulated Hausdorff locally compact abelian group, and $I \subset \mathbf{R}$ a non-empty bounded open interval. Let D be an invariant differential operator on $C^\infty(G)$, and define $p_D : \Omega(G) \to \mathbf{C}$ be*

$$p_D(c) := (Dc)(e) \tag{D.3.4.1}$$

for $e \in G$ the neutral element (on $\widehat{G} \subset \Omega(G)$ this recovers the symbol $p(D)$ from Definition D.1.6.10). Fix a Haar measure μ on G to define the Fourier-Mellin transform $M : \mathcal{S}_{\overline{I}}G) \to \mathcal{S}_I(G) \stackrel{M_I}{\to} \mathscr{H}(\Omega(G)_I)$ by $M(\phi) : c \mapsto \int_G \phi(g)c(g)\,\mathrm{d}\mu(g)$.

For every bounded subset $\mathcal{B} \subset \mathcal{S}_{\overline{I}}(G)$ and a fixed choice of $\omega \in \widehat{G^1}$ there exist constants $C_{\mathcal{B},D,I} > 0$ and $M_{\omega,\mathcal{B},D,I} > 0$ such that

$$|M(\phi)(c)| \leq \frac{C_{\mathcal{B},D,I}}{|p_D(c)|} \tag{D.3.4.2}$$

for all $\phi \in \mathcal{B}$ and all $c \in \Omega(G)_{I,\omega}$ such that $|p_D(c)| \geq M_{\omega,\mathcal{B},D,I}$.

PROOF. We let $a_1, \ldots, a_n : \mathbf{R} \to G$ be 1-parameter subgroups that give a basis for $\mathcal{L}(G)$ ($n \geq 1$ since G is $\mathbf{R}_{>0}$-modulated), so $D = \sum b_J \partial_a^J$ as a sum over finitely many multi-indices $J = (j_1, \ldots, j_n)$ with $b_J \in \mathbf{C}$ and $j_i \in \mathbf{N}$, where $\partial_a^J := \partial_{a_1}^{j_1} \cdots \partial_{a_n}^{j_n}$. Suppose there are N terms occurring in D, so for any $c \in \Omega(G)$ we have

$$|p_D(c)| \leq \sum |b_J||p_{\partial_a^J}(c)| \leq N \max_J |b_J|\,|p_{\partial_a^J}(c)|.$$

Note that N is independent of c, but the multi-index J_c for which $|b_J||p_{\partial_a^J}(c)|$ is maximized may depend on c. If we can settle the case of the operators ∂_a^J, and let $C'_{\mathcal{B},D,I}$ be the maximum of the constants $|b_J|C_{\mathcal{B},\partial_a^J,I}$ and $M_{\omega,\mathcal{B},D,I}$ be the maximum of the constants $N|b_J|M_{\omega,\mathcal{B},\partial_a^J,I}$ then for any $\phi \in \mathcal{B}$ and $c \in \Omega(G)_{I,\omega}$ with $|p_D(c)| \geq M_{\omega,\mathcal{B},D,I}$ we have

$$|M(\phi)(c)| \leq \frac{C'_{\mathcal{B},D,I}}{\max |b_J||p_{\partial_a^J}(c)|} \leq \frac{NC'_{\mathcal{B},D,I}}{|p_D(c)|}.$$

Thus, $C_{\mathcal{B},D,I} := NC'_{\mathcal{B},D,I}$ and $M_{\omega,\mathcal{B},D,I}$ do the job for D.

Now we may and do assume $D = \partial_{a_1}^{r_1} \cdots \partial_{a_n}^{r_n}$ for some exponents $r_i \in \mathbf{N}$. Next we reduce to the case that G is an elementary abelian Lie group. By second-countability, there is a cofinal decreasing *sequence* $\{k_m\}$ of compact subgroups of G with compactly generated orthogonal such that $\mathcal{S}_{\overline{I}}(G) = \varinjlim \mathcal{S}_{\overline{I}}(k_m, G)$ as a countable strict inductive limit, so $\mathcal{B} \subset \mathcal{S}_{\overline{I}}(k_{m_0}, G)$ for some m_0 by [**Scha**, Ch. II, 6.5]. Increasing m_0 if necessary allows us to also arrange that $\omega(k_{m_0}) = 1$. In particular, for $k = k_{m_0}$ we see that every $\phi \in \mathcal{B}$ is k-invariant and $\omega \in (\widehat{G/k})^1$. Clearly also $D(\phi)$ is k-invariant since D is an invariant differential operator, and $D(\phi) \in \mathcal{S}_{\overline{I}}(G)$ by Remark D.3.4.1, so $D(\phi) \in \mathcal{S}_{\overline{I}}(k, G) \simeq \mathcal{S}_{\overline{I}}(G/k)$. The quotient G/k is an elementary abelian Lie group and it is $\mathbf{R}_{>0}$-modulated because $k \subset G^1$.

For any 1-parameter subgroup $b : \mathbf{R} \to G$ with composite 1-parameter subgroup $\overline{b} : \mathbf{R} \to G/k$, we have

$$(\partial_b \phi)(x) = \lim_{t \to 0} \frac{\phi(b(t)x) - \phi(x)}{t} = (\partial_{\overline{b}} \overline{\phi})(x \bmod k)$$

for $\overline{\phi} : G/k \to \mathbf{C}$ the corresponding element of $\mathcal{S}_{\overline{I}}(G/k)$ (see Lemma D.1.5.4). Thus, $D(\phi) \in \mathcal{S}_{\overline{I}}(k, G)$ corresponding to $\overline{D}(\overline{\phi}) \in \mathcal{S}_{\overline{I}}(G/k)$ where $\overline{D} := \partial_{\overline{a}_1}^{r_1} \cdots \partial_{\overline{a}_n}^{r_n}$ as an invariant differential operator on $C^\infty(G/k)$. (The $\overline{a}_i$'s may merely span $\mathcal{L}(G/k)$ rather than be a basis for it, but that doesn't matter because once we reduce to studying $\overline{D}$ and $\mathcal{B} \subset \mathcal{S}_{\overline{I}}(G/k)$ we can go again through the reduction to a product among the first-order operators associated to a basis of $\mathcal{L}(G/k)$.)

By using Weil's coset integration formula from Theorem D.1.2.1 exactly as in the passage to elementary abelian Lie groups in the proof of Theorem D.3.3.13, we see from the k-invariance of ϕ that $M(\phi)(c) = 0$

except possibly when $c \in k^\perp = \widehat{G/k}$, and that if $c \in k^\perp$ then

$$M(\phi)(c) = M(\overline{\phi})(\overline{c})$$

where $\overline{c} \in \Omega(G/k)$ corresponds to c and the Fourier-Mellin transform on G/k is defined using the Haar measure associated to the initial choice of μ on G (used to define M for G) and the volume-1 Haar measure on k. Since it is also clear from the definitions that $p_D(c) = p_{\overline{D}}(\overline{c})$, we may rename G/k as G to reduce to the case that G is an elementary abelian Lie group.

Since G is $\mathbf{R}_{>0}$-modulated, any choice τ of continuous homomorphic section to the module $|\cdot|_G$ induces an isomorphism of Hausdorff locally compact abelian groups $G \simeq \mathbf{R}_{>0} \times G^1$, and hence the compact Hausdorff abelian group $G^1 = \ker(|\cdot|_G)$ is an elementary abelian Lie group: $G^1 \simeq \mathbf{T}^m \times F$ for an integer $m = n - 1 \geq 0$ and a finite abelian group F (with $\mathbf{T} = \mathbf{R}/\mathbf{Z}$). We have $F = \pi_0(G)$ and the 1-parameter subgroups of G are all contained in the identity component, so by treating the restrictions of ϕ to each of the finitely many connected components of G separately we may and do assume G is connected: $G = \mathbf{R}_{>0} \times \mathbf{T}^m$.

Now $\mathcal{S}_{\overline{I}}(G) \simeq \mathcal{S}_{\overline{I}}(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(\mathbf{T}^m) \simeq \mathcal{S}_{\overline{I}}(\mathbf{R}_{>0}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(\mathbf{T}^m)$ where both tensor factors are Fréchet spaces. Since $\mathcal{S}_{\overline{I}}(G)$ is Montel (Theorem D.3.3.7), the closure of $\mathcal{B}$ (which is also bounded [**Scha**, Ch. I, 5.1]) is compact. Thus, by [**Tre**, Cor. 2 to Thm. 45.2] there are compact subsets $K_1 \subset \mathcal{S}_{\overline{I}}(\mathbf{R}_{>0})$ and $K_2 \subset \mathcal{S}(\mathbf{T}^m)$ such that $\mathcal{B}$ is contained in the closure $\overline{\mathrm{cch}}(K_1 \otimes K_2)$ of the convex circled hull of the set of elementary tensors $f_1 \otimes f_2$ for $f \in K_1$ and $f_2 \in K_2$. If a constant $C_{\mathcal{B},D,I}$ can be found that works for all such elementary tensors then it also works for their circled hull as well as for all convex combinations from the circled hull, and then for the closure thereof. In other words, it suffices to treat the case $\phi = f_1 \otimes f_2$. Writing $c = c_1 \otimes \omega$ for $c_1 \in \Omega(\mathbf{R}_{>0})_I$, we have $M(\phi)(c) = M(f_1)(c_1)\, M(f_2)(\omega)$ (for a choice of Haar measures on $\mathbf{R}_{>0}$ and G^1 whose product measure is the chosen one on G).

Writing $D = D_1 \mathbin{\overline{\otimes}} D_2$ on $\mathcal{S}_{\overline{I}}(G) = \mathcal{S}_{\overline{I}}(\mathbf{R}_{>0}) \mathbin{\overline{\otimes}} \mathcal{S}(G^1)$ (where $D_1 = (t\partial_t)^{r_1}$ for some integer $r_1 \geq 0$ and D_2 is a product of $r_2 \geq 0$ operators – with repetition allowed – among the ∂_{z_j}'s for $(z_1, \ldots, z_m)$ the standard coordinates on $G^1 = \mathbf{T}^m$), clearly

$$p_D(c) = p_{D_1}(c_1) p_{D_2}(\omega).$$

Since $p_{D_2}(\omega)$ is a fixed value (as D and ω are fixed), keeping $|p_D(c)|$ above a positive constant M ensures $|p_{D_1}(c_1)| \geq M/|p_{D_2}(\omega)|$ as we vary such $c = c_1 \otimes \omega$. In this way we see it suffices to separately treat the case $G = \mathbf{R}_{>0}$ and an analogous result for the compact $G^1 = \mathbf{T}^m$: for any compact subset $K \subset \mathcal{S}(G^1)$ and the differential operator $D = \partial_{z_1}^{r_1} \cdots \partial_{z_m}^{r_m}$ on $G^1 = \mathbf{T}^m$, if $p_D(\omega) \neq 0$ then there is a constant $C_K > 0$ such that $|M(\phi)(\omega)| \leq C_K$ for all $\phi \in K$. Since $\mathcal{S}(\mathbf{T}^m) \simeq \mathcal{S}(\mathbf{T}^{m-1}) \mathbin{\widehat{\otimes}}_\pi \mathcal{S}(\mathbf{T})$, if $m > 1$ then we can go through an analogous argument to pass to the separate cases of $\mathbf{T}^{m-1}$ and $\mathbf{T}$. In other words, it suffices to treat the case $G = \mathbf{R}_{>0}$ and the analogous result as stated above for $G = \mathbf{T} = \mathbf{R}/\mathbf{Z}$. For $G = \mathbf{R}_{>0}$ we may assume $D = (t\partial_t)^r$ for some $r \geq 1$ and for $G = \mathbf{T}$ we may assume $D = \partial_z^r$ for some $r \geq 1$.

For $G = \mathbf{R}_{>0}$ we have $c(t) = t^s$ for $s \in \mathbf{C}_I$, so then $p_{(t\partial_t)^r}(c) = s^r$. Since $|s^r|/|\mathrm{Im}(s)|^r$ is uniformly bounded above on the region when $|\mathrm{Im}(s)|$ stays above a chosen $\delta > 0$, requiring $|p_D(c)| \geq (\delta + \max(|a|, |b|))^r$ where $I = (a, b)$ ensures $|\mathrm{Im}(s)| \geq \delta$, Thus, we can use Proposition B.2.2.3(2) provided the constant there (which depends on ϕ, as seen at the end of its proof) is bounded independently of ϕ varying in a compact subset $K \subset \mathcal{S}_{\overline{I}}(\mathbf{R})$. By construction the constant there can be taken to be

$$\|t^a (t\partial_t)^r(\phi)\|_{L^1(\mathbf{R}_{>0}, \mathrm{d}t/t)} + \|t^b (t\partial_t)^r(\phi)\|_{L^1(\mathbf{R}_{>0}, \mathrm{d}t/t)}$$

where $I = (a, b)$. This constant is bounded for ϕ varying through a compact subset of $\mathcal{S}_{\overline{I}}(\mathbf{R}_{>0})$ because $\mathcal{S}_{[a,b]}(\mathbf{R}_{>0}) \to \mathcal{S}(\mathbf{R}_{>0}) \times \mathcal{S}(\mathbf{R}_{>0})$ via $\phi \mapsto (t^a\phi, t^b\phi)$ is a topological embedding and the seminorm $f \mapsto \|(t\partial_t)^r(f)\|_{L^1(\mathbf{R}_{>0}, \mathrm{d}t/t)}$ on $\mathcal{S}(\mathbf{R}_{>0})$ is among a collection of seminorms defining the topology on $\mathcal{S}(\mathbf{R}_{>0})$ (see (D.1.10.7) applied to $\mathbf{R} \simeq \mathbf{R}_{>0}$).

Now consider $G = \mathbf{T} = \mathbf{R}/\mathbf{Z}$ equipped with the volume-1 Haar measure. We have $\omega(z) = e^{2\pi imz}$ for some $m \in \mathbf{Z}$, so for $D = \partial_z^r$ we have $p_D(\omega) = (2\pi im)^r$. Thus, we may assume $m \neq 0$ and then integrating by parts r times gives

$$M(\phi)(z^m) = \int_0^1 \phi(x)e^{2\pi imx}\,\mathrm{d}x = -\frac{1}{2\pi im}\int_0^1 (\partial_x\phi)(x)e^{2\pi imx}\,\mathrm{d}x = \cdots = \left(\frac{-1}{2\pi im}\right)^r \int_0^1 (\partial_x^r\phi)(x)e^{2\pi imx}\,\mathrm{d}x$$

since ϕ is 1-periodic. We are then done using the constant $C_K = (2\pi|m|)^{-r}\sup_{\phi\in K}\|D\phi\|_{L^1(\mathbf{T})}$ provided this is finite (as ϕ varies through a compact subset $K \subset \mathcal{S}(\mathbf{T})$). The seminorm $\phi \mapsto \|D(\phi)\|_{L^1(\mathbf{T})}$ on $\mathcal{S}(\mathbf{T})$ is among a collection of seminorms defining the topology on $\mathcal{S}(\mathbf{T})$ as in (D.1.10.10) (with $p = 1$ there), so it is bounded on compact subsets of $L^1(\mathbf{T})$. □

D.3.5. Poisson Summation Formula on $\mathbf{R}_{>0}$-modulated groups: isolated singularities. Let G be a Hausdorff locally compact abelian group, μ_G a choice of Haar measure on G. The Poisson Summation Formula (see (D.1.2.8)) holds everywhere on G –rather than only μ_G-almost everywhere – in the presence of continuity hypotheses as in Theorem D.1.2.7(3). We now give a class of functions with finitely many isolated singularities and smooth everywhere else for which the Formula holds everywhere over G upon redefining the function at singular points as the arithmetic mean of suitable "one-sided" limit values.

Now let $(G, |\cdot|_G)$ be an $\mathbf{R}_{>0}$-modulated group equipped with the canonical Haar measure μ_G that is the unique one such that, upon denoting by μ_1 the volume-1 Haar measure on G^1, the unique Haar measure obtained on $G/G^1 \simeq \mathbf{R}_{>0}$ (the isomorphism being the module $|\cdot|_G$) as per Theorem D.1.2.1 is $\mathrm{d}t/t$ on $\mathbf{R}_{>0}$.

Proposition D.3.5.1 (Poisson Summation Formula - jump singularities on $\mathbf{R}_{>0}$-modulated groups). *Let G be an $\mathbf{R}_{>0}$-modulated group with module $|\cdot|_G$. Let μ_G be the canonical Haar measure on G. Let $\gamma \in G$ be an element satisfying $|\gamma|_G > 1$. Let $\chi_0 \in \widehat{G}$ be a continuous (unitary) character, and let $\tau : \mathbf{R}_{>0} \to G$ be a choice of continuous homomorphic section to the module $|\cdot|_G$.*

Let $\phi : G \to \mathbf{C}$ be a function. Assume:

(1) *ϕ is bounded, and for every integer $N \geq 1$, there exist real numbers $T > 0$ and $C_N > 0$, such that*

$$|\phi(g)| \leq C_N e^{-N\,|\log|g|_G|} \tag{D.3.5.1}$$

for all $g \in G$ with $|\log|g|_G| > T$,

(2) *for every $g \in G^1$, the continuous function $\phi(\tau(t)g)$ of $t \in \mathbf{R}_{>0}$ is smooth away from finitely many points $\{t_1, \ldots, t_m\} \subset \mathbf{R}_{>0}$ and admits one-sided limits at each of them for each $g \in G^1$, and the derivatives $(t\partial_t)^k\phi(\tau(t)g)$ are continuous functions of*

$$(t, g) \in (\mathbf{R}_{>0} - \{t_1, \ldots, t_m\}) \times G^1$$

for each integer $k \geq 0$,

(3) *ϕ is $\chi_0|_{G^1}$-homogeneous with respect to the left-translation action by G^1.*

Then upon redefining ϕ on the finitely many fibres of $|\cdot|_G$ over $\{t_1, \ldots, t_m\}$ to be

$$\phi(\tau(t_k)g) := \frac{\lim_{t\to t_k^+}\phi(\tau(t)g) + \lim_{t\to t_k^-}\phi(\tau(t)g)}{2} \tag{D.3.5.2}$$

for $1 \leq k \leq m$ and for all $g \in G^1$, the following identity holds for all $x \in G$:

$$\sum_{n\in\mathbf{Z}} \chi_0(\gamma^n x)\phi(\gamma^n x) = \frac{1}{\log|\gamma|_G}\sum_{m\in\mathbf{Z}} \widehat{\phi}(\overline{\chi}_0|\cdot|_G^{-\frac{2\pi im}{\log|\gamma|_G}+it_0})|x|_G^{\frac{2\pi im}{\log|\gamma|_G}} \tag{D.3.5.3}$$

where $t_0 \in [0, 2\pi/\log|\gamma|_G)$ satisfies $\chi_0(\gamma) = |\gamma|_G^{it_0}$.

Remark D.3.5.2. We recall that for G an $\mathbf{R}_{>0}$-modulated group and $\gamma \in G$ such that $|\gamma|_G > 1$, upon choosing a holomorphic parametrization $\{\chi|\cdot|_G^s\}$ for the Riemann surface $\Omega(G)$ of continuous quasi-characters,

with $s \in \mathbf{C}$ and χ varying over a choice $\widetilde{G^1}$ of continuous (unitary) characters each lifting each point of the discrete set $\widehat{G^1}$, the continuous characters of G that are trivial on the discrete closed subgroup $\gamma^{\mathbf{Z}}$ are all parametrized as

$$\chi|\cdot|_G^{\frac{2\pi i m}{\log|\gamma|_G} - it(\chi)}$$

as χ varies over $\widetilde{G^1}$ and $m \in \mathbf{Z}$, with $t(\chi)$ the unique real number contained in $[0, 2\pi/\log|\gamma|_G)$ satisfying:

$$e^{it(\chi)\log|\gamma|_G} = \chi(\gamma). \tag{D.3.5.4}$$

In particular, $\chi' := \chi|\cdot|_G^{-it(\chi)}$ satisfies $\chi'(\gamma) = 1$, and hence χ' is trivial on $\gamma^{\mathbf{Z}}$.

Proof of Proposition D.3.5.1. We observe that for each $x \in G$, writing $x = \tau(t)g$ for unique $(t, g) \in \mathbf{R}_{>0} \times G^1$, both sides of (D.3.5.3) are independent of $g \in G^1$ (this uses hypothesis (3) for independence of g on the left side of (D.3.5.3)).

We define $\psi(x) := \chi_0(x)\phi(x)$, for $x \in G$; this is invariant under left-translation by G^1, so for $x \in G$ the value $\psi(x)$ only depends on $|x|_G$. Note that ψ also satisfies (D.3.5.1) since χ_0 is unitary. We have

$$\widehat{\psi}(\chi) = \int_G \psi(y)\overline{\chi}(y)\,\mathrm{d}\mu_G(y) = \int_{\mathbf{R}_{>0}} \psi(\tau(t))\overline{\chi}(\tau(t))\,\frac{\mathrm{d}t}{t} = \int_{G^1} \overline{\chi}(g)\,\mathrm{d}\mu_1(g) \tag{D.3.5.5}$$

and the rightmost side of (D.3.5.5) is nonzero only if $\overline{\chi}|_{G^1} = 1$. The identity in (D.3.5.3) therefore follows from Proposition B.4.1.2 applied to $\psi \circ \tau$ and (in the notation of the Proposition) $\chi = 1$, $\theta = |\gamma|_G$ upon rewriting the hypotheses on ϕ in terms of ψ. □

D.3.6. Approximation by smooth compactly supported functions. The following approximation result will be useful for us:

Lemma D.3.6.1 (Approximation by smooth compactly supported functions). *Let $(G, |\cdot|_G)$ be a second-countable locally compact Hausdorff $\mathbf{R}_{>0}$-modulated abelian group. Let $I \subset \mathbf{R}$ be a non-empty interval. Then $\mathcal{D}(G) := C_c^\infty(G)$ is a dense subspace of $\mathcal{S}_I(G)$ with continuous inclusion.*

We equip $\mathcal{D}(G)$ with the usual countable strict inductive limit topology (via any choice of countable exhaustion of G by compact subsets).

Proof. Each $\phi \in \mathcal{D}(G)$ certainly satisfies $|\cdot|_G^a\phi \in \mathcal{S}(G)$ for every $a \in \mathbf{R}$, so $\mathcal{D}(G) \subset \mathcal{S}_I(G)$.

For every $a \in \mathbf{R}$, multiplication by the smooth function $|\cdot|_G^a$ is continuous and continuously invertible on $\mathcal{D}(G)$. Thus, the composition of $T_a : \mathcal{S}_I(G) \to \mathcal{S}(G)$ with the injective linear inclusion $\mathcal{D}(G) \subset \mathcal{S}_I(G)$ is continuous for every $a \in \mathbf{R}$. By design of the projective topology on $\mathcal{S}_I(G)$, the inclusion

$$j_G : \mathcal{D}(G) \to \mathcal{S}_I(G)$$

is therefore continuous.

Upon choosing a continuous homomorphic section τ to $|\cdot|_G$, by Theorem D.3.3.7(1) (which uses second-countability of G) we have a linear topological isomorphism $\mathcal{S}_I(G) \simeq \mathcal{S}(G^1)\,\overline{\otimes}\,\mathcal{S}_I(\mathbf{R}_{>0}) = \mathcal{D}(G^1)\,\overline{\otimes}\,\mathcal{S}_I(\mathbf{R}_{>0})$ (the final equality using that G^1 is compact). We similarly have a linear topological isomorphism $\mathcal{D}(G) \simeq \mathcal{D}(G^1)\,\overline{\otimes}\,\mathcal{D}(\mathbf{R}_{>0})$ by [**Bru**, §5, Thm.3(a), p. 55]. The *continuous* inclusion $\mathcal{D}(G) \to \mathcal{S}_I(G)$ is thereby identified with the natural continuous map

$$\mathrm{id}_{\mathcal{D}(G^1)}\,\overline{\otimes}\,j_{\mathbf{R}_{>0}} : \mathcal{D}(G^1)\,\overline{\otimes}\,\mathcal{D}(\mathbf{R}_{>0}) \to \mathcal{D}(G^1)\,\overline{\otimes}\,\mathcal{S}_I(\mathbf{R}_{>0}) \tag{D.3.6.1}$$

since by continuity is suffices to check on the dense subspace $\mathcal{D}(G^1) \otimes \mathcal{D}(\mathbf{R}_{>0})$ (and so by linearity it suffices to check on elementary tensors).

To prove (D.3.6.1) has dense image, we claim it suffices to treat the case $G = \mathbf{R}_{>0}$. In other words, if the map between second tensor factors in (D.3.6.1) has dense image then we claim the map (D.3.6.1) has dense image. By the Hahn-Banach Theorem, it suffices to show that the induced map between continuous linear

duals is injective. The continuous dual is insensitive to passage to completion, so it is the same as showing that the natural map

$$\mathrm{id}_{\mathcal{D}(G^1)} \otimes j_{\mathbf{R}_{>0}} : \mathcal{D}(G^1) \otimes_\iota \mathcal{D}(\mathbf{R}_{>0}) \to \mathcal{D}(G^1) \otimes_\iota \mathcal{S}_I(\mathbf{R}_{>0})$$

induces an injection between continuous linear duals.

The continuous linear dual of an inductive tensor product $E \otimes_\iota F$ (for Hausdorff locally convex E and F) is the space of "separately continuous" bilinear forms $B : E \times F \to \mathbf{C}$ (i.e., $B(x,\cdot) : F \to \mathbf{C}$ and $B(\cdot, y) : E \to \mathbf{C}$ are continuous for all $x \in E$ and $y \in F$). In other words, if $B : \mathcal{D}(G^1) \times \mathcal{S}_I(\mathbf{R}_{>0}) \to \mathbf{C}$ is a separately continuous bilinear map that vanishes on $\mathcal{D}(G^1) \times \mathcal{D}(\mathbf{R}_{>0})$ then we want to show $B = 0$ if $\mathcal{D}(\mathbf{R}_{>0})$ is dense in $\mathcal{S}_I(\mathbf{R}_{>0})$. But such vanishing for B is immediate from such density due to continuity of B in the second variable for each fixed entry in the first variable.

We are reduced to the case $G = \mathbf{R}_{>0}$: proving $\mathcal{D}(\mathbf{R}_{>0})$ is dense in $\mathcal{S}_I(\mathbf{R}_{>0})$. By the definition of the topology, density can be checked by working with each $\mathcal{S}_J(\mathbf{R}_{>0})$ for each J in an increasing sequence of compact intervals exhausting I. Thus, we can assume I is compact and hence bounded. Since multiplication by $|\cdot|_{\mathbf{R}}^a$ for any real number $a \in \mathbf{R}$ is continuous and continuously invertible on $\mathcal{D}(\mathbf{R}_{>0})$, we can use such multiplication to move the bounded I arbitrarily far to the right. Thus, we may and do assume $I \subset (0, +\infty)$. Using $\exp : \mathbf{R} \simeq \mathbf{R}_{>0}$, the Hausdorff locally convex topological $\mathbf{C}$-vector space $\mathcal{D}(\mathbf{R}_{>0})$ is linearly and topologically isomorphic to $\mathcal{D}(\mathbf{R})$ endowed with its locally convex inductive limit topology, and similarly $\mathcal{S}_I(\mathbf{R}_{>0})$ is linearly and topologically isomorphic to the Hausdorff locally convex topological vector space $\mathcal{S}_I(\mathbf{R})$.

A family of continuous seminorms generating the Hausdorff locally convex topology on $\mathcal{S}_I(\mathbf{R})$ is

$$\nu_{a,k}(\phi) := \sup_{x \in \mathbf{R}} |e^{ax} \partial_x^k \phi(x)|$$

for $(a,k) \in I \times \mathbf{N}$ with $I \subset (0,\infty)$. Pick $\psi \in \mathcal{D}(\mathbf{R})$ with support contained in $[-2,2]$, and $\psi(\mathbf{R}) \subset \mathbf{R}_{\geq 0}$ with $\psi([-1,1]) = \{1\}$. Choose $\varphi \in \mathcal{S}_I(\mathbf{R})$.

For every integer $m \geq 1$, we define $\psi_m(x) := \psi(x/m)$ and define $\phi_m := \psi_m \cdot \phi$, an element of $\mathcal{D}(\mathbf{R})$. We now show that $\lim_{m \to +\infty} \phi_m = \phi$ in $\mathcal{S}_I(\mathbf{R})$, so $\mathcal{D}(\mathbf{R})$ would then be dense in the metric space $\mathcal{S}_I(\mathbf{R})$. For all $x \in \mathbf{R}$, every integer $m \geq 1$, and every pair $(a,k) \in I \times \mathbf{N}$,

$$\begin{aligned} \partial^k \phi_m(x) &= \sum_{0 \leq r \leq k} \left(\frac{1}{m}\right)^{k-r} \binom{k}{r} \partial_x^r \phi(x) \partial_x^{k-r} \psi(x/m) \\ &= \sum_{0 \leq r \leq k, r \neq k} \left(\frac{1}{m}\right)^{k-r} \binom{k}{r} \partial_x^r \phi(x) \partial_x^{k-r} \psi(x/m) + \partial_x^k \phi(x). \end{aligned} \tag{D.3.6.2}$$

We deduce that for every integer $m \geq 1$ and every pair $(a,k) \in I \times \mathbf{N}$,

$$\begin{aligned} |e^{ax} \partial^k \phi_m(x) - e^{ax} \partial_x^k \phi(x)| &\leq \left| e^{ax} \sum_{0 \leq r \leq k, r \neq k} \left(\frac{1}{m}\right)^{k-r} \binom{k}{r} \partial_x^r \phi(x) \partial_x^{k-r} \psi(x/m) \right| \\ &\leq \sum_{0 \leq r \leq k, r \neq k} \left(\frac{1}{m}\right)^{k-r} \binom{k}{r} |e^{ax} \partial_x^r \phi(x) \partial_x^{k-r} \psi(x/m)|. \end{aligned} \tag{D.3.6.3}$$

Now observe that for every integer $m \geq 1$ and every pair $(a,k) \in I \times \mathbf{N}$,

$$\sup_{x \in \mathbf{R}} |e^{ax} \partial^k \phi_m(x) - e^{ax} \partial_x^k \phi(x)| \leq \sum_{0 \leq r \leq k, r \neq k} \binom{k}{r} \sup_{x \in \mathbf{R}} \frac{|\partial^{k-r} \psi(x/m)|}{m^{k-r}} \nu_{a,r}(\phi) \leq \frac{C_k}{m} \sum_{0 \leq r \leq k, r \neq k} \binom{k}{r} \nu_{a,r}(\phi) \tag{D.3.6.4}$$

where $C_k := \max_{1\leq k\leq r, r\neq k} \sup_{x\in\mathbf{R}} |\partial^{k-r}\psi|$ and the right side in (D.3.6.4) tends to zero as $m \to +\infty$, proving that $\phi_m \to \phi$ in $\mathcal{S}_I(\mathbf{R})$. □

APPENDIX E

Topological stacks

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[1] Complete this Appendix, correcting it and shortening it.

In this appendix we review the theory of compactly generated weakly Hausdorff spaces. Our intent is to present a comprehensive treatment in a uniform and detailed manner, also carefully handling set-theoretic aspects and including comprehensive references to the literature to build the geometric foundations required to make sense of "periodic quotient stacks" associated to primes. This provides insight into the main trace calculation upon which Refined Explicit Formulas rest. It also makes the connection of our work to Deninger's work, though strictly speaking this is not logically needed for our main trace work (but it guided how we discovered our trace formula results).

We settle on some notation and terminology.

NOTATION E.0.0.1. We write:

(1) **Sets** for the large category whose objects are sets, and morphisms are functions between sets.
(2) **Spaces** for the large category whose objects are topological spaces, and morphisms are continuous functions between topological spaces.
(3) For a topological space X, X^{set} for the set underlying X.
(4) For topological spaces X, Y, $\mathrm{C}(X, Y)$ for the set of all continuous functions $f : X \to Y$.

TERMINOLOGY E.0.0.1. We call:

(1) A *space* any object of **Spaces**, i.e. any topological space. We will use this abbreviation whenever an adjective is to precede "topological space". For instance, "compactly generated topological space" will be abbreviated to "compactly generated space".
(2) A *map*, or *continuous map*, $f : X \to Y$ between spaces X and Y any continuous function $f : X \to Y$. We include the word *continuous* when there is potential for ambiguity.
(3) A *set-function* $f : X \to Y$ between sets X and Y is any function $f : X \to Y$. We denote by $\mathrm{Map}(X, Y)$ the set of all set-functions $X \to Y$.
(4) A *set-function* $f : X \to Y$ between spaces X and Y is any set-function $f : X^{\mathrm{set}} \to Y^{\mathrm{set}}$, not necessarily underlying a continuous function. For a continuous map $f : X \to Y$, we write f^{set} for its underlying set-function. We write $(\cdot)^{\mathrm{set}}$ for the resulting forgetful functor **Spaces** $\to$ **Sets**.
(5) A *compact* space is a topological space such that every open cover of K has a finite refinement.
(6) A *locally compact* space is a topological space such that every point has a fundamental system of compact neighborhoods.

Our main references for general topology throughout this appendix are [**Du**], [**Bou-GT**], [**Ke**], and [**SP**]. In particular, we refer to [**Bou-GT**, Ch. I ,§1, §5], [**Du**, Ch. III], and [**SP**, Tags 004O, 0405] for background material on point-set topology, to [**Bou-GT**, Ch. I, §8], [**Du**, Ch. VII, §1-3], and [**SP**, Tag 08ZD] for background material on separation axioms, [**Bou-GT**, Ch. I, §9], [**Du**, Ch. X, §1,6,9], [**Ke**, Ch. 5], and [**SP**, Tags 0059, 08ZQ] for background material around compactness, and [**Du**, Ch. XII, §1-3], and [**Ke**, Ch. 7] for background material on function spaces. We will further refer to [**SP**, Tag 005M] on proper maps, and [**SP**, Tags 08ZW, 0908] on profinite spaces and Stone-Čech compactifications.

We caution the reader about diverging terminologies within the above sources. Specifically, our use of the word *compact* follows [**Du**, **Ke**] but not [**Bou-GT**, **SP**], where the same adjective is replaced by *quasi-compact*. We further caution the reader that several sources within the literature that will be occasionally referred to throughout this work use the word *compact* to mean *compact and Hausdorff* (see, for example, [**LGL**, **Rezk**, **Ste**]). Throughout this work, we shall always clarify when a separation assumption is being made.

We further settle on the following:

TERMINOLOGY E.0.0.2. We let X, Y be topological spaces, and $f : X \to Y$ a continuous map.

(1) The *quotient topology* on $f(X)$ is the finest topology on $f(X)$ making the induced map $f : X \to f(X)$ continuous (i.e., $U \subset f(X)$ is open if and only if $f^{-1}(U)$ is open in X). This may not be the subspace

topology on $f(X)$. We call a map f for which it agrees with the subspace topology a *strict* map (see part (1) of [**SP**, Tag 0406]).

(2) The *image* of X under f, denoted $f(X)$ or $\mathrm{Im}(f)$, is the subset $f(X)$ of Y endowed with the *quotient topology* induced from the topology on X.

(3) The *pre-image* of a subspace $Z \subset Y$ along the continuous map $f : X \to Y$ is the subset $f^{-1}(Z) \subset X$ endowed with the subspace topology induced from the topology on X.

(4) We say f is an *immersion* if and only if it is injective and strict.

(5) We say f is *submersive* if and only if it is surjective and strict (see part (2) of [**SP**, Tag 0406]).

(6) We say f is *étale* if and only if it is a local homeomorphism.

We further clarify that when considering more than one topology on the same underlying set of a fixed topological space X, we shall use the terminology *intrinsic topology* on X to refer to the topology that X is equipped with from the outset of the discussion.

We caution the reader on differing nomenclature choices for (4) and (5) between this work and some of the sources we shall be referring to throughout. The class of morphisms we call immersions is referred to as "inclusions" in [**Rezk**, **Str**]. Other terms used to refer to submersive maps within the literature that we shall refer to throughout are *identification maps* (see, for example, [**Br4**, Ch. 4]), *quotient maps* (see [**Str**, Def. 2.18]) or *proclusive maps/proclusions* (see, for example, [**Rezk**, §9.1] and [**Ste**, Def. 2.5]). We shall make use of the preceding "Terminology" section occasionally also to make local clarifications later throughout this appendix as the need arises, so as to remove all terminological ambiguities.

E.1. Compactly generated spaces

This section is devoted to collecting background material on compactly generated spaces. Its purpose is to pave the way for a discussion in §E.2 of the theory of compactly generated weakly Hausdorff spaces, whose category serves as an ambient category within which we shall construct a small site of topological spaces. When arguments are not given here, they are referenced into other sources.

The class of compactly generated weakly Hausdorff topological spaces, abbreviated "CGWH", was introduced with the intent of constructing a large category of topological spaces that, while containing most examples of spaces of interest, such as locally compact Hausdorff topological spaces and CW complexes, also has good categorical properties, with particular emphasis on a cartesian closed monoidal category structure.

Compactly generated weakly Hausdorff spaces have so far had a prominent role to play within the literature around homotopy theory; see for example [**BG**, Ch.8] for an account and complements on [**Ste**], with basic applications to homotopy theory in [**BG**, Ch. 9]. We refer the reader to [**LMS**] for an example of the foundational use of compactly generated spaces in stable homotopy theory, starting with [**LMS**, Ch. I, §1] on the equivariant stable category, and to J.P. May's work [**May**] on iterated loop spaces. We further refer the reader to L.G. Lewis' Ph.D. thesis [**LGL**] and its appendix [**LGL**, Appendix A] containing a comprehensive account of compactly generated weakly Hausdorff spaces, which will also serve as one of our main sources for this section and the next.

The terminology often used to refer to such a category of topological spaces is "convenient", and the idea was axiomatized and studied systematically in the early 1960's. We refer the reader to R. Brown's D.Phil. thesis [**Br1**], with particular emphasis on [**Br1**, Ch. I, Thm. 2.36, Thm. 2.37] and the preceding discussion in [**Br1**, Ch. I, §2] on k-spaces where, preceding the introduction of the concept of closed monoidal categories by a few years, the author discusses exponential spaces in a manner that is a precursor to Steenrod's article [**Ste**] where a candidate convenient category of topological spaces was introduced and studied, and its monoidal structure and cartesian closure discussed. To the best of the author's knowledge, Brown's papers [**Br2**] and [**Br3**], both following Brown's thesis and pre-dating Steenrod's 1967 article, are the foothold of the search for

convenient categories of spaces and for the identification of their desired properties, whose development then gained traction after [**Ste**] and [**Vogt**].

The concept of "k-space", where the letter k is understood to refer to the german word "kompakt", predates Brown's thesis and the very idea of a convenient category. It appears to go back to Princeton lectures by Hurewicz from the late 1940's, and was first introduced in the literature in [**Ga**, Definition, p. 303]. It can be found in Kelley's book on general topology (see [**Ke**, Ch. 7, p. 130]) from 1955. It was after [**Br1**] and particularly in [**Br2**] that the importance of k-spaces, in their definition from [**Ke**, Ch. 7, p. 130] was observed. We refer the reader to [**Br2**], where the author advertised the category of k-spaces as a possible candidate for a convenient category, before an axiomatization of "convenience" was made in the literature (see [**Br2**], and [**Vogt**] for discussions of convenient categories and their properties). The reader is further referred to [**nLab1**, §6] for further historical information on the inception and development of these themes.

The defining properties of a convenient category $\mathcal{C}$ of topological spaces are usually assumed to be:

(1) $\mathcal{C}$ is a subcategory of **Spaces** that is full and replete (i.e., any object of **Spaces** isomorphic to an object of $\mathcal{C}$ is an object of $\mathcal{C}$),
(2) $\mathcal{C}$ contains all CW complexes and most examples of spaces of practical interest (see the beginning of [**Ste**, §1]),
(3) $\mathcal{C}$ is closed under small limits and colimits (see [**Vogt**, §2, Thm. 2.1]),
(4) for every object X in $\mathcal{C}$, closed subspaces of X are objects in $\mathcal{C}$ (see [**Vogt**, §2, Axiom 1]); it is sometimes required that open subspaces also satisfy this property (see [**Vogt**, §2, Axiom 1*]).
(5) $\mathcal{C}$ is cartesian closed (see [**Vogt**, §2, Axiom 2] for axioms in this direction, and [**Br3**, §1] on the exponential law; see also [**Ste**, §1, (1),(2)]) with respect to its cartesian monoidal structure (see [**MacL**, Ch. VII, §7-8]).

The reader is especially referred to [**ELS**], particularly to [**ELS**, §3, §5], and to [**ELS**, Thm. 5.15], where a comprehensive treatment of convenient categories, their stability with respect to internal exponentiation, and their general construction, alongside a uniform proof of cartesian closedness, is given. We further refer the reader to [**nLab1**, Def. 2.1] and the discussion in [**nLab1**, §2] for an overview around the concept of "convenient category" of topological spaces.

We will use [**ELS**] to highlight the relation between some of the differing definitions of convenient categories based on compactly generated spaces.

The notion of compactly generated spaces we adopt here is the one introduced in [**McC**, §2, p. 275] under the term "k-spaces", and in [**LGL**, Appendix A, §1, p. 148], which is also the notion in [**Str**, Def. 1.1] and [**Rezk**, §3.1] (in the latter, also under the term "k-spaces"), the only differences being with respect to terminology. The notion of k-space in [**Ke**, Ch. 7, p. 230] is defined using the immersions of compact subspaces as test continuous maps for closure of subsets and does not incorporate a Hausdorff condition on the test compact subspaces. It is therefore different from ours (see §E.2.4 and especially Lemma E.2.4.11 for an example illustrating the difference).

We further refer the reader to [**LM**, §1], where compact generation is also defined in terms of intersections with compact, possibly non-Hausdorff, subspaces (see the end of [**LM**, p. 1]), as in [**Ke**]. A definition is given in [**Vogt**, §5,(iii)] after an axiomatic development leading to a general method to construct large categories of topological spaces stable under certain categorical operations, and it is equivalent to the one adopted throughout this work. Another is introduced in [**Vogt**, §5, (iv)] which is related to the one in [**Ke**, Ch. 7, p. 230] via Lemma E.2.4.2. In Steenrod's paper (see [**Ste**, p.2]), a Hausdorff condition is built into the very definition of compact generation, to arrange for all compact subspaces of compactly generated spaces to be closed. In a similar vein, we shall eventually incorporate a weak Hausdorff condition, as in [**McC**, §2], [**LGL**, p. 148], and throughout loc.cit., which is sufficient to ensure that all compact subspaces of compactly

generated (weakly Hausdorff) spaces are closed. Weakening the Hausdorff condition in [**Ste**, §2] rendered itself necessary to arrange for better stability properties of the resulting convenient category, especially in regard to computation in **Spaces** of certain classes of colimits.

Other features of the category of (weakly Hausdorff) compactly generated spaces are the availability of a simple classification of important classes of maps (such as regular monomorphisms, which are shown to be the closed immersions, and regular epimorphisms, which are shown to be the submersive maps; see Definition E.2.3.14 for these notions of regularity), together with stability of submersive maps under arbitrary base change. Such stability leads to a transparent description of the canonical topology on small full subcategories of the category of compactly generated weakly Hausdorff spaces closed under pullbacks, contributing to rendering them a suitable choice (for our intended purposes) to underlie small sites.

The weak Hausdorff condition can be found in [**McC**, §2,p.275], where it is attributed to J.C. Moore. This appears to be the first source within the literature recording the concept. We refer the reader to the beginning of [**McC**, §2] for further motivation for the weakening of the Hausdorff condition from [**Ste**].

We may summarize the differences between the various concepts of compact generation for topological spaces as being mostly in regard to whether or not the test spaces for openness, equivalently closedness, of subsets is assumed to satisfy a non-trivial separation condition or just be compactness.

The main divergences between the theories are discernible from a number of features, such as the fact that the notion of compact generation adopted throughout this appendix is stable under passage to arbitrary open subspaces, which does not hold in other theories and especially without a separation condition on the compact test spaces (see §E.2.4 and especially Lemma E.2.4.12 for an example of this discrepancy). Stability of compact generation under passage to open subspaces of a space X may not hold when the compact test spaces are assumed to be compact *subspaces* of X, even under a Hausdorff assumption, as is done in [**Ste**]. It does hold for *regular* open subspaces (see [**Ste**, 2.4]).

The approach we adopt throughout this work is the approach of [**LGL**, Appendix A], [**McC**, §2], [**Str**, §1, §2], and [**Rezk**, §3.1] (we recall for the sake of clarity that in this latter reference, "compact spaces" is equivalent to "compact Hausdorff spaces" here, insofar as "quasi-compactness" is used in loc.cit. to refer to compact spaces in this work). The notion of compactly generated space in [**McC**, §2] and [**Rezk**, §5] is equivalent to our notion of compactly generated weakly Hausdorff space. Weakly Hausdorff spaces throughout this appendix, and "k-Hausdorff" spaces in [**Rezk**, §4] are different concepts, but they are equivalent conditions on compactly generated spaces (see [**Rezk**, Prop.11.4]) and so most results in loc.cit. can be referred to throughout this appendix with only minor terminological caveats for the reader, which will be stated as the need arises.

The advantages of using the category of compactly generated weakly Hausdorff spaces are several. As an instance, the fact that the categories of compactly generated spaces and of compactly generated weakly Hausdorff spaces are cartesian-closed monoidal categories implies that product functors have right-adjoints, and as such commute with all small colimits (see [**LGL**, Cor. 5.6]). This implies, in particular, compatibility of submersive maps with finite products (see [**LGL**, Prop. 5.8]), which does not hold in the category of all topological spaces (see Example E.1.2.18). We establish this here with and without separation assumptions, as a consequence of cartesian closure (Lemma E.2.2.10, respectively Lemma E.1.2.13), combined with the characterization of submersive maps as the effective epimorphisms in each category (Lemma E.2.3.8, respectively Lemma E.1.2.14); see Definition E.2.3.14 for this notion of effectivity. Within such categories, CW complexes are examples of objects, and products of CW complexes are again CW complexes.

The properties we are most interested in arranging for a convenient category of topological spaces are:

(1) stability of all objects under passage to closed subspaces,
(2) stability of all objects under passage to open subspaces,
(3) existence of finite limits and finite colimits,
(4) stability of submersive maps under arbitrary pull-backs,

(5) computability in **Spaces** of certain push-outs.

Arranging all of the above properties requires a delicate balancing act, which leads to the combination of compact generation with the weak Hausdorff separation condition in a natural manner. We will show by example how compact generation is sufficient to arrange for (1), (2), (3) and (5) to hold (see Lemma E.1.2.25 for (1) and (2), Lemma E.1.2.2 for (3) and (5)), but not for (4) (see Example E.1.2.16).

Imposing a Hausdorff condition on all objects, in combination with compact generation, as is done in [**Ste**], is sufficient to arrange for (1), (3), and (4) to hold (see [**Ste**, §2,2.4] for (1), see Lemma E.2.2.5 combined with the fact that finite limits and finite colimits of Hausdorff spaces are Hausdorff for (3), and see Lemma E.2.3.6 combined with (3) for (4)), but not for (2) (although see [**Ste**, §2,2.4]) and most importantly not for (5) (see Example E.2.3.12).

Combining compact generation with the weak Hausdorff separation condition arranges for all five properties to hold, and upon arriving at the category of compactly generated weakly Hausdorff spaces, in §E.3 we shall be selecting small full subcategories with enough closure properties to serve as small sites.

Our main sources throughout this section concerning compactly generated spaces are [**LGL**, §4, §5], [**McC**, §2], [**Str**, §2], [**Rezk**, §8, §11], and occasionally [**ELS**, §3, §5]. As mentioned in the incipit of [**Str**], the unpublished Ph.D. thesis of J. McClure contains an account of the theory of CGWH spaces as well.

E.1.1. Preparations on compactly generated spaces. We begin with a definition:

Definition E.1.1.1. Let X be a topological space.

(1) Let $Z \subset X$ be a subspace. We say Z is *k-closed* if and only if for every[2] compact Hausdorff topological space K and continuous map $g : K \to X$, $g^{-1}(Z)$ is a closed subspace of K.
(2) We denote by $k(X)$ the topological space whose underlying set is X^{set} and whose topology is the coarsest topology such that all the k-closed subsets of X are closed in $k(X)$. We call the topology on $k(X)$ the *compactly generated topology*, or *k-topology*, on X.

Remark E.1.1.2. On Definition E.1.1.1, we make the following observations.

(1) We note that the identity set-function $k(X) \to X$ is always continuous, which is to say that the topology of $k(X)$ is finer than the topology of X. Indeed, for any $Z \subset X$ closed, any compact Hausdorff space K, and any continuous map $g : K \to X$, the subspace $g^{-1}(Z) \subset K$ is closed. The closed subspaces of X are therefore closed in $k(X)$ as well.
(2) We note that, as a direct consequence of part (2) of Definition E.1.1.1, a subspace $Z \subset X$ is k-closed if and only if $Z \subset k(X)$ is closed.

Our definition of "compactly generated" space will rest on:

Lemma E.1.1.3. *Let X be a topological space. The following four conditions are equivalent:*

(1) *For all topological spaces Y and any set-function $f : X \to Y$, the following conditions are equivalent:*
   (a) *f is continuous.*
   (b) *For every compact Hausdorff space K and every continuous map $g : K \to X$, the composition $f \circ g$ is continuous.*
(2) *There exists a set S_X of compact Hausdorff spaces such that for all topological spaces Y and any set-function $f : X \to Y$, the following conditions are equivalent:*
   (a) *f is continuous.*
   (b) *For every element $K \in S_X$, and every continuous function $g : K \to X$, the composition $f \circ g$ is continuous.*

[2]Need to address, maybe with cross-reference, why no set-theoretic issues with this definition being over "all" pairs (K, g).

(3) *A subspace $U \subset X$ is open if and only if for every compact Hausdorff space K and every continuous function $g : K \to X$, the pre-image $g^{-1}(U)$ is open in K.*
(4) *There exists a set I, a compact Hausdorff space K_i for each $i \in I$, and for $Y := \coprod_{i\in I} K_i$ the I-indexed co-product a submersive map $q : Y \to X$.*

Before we can prove Lemma E.1.1.3, we need the following preparation:

LEMMA E.1.1.4 (On bouquets of traits and discrete two-point sets). *Let X be a topological space.*

(1) *Assume that there exist $x, y \in X$, with $x \neq y$, such that for every subset $A \subset X$ containing x, respectively y, A is closed. Then X is discrete and satisfies all four conditions in Lemma* E.1.1.3.
(2) *Assume that there exists a unique point $x \in X$ such that every subset $A \subset X$ containing x is closed in X.*
   (a) *The subspace $X - \{x\} \subset X$ is open and has the discrete topology.*
   (b) *Let $Z \subset X$ be the connected component of x, and V the closure of $X - Z$ in X. Then:*
      (i) *the natural continuous surjective map*

$$f : Z \coprod V \to X \tag{E.1.1.1}$$

is submersive (i.e., topologically a surjective quotient map);
      (ii) *for any compact Hausdorff space K and continuous map $g : K \to V$, the image $g(K) \subset V$ is compact Hausdorff and closed in V, with $K \to g(K)$ submersive;*
      (iii) *there exists a compact Hausdorff space K and a submersive map $K \to Z$.*
   (c) *If X satisfies condition* (1) *in Lemma* E.1.1.3 *then it also satisfies condition* (2).

PROOF. (1) : Note that $U := X - \{x\}$ is open in X since $\{x\}$ is closed. We first show U is discrete, by proving all subsets of U are closed in U. Let B be any subset of U and C be its closure in X. Since $B \cup \{x\}$ is closed (as it contains x), either $C = B$ or $C = B \cup \{x\}$. Either way, this closure meets U in B, so B is closed in U. It follows that $X - \{x\}$ is discrete. The same holds for $X - \{y\}$, and since $x \neq y$ we have that X is covered by the open subsets $X - \{x\}$ and $X - \{y\}$ each of which has all points open, so all points of X are open; i.e., X is discrete. Discrete spaces are easily seen to satisfy all four conditions in Lemma E.1.1.3 (since any continuous map to such a space from a compact Hausdorff space has finite image and clopen fibers, and all maps from a discrete space are continuous).

(2.a) is a consequence of the proof of (1). We prove (2.b). Let Z be the connected component of X containing x. Then Z is closed in X, and since Z and V are two closed subsets of X which cover X clearly (i) holds. Before we prove (ii) and (iii), we make a couple of observations about Z and V.

Firstly, we claim the only open subset $U \subset Z$ that is open in Z and contains x is Z itself. Indeed, any such U is closed in X (as all subsets of X containing x are closed by hypothesis) and so closed in Z, so U is clopen in the connected Z and non-empty (as it contains x), forcing $U = Z$ as desired.

Next consider the closure V of $X - Z$. Similarly to the argument in (1), since $(X - Z) \cup \{x\}$ is closed in X we see that either $V = X - Z$ (in which case the closed Z is also open and f from (E.1.1.1) is a homeomorphism) or $V = (X - Z) \cup \{x\}$. In particular, V meets $X - \{x\}$ in exactly $X - Z$.

(ii): We denote by K_0 the (quasi-)compact continuous image of K in V. We claim this image is closed in V. If K_0 contains x then K_0 is closed in X, hence in V. If not then K_0 is compact within $V - \{x\}$ which is discrete by (2.a). It follows that K_0 is finite, and since every point of V is closed in V it follows again that K_0 is closed in V (and even finite discrete). This says V is weakly Hausdorff: for every compact Hausdorff space K and continuous map $g : K \to V$, the image $g(K)$ is closed in V (see Definition E.2.1.1). We now make two (non-circular) forward references to §E.2: by Remark E.2.1.2 the subspace $K_0 \subset V$ is then also weakly Hausdorff, and by Lemma E.2.1.4 it is Hausdorff since it is a continuous image of a compact Hausdorff space. In particular, K_0 is compact Hausdorff. Since $g : K \to K_0$ is a continuous surjection between compact Hausdorff spaces, it is submersive.

(iii): We let S^1 be the unit circle, and $0 \in S^1$ a marked point (which we assume, upon parametrizing S^1 as $\mathbf{R}/\mathbf{Z}$ under the exponential submersion $\exp(2\pi it) : \mathbf{R} \to S^1$, to correspond to the image of $t = 0$). We consider the quotient of the set-theoretic coproduct $\coprod_{i \in I} S^1_i$, where S^1_i is S^1 and I is the index set $Z - \{x\}$, by the equivalence relation identifying all the marked points of the S^1_i's. We call 0 the image of each, equivalently any, of the marked points $0_i \in S^1_i$ in the quotient.

We call $\mathbf{H}_I$ the resulting quotient set, which we topologize as follows. For a subset $U \subset \mathbf{H}_I$ containing 0, we say it is open if and only if it is the union of all but finitely many of the images in $\mathbf{H}_I$ of the S^1_i's along with open subsets $U_i \subset S^1_i$ containing 0_i for the remaining finitely many $i \in I$. We say that a subset $U \subset \mathbf{H}_I$ which does not contain 0 is open if and only if it is a union of the images of open subsets $U_i \subset S^1_i$ for every $i \in I$ (the notation $\mathbf{H}_I$ intends to refer to the Hawaiian earring, which is the special case $I = \mathbf{N}$).

Then $\mathbf{H}_I$ is compact Hausdorff. The map g sending 0 to x and $S_i - \{0_i\}$ to $i \in Z - \{x\}$ is surjective and we claim it is continuous. Let $U \subset Z$ be an open subset. We want to show $g^{-1}(U)$ is open in $\mathbf{H}_I$. If U contains x then $U = Z$ (recall from the proof of (2.a) that Z has no proper open subset containing x), so we may focus on U not containing x. But then $g^{-1}(U)$ inside $\mathbf{H}_I$ is the union of the punctured circles $S_i - \{0_i\}$ for $i \in U$, and that is also open in $\mathbf{H}_I$.

To show g is submersive, let $C \subset Z$ be a subset whose pre-image $g^{-1}(C)$ is closed in $\mathbf{H}_I$. We want to show C is closed in Z, or equivalently in X. If $C \subset \{x\}$ then C is closed. If not, C meets the complement $Z - \{x\}$ of x. In particular, $g^{-1}(C)$ contains the image of $S^1_i - \{0_i\}$ for some $i \in I$. But $g^{-1}(C)$ is closed so it contains the closure in $\mathbf{H}_I$ of that ith punctured circle, hence it contains 0. It follows that $C = g(g^{-1}(C))$ contains $g(0) = x$, and hence C is closed in X (as is every subset of X containing x, by hypothesis).

(2.c) : For the purposes of proving that (1) implies (2) in Lemma E.1.1.3, by (2.i) and closedness of Z and V in X it is enough to separately treat Z and V in place of X. The implication holds for V by (2.ii) (by using the set of closed compact Hausdorff subspaces of V), and for Z by (2.iii). □

Remark E.1.1.5. Let W be a trait: the finite topological space $\{0, 1\}$ with 0 closed and 1 a generalization of 0 (equivalently, W is the spectrum of a discrete valuation ring). The proof of (2.iii) from Lemma E.1.1.4 shows that Z is homeomorphic to the glueing along $x \in Z$ of copies of W indexed by the set $Z - \{x\}$. It is a bouquet of traits bundled at x, indexed by $Z - \{x\}$.

Proof of Lemma E.1.1.3. We assume (1) and will deduce (4) holds. First we show that naturally $X = k(X)$. Let Y be the topological space $Y := k(X)$; i.e., the set X^{set} endowed with the k-topology on X. By (1) applied to this choice of Y and to $f = \mathrm{id} : X \to Y$ the identity set-function on the shared underlying set, f is continuou by design of the topology on Y. On the other hand, the topology of Y is finer than that of X by part (1) of Remark E.1.1.2. It follows that f^{-1} (also the identity as a set-function) is continuous, and hence the identity set-function is a homeomorphism; i.e., the two topologies on the set underlying X are the same.

If X is discrete, then the four conditions in Lemma E.1.1.3 all hold for X. By Lemma E.1.1.4, it is enough to treat the cases in which either X contains exactly one point $x \in X$ such that any subset $A \subset X$ containing x is closed, or contains no such point. In the former case, by Lemma E.1.1.4(2.c) all four conditions in Lemma E.1.1.3 hold for X. We therefore may and do now assume without loss of generality that for any point $x \in X$, there exists a non-closed subset $A \subset X$ such that $x \in A$ holds.

We now call $\Sigma \subset 2^X \times X$ the set of pairs (A, x) such that A is a non-closed subspace of X, $x \in A$ is a point. We claim that for every element (A, x) of Σ, there exists a compact Hausdorff space K and a continuous map $g : K \to X$, such that $g^{-1}(A)$ is not closed in K, and $g^{-1}(\{x\}) \neq \varnothing$. Indeed, as A is not closed in X, there exists K compact Hausdorff and $g : K \to X$ a continuous map, such that $g^{-1}(A)$ is not closed in K since X has the k-topology. If $g^{-1}(\{x\})$ is non-empty we are done, and otherwise define $K' := K \coprod \{x\}$, also compact Hausdorff, and extend g to a continuous map $g' : K' \to X$ defined as g on the K component and sending x to x. The pair (K', g') satisfies the required properties.

For any fixed element $(A,x) \in \Sigma$, we denote by $\mathscr{C}_{(A,x)}$ the class of all pairs (K,g), such that K is a compact Hausdorff space and $g : K \to X$ is a continuous map such that $g^{-1}(A)$ is not closed in K and $g^{-1}(\{x\})$ is non-empty. Since, by the foregoing, $\mathscr{C}_{(A,x)}$ is non-empty for every $(A,x) \in \Sigma$, by [**Jech**, Part I, §6, Lemma 6.3, p. 64] and specifically construction (6.4) in loc.cit. (see Remark E.1.1.6) there exists an ordinal α such that $\mathscr{C}_{(A,x)} \cap V_\alpha$ is also non-empty where V_α is the "von Neumann universe" associated to α. As V_α is a set, so is $\mathscr{C}_{(A,x)} \cap V_\alpha$. We define

$$\alpha_{(A,x)} := \inf\{\alpha \mid \mathscr{C}_{(A,x)} \cap V_\alpha \neq \varnothing\}, \quad \alpha := \sup_{(A,x)\in\Sigma} \alpha_{(A,x)} \tag{E.1.1.2}$$

Both ordinals in (E.1.1.2) exist, as the infimum and the supremum are both defined over *sets* of ordinals. Define

$$S'_X := \bigcup_{(A,x)\in\Sigma} \mathscr{C}_{(A,x)} \cap V_\alpha, \tag{E.1.1.3}$$

a non-empty set whose elements are pairs (K,g) with K compact Hausdorff and $g : K \to X$ continuous such that for every $(A,x) \in \Sigma$ there exists some $(K,g) \in S'_X$ such that $g^{-1}(A)$ is not closed in K and $g^{-1}(\{x\})$ is non-empty. We denote by $\sigma := (K,g)$ the elements of S'_X, and for every $\sigma \in S'_X$ we denote by K_σ the associated compact Hausdorff space and by $g_\sigma : K_\sigma \to X$ the associated continuous map. We further denote by:

$$q : \coprod_{\sigma\in S'_X} K_\sigma \to X \tag{E.1.1.4}$$

the map defined on K_σ to be g_σ, for each $\sigma \in S'_X$.

The map q is continuous, as for every $\sigma \in S'_X$, each g_σ is a continuous map $K_\sigma \to X$ by design of S'_X. We now check that q is surjective. Let $x \in X$ be a point, and A any non-closed subset of X containing x. Then (A,x) is an element of Σ. There exists some $\sigma \in S'_X$ such that $g_\sigma^{-1}(A)$ is not closed in K_σ, and $g_\sigma^{-1}(\{x\})$ is non-empty. It follows that x is in the image of g_σ, and hence q is surjective.

Suppose $U \subset X$ is a subspace such that $q^{-1}(U)$ is open in $\coprod_{\sigma\in S'_X} K_\sigma$. We have, setting $U_\sigma := g_\sigma^{-1}(U) \subset K_\sigma$,

$$q^{-1}(U) = \coprod_{\sigma\in S'_X} U_\sigma. \tag{E.1.1.5}$$

By assumption on U, $q^{-1}(U)$ is open and hence (E.1.1.5) forces $U_\sigma = g_\sigma^{-1}(U)$ to be open in K_σ for every $\sigma \in S'_X$. If U is not open in X then the complement $B := X - U$ would be non-closed. For any point $x \in B$, (B,x) is an element of Σ and so there exists some (K,g) in $\mathscr{C}_{(B,x)}$. As $\mathscr{C}_{(B,x)}$ is non-empty, by design of α from (E.1.1.2) we have $\mathscr{C}_{(B,x)} \cap V_\alpha \subset S'_X$ is also non-empty. There therefore exists $\sigma = (K,g)$ in S'_X such that $g^{-1}(B)$ is not closed in K, contradicting the openness of $g^{-1}(X - B) = U_\sigma$.

Now that q is submersive, a function $f : X \to Y$ is continuous if and only if $f \circ q$ is continuous. This assertion is in turn equivalent to $f \circ g_\sigma : K_\sigma \to Y$ being continuous for all $\sigma \in S'_X$ by the universal property of topological co-products. Hence, also (1) implies (3), and similarly (4) implies (3).

Next we show (1) implies (2). We now write $\mathscr{C}$ for the class of all compact Hausdorff spaces, and we define the class function $\phi : S'_X \to \mathscr{C}$ by $\phi(\sigma) = K_\sigma$. This is a class function (see [**Kunen**, Ch. I, §6]) since it is a binary relation on $S'_X \times \mathcal{C}$ such that $(\sigma, K) = (\sigma, K')$ implies $K = K'$. By the Axiom of Replacement (see [**Jech**, Part I, §1, (1.7)] and [**Kunen**, Ch. I, §6, Axiom 6]) the image of ϕ is a set, which we denote S_X. By design of S'_X, for every $K \in S_X$ there exists a continuous map $g_K : K \to X$ such that a set-function $f : X \to Y$ is continuous if and only if $f \circ g_K$ is continuous for all $K \in S_X$. In particular, for any set-function $f : X \to Y$, f is continuous if and only if $f \circ g$ is continuous for all $K \in S_X$ and all continuous maps $g : K \to X$. The set S_X therefore satisfies the requirements of part (2) of the Lemma.

Conversely, suppose that (2) holds. If part (a) of (1) holds, then so does part (b) of (1). Conversely, if part (b) of (1) holds then part (b) of (2) also holds and so by (2) we deduce that f is continuous, implying that part (a) of (1) holds. Part (1) therefore follows from part (2). We now show the equivalence of (2) and (4). If (2) holds then we choose $I = S'_X \coprod X$ where S'_X is the set of pairs (K, g) for $K \in S_X$ and $g : K \to X$ continuous. For every $i \in I$, either $i \in S'_X$, in which case $i = \sigma = (K_\sigma, g_\sigma)$, and we define $K_i := K_\sigma$ and $g_i := g_\sigma$. Otherwise, we have $i \in X$, in which case $i = x$ for a point $x \in X$, and we define $K_i := \{x\}$ and g_i the unique continuous map sending x to the point $x \in X$. We may therefore define the map $q : \coprod_{i \in I} K_i$ as the co-product of the g_i for all $i \in I$. The resulting map:

$$q : \coprod_{i \in I} K_i \to X$$

is a submersive map by [**Bou-GT**, Ch. I, §2, Prop. 6] or part (1) of [**SP**, Tag 08ZK], as it is surjective by design and, by (2) and the universal property of topological co-products, a function $f : X \to Y$ is continuous if and only if $f \circ q$ is. This proves (2) implies (4). The converse is obvious (using $S_X = I$), so (2) and (4) are equivalent.

If (4) holds then (3) does since it is enough to test openness of U on those compact Hausdorff spaces indexed by I. It remains to prove (3) implies (1). Irrespective of assuming that (3) holds, we see that part (b) of (1) follows from part (a). Assuming that (3) holds, we may further assume that part (b) of (1) holds. We let $V \subset Y$ be an open subspace, and call $U := f^{-1}(V)$, and observe that for any compact Hausdorff space K and any continuous map $g : K \to X$, $g^{-1}(U) = (f \circ g)^{-1}(V)$. As $f \circ g$ is continuous and V is open, we deduce that $g^{-1}(U)$ is open in K. As K and g are arbitrary, we deduce from (3) that U is open, and hence f is continuous. □

Remark E.1.1.6. Construction (6.4) from [**Jech**, Part I, §6] can be found on p. 65 of loc.cit. after [**Jech**, Part I, §6, Lemma 6.3] upon which it rests, and it is credited to Dana Scott and often referred to as "Scott's trick". It rests on the Axiom of Regularity. In particular, the construction of the set S_X in the proof of Lemma E.1.1.3 does not use any stronger version of the Axiom of Choice, and is therefore valid within ZFC, which is what this Remark is aimed at highlighting (in fact, it does not use the Axiom of Choice either).

Definition E.1.1.7. We define:

(1) a topological space X to be a *compactly generated space* if and only if it satisfies the equivalent conditions of Lemma E.1.1.3.
(2) the category **CG** whose objects are compactly generated spaces, and morphisms are continuous maps of topological spaces. We shall refer to **CG** as the *category of compactly generated spaces*.

Remark E.1.1.8. On the construction of $k(X)$ from Definition E.1.1.1 we observe that we can use the terminology introduced in Definition E.1.1.1 to rephrase Definition E.1.1.7. Specifically, a topological space X is compactly generated if and only if $k(X) = X$. Since $k(X)$ and X share the same underlying set, it makes sense to compare their topologies.

The following example exhibits compact generation as a non-vacuous condition on topological spaces.

Example E.1.1.9. Let X be an uncountable set equipped with the co-countable topology (i.e., the open subsets are $\emptyset$ and complements of countable subsets, so the closed subsets are X and the countable subsets). We claim X is not compactly generated. Indeed, we claim that $k(X)$ is discrete.

To see this, let Z be any subset of X and K any compact Hausdorff space, with $g : K \to X$ an arbitrary continuous map, so $g(K)$ is a compact subspace of X and inherits the co-countable topology from X. It is therefore finite discrete, and $Z \cap g(K)$ is open in $g(K)$. As such, its pre-image in K, which agrees with $g^{-1}(Z)$, is also open. This proves that any subset of X is k-open in X, and hence $k(X)$ is discrete.

If X was compactly generated then it would have to be discrete. As X is uncountable, the co-countable topology on it is not discrete. Any non-closed subspace $Z \subset X$ is therefore a k-closed non-closed subspace. For further discussion of examples for the co-countable topology, we refer to [**StSe**, Part II, 29, 21].

We refer to §E.2.4 for an example-driven illustration of the relations between the various definitions of compact generation present within the literature. This is located after §E.2.3 so as to facilitate the construction of examples resting on results discussed up to and including §E.2.3.

E.1.2. Categorical and topological stabilities of compactly generated spaces. Before moving forward, we recall several facts about the large category **CG** of compactly generated spaces and some important closure properties it enjoys. Our main sources for the materials collected in this subsection are [**LGL**, §2, §3, §5], [**McC**, §2], [**Str**, §1, §2], [**Rezk**, §8]. We shall refer to [**ELS**, §5] as well, concerning arguments towards the cartesian closure of **CG** as a cartesian monoidal category.

We first recall the relation between the large categories **CG** and **Spaces**. This latter, recall, is the category whose objects are topological spaces, and morphisms are continuous maps between topological spaces (see part (2) of Notation E.0.0.1). The forgetful functor

$$u_{\mathrm{cg}} : \mathbf{CG} \to \mathbf{Spaces} \tag{E.1.2.1}$$

sends a compactly generated space to itself as a topological space, and continuous maps to continuous maps.

LEMMA E.1.2.1. *The assignment $X \rightsquigarrow k(X)$ is a functor*

$$k : \mathbf{Spaces} \to \mathbf{CG} \tag{E.1.2.2}$$

that satisfies the following properties:

(1) *k is right-adjoint to the forgetful functor $u_{\mathrm{cg}} : \mathbf{CG} \to \mathbf{Spaces}$. Explicitly, any continuous $f : X \to Y$ with k-closed X uniquely factors continuously through the natural continuous bijection $k(Y) \to Y$.*
(2) *The unit $\eta : \mathrm{id} \to k \circ u_{\mathrm{cg}}$ of the adjunction (u_{cg}, k) is a natural isomorphism of functors; this is the inverse $X \to k(X)$ to the natural homeomorphism $k(X) \to X$ for k-closed X.*

PROOF. We check functoriality of the formation of $k(\cdot)$, from which the rest is clear (as $k(X) \to X$ via id_X is a homeomorphism for k-closed X). We have to show that if $f : X \to Y$ is a continuous map of topological spaces then any k-closed subset Z of Y has k-closed pre-image in X. For any continuous map $g : K \to X$ from a compact Hausdorff space K, the pre-image $g^{-1}(f^{-1}(Z))$ of $f^{-1}(Z)$ in K agrees with the pre-image $(f \circ g)^{-1}(Z)$ of Z to K under the composition with f, and it is therefore closed in K by continuity of the composition $f \circ g$. It follows $f^{-1}(Z)$ is k-closed. □

We now discuss stability properties of **CG** under several categorical operations, starting with formation of small limits and small colimits:

LEMMA E.1.2.2. *The category* **CG** *contains all small limits and all small colimits. Moreover, for any small category I and diagram $P : I \to \mathbf{CG}$:*

(1) *$\lim_I P$ in* **CG** *is computed as*

$$\lim_I P = k(\lim_I u_{\mathrm{cg}} \circ P) \tag{E.1.2.3}$$

upon evaluating $k(\cdot)$ on the limit of the small diagram $u_{\mathrm{cg}} \circ P$ in **Spaces**.
(2) *$\mathrm{colim}_I P$ in* **CG** *is computed in* **Spaces**, *i.e. the forgetful functor u_{cg} reflects and commutes with small colimits.*

PROOF. Assertion (1) is immediate from the universal property of $\lim_I$ in **Spaces** and Lemma E.1.2.1(1). For (2), the task amounts to showing that the topological colimit $\mathrm{colim}_I(u_{\mathrm{cg}} \circ P)$ is compactly generated. A

small colimit in **Spaces** is constructed as a quotient of a small co-product, so it is enough to check that small co-products and quotient maps preserve compact generation. For each $i \in I$, let J_i be a set as in part (4) of Lemma E.1.1.3 applied to the compactly generated space $(u_{\mathrm{cg}} \circ P)(i)$ and let $q_i : \coprod_{j \in J_i} K_j \to (u_{\mathrm{cg}} \circ P)(i)$ be a submersive map. Then

$$q := \coprod_{i \in I} q_i : \coprod_{i \in I} \coprod_{j \in J_i} K_j \to \coprod_{i \in I} (u_{\mathrm{cg}} \circ P)(i)$$

is a submersive map, and hence $\coprod_I (u_{\mathrm{cg}} \circ P)$ is compactly generated by Lemma E.1.1.3(4). Moreover, if $f : X \to Z$ is a quotient map with X compactly generated then Z inherits (4) in Lemma E.1.1.3 from X via composing with f since f is submersive. □

Remark E.1.2.3. On limits and colimits in **CG**, we make the following observations. Contrary to small colimits, small limits in **CG** are generally not computed in **Spaces**. This does not hold already for products: see Lemma E.2.4.9.

Both **Spaces** and **CG** contain finite products (by direct construction in **Spaces**, or by [SP, Tag 08ZT], and by Lemma E.1.2.2 for **CG**). The topology on the product $X \times Y$ internal to **CG** of two compactly generated spaces is not generally the product of the topologies on X and Y, as noted in Remark E.1.2.3. It is therefore our preference to introduce notation to distinguish the two products, and derivative functorial constructions.

Definition E.1.2.4. Let X, Y be compactly generated spaces. Upon writing:

$$X' := u_{\mathrm{cg}}(X), \quad Y' := u_{\mathrm{cg}}(Y)$$

for their underlying topological spaces, we define:

(1) $X \times Y$ to be the product of X and Y internal to **CG**.
(2) We define $X \times_0 Y$ to be the product of X' and Y' internal to **Spaces**, i.e. the set $X^{\mathrm{set}} \times Y^{\mathrm{set}}$ endowed with the product topology.
(3) For Z a compactly generated space and $f : X \to Z$, $g : Y \to Z$ continuous maps, upon writing $Z' := u_{\mathrm{cg}}(Z)$, and denoting by f, g the same continuous maps, we define $X \times^0_{f,Z,g} Y$ to be the fibre product $X' \times_{f,Z',g} Y'$ in **Spaces**, i.e. the pre-image $(f,g)^{-1}(\Delta_{Z'}(Z')) \subset X \times_0 Y$ of the continuous map:
$$(f,g) : X \times_0 Y \to Z \times_0 Z.$$
We refer to $X \times^0_{f,Z,g} Y$ as the *topological fibre product*.
(4) For I a set, and $P : I \to \mathbf{CG}$ a diagram, we denote by $\prod^0_I P$ the I-indexed product of $u_{\mathrm{cg}} \circ P$ in **Spaces**. We refer to $\prod^0_I P$ as the *topological product*.

Remark E.1.2.5. On Definition E.1.2.4, we make the following observations.

(1) Part (2) of Definition E.1.2.4 follows [Str, Def. 2.3]. We note that no notation has been introduced in Definition E.1.2.4 for I-indexed co-products since (by Lemma E.1.2.2(2)) the topological co-product $\coprod_I u_{\mathrm{cg}} \circ P$ of any I-indexed diagram $P : I \to \mathbf{CG}$ is compactly generated and computes the co-product in **CG**. The notation $\coprod_I$ is, therefore, un-ambiguous.
(2) For compactly generated X, Y, we have $X \times Y = k(X \times_0 Y)$ by Lemma E.1.2.2(1). More generally, for any set I and I-indexed diagram $P : I \to \mathbf{CG}$, the product $\prod_I P(i)$ in **CG** is computed as:
$$k\left(\prod_I (u_{\mathrm{cg}} \circ P)(i)\right)$$
by Lemma E.1.2.2(1).

An important property of **CG** is that it has an internal Hom-space bifunctor endowing **CG** with a cartesian-closed monoidal structure (we refer the reader to [Kel, §1.5], and [MacL, Ch. VII, §7, §8], for

foundational material on monoidal categories; the term "cartesian" refers to the $\otimes$ bifunctor, denoted $\square$ in [**MacL**, Ch. VII, §7, §8], being the cartesian product). We briefly discuss this aspect and some consequences of direct use for this appendix, with a view towards the next section on compactly generated weakly Hausdorff spaces, where the weak Hausdorff condition on objects will strengthen consequences of cartesian closure.

Our standard references for closed symmetric monoidal categories are [**MacL**, Ch. VII, §1] for monoidal categories in general, [**MacL**, Ch. VII, §7] and [**EK**, Ch. III] for closed categories and closed symmetric monoidal categories, and [**MacL**, Ch. VII, §8] for the category of compactly generated Hausdorff spaces. We do not incorporate any separation condition into the definition of the objects of **CG**, and we will impose only a weakening of the Hausdorff separation condition when we introduce the full subcategory of weakly Hausdorff compactly generated spaces, but the proof we give for cartesian closedness of **CG** follows the same general pattern of [**Ste**, §5] and [**MacL**, Ch. VII, §8].

A reference pinpointing the general structures underlying cartesian closedness of convenient categories of spaces is [**ELS**, §5]. We further refer the reader to Mac Lane's paper [**MacL2**] where associativity and commutativity coherences in monoidal categories are derived from the basic defining axioms of coherence within a monoidal category (see diagrams (3), (4), (7) in [**MacL**, Ch. VII, §1]). The theme of how to derive general coherences from (3),(4) and (7) of [**MacL**, Ch. VII, §1] is reprised in [**MacL**, Ch. VII, §2] and developed in [**MacL2**], with particular emphasis on [**MacL2**, Thm. 4.2], where coherence of the associativity and commutativity natural isomorphisms of a monoidal category are reduced to the pentagonal [**MacL2**, §3, 3.5] and hexagonal [**MacL2**, §4, 4.5] axioms. We discuss the closed monoidal category structure on **CG** so as to include self-contained arguments for **CG**, compatibly with the definition of compact generation in Definition E.1.1.7. Our main references for results on function spaces within **CG** and the resulting closed monoidal category structure on **CG** (and later **CGWH**) are [**LGL**, §5], [**ELS**, §5], [**Str**, §2], and [**Rezk**, §8].

DEFINITION E.1.2.6. Let X and Y be compactly generated spaces.

(1) For any compact Hausdorff space K, continuous map $g : K \to X$, and open subspace $U \subset Y$, let $N(g, K, U)$ denote the subset of $\mathrm{C}(X, Y)$ consisting of continuous maps $f : X \to Y$ such that $(f \circ g)(K) \subseteq U$:

$$N(g, K, U) = \mathrm{C}(X, Y) \times_{g^*, \mathrm{C}(K,Y), j_*} \mathrm{C}(K, U) \tag{E.1.2.4}$$

where $g^* : \mathrm{C}(X, Y) \to \mathrm{C}(K, Y)$ is $f \mapsto f \circ g$ and $j_* : \mathrm{C}(K, U) \to \mathrm{C}(K, Y)$ is $f \mapsto j \circ f$ for $j : U \to Y$ the open immersion. (Note that $N(g, \emptyset, \emptyset) = \mathrm{C}(X, Y)$ for the unique $j : \emptyset \to X$.)

We denote by $\mathcal{B}$ the set of subsets $N \subset \mathrm{C}(X, Y)$ of the form $N = N(g, K, U)$ for some compact Hausdorff space K, continuous map $g : K \to X$, and open subspace $U \subset Y$.

(2) The *compact-open topology* on $\mathrm{C}(X, Y)$ is the unique topology on $\mathrm{C}(X, Y)$ admitting $\mathcal{B}$ as a sub-base (see [**SP**, Tag 08ZI] and [**SP**, Tag 08ZJ], as well as [**Du**, Ch. III, Thm. 3.1] and the discussion after [**Ke**, Ch. 1, Thm. 12, p. 48]). In other words, finite intersection among members of $\mathcal{B}$ are a base for the topology. We call the sub-base $\mathcal{B}$ the *standard sub-base* for the compact-open topology on $\mathrm{C}(X, Y)$. (In Lemma E.1.2.8 we give a set-theoretic consistency check, showing that the compact-open topology can be defined using a *set* of triples (g, K, U) rather than just a class.)

REMARK E.1.2.7. We note that the compact-open topology referred to above is not the usual notion of topology on function spaces under the same name. Indeed, for topological spaces X and Y, one usually defines the compact-open topology on $\mathrm{C}(X, Y)$ as the topology having as sub-base the collection of subsets $N(g, K, U)$ for $g : K \to X$ an immersion of a compact space K. Throughout this appendix, the terminology "compact-open topology" will solely be referring to the topology introduced in Definition E.1.2.6(2), generating a finer topology. The topology defined in Definition E.1.2.6 is a topology on $\mathrm{C}(X, Y)$, for compactly generated X and Y, such that $k(\mathrm{C}(X, Y))$ defines an internal mapping functor promoting **CG** to a closed monoidal category with respect to the cartesian monoidal structure (we refer the reader to [**MacL**, Ch. VII] for background on cartesian monoidal categories, and to [**MacL**, Ch. III, §1, Prop. 5] for the construction of a canonical monoidal structure

on categories with finite products and a final object). Proving this fact is the purpose of the next results (see Lemma E.1.2.12 and Proposition E.1.2.13).

The reader may wonder whether the k-topology on the compact-open topology on $\mathrm{C}(X,Y)$ is the only topology promoting the set-valued bifunctor of continuous maps between compactly generated spaces to an internal Hom-functor promoting **CG** to a cartesian closed monoidal category. We refer the reader to [**ELS**, §5] for a discussion pertaining uniqueness aspects of the k-topology associated to the compact-open topology being considered (see especially [**ELS**, §5, Thm. 5.15]). We follow [**LGL**, §5] throughout the discussion on function spaces of compactly generated spaces, and later on compactly generated spaces incorporating a separation condition (see Proposition E.2.2.10).

LEMMA E.1.2.8. *Let X and Y be compactly generated spaces. There exists a set $S_{X,Y}$, and for any $\sigma \in S_{X,Y}$ a triple $(g_\sigma, K_\sigma, U_\sigma)$ with K_σ a compact Hausdorff space, $g_\sigma : K_\sigma \to X$ a continuous map, and $U_\sigma \subset Y$ an open subspace such that the coarsest topology on $\mathrm{C}(X,Y)$ with the property that every element of the set*

$$\{N(g_\sigma, K_\sigma, U_\sigma), \sigma \in S_{X,Y}\} \tag{E.1.2.5}$$

is open in $\mathrm{C}(X,Y)$ is the compact-open topology on $\mathrm{C}(X,Y)$.

PROOF. We fix a subset $N \subset \mathrm{C}(X,Y)$ of the form $N = N(g,K,U)$ as in (E.1.2.4) for a compact Hausdorff space K, a continuous map $g : K \to X$, an open subspace $U \subset Y$. We call $\mathscr{C}_N$ the class of those triples (g,K,U) such that K is a compact Hausdorff space, $g : K \to X$ is a continuous map, and $U \subset Y$ is an open subspace satisfying

$$N(g,K,U) = N.$$

By the choice of N, $\mathscr{C}_N$ is non-empty. There therefore exists an ordinal α such that $\mathscr{C}_N \cap V_\alpha$ is non-empty. By [**Jech**, Part I, §6, Lemma 6.3] (see construction (6.4) in [**Jech**, Part I, §6, p. 65]), upon defining

$$\alpha_N := \inf\{\alpha \mid \mathscr{C}_N \cap V_\alpha \neq \varnothing\}, \quad \alpha := \sup\{\alpha_N \mid N \in \mathcal{B}\} \tag{E.1.2.6}$$

we have $\mathcal{C}_N \cap V_\alpha \neq \varnothing$ for every $N \in \mathcal{B}$, where $\mathcal{B}$ is the standard sub-base for the compact-open topology on $\mathrm{C}(X,Y)$ (see Definition E.1.2.6(2)). We note that the ordinals α_N and α from (E.1.2.6) indeed exist, as the infimum and supremum are defined over *sets* of ordinals. We call $S_N := \mathcal{C}_N \cap V_\alpha$, a set. We further call $S_{X,Y}$ the set:

$$S_{X,Y} := \bigcup_{N \in \mathcal{B}} S_N.$$

By design of α, for every $N \in \mathcal{B}$ there exists $(g,K,U) \in S_{X,Y}$ such that $N = N(g,K,U)$. □

CONSTRUCTION E.1.2.9. We let X, Y be compactly generated spaces, and endow the set $\mathrm{C}(X,Y)$ with the compact-open topology of Definition E.1.2.6. Define:

$$\mathscr{C}ont_k(X,Y) := k(\mathrm{C}(X,Y)).$$

We now check that $\mathscr{C}ont_k$ is bifunctorial on **CG** (in analogy with [**LGL**, §5, Thm. 5.5], see especially the discussion after [**LGL**, §5, Lemma 5.2], [**MacL**, Ch. VIII, Thm. 3], [**Str**, Lemma 2.10] and [**Rezk**, Prop. 8.2]).

LEMMA E.1.2.10. *Let X, Y be compactly generated spaces. Then formation of:*

$$\mathscr{C}ont_k(X,Y)$$

is contravariantly functorial in **CG** *with respect to X, and covariantly functorial in* **CG** *with respect to Y.*

PROOF. For $f : Y \to Y'$ a continuous map in **CG**, $U \subset Y'$ an open subspace with $j' : U \to Y'$ the open immersion, K a compact Hausdorff space, and $a : K \to X$ a continuous map, we claim

$$f_* : \mathrm{C}(X,Y) \to \mathrm{C}(X,Y'), \quad g \rightsquigarrow \phi_f(g) := f \circ g \tag{E.1.2.7}$$

is continuous. Clearly it is enough to check that the pre-image under f_* of every element of the standard sub-base for the compact-open topology on $\mathrm{C}(X, Y')$ is an element of the standard sub-base for the compact-open topology on $\mathrm{C}(X, Y)$. By design of $N(a, K, U)$, we have

$$(f_*)^{-1}(N(a, K, U)) = N(a, K, f^{-1}(U)). \tag{E.1.2.8}$$

Applying k then gives a continuous map

$$\mathscr{C}ont_k(X, Y) \to \mathscr{C}ont_k(X, Y')$$

This proves that $\mathscr{C}ont_k(X, Y)$ is covariantly functorial with respect to Y.

Now for $f : X' \to X$ be a continuous map in **CG**, K a compact Hausdorff space, $a : K \to X'$ a continuous map, and $U \subset Y$ an open subspace, let $f^* : \mathrm{C}(X, Y) \to \mathrm{C}(X', Y)$ be $g \mapsto g \circ f$. By a similar token as in the foregoing we have

$$(f^*)^{-1}(N(a, K, U)) = N(f \circ a, K, U),$$

giving continuity of f^* with respect to the compact-open topology on source and target. Applying k to f^* gives

$$f^* : \mathscr{C}ont_k(X, Y) \to \mathscr{C}ont_k(X', Y)$$

as a continuous map. This proves that $\mathscr{C}ont_k(X, Y)$ is contravariantly functorial with respect to X. □

The conclusion of Lemma E.1.2.10 is that $\mathscr{C}ont_k$ is a bifunctor:

$$\mathscr{C}ont_k : \mathbf{CG}^{\mathrm{opp}} \times \mathbf{CG} \to \mathbf{CG}.$$

Our aim is to now prove that $\mathscr{C}ont_k$ is compatible with the internal product in **CG**, and, specifically, satisfies a natural Hom-$\otimes$ adjunction, analogously to [**MacL**, Ch. VII, §8, (2)] and [**Ste**, Thm.5.6], and as proved in [**LGL**, Thm. 5.5], [**Str**, Prop. 2.20], and [**Rezk**, Prop. 8.8]. We provide an argument here as well for the sake of completeness, and to further highlight relations among the various approaches within the literature. We will mostly rely upon [**LGL**, §5], [**Str**, §2] and [**Rezk**, §8].

Remark E.1.2.11. We highlight here how the approaches to compactly generated function spaces in the above sources relate to one another. Throughout this work, we proceed as in [**LGL**, §5], and, specifically, we define the compact-open topology on function spaces in the sense of Definition E.1.2.6, following [**LGL**, §5, p. 160]. This is the approach adopted in [**Str**, §2] as well. We caution the reader that the topology defined on function spaces in [**Rezk**, §8, 8.1] is not generally equivalent to the compact-open topology as defined in [**LGL**, §5, p. 160] and [**Str**, Def. 2.8], and throughout this appendix. It is in fact finer (see [**Rezk**, Lemma 8.4]). However, applying k removes this discrepancy: by [**ELS**, Thm. 5.15], the compactly generated function spaces associated to either topology are the same. Equivalently, the functor $k(\cdot)$ identifies the topology defined in [**Rezk**, §8, 8.1] on function spaces with the k-topology on the compact-open topology defined in [**LGL**, §5, p. 160], [**Str**, Def. 2.8], and this appendix.

We have the following:

Lemma E.1.2.12. *Let X, Y, Z be compactly generated spaces. The natural set-map*

$$\theta_{X,Y,Z} : \mathscr{C}ont_k(X \times Y, Z) \to \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z))$$

defined by $f \mapsto (x \mapsto f(x, \cdot))$ is a homeomorphism. In particular,

$$\theta^{-1}_{\mathscr{C}ont_k(X,Y),X,Y}(\mathrm{id})$$

is the evaluation map

$$\mathscr{C}ont_k(X, Y)^{\mathrm{set}} \times X^{\mathrm{set}} \to Y^{\mathrm{set}}$$

sending a pair (f, x) to $f(x)$.

PROOF. We define the "evaluation" set-function

$$\mathrm{ev} : \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}}) \times Y^{\mathrm{set}} \to Z^{\mathrm{set}}, \quad (f, y) \mapsto \mathrm{ev}(f, y) := f(y) \tag{E.1.2.9}$$

and the set-function:

$$\psi : \mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}})) \to \mathrm{Map}(X^{\mathrm{set}} \times Y^{\mathrm{set}}, Z^{\mathrm{set}}), \quad f \mapsto \psi(f) := \mathrm{ev} \circ (f, \mathrm{id}_Y) \tag{E.1.2.10}$$

natural with respect to X, Y and Z. Explicitly, $\psi(f)(x, y) = f(x)(y)$.

We check that

$$\psi(\mathrm{C}(X, \mathrm{C}(Y, Z)) \subset \mathrm{C}(X \times Y, Z) \tag{E.1.2.11}$$

as subsets of $\mathrm{Map}(X^{\mathrm{set}} \times Y^{\mathrm{set}}, Z^{\mathrm{set}})$. To this end, define the set-function

$$\eta : \mathrm{Map}(X^{\mathrm{set}}, \mathrm{Mat}(Y^{\mathrm{set}}, Z^{\mathrm{set}})) \to \mathrm{Map}(X^{\mathrm{set}} \times Y^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}}) \times Y^{\mathrm{set}}) \tag{E.1.2.12}$$

by

$$f \mapsto \eta(f) := (f, \mathrm{id}_Y).$$

If $f \in \mathrm{C}(X, \mathrm{C}(Y, Z))$, where continuity of f is with respect to the intrinsic topology on X and the compact-open topology on $\mathrm{C}(Y, Z)$ then f factors uniquely through $k(\mathrm{C}(Y, Z)) \to \mathrm{C}(Y, Z)$ since X is in **CG**, so $\eta(f)$ is continuous as a map

$$X \times Y \to k(\mathrm{C}(Y, Z)) \times Y = k(\mathrm{C}(Y, Z) \times_0 Y)$$

by functoriality of the k-topology. As ψ from (E.1.2.10) is by definition $\psi = \mathrm{ev} \circ \eta$, with ev the evaluation set-function from (E.1.2.9), to prove the containment in (E.1.2.11) it is enough to check that the restriction of ev to the set-theoretic product $\mathrm{C}(Y, Z) \times Y^{\mathrm{set}}$ is continuous with respect to the topology of $k(\mathrm{C}(Y, Z) \times_0 Y)$ and the intrinsic topology on Z.

We therefore let $U \subset Z$ be an open subspace, K a compact Hausdorff space, and $a : K \to k(\mathrm{C}(Y, Z) \times_0 Y)$ a continuous map. It is enough to check that $(\mathrm{ev} \circ a)^{-1}(U)$ is open in K. We define the continuous composition

$$a_1 : K \xrightarrow{a} k(\mathrm{C}(Y, Z) \times_0 Y) \xrightarrow{\mathrm{id}} \mathrm{C}(Y, Z) \times_0 Y \xrightarrow{\mathrm{pr}_1} \mathrm{C}(Y, Z) \tag{E.1.2.13}$$

We further define the continuous composition

$$a_2 : K \xrightarrow{a} k(\mathrm{C}(Y, Z) \times_0 Y) \xrightarrow{\mathrm{id}} \mathrm{C}(Y, Z) \times_0 Y \xrightarrow{\mathrm{pr}_2} Y \tag{E.1.2.14}$$

We have:

$$\begin{aligned} (\mathrm{ev} \circ a)^{-1}(U) &= a^{-1}(\mathrm{ev}^{-1}(U)) \\ &= \{t \in K \mid (a_1(t), a_2(t)) \in \mathrm{ev}^{-1}(U)\} \\ &= \{t \in K \mid a_1(t)(a_2(t)) \in U\}. \end{aligned} \tag{E.1.2.15}$$

Let t_0 be an element in $a^{-1}(\mathrm{ev}^{-1}(U)) \subset K$. The map $a_1(t_0) : Y \to Z$ is continuous, and upon composing it with $a_2 : K \to Y$ we obtain a continuous map $g := a_1(t_0) \circ a_2 : K \to Z$. As $g^{-1}(U)$ is open in K and contains t_0, and K is compact Hausdorff, there exists open $V \subset g^{-1}(U)$ containing t_0 and having compact closure $K' := \overline{V}$ in K. It follows that

$$a_1(t_0) \in N(a_2, K', U) \subset \mathrm{C}(Y, Z). \tag{E.1.2.16}$$

Since $a_1 : K \to \mathrm{C}(Y, Z)$ is continuous with respect to the compact-open topology on $\mathrm{C}(Y, Z)$ and the topology on K, the subset $a_1^{-1}(N(a_2, K', U))$ is open in K. We claim that

$$a_1^{-1}(N(a_2, K', U)) \cap K' \subset (\mathrm{ev} \circ a)^{-1}(U). \tag{E.1.2.17}$$

Let $t \in a_1^{-1}(N(a_2, K', U)) \cap K'$ be an element. Then $a_1(t)(a_2(t))$ is an element of U, and hence

$$t \in \{t' \in K \mid a_1(t')(a_2(t')) \in U\} = (\mathrm{ev} \circ a)^{-1}(U)$$

by (E.1.2.15). This proves the containment claimed in (E.1.2.17).

We have found the open neighborhood $a_1^{-1}(N(a_2, K', U)) \cap V$ of t_0 in K contained in $(\mathrm{ev} \circ a)^{-1}(U)$. Indeed, t_0 is contained in V by design of V and is contained in $a_1^{-1}(N(a_2, K', U))$ by (E.1.2.16). As t_0 is an arbitrary element of $(\mathrm{ev} \circ a)^{-1}(U)$, we deduce that $(\mathrm{ev} \circ a)^{-1}(U)$ is open as desired. We have shown ψ defines a visibly injective set-function:

$$\psi_{X,Y,Z} : \mathrm{C}(X, \mathrm{C}(Y, Z)) \to \mathrm{C}(X \times Y, Z). \tag{E.1.2.18}$$

To show that $\psi_{X,Y,Z}$ is surjective, we build a continuous inverse as follows. We define the set-function

$$i : X^{\mathrm{set}} \to \mathrm{Map}(Y^{\mathrm{set}}, X^{\mathrm{set}} \times Y^{\mathrm{set}}), \quad x \mapsto (y \mapsto (x, y)) \tag{E.1.2.19}$$

and

$$\psi' : \mathrm{Map}(X^{\mathrm{set}} \times Y^{\mathrm{set}}, Z^{\mathrm{set}}) \to \mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}})), \quad h \mapsto \psi'(h) := h_{**}(i) \tag{E.1.2.20}$$

where h_* is the set-function

$$h_* : \mathrm{Map}(Y^{\mathrm{set}}, X^{\mathrm{set}} \times Y^{\mathrm{set}}) \to \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}}), \quad g \mapsto h_*(g) := h \circ g$$

and h_{**} is the set-function

$$h_{**} : \mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, X^{\mathrm{set}} \times Y^{\mathrm{set}})) \to \mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}})), \quad g \mapsto h_{**}(g) := h_* \circ g. \tag{E.1.2.21}$$

It is clear that $\psi \circ \psi' = \mathrm{id}$ on $\mathrm{Map}(X^{\mathrm{set}} \times Y^{\mathrm{set}}, Z^{\mathrm{set}})$ and $\psi' \circ \psi = \mathrm{id}$ on $\mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}}))$. To check that $\psi_{X,Y,Z}$ is surjective, it is enough to check the containment

$$\psi'(\mathrm{C}(X \times Y, Z)) \subset \mathrm{C}(X, \mathrm{C}(Y, Z))$$

as subsets of $\mathrm{Map}(X^{\mathrm{set}}, \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}}))$.

In other words, we must check that for any $h : X \times Y \to Z$ continuous with respect to the compactly generated topologies on $X \times Y$ and Z, the set-function $\psi'(h) : X^{\mathrm{set}} \to \mathrm{Map}(Y^{\mathrm{set}}, Z^{\mathrm{set}})$ takes values in $\mathrm{C}(Y, Z)$ and $\psi'(h)$ is continuous with respect to the compactly generated topology on X and the compact-open topology on $\mathrm{C}(Y, Z)$.

For any $x \in X$, $\psi'(h)(x) = h(x, \cdot) = h \circ i_x$ as a set-function $Y \to Z$ for the visibly continuous x-insertion map $i_x : Y \to X \times_0 Y$. Since Y is compactly generated, i_x is also continuous as a map $Y \to X \times Y$ by Lemma E.1.2.1(1). Since h is itself a continuous map $X \times Y \to Z$, the composition $h \circ i_x : Y \to Z$ is continuous, and hence $\psi'(h)(x) \in \mathrm{C}(Y, Z)$. We now check $\psi'(h) : X \to \mathrm{C}(Y, Z)$ is continuous upon endowing X with its intrinsic compactly generated topology and $\mathrm{C}(Y, Z)$ with the compact-open topology. By Lemma E.1.2.10, h_* is continuous as a map

$$k(\mathrm{C}(Y, X \times Y)) \to k(\mathrm{C}(Y, Z)).$$

Upon writing

$$W_1 := k(\mathrm{C}(Y, X \times Y)), \quad W_2 := k(\mathrm{C}(Y, Z)) \tag{E.1.2.22}$$

we similarly get

$$h_{**} : \mathrm{C}(X, W_1) \to \mathrm{C}(X, W_2) \tag{E.1.2.23}$$

induced by (E.1.2.21) with this map h_{**} also continuous. Provided that $i : X^{\mathrm{set}} \to \mathrm{Map}(Y^{\mathrm{set}}, X^{\mathrm{set}} \times Y^{\mathrm{set}})$ is valued in $\mathrm{C}(Y, X \times Y)$ and continuous with respect to the intrinsic topology on X and the compact-open topology on $\mathrm{C}(Y, X \times Y)$, i is also continuous with respect to the intrinsic compactly generated topology on X and the k-topology W_1 associated to the compact-open topology on $\mathrm{C}(Y, X \times Y)$. Then $\psi'(h) = h_{**}(i)$ is continuous as desired. It is therefore enough to check that $i : X^{\mathrm{set}} \to \mathrm{Map}(Y^{\mathrm{set}}, X^{\mathrm{set}} \times Y^{\mathrm{set}})$ is valued in $\mathrm{C}(Y, X \times Y)$ and continuous with respect to the compactly generated topology on X and the compact-open topology on $\mathrm{C}(Y, X \times Y)$.

The containment $i(X) \subset \mathrm{C}(Y, X \times Y)$ is a consequence of continuity of i_x for every $x \in X$ as discussed above. We now prove that $i : X \to \mathrm{C}(Y, X \times Y)$ is itself continuous. To this end, we let K be a compact Hausdorff space, $a : K \to Y$ a continuous map, and $U \subset X \times Y$ an open subspace, and check that $i^{-1}(N(a, K, U))$ is open in X. Since X is compactly generated, it is enough to show that for any compact Hausdorff space K' and continuous map $b : K' \to X$, $b^{-1}(i^{-1}(N(a, K, U)))$ is open in K'. We have:

$$b^{-1}(i^{-1}(N(a,K,U))) = \{t \in K' \mid b(t) \in i^{-1}(N(a,K,U))\} = \{t \in K' \mid \{t\} \times K \subset (b,a)^{-1}(U)\}. \tag{E.1.2.24}$$

Since $(b, a) : K' \times K \to X \times Y$ is continuous, $W := (b, a)^{-1}(U)$ is open in $K' \times K$. We caution the reader that in this last assertion the product $X \times Y$ is the one for **CG**. However, as $K' \times_0 K$ is compact Hausdorff, it is compactly generated by Lemma E.1.1.3(4), and hence $K' \times K = K' \times_0 K$. We now apply [**SP**, Tag 005M] to $A := \{t\}$ for $t \in b^{-1}(i^{-1}(N(a, K, U)))$ and $B := K$ to get an open $V \subset K'$ containing t such that $V \times_0 K \subset W$. This means that V is contained in $(i \circ b)^{-1}(N(a, K, U))$. We have found an open neighborhood V of $t \in (i \circ b)^{-1}(N(a, K, U))$ entirely contained in $(i \circ b)^{-1}(N(a, K, U))$. As t is arbitrary, we deduce that $(i \circ b)^{-1}(N(a, K, U))$ is open in K', as desired. This proves continuity of i, which is what we needed to show to prove continuity of $\psi'(h) = h_{**}(i)$.

We have built the desired set-theoretic bijection

$$\psi_{X,Y,Z} : \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z)) \to \mathscr{C}ont_k(X \times Y, Z) \tag{E.1.2.25}$$

between objects in **CG** and it remains to show this bijection or its inverse

$$\psi'_{X,Y,Z} : \mathscr{C}ont_k(X \times Y, Z) \to \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z))$$

is a homeomorphism. It suffices to show this inverse map is continuous, since then applying $\mathrm{C}(T, \cdot)$ makes sense for any compactly generated T and yields the set-map $\psi'_{T\times X,Y,Z}$ that we know a bijection (so by Yoneda's Lemma, the map $\psi'_{X,Y,Z}$ in **CG** is an isomorphism).

We proved in the discussion after (E.1.2.10) that the evaluation morphism from (E.1.2.9) induces a continuous map

$$\mathrm{ev}_{Y,Z} : \mathscr{C}ont_k(Y, Z) \times Y \to Z. \tag{E.1.2.26}$$

for Y, Z compactly generated. Retaining the notation from (E.1.2.22), for a continuous $h : X \times Y \to Z$ we have a continuous map $h_{**} : \mathscr{C}ont_k(X, W_1) \to \mathscr{C}ont_k(X, W_2)$ from (E.1.2.23) and for $W'_1 := \mathscr{C}ont_k(X, W_1)$ and $W'_2 := \mathscr{C}ont_k(X, W_2) = \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z))$ and can write $\psi'(h) = \mathrm{ev}_{W'_1,W'_2}(h_{**}, i)$ exhibiting the point $\psi'(h) \in W'_2$ as a continuous function of h_{**} (since $i \in \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, X \times Y)) = W'_1$).

More precisely, $\psi'_{X,Y,Z}$ is the composition of three set-maps

$$\mathscr{C}ont_k(X \times Y, Z) \to \mathscr{C}ont_k(W_1, W_2) \to \mathscr{C}ont_k(W'_1, W'_2) \to W'_2 = \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z)) \tag{E.1.2.27}$$

where the first two steps are respectively $h \mapsto \mathscr{C}ont_k(Y, h) = h_*$ and $f \mapsto \mathscr{C}ont_k(X, f)$ and the last step is the continuous $\mathrm{ev}_{W'_1,W'_2}(\cdot, i)$ (with $i \in W'_1$). Thus, for continuity of $\psi'_{X,Y,Z}$ it suffices to show for all compactly generated X_1, X_2, X_2 that

$$\mathscr{C}ont_k(X_2, X_3) \to \mathscr{C}ont_k(\mathscr{C}ont_k(X_1, X_2), \mathscr{C}ont_k(X_1, X_3)) \tag{E.1.2.28}$$

defined by $g \mapsto g_* := \mathscr{C}ont_k(X_1, g)$ makes sense (i.e., g_* is continuous) and is continuous, since the first step in (E.1.2.27) is recovered by using $X_1 = Y$, $X_2 = X \times Y$, $X_3 = Z$ and the second step in (E.1.2.27) is recovered by using $X_1 = X$, $X_2 = W_1$, $X_3 = X_2$. For continuity of g_*, so (E.1.2.28) makes sense, by functoriality of $k(\cdot)$ it suffices to show $\mathrm{C}(X_1, X_2) \to \mathrm{C}(X_1, X_3)$ is continuous for the compact-open topologies; this holds because $g_*^{-1}(N(a, K, U)) = N(a, K, g^{-1}(U))$. [3] □

[3]The continuity of (E.1.2.28) requires explanation, and was not justified before: all that was said was "a similar argument" to the continuity of $\psi'(h)$ in h_{**} works, but that is too breezy.

We recall, as a consequence of [MacL, Ch.I II, §1, Prop. 5], that the internal product in **CG**, alongside the final object of **CG** (a point endowed with the discrete topology), determine a canonical (cartesian) symmetric monoidal structure on **CG**, whose structure morphisms are provided by the cited Proposition. We refer to it via the succint notation $(\mathbf{CG}, \times, *)$, where $*$ denotes the final object of **CG**, expecting no confusion to arise.

PROPOSITION E.1.2.13. *The bifunctor:*

$$\mathscr{C}ont_k : \mathbf{CG}^{\mathrm{opp}} \times \mathbf{CG} \to \mathbf{CG}$$

endows the symmetric monoidal category $(\mathbf{CG}, \times, *)$ *with the structure of a cartesian closed symmetric monoidal category enriched over itself.*

The Proposition is proved in [Str, Prop. 2.12], resting on [Str, Prop. 2.11] which is in turn a consequence of the Tube Lemma (recalled in [Str, Lemma 2.5]) that was used in the proof of Lemma E.1.2.12 (for the continuity of i).

PROOF. By Lemma E.1.2.12 there exists a natural homeomorphism

$$\theta_{X,Y,Z} : \mathscr{C}ont_k(X \times Y, Z) \to \mathscr{C}ont_k(X, \mathscr{C}ont_k(Y, Z)).$$

By the very definition of closed monoidal category (see [MacL, Ch. VII, §1] and [EK, Ch. III]), the cartesian (symmetric) monoidal structure $(\times, *)$ uniquely determined on **CG** by [MacL, Ch. III, §1, Prop. 5] is therefore closed, the internal-Hom bifunctor being given by $\mathscr{C}ont_k(\cdot, \cdot)$. □

As a direct application of Proposition E.1.2.13, we now prove that products in **CG** of submersive maps are submersive. We begin with a characterization of submersive maps in **CG**, and further refer the reader to [LGL, §5, Lemma 5.7] and its proof:

LEMMA E.1.2.14. *Let X, Y be compactly generated spaces, and $f : X \to Y$ a continuous map. The following conditions are equivalent:*

(1) *There exists a compactly generated space R, two maps $s, t : R \rightrightarrows X$, and a homeomorphism $c : \mathrm{coeq}(s,t) \to Y$, such that the canonical map $p : X \to \mathrm{coeq}(s,t)$ satisfies $c \circ p = f$.*
(2) *f is the co-equalizer in* **CG** *of the diagram $p_1, p_2 : X \times_{f,Y,f} X \rightrightarrows X$, where p_1, p_2 are the two projections.*
(3) *f is a submersive map.*

PROOF. We have (1) $\Rightarrow$ (3) since p is submersive in (1). We now assume that (3) holds, and prove that (2) holds. We choose $R := X \times_{f,X,f} X$, with the two maps to X defined to be the two projections $p_1, p_2 : R \rightrightarrows X$. We check that if f is submersive then $f : X \to Y$ is the co-equalizer of p_1, p_2 in **CG**. By Lemma E.1.2.2(2), we may compute the co-equalizer of the two projections p_1, p_2 in **Spaces**. We call $X' := u_{\mathrm{cg}}(X)$ and $Y' = u_{\mathrm{cg}}(Y)$, and retain the notation f for the continuous map $u_{\mathrm{cg}}(f)$, and R' the topological fibre product $R' := (f,f)^{-1}(\Delta_X(X))$. Its topology is the subspace topology of the product topology on $X \times_0 X$. We denote the two projections by $p_1', p_2' : R' \rightrightarrows X'$. The quotient X'/R' is the topological co-equalizer of p_1', p_2'. Part (2) of Lemma E.1.2.2 implies that we have $X'/R' = k(X'/R') = \mathrm{coeq}(p_1, p_2)$, where $p_1 := k(p_1')$ and $p_2 := k(p_2')$. This latter co-equalizer is the co-equalizer in **CG**. The evident map $c' : X'/R' \to Y'$ is a homeomorphism by design of R' since f is submersive. Since $X \times_{f,Y,f} X = k(X' \times^0_{f,Y',f} X') = k(R')$ by Lemma E.1.2.2(1), then $c := k(c')$ is a homeomorphism $\mathrm{coeq}(p_1, p_2) \to Y$ such that $c \circ k(\varphi) = f$ for the natural map $\varphi : X' \to X'/R'$, and so (2) is proved. If (2) holds, then we may set $R := X \times_{f,Y,f} X$, and $(s,t) = (p_1, p_2)$, with $p_1, p_2 : R \rightrightarrows X$ the two projections, to deduce that (1) holds. □

REMARK E.1.2.15. We have shown in Lemma E.1.2.14 that submersive maps are exactly the regular epimorphisms in **CG**, and the effective epimorphisms in **CG**. We caution the reader that, despite the Lemma, **CG** is not a regular category. We refer the reader to [Bor2, Ch. 2] for the foundational material on regular categories, and specifically for [Bor2, Ch. 2, Def. 2.1.1]. Stability of regular epimorphisms under arbitrary

pull-backs is a defining requirement of regularity. For regularity to hold for **CG**, regular epimorphisms should therefore be stable under arbitrary base change internally in **CG**. This does not hold, as Example E.1.2.16 below shows. To obtain a *regular* large category of topological spaces we will be incorporating a separation condition on objects in the next section, §E.2. Nonetheless, Lemma E.1.2.17 on preservation of submersiveness under direct products in **CG** holds with no separation conditions (see also [**LGL**, §5, Prop. 5.8]).

The following example is [**Bor2**, Counterexample 2.4.5], as a counterexample to regularity for the category **Spaces**. It can be adapted as below to serve the same purpose for **CG**:

EXAMPLE E.1.2.16. Let $X := \{1,2,3,4\}$ endowed with the topology $\tau_X := \{\varnothing, \{1,2\}, X\}$, $Y := \{a,b,c\}$ endowed with the topology $\tau_Y := \{\varnothing, Y\}$, and $Z := \{\alpha,\beta,\gamma\}$ endowed with the topology $\tau_Z := \{\varnothing, \{\alpha,\beta\}, Z\}$. We define $f : X \to Y$ to be $f(1) = a$, $f(2) = b = f(3)$, $f(4) = c$, and $g : Z \to Y$ to be $g(\alpha) = a$, $g(\beta) = c = g(\gamma)$. We have $P := X \times^0_{f,Y,g} Z = \{(1,\alpha),(4,\beta),(4,\gamma)\}$ endowed with the topology $\tau := \{\varnothing, \{(1,\alpha)\}, \{(1,\alpha),(4,\beta)\}, P\}$. The map f is submersive, whereas the first projection $f_Z : P \to Z$ is not, since $f_Z^{-1}(\{\alpha\}) = \{(1,\alpha)\}$ is open, while $\{\alpha\}$ is not. Since X, Y, Z and P are finite topological spaces, they are all compactly generated (for example by [**Str**, Prop. 1.6], and the topological fibre product P is therefore a fibre product in **CG** by Lemma E.1.2.2(1). This is an example of a submersive map in **CG** which is not stable under arbitrary base change internally in **CG**.

Here is the promised result on stability of submersive maps under products in **CG**:

LEMMA E.1.2.17. *Let X, Y, X', Y' be compactly generated spaces, and $f : X \to X'$, $g : Y \to Y'$ submersive maps. Then:*

$$(f,g) : X \times Y \to X' \times Y'$$

is a submersive map.

PROOF. We may factor (f,g) as $(f,\mathrm{id}) : X \times Y \to X' \times Y$, followed by $(\mathrm{id},g) : X' \times Y \to X' \times Y'$. It is therefore enough to treat the case $Y = Y'$ and $g = \mathrm{id}$. By Lemma E.1.2.14, X' is the co-equalizer of the two projections $X \times_{f,X',f} X \rightrightarrows X$ in **CG**. Lemma E.1.2.12 implies that the product functor $(\cdot) \times Y$ is a left-adjoint, and hence commutes with representable colimits. As co-equalizers are small colimits, all representable by Lemma E.1.2.2(2), we conclude. □

The above Lemma E.1.2.17 does not hold in **Spaces**, as the following example shows. The existence of examples in **Spaces** such as Example E.1.2.18 is one of the reasons why we shall be relying on the category of compactly generated spaces (and its variant incorporating a weak Hausdorff condition on objects, to be discussed in §E.2) to select better categories of topological spaces to serve as a small site.

EXAMPLE E.1.2.18. We let $X := \mathbf{Q}$ be endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$, and we let X' be the quotient of X obtained by identifying the subset of integers $\mathbf{Z} \subset \mathbf{Q}$ to $\{0\}$. The continuous map $f : X \to X'$ is submersive by construction. On the other hand, the map $(f,\mathrm{id}) : X \times_0 X \to X' \times_0 X$ of topological products is not submersive (see [**Br4**, Ch. 4, Example 4] for a proof of this assertion).

E.1.2.1. *Examples and permanence properties of compactly generated spaces.* Building on the previous section and especially on Lemma E.1.2.1, Lemma E.1.2.2, Lemma E.1.2.10 and Proposition E.1.2.13, we collect here results that exhibit familiar classes of topological spaces to be compactly generated, and will establish when formation of products and, on occasion, passage to subspaces, preserves compact generation by preserving some stronger property of topological spaces. In particular, we will collect sufficient conditions for constructions in **CG** to be computable in **Spaces**, for special classes of compactly generated spaces. This includes all topological spaces of interest for using this appendix to study periodic orbits in our main work.

PROPOSITION E.1.2.19. *We have:*

(1) *all locally compact Hausdorff spaces are compactly generated,*
(2) *the product in* **Spaces** *of a compactly generated space and a locally compact Hausdorff space is compactly generated* (*so it computes the product in* **CG**).

We refer the reader to [**Str**, Prop. 2.6].

PROOF. We call I the set of all compact Hausdorff subspaces of X, and for $i \in I$ let K_i be the compact Hausdorff subspace $i \subset X$. Since X is locally compact Hausdorff, the interiors of the K_i cover X. Upon calling $g_i : K_i \to X$ the immersion, we have $g_i^{-1}(Z) = Z \cap K_i$ for every subspace $Z \subset X$. It follows from the local nature of closedness that a subspace Z such that $g_i^{-1}(Z)$ is closed in K_i for every $i \in I$ is itself closed in X. We now consider the continuous map:

$$q := \coprod_{i \in I} g_i : \coprod_{i \in I} K_i \to X \tag{E.1.2.29}$$

We have shown that every subspace Z of X such that $q^{-1}(Z)$ is closed in $\coprod_{i \in I} K_i$, is closed in X. This proves (1). To prove (2), we use a related method along with Lemma E.1.1.3(3) to reduce to checking the product against a compact Hausdorff space is compactly generated.[4] Hence, (E.1.2.29) is submersive and so X is compactly generated. □

REMARK E.1.2.20. On the above Lemma E.1.2.19, we observe that the Hausdorff assumption in both part (1) and part (2) of the Lemma cannot be dropped altogether. We refer the reader to Lemma E.2.4.9, for a counter-example, given by the one-point compactification of the rationals endowed with their subspace topology inherited from the Euclidean topology on $\mathbf{R}$. We refer the reader to [**StSe**, Part II, 35]. This is an example of a compact topological space (in fact, even weakly Hausdorff, as will be discussed in Lemma E.2.4.9; we refer the reader to Definition E.2.1.1 for the weak Hausdorff separation condition and to §E.2 for the related material) such that its compact topological self-product is not compactly generated.

Another class of topological spaces that can be seen to be compactly generated, is that of sequential spaces. We recall that a topological space X is *sequential* if and only if any sequentially closed subset $Z \subset X$ (i.e. such that for any sequence $\{z_n\}_{n \geq 0} \subset Z$ converging to a point $x \in X$, we have $x \in Z$) is closed in Z (see [**Fr**, Prop. 1.1] and the definition on [**Fr**, §1, p. 109] before [**Fr**, Prop. 1.2]). Examples of sequential spaces are all topological spaces satisfying the first axiom of countability. In fact, any sequential space receives a submersive map from a first-countable topological space (see [**Fr**, Cor. 1.14]).

We have the following:

PROPOSITION E.1.2.21. *We have:*

(1) *all sequential topological spaces are compactly generated,*
(2) *the product in* **Spaces** *of two first-countable topological spaces is compactly generated and computes the product in* **CG**.

PROOF. We prove (1). We let X be a sequential space, and $Z \subset X$ a k-closed subset . We are to show that Z is sequentially closed, hence closed, in X. Let $\{z_n\}_{n \geq 0} \subset Z$ be a sequence converging to a point $x \in X$. We endow $\mathbf{N}$ with the discrete topology, and call K the one-point compactification of $\mathbf{N}$. This K is compact Hausdorff, and we may define the set-function $g : K \to X$ by assigning $g(n) := z_n$ for all integers $n \geq 0$, and $g(\infty) := x$. Since the sequence $\{z_n\}_{n \geq 0}$ is convergent to x, g is continuous. It follows that $g^{-1}(Z)$ is closed in K. Since $\{z_n\}_{\geq 0}$ is contained in Z, we deduce the containment $\mathbf{N} \subset g^{-1}(Z)$. As $\mathbf{N} \subset K$ is dense and $g^{-1}(Z)$ is closed in K, we deduce that we have $K = g^{-1}(Z)$, and hence $\infty \in g^{-1}(Z)$. It follows $x \in Z$, and Z is sequentially closed, hence closed in X. By part (3) of Lemma E.1.1.3, X is compactly generated, proving (1).

[4] The compact generatedness has to be justified; was not explained.

Part (2) follows from the fact that, for first-countable spaces X, Y, and any $(x, y) \in X \times_0 Y$, the set of open subspaces $U \times_0 V$ as U ranges over the elements of a countable base at x, and V ranges over the elements of a countable local base at y, is a countable base at (x, y), therefore proving that $X \times_0 Y$ is first-countable. As first-countable topological spaces are sequential (see part (2) of [**Fr**, Cor. 1.14]), part (1) applies and implies that $X \times_0 Y$ is compactly generated. It follows that we have $k(X \times_0 Y) = X \times_0 Y$, and hence $X \times_0 Y$ computes in **Spaces** the product $X \times Y$ in **CG** by part (1) of Lemma E.1.2.2. □

Remark E.1.2.22. Another way to see that a sequential space X is compactly generated is to directly exhibit a submersive map from a locally compact Hausdorff space. To do this, we consider the set I of all convergent sequences in X. For $i \in I$, we denote by $\mathbf{x}(i)$ the associated sequence $\mathbf{x}(i) = \{x_n(i), n \geq 0\} \subset X$. We call K the one-point compactification of $\mathbf{N}$ (with $\mathbf{N}$ endowed with the discrete topology), we define $K_i := K$ for every $i \in I$, and we define:

$$q : \coprod_{i \in I} K_i \to X$$

by sending any non-negative integer $n \in K_i$, to $x_n(i)$ and $\infty \in K_i$ to a choice of limit $\lim_{n \to \infty} x_n(i)$ in X. As every point of X is the limit of a constant sequence in X, q is surjective. The argument at the beginning of the proof of Proposition E.1.2.21 shows that a subspace $Z \subset X$ is closed if and only if $q^{-1}(Z)$ is, and hence q is submersive.

Remark E.1.2.23. We note that the topological product of two sequential spaces may not be sequential. We refer the reader to Example E.1.2.24 below for an example. Another example exhibiting this phenomenon can be found in [**ABM**, §2] (see, specifically, [**ABM**, §2, Property IV]). This is the reason for the stronger assumption in part (2) of Proposition E.1.2.21.

Example E.1.2.24. As in Example E.1.2.18, we let $X := \mathbf{Q}$ be endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$, and we let X' be the quotient of X obtained by identifying the subset of all integers $\mathbf{Z} \subset \mathbf{Q}$ to $\{0\}$. The product space $X' \times_0 X$ is an example of:

(1) a product of compactly generated spaces which is not compactly generated,
(2) a product of sequential spaces which is not sequential (see [**Fr**, §1, p. 109] before [**Fr**, Prop. 1.2]).

Indeed, we note that $X \times_0 X$ is compactly generated by Lemma E.1.2.21. Moreover, Lemma E.1.2.17 implies that $X' \times_0 X$ is not compactly generated: if it were then the topological product map (f, id) would be submersive by the Lemma, contradicting Example E.1.2.18.

We prove the following basic permanence properties for the condition of compact generation for topological spaces.

Lemma E.1.2.25. *Let X be a compactly generated topological space, and $T \subset X$ be a subspace. Assume that one of the following conditions holds:*

(1) *T is closed in X with respect to the subspace topology.*
(2) *T is open in X with respect to the subspace topology.*
(3) *T is locally closed in X with respect to the subspace topology.*

Then T is compactly generated.

The Lemma is also proved in [**Str**, Lemma 2.26].

Proof. By Lemma E.1.1.3, there exists a set I, compact Hausdorff topological spaces K_i for every $i \in I$, and a submersive map:

$$q : \coprod_{i \in I} K_i \to X.$$

If T is closed, then $q' : q^{-1}(T) \to T$ is a submersive map. As $q^{-1}(T)$ is a closed subspace of $\coprod_{i\in I} K_i$, it is of the form $\coprod_{i\in I} C_i$ for closed subspaces $C_i \subseteq K_i$. As each K_i is compact Hausdorff, so is each C_i, so T is compactly generated, proving (1). If T is open in X, then $q' : q^{-1}(U) \to U$ is also a submersive map and since $q^{-1}(U) \subseteq \coprod_{i\in I} K_i$ is open, it is locally compact Hausdorff. As every locally compact Hausdorff topological space is compactly generated by Proposition E.1.2.19, proving part (2) is now accomplished as follows. As $q^{-1}(U)$ is compactly generated, we may find, by part (4) in Lemma E.1.1.3, a set I, for every $i \in I$, a compact Hausdorff space K'_i, and a submersive map $p : \coprod_{i\in I} K'_i \to q^{-1}(U)$. Since $q' \circ p$ is submersive for $q' = q|_{q^{-1}(U)} : q^{-1}(U) \to U$, Lemma E.1.1.3 implies that U is compactly generated, as desired. Part (3) follows by combining (1) and (2). □

We complement the above result with a stability result for closed immersions under products, which we can now prove upon combining Lemma E.1.2.17 and the above Lemma E.1.2.25:

LEMMA E.1.2.26. *Let X, Y, X', Y' be compactly generated spaces, and $f : X \to X'$, $g : Y \to Y'$ closed immersions. Then:*

$$(f, g) : X \times Y \to X' \times Y'$$

is a closed immersion.

PROOF. We may factor (f, g) as $(f, \mathrm{id}) : X \times Y \to X' \times Y$, followed by $(\mathrm{id}, g) : X' \times Y \to X' \times Y'$. It is therefore enough to treat the case $Y = Y'$ and $g = \mathrm{id}$. Thus, we are reduced to the case f is the inclusion of a closed subspace of X'. The subspace $X \times_0 Y \subset X' \times_0 Y$ is closed with respect to the subspace topology. By part (2) of Remark E.1.1.8, $X \times_0 Y$ is closed with respect to the subspace topology of the k-topology on $k(X' \times_0 Y) = X' \times Y$ as well, and by part (2) of Lemma E.1.2.25, $X \times_0 Y$ is compactly generated and hence computes $X \times Y$, exhibiting $X \times Y$ as a closed subspace of $X' \times Y$, as desired. □

Here is a criterion to detect immersions of compactly generated spaces, for later use:

LEMMA E.1.2.27. *Let X, Y be compactly generated topological spaces, and $f : X \to Y$ a continuous injective map. The following are equivalent:*

(1) *f is an immersion.*
(2) *For any compactly generated topological space T and any set-function $g : T \to X$, g is continuous if and only if the composition $f \circ g$ is continuous.*

PROOF. That (1) ⇒ (2) is obvious. Conversely, suppose for all continuous $g : T \to X$ that $f \circ g$ is continuous. We will deduce (1). Let $i : X \to f(X)$ be the continuous bijective map induced by f, with $f(X)$ being endowed with the subspace topology from the topology of Y. We call $i' : f(X) \to Y$ the immersion, and we have $i' \circ i = f$ and $f \circ i^{-1} = i'$. By the foregoing (especially (2)), both i and i^{-1} are continuous. Hence f is an immersion. □

As an application of Lemma E.1.2.27, we conclude this subsection by determining when the continuous push-forward between function spaces along an immersion is an immersion.

LEMMA E.1.2.28. *Let X, Y be compactly generated spaces, $i : Z \to Y$ an immersion. Then the induced map*

$$i_* : \mathscr{C}ont_k(X, Z) \to \mathscr{C}ont_k(X, Y)$$

is an immersion. If i is a closed immersion then i_* is a closed immersion.

PROOF. Let $g : T \to \mathscr{C}ont_k(X, Z)$ be an arbitrary set-function from a compactly generated topological space T such that $i_* \circ g$ is continuous. By Lemma E.1.2.12, the associated set-map $h : T \times X \to Y$ is continuous, and the associated set-map corresponding to g satisfies $i \circ h' = h$. As i is an immersion, g is continuous. By Lemma E.1.2.27, we deduce that i_* is an immersion.

We now assume that i is a closed immersion. We may assume $Z = i(Z)$. We now observe that in this case we have:

$$\mathscr{C}ont_k(X,Z) = \bigcap_{x\in X} (\mathscr{C}ont_k(X,Y) - N(i_x, \{x\}, Y - Z))$$

with $i_x : \{x\} \to X$ the inclusion in X of the point x, which is an infinite intersection of closed subspaces, hence closed in $\mathscr{C}ont_k(X,Y)$. As the image of i_* is, therefore, closed, i_* is a closed immersion. □

E.2. Compactly generated weakly Hausdorff spaces

In this section we introduce a weakening of the Hausdorff separation condition, and discuss its effects on compactly generated spaces. Specifically, we shall establish improved closure properties for the resulting full subcategory of **CG**. Sources for this section are [**LGL**, §3], [**McC**, §2], [**Rezk**, §4] and [**Str**, §1].

E.2.1. Preparations on compactly generated weakly Hausdorff spaces. We begin with a definition, introducing the weak Hausdorff separation condition:

Definition E.2.1.1. A topological space X is *weakly Hausdorff* if and only if for every compact Hausdorff topological space K and continuous map $f : K \to X$, $f(K)$ is a closed subspace of X.

Remark E.2.1.2. Clearly subspaces of weakly Hausdorff spaces are weakly Hausdorff.

Remark E.2.1.3. We recall that if X is Hausdorff then any compact subspace is closed. In particular, Hausdorff spaces are weakly Hausdorff. For an example of a weakly Hausdorff (compact) topological space that is not Hausdorff, we refer the reader to Lemma E.2.4.9.

Lemma E.2.1.4. *Let K be a compact weakly Hausdorff topological space. Assume that there exists a compact Hausdorff topological space K' and a surjective continuous map $q : K' \to K$. Then K is Hausdorff.*

Proof. Since K is weakly Hausdorff, every point of K is closed and so q has compact fibers over K. Indeed, for every $x \in K$, $q^{-1}(\{x\})$ is closed in K', and this latter is a compact Hausdorff topological space. It follows $q^{-1}(\{x\})$ is compact. It is moreover Hausdorff. For any $C \subset K'$ closed, C is compact Hausdorff in K' and so $q(C)$ is compact in K, and closed since K is weakly Hausdorff by assumption. We deduce that q is a closed map, and so q is submersive. Let $x, x' \in K$ be distinct points. Since K is weakly Hausdorff, its points are closed. As K' is normal and $q^{-1}(\{x\})$ and $q^{-1}(\{x'\})$ are closed, there exist disjoint open subspaces $U, U' \subset K'$ such that $q^{-1}(\{x\}) \subset U$ and $q^{-1}(\{x'\}) \subset U'$. The subspaces of K' given by $K' - U$ and $K' - U'$ are closed, and as q is a closed map, then $C := q(K' - U)$ and $C' := q(K' - U')$ are closed. It follows that $K - C$ and $K - C'$ are disjoint open subspaces of K containing x and x' respectively. We deduce that K is Hausdorff. □

Lemma E.2.1.5. *If X is a weakly Hausdorff topological space then so is $k(X)$.*

Proof. Let K be a compact Hausdorff topological space, and $g' : K \to k(X)$ a continuous map. Compose with the continuous bijection $k(X) \to K$ to define a continuous $g : K \to X$. Since X is weakly Hausdorff, $g(K)$ is closed in X. By continuity of $k(X) \to X$, it follows that $g'(K)$ is closed in $k(X)$. We deduce $k(X)$ is weakly Hausdorff. □

Lemma E.2.1.6. *Let X be a topological space. We consider the following conditions:*

(1) *X is weakly Hausdorff,*
(2) *the image $\Delta \subset X \times_0 X$ of the diagonal morphism $\Delta_X : X \to X \times_0 X$, is k-closed,*
(3) *for every compact Hausdorff space K and every continuous map $g : K \to X$, the topological fibre product $K \times^0_{g,X,g} K$ is closed in $K \times_0 K$.*

Then (1) ⇒ (2) and (2) ⇔ (3). Moreover, if X is compactly generated then (1) ⇔ (2).

We refer the reader to [**Str**, Prop. 2.14] and to [**Rezk**, Prop. 4.2]. We further refer the reader to Lemma E.2.4.9, exhibiting an example of a weakly Hausdorff and compactly generated space X whose topological diagonal subspace is *not* closed.

PROOF. Suppose X is weakly Hausdorff. For any compact Hausdorff topological space K and continuous map $g : K \to X \times_0 X$, we call $g_1, g_2 : K \rightrightarrows X$ the composition of g with the first and second projection, respectively. We claim that

$$g^{-1}(\Delta) = \{x \in K \mid g_1(x) = g_2(x)\} \tag{E.2.1.1}$$

is closed. As $g_1(K)$ and $g_2(K)$ are compact subspaces in X, since X is weakly Hausdorff their union $C := g_1(K) \cup g_2(K)$ is therefore closed in X, and $g(K)$ is contained in $C \times_0 C \subset X \times_0 X$. We call D the image of the topological diagonal $\Delta_C : C \to C \times_0 C$. Clearly

$$g^{-1}(D) = g^{-1}(\Delta) \tag{E.2.1.2}$$

as subspaces of K. Since $K \coprod K$ is compact and Hausdorff, defining $h : K \coprod K \to C$ to be g_1 on the first copy of K and g_2 on the second makes h a continuous surjection from a compact Hausdorff space. By Lemma E.2.1.4, C is compact Hausdorff. Thus, D is a closed subspace of C, so by (E.2.1.2) the subspace $g^{-1}(\Delta)$ of K from (E.2.1.1) is a closed subspace of K since g is continuous. This proves the claim, which in turn proves that Δ is k-closed in $X \times_0 X$, establishing (2) from (1).

We now assume that (2) holds, and prove (3). Let K be a compact Hausdorff space, $g : K \to X$ a continuous map. Then we have:

$$K \times^0_{g,X,g} K = (g, g)^{-1}(\Delta) \subset K \times_0 K \tag{E.2.1.3}$$

as subspaces of $K \times_0 K$. Since Δ is k-closed in $X \times_0 X$ and $(g, g) : K \times_0 K \to X \times_0 X$ is continuous with $K \times_0 K$ compact Hausdorff (hence its topology is its k-topology), we see $(g, g)^{-1}(\Delta)$ is closed, which proves (3).

We now assume (3) and prove that (2) holds. Let K be a compact Hausdorff space and $g : K \to X \times_0 X$ a continuous map. Upon denoting g_1 and g_2 the compositions of g with the first, respectively the second, projections $p_1, p_2 : X \times_0 X \rightrightarrows X$, the map g factors as a composition of the continuous maps $\Delta_K : K \to K \times_0 K$, and $(g_1, g_2) : K \times_0 K \to X \times_0 X$. We need to show that $(g_1, g_2)^{-1}(\Delta)$ is a closed subspace of $K \times_0 K$. Let $K' := K \coprod K$ and define $h : K' \to X$ to be g_1 on the first copy of K, and g_2 on the second copy of K. Clearly K' is compact and Hausdorff. Then by (3) we have $K' \times^0_{h,X,h} K'$ is closed in $K' \times_0 K'$. Clearly $K \times^0_{g_1,X,g_2} K$ splits off as an open and closed subspace of $K' \times^0_{h,X,h} K'$, and as such recovers its natural embedding into $K \times_0 K$ as an open and closed subspace of $K' \times_0 K'$. We conclude that $K \times^0_{g_1,X,g_2} K$ is closed in $K \times_0 K$. We have:

$$(g_1, g_2)^{-1}(\Delta) = K \times^0_{g_1,X,g_2} K$$

as subspaces of $K \times_0 K$, and so indeed $(g_1, g_2)^{-1}(\Delta)$ is closed in $K \times_0 K$.

It remains to check that (2) implies (1) when X is compactly generated. Suppose K is a compact Hausdorff space and $g : K \to X$ a continuous map. Then for any compact Hausdorff space K' and $g' : K' \to X$ any continuous map,

$$(g')^{-1}(g(K)) = p_2((g, g')^{-1}(\Delta))$$

as subspaces of K' where $p_2 : K \times_0 K' \to K'$ is the second projection. As Δ is k-closed by (2), $(g, g')^{-1}(\Delta) \subset K \times_0 K'$ is closed, and its image in K' under p_2 is closed, as p_2 is a closed map by [**SP**, Tag 005P]. This proves that $g(K)$ is k-closed in X. As X is compactly generated, $g(K)$ is closed by Remark E.1.1.8. □

REMARK E.2.1.7. We note that part (2) of Lemma E.2.1.6 suggests a weakening of the notion of weakly Hausdorff space. Spaces whose diagonal subspace is k-closed are called *k-Hausdorff*. Weakly Hausdorff and k-Hausdorff are not equivalent concepts (see Example E.2.1.10). However, they are for compactly generated

spaces, which is the content of the last assertion in Lemma E.2.1.6. We refer the reader to [**Rezk**, §4] for background material on k-Hausdorff spaces and how it relates to compactly generated weakly Hausdorff spaces. Throughout this appendix, we limit ourselves to the weak Hausdorff property, as it is equivalent to the correct separation condition to impose on compactly generated spaces (namely, the k-Hausdorff separation condition) to obtain the desired improvements to the category **CG**, and because it can be directly compared to other separation axioms. It is stronger than T_1 (see Example E.2.1.8) and weaker than T_2 (see Example E.2.1.9 below, which can be arranged to be compactly generated: see Lemma E.2.4.5). As a separation axiom, it is sometimes referred to as t_2.

Example E.2.1.8. Let X be an infinite set endowed with the co-finite topology. Then X is T_1, but not weakly Hausdorff (it violates Lemma E.2.1.6(2)).

Example E.2.1.9. Let Y be any Hausdorff space that is *not* locally compact. Then the one-point compactification $X := Y^+$ is an example of a weakly Hausdorff topological space that is not Hausdorff. Examples are [**StSe**, Part II, 35], which is also Example E.2.4.4, or a product of an uncountable number of copies of $\mathbf{R}$, respectively of $[0,1]$. Another example is [**StSe**, Part II, 99].

Example E.2.1.10. We let X be the one-point compactification of the Arens-Fort space (see [**StSe**, Part II, 26]). Then X is not weakly Hausdorff. Indeed, we let $(0,0)$ be the origin in the Arens-Fort space, the unique non-isolated point in the Arens-Fort space. Then $X - \{(0,0)\}$ is homeomorphic to the one-point compactification of a discrete space (see [**StSe**, Part II, 26,(8)]) and it is compact Hausdorff. It is, however, non-closed in X. The space X is, nonetheless, k-Hausdorff. This is [**PB**, Space S000165].

The following lemma concludes the preparations:

Lemma E.2.1.11. *Let X be a compactly generated space, and $R \subset X \times X$ an equivalence relation. Then the following conditions are equivalent:*

(1) *the quotient space X/R is weakly Hausdorff,*
(2) *R is a closed subspace of $X \times X$.*

We refer the reader to [**Str**, Prop. 2.17, Prop. 2.20], and [**Str**, Cor. 2.21].

Proof. We call $\overline{\Delta}$ the diagonal subspace in $(X/R) \times_0 (X/R)$. Since X/R inherits compact generatedness from X, the map $q : X \times X \to (X/R) \times (X/R)$ is a submersive map by Lemma E.1.2.17 applied to $X = Y$, $X' = Y' = X/R$, and $f = g$ chosen to be the quotient map $X \to X/R$. As $q^{-1}(\overline{\Delta}) = R \subset X \times X$, we see that (2) is equivalent to the statement that $\overline{\Delta}$ is closed in $(X/R) \times (X/R)$, since q is a submersive map. This latter assertion is in turn equivalent to $\overline{\Delta}$ being k-closed in $(X/R) \times_0 (X/R)$. By the equivalence of part (1) and part (2) of Lemma E.2.1.6, which holds since X is compactly generated, we conclude. □

Example E.2.1.12. As a simple example of a non-closed $R \subset X \times X$ resulting in a non-weakly Hausdorff (compactly generated) quotient, we take X to be the real line $\mathbf{R}$ endowed with its Euclidean topology, and R to be the subset of $\mathbf{R} \times_0 \mathbf{R}$ of those elements (x, y) such that $x = 0$ if and only if $y = 0$, and if $y \neq 0$ then $x/y \in \mathbf{Q}^\times$. Then R is not closed in the compactly generated Hausdorff $\mathbf{R} \times_0 \mathbf{R}$. We note that $\mathbf{R} \times_0 \mathbf{R} = \mathbf{R} \times \mathbf{R}$ (Proposition E.1.2.19(2)). The quotient X/R is homeomorphic to the quotient of $\mathbf{R}$ by the left-multiplication action by $\mathbf{Q}^\times$, with $\mathbf{Q}^\times$ being endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$. This quotient is then not weakly Hausdorff.

The following important structural result holds specifically for compactly generated weakly Hausdorff spaces.

Proposition E.2.1.13. *Let X be a compactly generated topological space. The following conditions are equivalent:*

(1) *There exists a subset I of the set of all closed compact Hausdorff* **subspaces** *of X such that upon denoting by $i_K : K \to X$ the closed immersion of K in X for each $K \in I$, the continuous map*

$$q := \coprod_{K\in I} i_K : \coprod_{K\in I} K \to X$$

is a submersive map.

(2) *X is weakly Hausdorff.*

Remark E.2.1.14. By Lemma E.1.1.3(4), condition (1) above characterizes the combined property "compactly generated and weakly Hausdorff" *without* needing to assume compact generatedness.

Proof. We assume (1). Upon calling $Y := \coprod_{K\in I} K$, then $Y \times^0_{q,X,q} Y = \coprod_{(K,K')\in I\times I} K \times^0_{i_K,X,i_{K'}} K'$. We note that as subspaces of $K \times_0 K'$,

$$K \times^0_{i_K,X,i_{K'}} K' = (i_K, i_{K'})^{-1}(\Delta_X(X))$$

and since $(i_K, i_{K'}) : K \times_0 K' \to X \times_0 X$ is a closed immersion (as $K, K' \subset X$ are closed by hypothesis) $K \times^0_{i_K,X,i_{K'}} K'$ is a closed subspace of $\Delta_X(X)$. As the topological diagonal Δ_X is an immersion, it is a homeomorphism of X onto the diagonal subspace in $X \times_0 X$. We call $\delta : X \to \Delta_X(X)$ the homeomorphism induced by the topological diagonal morphism Δ_X. Under δ^{-1}, $K \times^0_{i_K,X,i_{K'}} K'$ is mapped homeomorphically onto $K \cap K'$ as a subspace of X. As both K and K' are closed in X, $K \cap K'$ is a closed subspace of X, and hence a closed subspace of both K and K' by [**Bou-GT**, Ch. I, §3, Prop. 1]. As K and K' are compact Hausdorff, so is $K \cap K'$. It follows that $K \times^0_{i_K,X,i_{K'}} K'$ is compact Hausdorff, and the immersion

$$K \times^0_{i_K,X,i_{K'}} K' \to K \times_0 K'$$

is closed. We deduce that $Y \times^0_{q,X,q} Y$ is a closed subspace of

$$Y \times_0 Y = \coprod_{(K,K')\in I\times I} (K \times_0 K').$$

Since Y is locally compact Hausdorff, Lemma E.1.2.19 implies that $k(Y \times_0 Y) = Y \times_0 Y$; since $Y \times^0_{q,X,q} Y$ is a closed subspace of $Y \times_0 Y$, it is compactly generated by Lemma E.1.2.25(1). By Lemma E.2.1.11, the quotient of Y by the closed equivalence relation $R := Y \times^0_{q,X,q} Y = Y \times Y$ is weakly Hausdorff. By Lemma E.1.2.14, such a quotient is homeomorphic to X. This proves (2).

Conversely, we assume that (2) holds. We let I be the set of all closed compact Hausdorff subspaces of X. We call:

$$q := \coprod_{K\in I} i_K : Y := \coprod_{K\in I} K \to X \tag{E.2.1.4}$$

the continuous topological co-product of the closed immersions $i_K : K \to X$. We claim that q from (E.2.1.4) is a submersive map. By Lemma E.1.1.3(4), there exists a set J of compact Hausdorff spaces, and continuous maps $g_K : K \to X$ for every $K \in J$, such that the induced continuous map

$$q' : \coprod_{K\in J} g_K : Y' := \coprod_{K\in J} K \to X \tag{E.2.1.5}$$

is submersive. As $g_K(K)$ endowed with the subspace topology is weakly Hausdorff since X is, and it receives a continuous surjection from the compact Hausdorff space K, it is itself compact Hausdorff by Lemma E.2.1.4. We define the set-function $J \to I$ by sending $K \in J$ to $g_K(K) \in I$. This induces a continuous map

$$g : \coprod_{K\in J} K \to \coprod_{K\in I} K$$

defined as the identity on every $K \in J$, and such that $q \circ g = q'$. As q' is submersive, so is q from (E.2.1.4). □

In light of Proposition E.2.1.13, we see that compactly generated weakly Hausdorff spaces are closely related to locally compact Hausdorff spaces. As for general compactly generated spaces, they are quotients of locally compact Hausdorff spaces, and the Proposition shows that the weak Hausdorff condition on an object X means that we may choose, as the set S_X from Lemma E.1.1.3 of those compact Hausdorff spaces that compactly generate X, the set of compact Hausdorff *subspaces*. We can further characterize those objects of **CGWH** that are locally compact Hausdorff by rephrasing the locally compact condition on X as a stronger smallness condition on S_X.

We need the following:

LEMMA E.2.1.15. *Let $\{X_n\}_{n\geq 0}$ be a directed sequence of topological spaces. Assume:*

(1) *for every integer $n \geq 0$, X_n is a T_1-space;*
(2) *for $X = \mathrm{colim}_{n\geq 0} X_n$ in* **Spaces**, *the canonical maps $f_n : X_n \to X$ have finite fibres.*

Then for every compact subspace $K \subset X$, there exists an integer $n \geq 0$ such that K is contained in the image of X_n.

We also refer the reader to [**VLH**, Lemma 2.4].

PROOF. [5] We let $K \subset X$ be a compact subspace. For every integer $n \geq 1$ such that $(f_n(X_n) - f_{n-1}(X_{n-1})) \cap K \neq \varnothing$, we choose x_n in the intersection. The sequence $F := \{x_n\}_{n\geq 0}$ is such that for every subset $E \subset F$, $E \cap f_n(X_n)$ is a finite[6] set for every $n \geq 0$. As $f : X_n \to f_n(X_n)$ has finite fibres, $f_n^{-1}(E)$ is a finite set, hence closed in X_n since X_n is T_1. By design of the topology on X, E is closed as a subspace of X. Since E was an arbitrary subset of F, we deduce that the subspace topology on F inherited from the topology of X is the discrete topology. As K is a subspace of X and F is contained in K, F is discrete as a subspace of K. Since K is compact, F is finite[7] and therefore entirely contained in $f_N(X_N)$ for large enough $N \geq 0$. In particular, we have $(X - f_N(X_N)) \cap K = \varnothing$, and hence $K \subset f_N(X_N)$, as desired. □

We recall that a topological space is *hemicompact* if the set of all of its compact subspaces has a countable co-final subset.

LEMMA E.2.1.16. *Let X be a topological space. Assume:*

(1) *X is hemicompact.*
(2) *X is first-countable.*

Then X is locally compact.

PROOF. We let $\{K_n\}_{n\geq 0}$ be a sequence of compact subspaces such that every compact subspace $K \subset X$ is contained in K_n for some integer $n \geq 0$. Let $x \in X$ be a point, and $\{U_m\}_{m\geq 0}$ a countable local base at x. We may and do assume that $U_{m+1} \subseteq U_m$ holds for every $m \geq 0$. Suppose that for every integer $n \geq 0$, $(X - K_n) \cap U_n \neq \varnothing$. We choose x_n in the intersection and consider the union of the sequence and its limit point $F := \{x_n\}_{n\geq 0} \cup \{x\}$. Then $F \subset X$ is compact, and hence contained in K_N, for some integer $N \geq 0$. However, since $x_N \in F$ is an element of U_N, it cannot be an element of K_N, a contradiction. It follows that there exists an integer $n \geq 0$ such that U_n is contained in K_n, and hence K_n is a compact neighborhood of x in X. As $x \in X$ is arbitrary, X is locally compact. □

COROLLARY E.2.1.17. *Let X be a weakly Hausdorff space. If X is first-countable and countably compactly generated*[8] *then it is locally compact.*

[5] This proof needs correction; either make a simple fix or give a literature reference for a correct proof.
[6] why?
[7] False: $\{1/2^n\}_{n\geq 1}$ as a subspace of $[0,1]$ has the discrete topology but is infinite
[8] Need to remind the reader what "countably compactly generated" means

PROOF. By Proposition E.2.1.13, we may find a rising sequence $\{K_n\}_{n\geq 0}$ of compact Hausdorff subspaces such that $X = \mathrm{colim}_{n\geq 0} K_n$. By Lemma E.2.1.15, X is hemicompact. By Lemma E.2.1.16, it is locally compact. □

E.2.2. Categorical and topological stabilities of compactly generated weakly Hausdorff spaces. We now define the category of compactly generated weakly Hausdorff spaces, as a full subcategory of **CG**, the category of compactly generated spaces.

DEFINITION E.2.2.1. We call **CGWH** the category whose objects are compactly generated weakly Hausdorff topological spaces, and morphisms are continuous maps of topological spaces. We refer to **CGWH** as the category of compactly generated weakly Hausdorff spaces.

Our aim is to now collect results that show that the full subcategory **CGWH** of **CG** retains many of the good properties of **CG**, such as closure under small limits and small colimits, and the cartesian-closed monoidal structure, and has additional features that will become convenient later. One disadvantage of **CGWH** is that small colimits in **CGWH** cannot be computed in **Spaces**. In other words, the forgetful functor to **Spaces** does not reflect small colimits. However, all the small colimits of interest for the purposes of this appendix will indeed be reflected by the forgetful functor, and can be computed by their topological counterpart. One important advantage of **CGWH** is that submersive maps will turn out to be exactly the universally effective epimorphisms of **CGWH**, therefore providing a direct and concrete description of the canonical topology on **CGWH**, and rendering the development of a theory of stacks in groupoids over a site of compactly generated weakly Hausdorff spaces efficient.

In this subsection, we study analogs for **CGWH** of the closure properties of **CG**. We begin by comparing the two categories. We consider the forgetful functor:

$$u_{\mathrm{wh}} : \mathbf{CGWH} \to \mathbf{CG} \tag{E.2.2.1}$$

sending a compactly generated weakly Hausdorff topological space to itself as a compactly generated space, and sending continuous maps to continuous maps. We further define:

$$u_{\mathrm{cgwh}} : \mathbf{CGWH} \to \mathbf{Spaces} \tag{E.2.2.2}$$

to be the forgetful functor sending a compactly generated weakly Hausdorff space to itself as an ordinary topological space, and sending continuous maps to continuous maps. Since $u_{\mathrm{cg}} \circ u_{\mathrm{wh}} = u_{\mathrm{cgwh}}$, the notation being used should be un-ambiguous.

We now discuss the existence of a left-adjoint

$$h : \mathbf{CG} \to \mathbf{CGWH}$$

to the forgetful functor. In other words, $h(X) \in \mathbf{CGWH}$ is an initial object receiving a continuous map from $X \in \mathbf{CG}$. Its construction and properties are briefly recalled below. (We refer the reader to [**Str**, Prop. 2.22].)

DEFINITION E.2.2.2. Let X be a compactly generated space. We define:

$$R_X \subset X \times X \tag{E.2.2.3}$$

to be the intersection of all closed equivalence relations on X (for the k-topology on $X \times X$). We define the topological space:

$$h(X) := X/R_X \tag{E.2.2.4}$$

endowed with the quotient topology. We call:

$$q_X : X \to h(X) \tag{E.2.2.5}$$

the quotient map.

REMARK E.2.2.3. For X a compactly generated space, since the equivalence relation $R_X \subset X \times X$ from (E.2.2.3) is *closed*, the quotient space $h(X)$ is compactly generated weakly Hausdorff by Lemma E.2.1.11.

We now check that formation of R_X from (E.2.2.3) is natural in X, and therefore $h(X)$ defines a functor:

$$h : \mathbf{CG} \to \mathbf{CGWH}. \tag{E.2.2.6}$$

We begin by proving an analog of Lemma E.1.2.1; see [**Str**, Prop. 2.22] and [**Str**, Cor. 2.23].

LEMMA E.2.2.4. *Let X be a compactly generated space, and $\Sigma \subset 2^{X\times X}$ the set of all subspaces $R \subset X \times X$ such that:*

(a) *R^{set} is an equivalence relation on $X^{\mathrm{set}} \times X^{\mathrm{set}}$.*
(b) *R is closed in $X \times X$.*

Then, upon calling:

$$R_X := \bigcap_{R\in\Sigma} R \tag{E.2.2.7}$$

we have that R_X^{set} is an equivalence relation on $X^{\mathrm{set}} \times X^{\mathrm{set}}$, and R_X is closed in $X \times X$. Moreover:

(1) *formation of $h(X)$ is natural in X,*
(2) *the functor $h : \mathbf{CG} \to \mathbf{CGWH}$ is left-adjoint to the forgetful functor $u_{\mathrm{wh}} : \mathbf{CGWH} \to \mathbf{CG}$,*
(3) *the natural map $\varepsilon_X : X \to h(X)$ is an isomorphism when X is also weakly Hausdorff,*
(4) *applying h to the natural map $Y \to u_{\mathrm{wh}}(h(Y))$ for $Y \in \mathbf{CG}$ yields $\varepsilon_{h(Y)}$ and so is an isomorphism.*

PROOF. We note that, as $X \times X$ is closed in itself, Σ is non-empty. The set-theoretic intersection of equivalence relations on a set is an equivalence relation, so R_X is a closed equivalence relation in $X \times X$. We now check part (1) and part (2).

Let Y be a compactly generated weakly Hausdorff space, and $f : X \to Y$ a morphism in $\mathbf{CG}$. We claim that $X \times_{f,Y,f} X$ is closed in $X \times X$. Indeed, we have:

$$X \times^0_{f,Y,f} X = (f, f)^{-1}(\Delta_Y(Y)) \tag{E.2.2.8}$$

and $\Delta_Y(Y) \subset Y \times_0 Y$ is k-closed by part (2) of Lemma E.2.1.6. As f is continuous, $(f, f)^{-1}(\Delta_Y(Y))$ is k-closed by functoriality of the k-topology (see Lemma E.1.2.1). Moreover, $(f, f)^{-1}(\Delta_Y(Y))$ is an equivalence relation on $X \times_0 X$, since $\Delta_Y(Y)$ is in Y. From (E.2.2.8) we deduce that

$$X \times_{f,Y,f} X = k(X \times^0_{f,Y,f} X) = k((f, f)^{-1}(\Delta_Y(Y)))$$

is closed in $X \times X$, since $(f, f)^{-1}(\Delta_Y(Y))$ is k-closed, as seen in the foregoing. It follows that we have the containment $R_X \subset X \times_{f,Y,f} X$, which gives a unique map:

$$\psi : h(X) \to \mathrm{coeq}(p_1, p_1 : X \times_{f,Y,f} X \rightrightarrows X) \tag{E.2.2.9}$$

with p_1, p_2 the two projections, the co-equalizer being formed in $\mathbf{CG}$, and such that $\psi \circ q_X$ agrees with the canonical submersive map:

$$f' : X \to \mathrm{coeq}(p_1, p_2) \tag{E.2.2.10}$$

that $\mathrm{coeq}(p_1, p_2)$ receives from X.

The space Y further receives a unique map ϕ from $\mathrm{coeq}(p_1, p_1)$ such that the submersive map from (E.2.2.10) satisfies

$$q \circ f' = f \tag{E.2.2.11}$$

In light of (E.2.2.11) and since the map ψ from (E.2.2.9) satisfies $\psi \circ q_X = f'$, we deduce:

$$q \circ \psi \circ q_X = f$$

which means that f factors through q_X. We further observe that both q and ψ are unique: q by universality of topological quotients and Lemma E.1.2.14, and ψ by the universal property of co-equalizers and the definition of the map f' of (E.2.2.10). We have shown that for any compactly generated spaces X, Y with Y weakly Hausdorff, and any continuous map $f : X \to Y$, f factors uniquely through $q_X : X \to h(X)$ from (E.2.2.5). This proves functoriality of $h(\cdot)$ and part (2). Part (3) is clear from the universal property, and (4) is part of the formalism of adjunction. □

LEMMA E.2.2.5. *The category* **CGWH** *contains all small limits and all small colimits. Moreover, for any small category I and diagram $P : I \to$* **CGWH***:*

(1) $\lim_I P$ *in* **CGWH** *is computed in* **CG**, *i.e. the forgetful functor u_{wh} reflects and commutes with small limits.*
(2) $\mathrm{colim}_I P$ *in* **CGWH** *is computed as:*

$$\mathrm{colim}_I P = h(\mathrm{colim}_I u_{\mathrm{wh}} \circ P) \tag{E.2.2.12}$$

upon evaluating $h(\cdot)$ on the colimit of the small diagram $u_{\mathrm{wh}} \circ P$ in **CG**.

PROOF. Assertion (2) is immediate from h being a left-adjoint to u_{wh}. For (1), we just have to show that the limit in **CG** of the diagram $u_{\mathrm{wh}} \circ P$ is weakly Hausdorff. To see this, we use Lemma E.2.1.6. We observe that the diagonal subspace $\Delta \subset (\lim_I (u_{\mathrm{wh}} \circ P)) \times_0 (\lim_I (u_{\mathrm{wh}} \circ P))$, as a subspace of $\prod^0_{i \in I} (u_{\mathrm{wh}} \circ P)(i)$, is the intersection:

$$\bigcap_{i \in I} p_i^{-1}(\Delta_i)$$

where $p_i : \lim_I (u_{\mathrm{wh}} \circ P) \to (u_{\mathrm{wh}} \circ P)(i)$ is the projection, and $\Delta_i \subset (u_{\mathrm{wh}} \circ P)(i) \times_0 (u_{\mathrm{wh}} \circ P)(i)$ is the diagonal subspace. This latter is k-closed by Lemma E.2.1.6(2), as $(u_{\mathrm{wh}} \circ P)(i)$ is weakly Hausdorff. It follows that $p_i^{-1}(\Delta_i)$ is also k-closed within $\prod^0_{i \in I} (u_{\mathrm{wh}} \circ P)(i)$. The intersection $\Delta = \bigcap_{i \in I} p_i^{-1}(\Delta_i)$ is therefore also k-closed, and by Lemma E.2.1.6 we conclude. □

The following remark is the analog for **CGWH** of Remark E.1.2.3.

REMARK E.2.2.6. On the construction and properties of $h(X)$, we observe:

(1) the functor $h(\cdot)$, by its very design, generally does not preserve underlying sets, contrary to the functor $k(\cdot)$ that only modifies the topology on spaces,
(2) small limits in **CGWH** can be computed in **CG** by Lemma E.2.2.5(1), and by Lemma E.1.2.2(1) their underlying set is the limit of the underlying sets of the objects within the diagram in **CGWH**,
(3) if $P : I \to$ **CGWH** is a diagram such that the limit of the induced diagram $u_{\mathrm{cg}} \circ u_{\mathrm{wh}} \circ P = u_{\mathrm{cgwh}} \circ P$ in **Spaces** is compactly generated weakly Hausdorff then $\lim_I (u_{\mathrm{cg}} \circ u_{\mathrm{wh}} \circ P) = \lim_I (u_{\mathrm{cgwh}} \circ P)$ computes $\lim_I P$ in **CGWH**,
(4) contrary to the case of colimits in **CG**, the underlying set of a small colimit in **CGWH** is not the colimit of the underlying sets of the objects within the diagram in **CGWH** unless the colimit computed in **CG** is already weakly Hausdorff; see Example E.2.2.7 below.

EXAMPLE E.2.2.7. The following is an example of a small (filtered) colimit in **CGWH** of a small diagram $P : I \to$ **CGWH**, such that the set-theoretic colimit $\mathrm{colim}_{i \in I} P(i)^{\mathrm{set}}$ is not the set underlying $\mathrm{colim}_I P$. We let $I := \mathbf{N}_{>0}$ be the positive integers with their natural order, defining $\mathrm{Map}_I(i, j)$ to be the singleton set if $i \leq j$, and the empty set otherwise. We define $P(i) := S^1$, the unit circle, and for any morphism $i \to j$ in I, we set $P(i \to j)$ to be the the map raising $z \in S^1$ to the 2^{j-i}-th power. Then the colimit in **Spaces** of the resulting diagram of underlying topological spaces is not T_1, even though each $P(i)$ is compact Hausdorff. In particular, the colimit $\mathrm{colim}_I P$ in **CGWH** will have as its underlying set a non-trivial quotient of the set-theoretic colimit of the underlying sets of the $P(i)$'s. The reader is referred to [**Du**, Appendix 2, Example 2, p. 422].

REMARK E.2.2.8. In Definition E.1.2.4, we introduced notation to distinguish arbitrary small topological products, and topological fibre products, from the internal products and fibre products formed in **CG**. The definition omits any mention of co-products, because of part (2) of Lemma E.1.2.2, asserting that colimits in **CG** can be computed in **Spaces**, leaving no ambiguity to the meaning of the notation $\coprod$. However, the analog for compactly generated *weakly Hausdorff* spaces of part (2) of Lemma E.1.2.2 does not hold. In other words, not all colimits in **Spaces** of compactly generated weakly Hausdorff spaces will compute colimits internal to **CGWH**, a phenomenon witnessed by Example E.2.2.7. Co-products in **Spaces**, however, do, leaving once again no ambiguity to the use of the notation $\coprod$. We will therefore continue to have no necessity for dedicated notation for topological co-products. This is (2) in the next lemma.

We have the following:

COROLLARY E.2.2.9. *Let I be a small category, and $P : I \to \mathbf{CGWH}$ a diagram. Then:*

(1) *The product $\prod_I u_{\mathrm{wh}} \circ P$ in **CG** computes the product $\prod_I P$ in **CGWH**.*
(2) *The co-product $\coprod_I u_{\mathrm{cgwh}} \circ P$ in **Spaces** computes the co-product $\coprod_I P$ in **CGWH**.*

PROOF. Part (1) is a consequence of Lemma E.2.2.5(1). To see part (2), we observe that $\coprod_I u_{\mathrm{cgwh}} \circ P$ in **Spaces** computes the co-product $\coprod_I u_{\mathrm{wh}} \circ P$ in **CG** by Lemma E.1.2.2(2). We need to check that the compactly generated (Lemma E.1.2.2(2))

$$X = \coprod_I u_{\mathrm{wh}} \circ P \tag{E.2.2.13}$$

is weakly Hausdorff. This is clear from (2) $\Rightarrow$ (1) in Lemma E.2.1.6 for $X \in \mathbf{CG}$. □

NOTATION E.2.2.1. When X and Y are compactly generated weakly Hausdorff spaces, we will, henceforth, denote by $\mathscr{C}ont(X,Y)$ the compactly generated function space $\mathscr{C}ont_k(X,Y)$.

The above change in notation is justified by the following Lemma, for which we refer the reader to [**Str**, Prop.2.24].

PROPOSITION E.2.2.10. *If X and Y are compactly generated topological spaces and Y is weakly Hausdorff, then $\mathscr{C}ont_k(X,Y)$ is a compactly generated weakly Hausdorff topological space. The cartesian symmetric monoidal category $(\mathbf{CGWH}, \times, *)$ inherits from the cartesian symmetric monoidal category $(\mathbf{CG}, \times, *)$ the cartesian-closed monoidal category structure.*

PROOF. As $(\mathbf{CGWH}, \times, *)$ is a full symmetric monoidal subcategory of $(\mathbf{CG}, \times, *)$, the last assertion follows from the first. It is therefore enough to check that for X and Y compactly generated spaces, with Y weakly Hausdorff, then the compactly generated function space $\mathscr{C}ont_k(X,Y)$ is weakly Hausdorff. We now let $W := \mathscr{C}ont_k(X,Y)$, and for any $x \in X$ let $e_x : W \to Y$ be given by $e_x(f) := f(x)$. Since e_x is obtained by $\mathscr{C}ont_k$-functoriality with respect to $\{x\} \to X$, e_x is continuous. We now observe that as subspaces of $W \times_0 W$:

$$(e_x, e_x)^{-1}(\Delta_Y(Y)) = \{(f,g) \in W \times W \mid f(x) = g(x)\} \subset W \times_0 W \tag{E.2.2.14}$$

We claim $(e_x, e_x)^{-1}(\Delta_Y(Y))$ is k-closed. Since Y is weakly Hausdorff, $\Delta_Y(Y)$ is k-closed in $Y \times_0 Y$ by Lemma E.2.1.6(2). By functoriality of the k-topology (Lemma E.1.2.1), $(e_x, e_x)^{-1}(\Delta_Y(Y))$ is also k-closed. It follows from (E.2.2.14) that $(e_x, e_x)^{-1}(\Delta_Y(Y))$ is closed in $W \times W$ for every $x \in X$, and hence $\bigcap_{x \in X}(e_x, e_x)^{-1}(\Delta_Y(Y))$ is also closed in $W \times W$, being an intersection of closed subspaces. We now observe that

$$\Delta_W(W) = \bigcap_{x \in X} (e_x, e_x)^{-1}(\Delta_Y(Y))$$

so by Lemma E.2.1.6(2) we get $\mathscr{C}ont_k(X,Y)$ is weakly Hausdorff. □

E.2.3. Permanence properties of compactly generated weakly Hausdorff spaces. Examples of compactly generated weakly Hausdorff topological spaces are all locally compact Hausdorff topological spaces by (1) in the next lemma.

PROPOSITION E.2.3.1. *We have:*

(1) *all locally compact Hausdorff topological spaces X are compactly generated weakly Hausdorff,*
(2) *the product in* **Spaces** *of a compactly generated weakly Hausdorff space and a locally compact Hausdorff space is compactly generated weakly Hausdorff and computes the product in* **CGWH**.

PROOF. By Lemma E.1.2.19(1), X is compactly generated; it is weakly Hausdorff since it is Hausdorff. By Lemma E.1.2.19(2), $X \times_0 Y$ is compactly generated and the product $X \times Y$ in **CG** can be computed in **Spaces**. By Lemma E.2.2.5(1), the product $X \times Y$ in **CG** computes the product in **CGWH**, and the Proposition is proved. □

REMARK E.2.3.2. On the above Lemma E.2.3.1, we make the following observations:

(1) it will allow us to compute internal products of the locally compact Hausdorff topological spaces intervening in our constructions of topological stacks of interest by working within **Spaces**, therefore leaving no ambiguity as to where products are being formed and allowing access to product topologies of the spaces involved.
(2) The Hausdorff assumption in part (2) of the Lemma cannot be dropped altogether. We refer the reader to Lemma E.2.4.9, for a counter-example.

We conclude this subsection by reviewing a number of permanence properties enjoyed by **CGWH**, and with a result providing a concrete description of the (regular) monomorphisms and (regular) epimorphisms in **CGWH**. We first prove the following analog of Lemma E.1.2.27, now characterizing immersions in **CGWH** rather than in **CG**:

LEMMA E.2.3.3. *Let X, Y be compactly generated weakly Hausdorff spaces and $f : X \to Y$ a continuous injective map. The following are equivalent:*

(1) *f is an immersion.*
(2) *For any compactly generated weakly Hausdorff topological space T and any set-function $g : T \to X$, g is continuous if and only if $f \circ g$ is continuous.*

For the proof of this result, also see [**Str**, Cor. 2.29].

PROOF. By Lemma E.1.2.27, f is an immersion if and only if for any compactly generated space T and any set-function $g : T \to X$, $f \circ g$ is continuous if and only if g is. We claim that it is enough to verify this condition when T is, in addition, weakly Hausdorff. Indeed, let T be any compactly generated space and $g : T \to X$ any set-function. If g is continuous, it factors through the continuous quotient map $T \to h(T)$ by Lemma E.2.2.4(2), as X is weakly Hausdorff. If $f \circ g$ is continuous, then $f \circ g$ factors through $h(T)$ by Lemma E.2.2.4(2), as Y is weakly Hausdorff. In this latter case, g itself factors through $h(T)$, as f is continuous and injective. Either way, we may replace T with $h(T)$ and assume that T is weakly Hausdorff. □

LEMMA E.2.3.4. *Let $g : X \to Y$ be a continuous map of compactly generated weakly Hausdorff topological spaces. If f is a continuous section to g, then f is a closed immersion and g is a submersive map.*

PROOF. Since g is a continuous surjection, the existence of f clearly implies that g is a quotient map, hence submersive, and f is an immersion. We now observe that we have $f(Y) = \{x \in X \mid f(g(x)) = x\}$. It follows that we have:

$$f(Y) = (f \circ g, \mathrm{id})^{-1}(\Delta_X(X)),$$

and as $\Delta_X(X) \subset X \times_0 X$ is k-closed in $X \times_0 X$ by part (2) of Lemma E.2.1.6, $f(Y)$ is also k-closed in X, hence closed, as X is compactly generated. It follows that f is a closed immersion. □

We now prove permanence properties for (closed) immersions under composition.

LEMMA E.2.3.5. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of compactly generated weakly Hausdorff topological spaces.*

(1) *If $g \circ f$ is an immersion then so is f.*
(2) *If $g \circ f$ is a closed immersion then so is f.*
(3) *If $g \circ f$ is an open immersion then f is a locally closed immersion.*

PROOF. Assertion (1) holds with general topological spaces (for each open $U \subset X$ and an open $V \subset Z$ such that $(g \circ f)^{-1}(V) = U$, we have $f^{-1}(W) = U$ for the open $W = g^{-1}(V) \subset Y$). For (2) and (3), we can replace Z with the subspace $(g \circ f)(X) \subset Z$ to reduce to the case that $Z = X$; i.e., f is a section to g. Then we are done by Lemma E.2.3.4. □

We now survey a number of permanence properties of certain classes of morphisms in **CGWH**. For the proof of the following lemma, we refer the reader to [**Str**, Prop. 2.32].

LEMMA E.2.3.6. *The following classes of morphisms in* **CGWH** *are stable under finite compositions and arbitrary base-change, and are local on the target:*

(1) *Injective maps.*
(2) *Surjective maps.*
(3) *Immersions.*
(4) *Closed immersions.*
(5) *Open immersions.*
(6) *Submersive maps.*

We note that by Example E.1.2.16, part (6) of the Lemma does not hold if we drop the weak Hausdorff assumption. Stability of submersive maps under arbitrary base change is an advantage of the category **CGWH** over the category **CG**.

PROOF. As fibre products in **CGWH** can be computed in **CG** by part (1) of Lemma E.2.2.5, and since the forgetful functor **CG** $\to$ **Sets** preserves fibre products by Lemma E.1.2.2(1), checking (1) and (2) is reduced to showing that injective, respectively surjective, set-functions remain injective, respectively surjective, after arbitrary set-theoretic fibre product, and are stable under finite compositions, which is indeed the case.

Obviously immersions, closed immersions, open immersions, and submersions are stable under finite compositions. Being an injective, respectively surjective, continuous map is local on the target, and being an immersion is also local on the target. Closed, respectively open, immersions are also local on the target because closed, respectively open maps are. Submersiveness is also local on the target. We are now to check that immersions, closed immersions, open immersions and submersive maps are stable under arbitrary base change in the category **CGWH**. To this end, we consider the following cartesian diagram in **CGWH**:

$$\begin{array}{ccc} X_T := X \times_{f,Y,g} T & \xrightarrow{f_T} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{E.2.3.1}$$

where we assume that f is an immersion. We let P be an arbitrary compactly generated weakly Hausdorff space, with $h : P \to X_T$ an arbitrary set-function such that $f_T \circ h$ is continuous. It follows that $g \circ f_T \circ h = f \circ p \circ h$ is

continuous. As f is an immersion, we deduce that $p \circ h$ is continuous. Since $f_T \circ h$ is continuous by assumption and $p \circ h$ is continuous by the foregoing, the universal property of fibre products yields continuity of the unique map $h' : P \to X_T$. The underlying set-functions of h and h' are the same set-function, so h is continuous. By Lemma E.2.3.3, f_T is an immersion. Set-theoretically, we have:

$$f_T(X_T) = g^{-1}(f(X)). \tag{E.2.3.2}$$

As a subspace of T, $g^{-1}(f(X))$ is closed, respectively open when f is a closed, respectively open immersion. It follows by (E.2.3.2) that closed immersions, respectively open immersions, are stable under arbitrary base-change in **CGWH**.

We now prove that submersive maps are stable under arbitrary base-change. To this end, we consider the cartesian diagram from (E.2.3.1), where f is now assumed to be a submersive map. By Lemma E.1.2.17, combined with part (1) of Lemma E.2.2.5, asserting that products in **CGWH** can be computed in **CG**, the map $(f, \mathrm{id}) : X \times T \to Y \times T$ is also a submersive map. Moreover, the following diagram in **CGWH** is also cartesian:

$$\begin{array}{ccc} X_T := X \times_{f,Y,g} T & \xrightarrow{(f,\mathrm{id})_T = f_T} & T \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle (g,\mathrm{id})} \\ X \times T & \xrightarrow[(f,\mathrm{id})]{} & Y \times T \end{array} \tag{E.2.3.3}$$

and (g, id) is a closed immersion by Lemma E.2.3.4, as it is a continuous section to the second projection $Y \times T \to T$. Upon renaming in (E.2.3.3) the maps (g, id) as g, (f, id) as f, p' as p, we turn (E.2.3.3) into (E.2.3.1) and are reduced to the case g is a closed immersion. Then $p^{-1}(f^{-1}(g(T)))$ is closed in X_T, hence it belongs to **CGWH** and categorically it coincides with X_T, so then $X_T \to T$ is the restriction of the submersive $f : X \to Y$ over a closed subset $T \subset Y$. Hence, $X_T \to T$ is a submersion as desired. □

In a similar vein, we can descend some of the properties of continuous maps in **CGWH** in Lemma E.2.3.6, along submersive maps:

Lemma E.2.3.7. *Let X, Y, T be compactly generated weakly Hausdorff spaces, $f : X \to Y$ and $g : T \to Y$ two continuous maps. We consider the following cartesian diagram in* **CGWH***:*

$$\begin{array}{ccc} X_T := X \times_{f,Y,g} T & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{E.2.3.4}$$

Assume that g is submersive.

(1) *If f' is injective then so is f.*
(2) *If f' is surjective then so is f.*
(3) *If f' is an immersion then so is f.*
(4) *If f' is a closed immersion then so is f.*
(5) *If f' is an open immersion then so is f.*
(6) *If f' is a submersive map then so is f.*

Proof. We prove (1). By part (1) of Lemma E.2.2.5, the fibre product is computed in **CG**, and hence its underlying set is a fibre product in **Sets** by Lemma E.1.2.2(1). Let x, x' be points in X such that $f(x) = f(x')$. As g is submersive, it is surjective, and hence there exists $t \in T$ such that $g(t) = f(x) = f(x')$. This gives

$z, z' \in X_T$ such that $f'(z) = f'(z') = t$, and $g'(z) = x$, $g'(z') = x'$. Since f' is injective, we have $z = z'$, and hence $x = x'$, so f is injective. Part (2) follows from surjectivity of $g \circ f'$.

We prove (3). As f' is an immersion, f is injective by part (1). Let Z be an arbitrary compactly generated weakly Hausdorff space, and $h : Z \to X$ a set-function such that $f \circ h$ is continuous. We consider the following diagram:

(E.2.3.5)
$$\begin{array}{ccccc} Z_T := Z \times_{f\circ h, Y, g} T & \overset{k}{\overset{\frown}{\longrightarrow}} & & & Y \\ \downarrow{\scriptstyle g''} & & \underset{h'}{\longrightarrow} X_T & \xrightarrow[f']{} & \downarrow{\scriptstyle g} \\ Z & & \xrightarrow[h]{} X & \xrightarrow[f]{} & Y \end{array}$$

where the two solid squares are cartesian. By part (6) of Lemma E.2.3.6, g'' is submersive. Let h' be the set-function obtained as the set-theoretic base change of h along g'. We have $f' \circ h' = k$, which is continuous. As f' is an immersion by assumption, h' is continuous. It follows that $h \circ g'' = g' \circ h'$ is continuous. Since g'' is submersive, h is continuous and so by Lemma E.2.3.3 we see f is an immersion.

We prove (4) and (5). By (3), f is an immersion. The image of f' endowed with the subspace topology is closed, respectively open, in Y. The image $f(X)$ of f endowed with the subspace topology is such that $g^{-1}(f(X)) = f'(X_T)$. Since g is submersive, $f(X)$ is closed, respectively open, and hence the immersion f is a closed, respectively open, immersion. We prove (6). Since f' and g are submersive, their composition is submersive by Lemma E.2.3.6(6), and hence f is submersive. □

We discuss properties of submersive maps within the category **CGWH**. Here is an analog in **CGWH** of Lemma E.1.2.14 for **CG**.

LEMMA E.2.3.8. *Let X, Y be compactly generated weakly Hausdorff spaces, and $f : X \to Y$ a continuous map. The following conditions are equivalent:*

(1) *There exists a compactly generated weakly Hausdorff space R, two maps $s, t : R \rightrightarrows X$ such that $f \circ s = f \circ t$, and a continuous $c : \mathrm{coeq}(s,t) \to Y$ such that the canonical map $p : X \to \mathrm{coeq}(s,t)$ satisfies $c \circ p = f$.*
(2) *f is the co-equalizer in **CGWH** of the diagram $p_1, p_2 : X \times_{f,Y,f} X \rightrightarrows X$, where p_1, p_2 are the two projections.*
(3) *f is a submersive map.*

PROOF. We assume that (1) holds and deduce (3). By Lemma E.2.2.5(2), the co-equalizer of (s,t) is computed as:

$$h(\mathrm{coeq}_{\mathbf{CG}}(u_{\mathrm{wh}}(s), u_{\mathrm{wh}}(t)))$$

where $\mathrm{coeq}_{\mathbf{CG}}(u_{\mathrm{wh}}(s), u_{\mathrm{wh}}(t))$ denotes the co-equalizer in **CG** of $u_{\mathrm{wh}}(s)$ and $u_{\mathrm{wh}}(t)$. For the canonical projection $p : X \to \mathrm{coeq}(s,t)$, the canonical projection $p' : u_{\mathrm{wh}}(X) \to \mathrm{coeq}_{\mathbf{CG}}(u_{\mathrm{wh}}(s), u_{\mathrm{wh}}(t))$, and the submersive map

(E.2.3.6)
$$q : \mathrm{coeq}_{\mathbf{CG}}(u_{\mathrm{wh}}(s), u_{\mathrm{wh}}(t)) \to h(\mathrm{coeq}_{\mathbf{CG}}(u_{\mathrm{wh}}(s), u_{\mathrm{wh}}(t)))$$

from (E.2.2.5), we have

(E.2.3.7)
$$q \circ p' = u_{\mathrm{wh}}(p).$$

Since p' is submersive by the equivalence of part (1) and part (3) of Lemma E.1.2.14, so is $u_{\mathrm{wh}}(p)$ because q is submersive. Since X and $\mathrm{coeq}(s,t)$ are compactly generated weakly Hausdorff, $u_{\mathrm{wh}}(p)$ and p are the same continuous map, which is therefore submersive. As c is a homeomorphism, $f = c \circ p$ is also submersive and so (3) holds.

We now assume that (3) holds, and verify that (2) holds. Since we have $X \times^0_{f,Y,f} X = (f,f)^{-1}(\Delta_Y(Y))$ as subspaces of $X \times_0 X$, and $\Delta_Y(Y)$ is k-closed as a subspace of $Y \times_0 Y$ by Lemma E.2.1.6, we deduce that $X \times^0_{f,Y,f} X$ is a k-closed subspace of $X \times_0 X$. Thus, $X \times_{f,Y,f} X \subset X \times X$ is closed. The map $u_{\mathrm{wh}}(f)$ is the co-equalizer of its kernel pair in **CG** by (3) $\Rightarrow$ (2) in Lemma E.1.2.14 since $u_{\mathrm{wh}}(f) = f$ *topologically* and we're assuming f is submersive. The kernel pair of f in **CGWH** is computed in **CG** by part (1) of Lemma E.2.2.5. Since $X \times_{f,Y,f} X \subset X \times X$ is closed, then by Lemma E.2.1.11, upon calling $p_1, p_2 : X \times_{f,Y,f} X \rightrightarrows X$ the two projections, $\mathrm{coeq}(p_1,p_2)$ is computed in **CG** and (2) is proved. Since (2) implies (1) upon choosing $R := X \times_{f,Y,f} X$ and $(s,t) = (p_1,p_2)$, the Lemma is proved. □

We now approach a result concerning the computability in **Spaces** of certain push-outs in **CGWH**. We begin with the following:

LEMMA E.2.3.9. *Let $f : X \to Y, g : Z \to Y$ be morphisms in* **CGWH**. *Upon denoting by $f' : X' \to Y'$ and $g' : Z' \to Y'$ the associated continuous maps in* **Spaces**, *we consider the following cartesian diagram in* **Spaces**:

$$\begin{array}{ccc} X'_{Z'} := X' \times^0_{f',Y',g'} Z' & \xrightarrow{f'_{Z'}} & Z' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

If $f'_{Z'}$ is a closed immersion (such as when f' is) then $X'_{Z'}$ is compactly generated weakly Hausdorff, and it computes the fibre product $X \times_{f,Y,g} Z$ of f and g in **CGWH**.

PROOF. Since $f'_{Z'}$ is an immersion, $X'_{Z'}$ is weakly Hausdorff since Z' is. It also lies in **CG** since $f'_{Z'}$ is a closed immersion. By Lemma E.2.2.5(1), $k(X'_{Z'}) = u_{\mathrm{wh}}(X) \times_{u_{\mathrm{wh}}(f),u_{\mathrm{wh}}(Y),u_{\mathrm{wh}}(g)} u_{\mathrm{wh}}(Z)$ is weakly Hausdorff and computes $X \times_{f,Y,g} Z$, and by the foregoing we have $k(X'_{Z'}) = X'_{Z'}$, so the lemma is proved. □

We now prove the promised result on push-outs:

LEMMA E.2.3.10. *Let X, Z, P be compactly generated weakly Hausdorff spaces. Let Y be the push-out in* **CGWH** *of X, Z along two continuous maps $p : P \to Z$ and $q : P \to X$ from P making the diagram*

$$\begin{array}{ccc} P & \xrightarrow{p} & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute in **CGWH** *and assume the map p is a closed immersion. Then f is a closed immersion and the push-out in* **CGWH** *can be computed in* **Spaces**. *Moreover:*

(1) *the fibre product $X \times_{f,Y,g} Z$ of f and g in* **CGWH** *can be computed in* **Spaces**,
(2) *in* **CGWH**, *the natural map*

$$P \to X \times_{f,Y,g} Z$$

is a homeomorphism.

PROOF. By Corollary E.2.2.9, co-products in **CGWH** are topological, and products in **CGWH** are computed in **CG**. We observe that Y is also the push-out in **CGWH** of the following two maps:

$$p \coprod \mathrm{id}_X : P \coprod X \to Z \coprod X, \quad \phi_q : P \coprod X \to X \tag{E.2.3.8}$$

where ϕ_q is defined to be q on P and the identity on X. Since ϕ_q is submersive and $p \coprod \mathrm{id}_X$ is a closed immersion, we may rename $P \coprod X$ as P, $Z \coprod X$ as Z, ϕ_q as q and $p \coprod \mathrm{id}_X$ as p, and assume that q is, in addition, submersive.

We define:

$$Y' := X' \coprod_{q',P',p'} Z', \tag{E.2.3.9}$$

with associated continuous maps $f' : X' \to Y'$ and $g' : Z' \to Y'$. Given $R' := \Delta_{X'}(X') \coprod P' \coprod P' \coprod \Delta_{Z'}(Z')$, with the continuous map $r : R' \to (X' \coprod Z') \times_0 (X' \coprod Z')$ defined as $\Delta_{X'}$ on the first component, $(p', q') \circ \Delta_{P'}$ on the second, $(q', p') \circ \Delta_{P'}$ on the third and $\Delta_{Z'}$ on the fourth, we call R the set-theoretic image $r(R')$ of R' in $(X' \coprod Z') \times_0 (X' \coprod Z')$. We endow $r(R')$ with the subspace topology. We now describe R explicitly. Calling:

$$W_1 := \{(a, x) \mid a \in p'(P'), x = q'(b), p'(b) = a\},\ W_2 := \{(a, b) \in X' \times_0 Z' \mid (b, a) \in W_1\} \tag{E.2.3.10}$$

where W_1 is the set-theoretic image in $Z' \times_0 X'$ of P' under $(p', q') \circ \Delta_{P'}$, and W_2 is the set-theoretic image in $X' \times_0 Z'$ of P' under $(q', p') \circ \Delta_{P'}$. We endow W_1 and W_2 with the subspace topologies, and we have:

$$R = \Delta_{X'}(X') \coprod W_1 \coprod W_2 \coprod \Delta_{Z'}(Z') \tag{E.2.3.11}$$

as subspaces of $(X' \coprod Z') \times_0 (X' \coprod Z')$. We now note that X', Z' and P' are weakly Hausdorff by assumption.

By part (2) of Lemma E.2.1.6, $\Delta_{X'}(X')$ and $\Delta_{Z'}(Z')$ are k-closed in $X' \times_0 X'$, respectively $Z' \times_0 Z'$. Since these products are closed in $(X' \coprod Z') \times_0 (X' \coprod Z')$, we deduce that $\Delta_{X'}(X')$ and $\Delta_{Z'}(Z')$ are k-closed in $(X' \coprod Z') \times_0 (X' \coprod Z')$. We now check that W_1 from (E.2.3.10) is k-closed in $Z' \times_0 X'$. If so, then W_2 from (E.2.3.10) is also k-closed in $X' \times_0 Z'$, and we shall deduce from (E.2.3.11) that R is k-closed in $(X' \coprod Z') \times_0 (X' \coprod Z')$. To check that W_1 is k-closed in $Z' \times_0 X'$, first observe that the set-theoretic image of $(p', q') : P' \times_0 P' \to Z' \times_0 X'$ is $p'(P') \times_0 X'$, since q' is surjective. By submersiveness of q', the topology on such a set-theoretic image is un-ambiguous, since the quotient topology is equivalent to the subspace topology, which in turn is the product topology of the subspace topology on $p'(P')$ with the intrinsic topology on X', since p' is an immersion. Equivalently, the map (p', q') is strict.

Since p' is a closed immersion, $p'(P')$ is a closed subspace of Z', and $p'(P') \times_0 X'$ is a closed subspace of $Z' \times_0 X'$. We now note that (p', q') is injective, and hence, being strict, it is a topological embedding (i.e. an immersion). It follows that (p', q') is a closed map. The image W_1 of the k-closed $\Delta_{P'}(P') \subset P' \times_0 P'$ is k-closed in $Z' \times_0 X'$ since it is contained in the closed $P' \times_0 X'$ as the graph $\Gamma_{q'}(P') = (q', \mathrm{id}_{X'})^{-1}(\Delta_{X'}(X'))$ with $\Delta_{X'}(X') \subset X' \times_0 X'$ a k-closed subset since X' is weakly Hausdorff. We have proved that R is k-closed in $(X' \coprod Z') \times_0 (X' \coprod Z')$.

Since, by Corollary E.2.2.9, products in **CGWH** are computed in **CG** and co-products in **CGWH** are computed in **Spaces**, and since by Lemma E.1.2.2 products in **CG** are computed upon evaluating the functor $k(\cdot)$ of Lemma E.1.2.1 on topological products, we have:

$$k((X' \coprod Z') \times_0 (X' \coprod Z')) = (X' \coprod Z') \times (X' \coprod Z')$$

and this topological space is weakly Hausdorff since X' and Z' are, we therefore deduce that $k(R)$ is closed in $(X' \coprod Z') \times (X' \coprod Z')$, and that, by Lemma E.2.1.11, $k(Y')$ is therefore weakly Hausdorff. As Y' is compactly generated, as a quotient of the compactly generated space $X' \coprod Z'$, we deduce that we have $k(Y') = Y'$, and hence $Y' = u_{\mathrm{cgwh}}(Y)$. This proves that the push-out of (p, q) in **CGWH** is computed in **Spaces**.

We now check, using the above explicit description of R, that f' is a closed immersion. Since f' is a continuous injection, it is the same to show f' is closed. Calling $t : X' \coprod Z' \to Y'$ the quotient map by R and $i : X' \to X' \coprod Z'$ the inclusion, we have $f' = t \circ i$ and for any $C \subset X'$ closed we have $t^{-1}(f'(C)) = C \coprod (q')^{-1}(C)$ which is closed, hence $f'(C)$ is closed as t is a submersive map.

To prove part (1), we observe that the fibre product of f and g in **CGWH** can be computed in **Spaces** by Lemma E.2.3.9, which is now applicable as f' is a closed immersion. To prove part (2), we observe that to check that $P = X \times_{f,Y,g} Z$ in **CGWH**, by part (1) and the assertion that the push-out of f, g in **CGWH** can

be computed in **Spaces** (proved in the foregoing) it is enough to check that the push-out in **Spaces** of the continuous maps f, g, and the topological fibre product of f, g, agree, which is seen by direct inspection. □

The following example is [**LGL**, Appendix A, p. 168], and it exhibits a push-out in **CGWH** whose underlying topological space is not a push-out in **Spaces**:

EXAMPLE E.2.3.11. We let $P := [-1, 1] - \{0\}$ endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$, we call $p : P \to [-1, 1] =: Z$ the open immersion, with $[-1, 1]$ also endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$, and $q : P \to \{-1, 1\} =: X$ the submersive map identifying $[-1, 0)$ to -1 and $(0, 1]$ to 1. The push-out $Y' := X \coprod^0_{q,P,p} Z$ in **Spaces** is a space with three points, only one of the three being a closed point (in the closure of the other two points). The space $Y := h(Y')$ is therefore made of a single closed point and computes the push-out $X \coprod_{q,P,p} Z$ in **CGWH**. We note that the push-out of p is not only not an immersion, but not even injective. In other words, Lemma E.2.3.10 does not hold if one drops the assumption on p being a closed immersion.

We refer the reader to Example E.2.2.7 (which is [**Du**, Appendix 2, Example 2, p. 422]) for an example of a colimit in **Spaces** of a small directed system of compactly generated and Hausdorff spaces that is not weakly Hausdorff and therefore does not compute the colimit internal to **CGWH**.

The following example illustrates the advantage of working within the category **CGWH** as opposed to the category of compactly generated *Hausdorff* spaces considered in [**Ste**]:

EXAMPLE E.2.3.12. We let Z be a compactly generated non-regular Hausdorff topological space (for an explicit example of a non-regular Hausdorff topological space that is compactly generated, we refer the reader to [**StSe**, Part II, 78]), and $p : P \to Z$ the closed immersion of a closed subspace witnessing the non-regularity of Z. We let X be a point, and $q : P \to X$ the quotient map collapsing P to a single point. While Lemma E.2.3.10 does still ensure that in **Spaces** the push-out $X \coprod_{q,P,p} Z$ is compactly generated weakly Hausdorff and computes the push-out $X \coprod_{q,P,p} Z$ in **CGWH**, the topological push-out is non-Hausdorff, even though X, P, Z are compactly generated and Hausdorff and p is a closed immersion. Indeed, such push-out is Z/P, and any point $z \in Z$ that cannot be separated from the closed subspace P, has image in Z/P that cannot be separated from the point induced as the image of P in the quotient.

This clarifies the advantage of weakening the Hausdorff condition considered in [**Ste**] to a weak Hausdorff separation condition. Namely, while arbitrary topological colimits of compactly generated weakly Hausdorff spaces may not compute the internal colimit in **CGWH**, as illustrated by Example E.2.3.11, push-outs in **CGWH** along closed immersions can indeed be computed in **Spaces**. Within the category of compactly generated Hausdorff spaces of [**Ste**], not even these can be computed in **Spaces**.

REMARK E.2.3.13. We observe that the submersive map $f : Z \to Y := X \coprod^0_{q,P,p} Z$ in **Spaces** obtained as the push-out in **Spaces** of the submersive map q in Example E.2.3.12, is not open.[9] This illustrates the additional degree of flexibility of incorporating the weak Hausdorff separation assumption on compactly generated spaces, as opposed to the Hausdorff condition. Indeed, quotients of compactly generated weakly Hausdorff spaces by closed equivalence relations are compactly generated weakly Hausdorff by Lemma E.2.1.11, while the analogous result does not hold upon replacing the weak Hausdorff condition with the Hausdorff condition. It only holds if, in addition,[10] the quotient map is open.

We conclude this section by discussing a result that provides a concrete characterization of several important classes of morphisms in **CGWH**. In light of Example E.1.2.16, this is, in turn, an advantage of the category **CGWH** over the category **CG**. First we recall a definition:

[9] why not open?

[10] why sufficient? Do we care? Maybe omit this final sentence.

DEFINITION E.2.3.14. In a category $\mathcal{C}$, an epimorphism is called *regular* if and only if it is the co-equalizer of a pair of morphisms, and a monomorphism is called *regular* if and only if it is the equalizer of a pair of morphisms.

If $\mathcal{C}$ contains all small limits and colimits, an epimorphism in $\mathcal{C}$ is called *effective* if and only if it is the co-equalizer of its own kernel pair, and is called *universally effective* if this property holds after arbitrary pull-back. A monomorphism $X \to Y$ is called *effective* if and only if $X \to Y \times_Q Y$ is an isomorphism, for the pushout Q of the two copies $X \rightrightarrows Y$ of the given monomorphism.[11] We say $\mathcal{C}$ is *regular* if every pull-back of a regular epimorphism is a regular epimorphism (see [**Bor2**, Def. 2.1.1]).

PROPOSITION E.2.3.15. *We have:*

(1) *The monomorphisms in* **CGWH** *are the injective continuous maps.*
(2) *The epimorphisms in* **CGWH** *are the continuous maps with dense image.*
(3) *The regular monomorphisms in* **CGWH** *are the closed immersions.*
(4) *The regular epimorphisms in* **CGWH** *are the submersive maps.*
(5) *The effective monomorphisms in* **CGWH** *are the closed immersions.*
(6) *The effective epimorphisms in* **CGWH** *are the submersive maps.*
(7) *The universally effective epimorphisms in* **CGWH** *are the submersive maps.*
(8) *The isomorphisms in* **CGWH** *are the homeomorphisms.*
(9) *Monomorphisms in* **CGWH** *are stable under products, co-products and compositions.*
(10) *Regular monomorphisms in* **CGWH** *are stable under products, co-products and compositions.*
(11) *Epimorphisms in* **CGWH** *are stable under finite products, co-products and compositions.*
(12) *Regular epimorphisms in* **CGWH** *are stable under finite products, co-products and compositions.*
(13) *Every morphism in* **CGWH** *is universally the composition of a regular epimorphism followed by a monomorphism.*
(14) **CGWH** *is a regular category.*

Within **CGWH**, it is our convention to refer to a pull-back as a *base-change*. Regularity for **CGWH** implies several structural properties for continuous maps between compactly generated weakly Hausdorff spaces that will not hold for general topological spaces, following from [**Bor2**, Ch.2,Thm.2.1.3]. In particular, within **CGWH** *any continuous map* factors as a submersive map followed by a continuous injective map, and the factorization is unique up to homeomorphism, which implies that it is preserved by base-change. While we will not rely on regularity of **CGWH**, but only indirectly and with no necessity to appeal to the general theory around the concept, we will use stability of submersive maps under arbitrary base change systematically.

Lemma E.2.3.8 identifies the regular epimorphisms in **CGWH** as exactly the submersive maps, since Lemma E.2.3.6(6) proves stability of submersive maps under arbitrary base change, and such stability implies regularity of **CGWH**, so we are implicitly using regularity of **CGWH**. This should further motivate the choice of **CGWH** as the category of "convenient" topological spaces of reference.

The assertion analogous to Proposition E.2.3.15(7) does not hold for **CG** (and hence for **Spaces**) as illustrated by Example E.1.2.16, which exhibits an example of a submersive map whose base change in **CG** is not submersive.

We further mention that, in fact, **CGWH** is also a co-regular category: see [**Str**, §3, Thm. 3.1] for a discussion on the co-regularity of **CGWH**, which ultimately amounts to combining [**Bou-GT**, §3,3.5] with Lemma E.2.3.10 and Lemma E.2.2.5.

Before we prove Proposition E.2.3.15, we introduce the following auxiliary construction, to be used in the argument.

[11] Check this definition is correct.

Construction E.2.3.16. Let X be a compactly generated weakly Hausdorff space, $Z \subset X$ a closed subspace, and denote by $i : Z \to X$ the closed immersion. We denote by $\{*\}$ a point endowed with the discrete topology. We let $a : Z \to \{*\}$ be the unique map to a point in **CGWH**, and extend i to a closed immersion $i : \{*\} \coprod Z \to \{*\} \coprod X$. We call

$$(X/Z)_* := \{*\} \coprod_{a, \{*\} \coprod Z, i} (\{*\} \coprod X) \tag{E.2.3.12}$$

the push-out formed in **CGWH**, and we denote by x the continuous map $\{*\} \to (X/Z)_*$. We denote by q the map

$$q : X \to (X/Z)_* \tag{E.2.3.13}$$

obtained as the composition of the open and closed immersion $X \to \{*\} \coprod X$ followed by the map $b : \{*\} \coprod X \to (X/Z)_*$ induced in the quotient by a. We further define the continuous map

$$p : X \to (X/Z)_* \tag{E.2.3.14}$$

to be the constant map obtained as the composition of the unique continuous map $X \to \{*\}$ followed by x.

We now collect some properties of $(X/Z)_*$, q, p and x to be used in the proof of Proposition E.2.3.15.

Lemma E.2.3.17. *Let X be a compactly generated weakly Hausdorff space, $Z \subset X$ a closed subspace. We let $(X/Z)_*$ be as in* (E.2.3.12)*, q the map from* (E.2.3.13) *and p the map from* (E.2.3.14)*.*

(1) *The map q is a closed submersion when Z is non-empty.*
(2) *The push-out defining $(X/Z)_*$ in* (E.2.3.12) *can be computed in* **Spaces***, and we have:*

$$\{*\} \coprod Z = \{*\} \times_{x,(X/Z)_*,q} (\{*\} \coprod X) \tag{E.2.3.15}$$

in **CGWH**. *Moreover, $\{*\} \coprod Z = \{*\} \times_{x,(X/Z)_*,q} (\{*\} \coprod X)$ is compactly generated and computes in* **Spaces** *the fibre product in* **CGWH**.
(3) *If Z is non-empty, we have*

$$Z = \{*\} \times_{x,(X/Z)_*,q} X \tag{E.2.3.16}$$

in **CGWH**. *Moreover, $\{*\} \times^0_{x,(X/Z)_*,q} X$ is compactly generated and computes in* **Spaces** *the fibre product in* **CGWH**.
(4) *We have:*

$$X = \{*\} \times_{x,(X/Z)_*,p} X \tag{E.2.3.17}$$

Moreover, we have $p^{-1}(\{x\}) = X$ and, if Z is non-empty, $q^{-1}(\{x\}) = Z$.

Proof. The following diagram is useful to consult during the proof that follows (with $i : \{*\} \coprod Z \to \{*\} \coprod X$ the natural closed immersion):

$$\begin{array}{ccccc} Z & \longrightarrow & \{*\} \coprod Z & \xrightarrow{a} & \{*\} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ X & \longrightarrow & \{*\} \coprod X & \xrightarrow{b} & (X/Z)_* \end{array} \qquad X \xrightarrow{q} (X/Z)_*$$

By Lemma E.2.3.10(1), the push-out defining $(X/Z)_*$ can be computed in **Spaces**. This gives the first assertion in (2), which we shall now use to prove the assertion in (1). The map $b : \{*\} \coprod X \to (X/Z)_*$ induced in the quotient by a is submersive since Lemma E.2.3.10 ensures $(X/Z)_*$ is a topological quotient of $(\{*\} \coprod X) \coprod \{*\}$.

In particular, given an arbitrary topological space T and a set-function $g : (X/Z)_* \to T$, g is continuous if and only if $g \circ b$ is. On the other hand, $g \circ b$ is continuous on $\{*\} \coprod X$ if and only if it is on X. As the image of x is in the image of q when Z is non-empty, we deduce that q is surjective when Z is non-empty, so it is submersive. To verify q is also closed, by submersiveness it suffices to show $q^{-1}(q(C))$ is closed in X for all closed $C \subset X$. This holds because for any subset $C \subset X$ we have $q^{-1}(q(C)) = C$ when C is disjoint from Z and $q^{-1}(q(C)) = C \cup Z$ when C meets Z.[12]

We next prove the rest of (2). By Lemma E.2.3.10(2), we have the identification in (E.2.3.15) and that such identification remains valid in **Spaces**. We prove (3). The same argument as for (2) applies to (E.2.3.16) since if Z is non-empty then $(X/Z)_*$ is also the push-out of the closed immersion $i : Z \to X$ and the unique map $Z \to \{*\}$. Part (4) follows from the very definition of p from (E.2.3.14), and the last assertion $p^{-1}(\{x\}) = X$ and $q^{-1}(\{x\}) = Z$ is a reformulation of (3) and (4). □

We now prove Proposition E.2.3.15.

Proof of Proposition E.2.3.15. We prove (1). Let $f : X \to Y$ be a monomorphism. Then $f_* : \mathrm{C}(T, X) \to \mathrm{C}(T, Y)$ is injective for all T in **CGWH**. Upon choosing $T = \{*\}$ a point, we have $\mathrm{C}(T, X) = X^{\mathrm{set}}$ and $\mathrm{C}(T, Y) = Y^{\mathrm{set}}$, so we deduce f is injective. Conversely if f is injective then f_* is also injective, so (1) is proved. We prove (2). Let $f : X \to Y$ be a continuous map of compactly generated weakly Hausdorff spaces, T a compactly generated weakly Hausdorff space, and $g, g' : Y \rightrightarrows T$ arbitrary continuous maps co-equalized by f. Then we have:

$$\mathrm{eq}(g, g') := Y \times_{g,T,g'} Y = (g, g')^{-1}(\Delta_T(T)) \subset Y.$$

The second identification holds as subspaces of $Y \times Y$, the product indeed being internal to **CGWH**, as the subspace topology on $(g, g')^{-1}(\Delta_T(T)) \subset Y \times Y$ is compactly generated since $(g, g')^{-1}(\Delta_T(T))$ is k-closed in $Y \times_0 Y$ because $\Delta_T(T)$ is k-closed in $T \times_0 T$ and (g, g') is continuous. We observe that the image of f is contained in $\mathrm{eq}(g, g')$. Since $\mathrm{eq}(g, g')$ is closed, $g = g'$ if f has dense image. We now assume conversely that f is an epimorphism. This forces X to be non-empty if Y is non-empty (as we may assume) since the two natural maps $Y \rightrightarrows Y \coprod Y$ are distinct for non-empty Y. In particular, the closure Z of the image of f is non-empty. Consider the maps $p, q : Y \rightrightarrows (Y/Z)_*$ provided by Construction E.2.3.16. The map f equalizes p and q, and since it is an epimorphism we deduce $p = q$ and hence, in the notation of Lemma E.2.3.17, $p^{-1}(\{x\}) = Y$ and $q^{-1}(\{x\}) = Z$ must be equal, forcing $Z = Y$. This means that f has dense image, and (2) is proved.

We prove (3). Suppose $i : Z \to X$ is a closed immersion. Then i is the equalizer of $(p, q) : X \to (X/Z)_*$ from Construction E.2.3.16 (even when Z is empty), and hence it is a regular monomorphism. Conversely, regular monomorphisms are closed immersions, as they are continuous pre-images of diagonal subspaces, and these are closed in every object of **CGWH**. Part (4) follows from Lemma E.2.3.8. We now prove (5). Since **CGWH** contains equalizers and co-kernel pairs, a monomorphism in **CGWH** is effective if and only if it is regular, and hence (5) follows from (3). Part (6) follows from Lemma E.2.3.8, and (7) follows from (6) and Lemma E.2.3.6(6).

To prove (8) it is enough to observe that isomorphisms are regular epimorphisms and monomorphisms, and by (4) and (1) these are the bijective submersive maps, hence the homeomorphisms. Part (9) follows from (1), and (10) follows from (3). For (11) and (12), the only delicate aspect[13] is stability of regular epimorphisms under finite products. By (4), this follows from Lemma E.2.3.6(6). Finally, part (13) is a formal consequence of the fact that **CGWH** is closed under finite limits (see Lemma E.2.2.5(1)) and finite colimits (see Lemma E.2.2.5(2)), and that regular epimorphisms are stable under arbitrary base change (see (4) and Lemma E.2.3.6).

[12] check this verification of closedness of q, attentive to effect of the functor h, since closedness proof was omitted before.

[13] Why for (11) – just usual epimorphism – is this the case? Can you use (4) somehow? Must clarify.

These three conditions say that **CGWH** is a *regular category* (see [**vO**, Def. 4.1] and [**Bor2**, Ch. 2, Def. 2.1.1]), so by [**vO**, Prop. 4.2][14] this proves (14). □

E.2.4. Counter-examples on compact generation and separation conditions. In this subsection we collect a number of examples illustrating the necessity of assumptions being made throughout the previous sections. We also highlight key differences between the definition of compact generation for spaces adopted throughout this work, and other definitions present within the literature. We focus specifically on the following two, apart from the one adopted throughout this appendix (see Definition E.1.1.7):

Definition E.2.4.1. Let X be a topological space. We say:

(1) X is a *Kelley compactly generated space* if and only if for any subspace $U \subset X$, U is open if and only if for any compact subspace $K \subset X$, $U \cap K$ is open in K.
(2) X is a *Steenrod compactly generated space* if and only if for any subspace $U \subset X$, U is open if and only if for any compact Hausdorff subspace $K \subset X$, $U \cap K$ is open in K.

We have the following simpler analog for Kelley compact generation of Lemma E.1.1.3, which we record here to aid in the construction of examples:

Lemma E.2.4.2. *Let X be a topological space. Then the following conditions are equivalent:*

(1) *X is Kelley compactly generated.*
(2) *For any topological space Y and set-function $f : X \to Y$, the following conditions are equivalent:*
   (a) *f is continuous.*
   (b) *For any compact subspace $K \subset X$ immersed through $i : K \to X$, the composition $f \circ i$ is continuous.*
(3) *A subspace $U \subset X$ is open if and only if for any compact topological space K and any continuous map $g : K \to X$, $g^{-1}(U)$ is open in K.*
(4) *There exists a set I, a compact topological space K_i for each $i \in I$, and for $Y := \coprod_{i\in I} K_i$ the I-indexed co-product topological space, a submersive map $q : Y \to X$.*

Proof. We prove that (1) and (2) are equivalent. For any topological space Y and set-function $f : X \to Y$, if f is continuous then part (b) of (2) also holds. Conversely, if part (b) of (2) holds for all (K, i) then for any open subspace $V \subset Y$, by writing $U := f^{-1}(V)$, for any compact subspace $K \subset X$ immersed through $i : K \to X$ we see that $(f \circ i)^{-1}(V)$ is open in K by continuity of $f \circ i$ and we have $U \cap K = (f \circ i)^{-1}(V)$. By (1), U is open in X, forcing f to be continuous. This proves (2) assuming (1). Conversely, if (2) holds, then we choose Y to be the topological space whose underlying set is X^{set}, and whose topology is the coarsest topology such that every subset Z for which $K \cap Z$ is closed in K is itself closed in Y. We let f be the identity set-function. By (2), f is continuous. Since the topology of Y on X^{set} is finer than that of X, f^{-1} is also continuous, and hence f is a homeomorphism. This shows (2) implies (1).

Since (3) implies (1) as a consequence of the definitions, we prove the converse. If (1) holds, we let K be an arbitrary compact topological space and $g : K \to X$ an arbitrary continuous map. Then g factors as $g = i \circ g'$ where i is the immersion of the image $K' := g(K) \subset X$, and $g' : K \to K'$ is continuous and surjective. Clearly K' is a compact subspace of X. If the intersection of U with any compact subspace $K' \subset X$ is open, then $g^{-1}(U) = (g')^{-1}(U \cap K')$ is open in K and hence U is open. Thus, (1) and (3) are therefore equivalent.

We now assume (2). We let I be the set of all compact subspaces of X. For $i \in I$, we denote K_i the corresponding compact subspace of X, and $g_i : K_i \to X$ the immersion. Upon writing $Y := \coprod_{i\in I} K_i$ and $q := \coprod_{i\in I} g_i$, the map $q : Y \to X$ is continuous and surjective. Moreover, for any topological space Z and any set-function $f : X \to Z$, by part (2) the map f is continuous if and only if $f \circ q$ is continuous. By [**Bou-GT**, Ch. I, §4, Prop. 6], q is submersive, and (4) is proved. Conversely, we assume that (4) holds. We let $U \subset X$ be

[14] why citing for verifying a *definition*?

a subspace such that for any compact subspace $K \subset X$, $U \cap K$ is open in K. Then, $U \cap q(K_i)$ is open in the compact subspace $q(K_i)$ for every $i \in I$, which forces $q^{-1}(U)$ to be open in Y. As q is submersive, U is open in X, hence X is Kelley compactly generated. □

We will now show by example that the two concepts of compact generation introduced in Definition E.2.4.1 are not equivalent to one another, nor are they equivalent to compact generation as defined in Definition E.1.1.7 and throughout this appendix. We first establish a result in the affirmative:

LEMMA E.2.4.3. *Let X be a topological space. We have:*

(1) *If X is Steenrod compactly generated, then it is compactly generated.*
(2) *If X is compactly generated, then it is Kelley compactly generated.*

PROOF. Assume that X is Steenrod compactly generated, and let $U \subset X$ be a subspace. We assume that for any compact Hausdorff space K and any continuous map $g : K \to X$, $g^{-1}(U)$ is open in K. In particular, this holds as (K, g) range over all compact Hausdorff subspaces of X, g being the immersion of such subspace. As X is Steenrod compactly generated, U is open, proving that X is compactly generated in the sense of Definition E.1.1.7, and (1) follows. If X is compactly generated, then by part (4) of Lemma E.1.1.3 there exists a set I and for every $i \in I$ a compact Hausdorff space K_i, with a submersive map $q : \coprod_{i \in I} K_i \to X$. By the equivalence of parts (4) and (1) of Lemma E.2.4.2 above, X is Kelley compactly generated. □

We now examine the following example:

EXAMPLE E.2.4.4. We let $Y := \mathbf{Q}$ be the rational numbers equipped with the subspace topology induced by the inclusion $\mathbf{Q} \subset \mathbf{R}$, the reals being endowed with their Euclidean topology. We define $X := Y^+$ to be the one-point compactification of Y. Since every neighborhood of the point $X - Y$ has non-empty intersection with every non-empty open subspace of X, X is non-Hausdorff. It is, however, weakly Hausdorff, as any compact subspace of X is closed. We refer the reader to [**StSe**, Part II, 35] for a discussion of further properties of this example.

The following Lemma illustrates how the assumption in Lemma E.2.1.4 concerning the existence of a continuous surjection from a compact Hausdorff space, cannot be dropped. It also illustrates how the weak Hausdorff separation condition is genuinely weaker than the Hausdorff separation condition.

LEMMA E.2.4.5. *There exists an example of a compact, compactly generated, weakly Hausdorff space that is not Hausdorff.*

PROOF. We let X be the topological space from Example E.2.4.4. Then X is compact and weakly Hausdorff. It is also compactly generated by Lemma E.1.2.21, since it is sequential (see [**StSe**, Part II, 35]). It is, however, non-Hausdorff. □

We consider the following:

EXAMPLE E.2.4.6. The Sierpiński space S is the topological space whose underlying set is $\{0, 1\}$ and whose topology τ is the collection of subsets of S:

$$\{\varnothing, \{1\}, S\}.$$

LEMMA E.2.4.7. *There exists a compactly generated space that is not Steenrod compactly generated.*

PROOF. The space S from Example E.2.4.6 is compactly generated by Lemma E.1.2.21, as every finite space is sequential. On the other hand, it is not Steenrod compactly generated, as the only compact Hausdorff subspaces K of S are the one-element sets $\{0\}$ and $\{1\}$, and any subset $Z \subset S$ satisfies $Z \cap K$ is closed in K. As the topology on S is not discrete, S is not Steenrod compactly generated. □

We point out in the following Lemma that for Hausdorff topological spaces there is no discrepancy among the various notions of compact generation reviewed here, and hence no potential for ambiguity as to which one is being referred to:

LEMMA E.2.4.8. *Let X be a Hausdorff topological space. Then the following conditions are equivalent:*

(1) *X is Kelley compactly generated.*
(2) *X is compactly generated.*
(3) *X is Steenrod compactly generated.*

PROOF. We just need to show (1) implies (3). We'll show (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (1). By Lemma E.2.4.2(4), there exists a set I, for every $i \in I$ a compact space K_i, and a submersive map $q : \coprod_{i\in I} K_i \to X$. As X is Hausdorff, the compact $K'_i := q(K_i)$ is Hausdorff too. We denote by $g_i : K'_i \to X$ the immersion, and by $q' := \coprod_{i\in I} g_i$ the co-product map $\coprod_{i\in I} K'_i \to X$. The map q' is continuous and surjective by design. As q is submersive, so is q', and hence X is compactly generated by Lemma E.1.1.3. We now assume that (2) holds, and prove (3). By Lemma E.1.1.3(4), there exists a set I and for every $i \in I$ a compact Hausdorff space K_i, and a submersive map $q : \coprod_{i\in I} K_i \to X$. We write $K'_i := q(K_i)$. This is compact and Hausdorff. Upon calling $g_i : K'_i \to X$ the immersion, then:

$$q' := \coprod_{i\in I} g_i : \coprod_{i\in I} K'_i \to X$$

is submersive, since q is. Let $U \subset X$ be a subspace such that its intersection with every compact Hausdorff subspace K of X is open in K. In particular, $U \cap K'_i$ is open in K'_i for every $i \in I$. This implies that $(q')^{-1}(U)$ is open in $\coprod_{i\in I} K'_i$, and as q' is submersive, U is open in X. We deduce that X is Steenrod compactly generated. The Lemma is proved. □

The following lemma illustrates how, upon dropping the Hausdorff assumption in Lemma E.2.1.4, topological products of a compact weakly Hausdorff space and a compactly generated (weakly) Hausdorff space may not be compactly generated.

LEMMA E.2.4.9. *There exists an example of a topological space X such that:*

(1) *X is compact, compactly generated weakly Hausdorff (and not Hausdorff),*
(2) *$X \times_0 X$ is not compactly generated.*

PROOF. We let Y be the rational numbers endowed with the subspace topology inherited from the Euclidean topology on $\mathbf{R}$, and $X := Y^+$ the one-point compactification as in Example E.2.4.4. Then X is weakly Hausdorff and compact. It is also compactly generated, as it is sequential (see [**StSe**, Part II, 35]). If $X \times_0 X$ was compactly generated then $X \times X = k(X \times_0 X)$ would coincide with $X \times_0 X$ (i.e., k-closed is the same as closed in $X \times_0 X$). Since X is weakly Hausdorff, the diagonal subspace $\Delta_X(X)$ would then be closed in $X \times_0 X$ (see Lemma E.2.1.6), so X would be Hausdorff, a contradiction. □

We consider the following:

EXAMPLE E.2.4.10. We let Y be a topological space such that:

(1) Y is T_1.
(2) Every compact subspace of Y is finite and discrete.
(3) Y is non-discrete.

Then the one-point compactification $X := Y^+$ is an example of a compact non-discrete space equipped with an open subspace all of whose compact subspaces are finite. Examples of spaces such as Y can be found in [**StSe**, Part II, 25, 26].

LEMMA E.2.4.11. *There exists an example of a Kelley compactly generated space that is not compactly generated.*

PROOF. Indeed, the space X is Example E.2.4.10 is an example of a Kelley compactly generated space (since Y is T_1 and $X - Y$ is a closed point) and we claim it is not compactly generated. If it was compactly generated then by Lemma E.1.2.25(2) every open subspace of X would have to be compactly generated, whereas Y is not (since $k(Y)$ is discrete but Y is not discrete). □

LEMMA E.2.4.12. *There exists an example of a topological space X and an open subspace $U \subset X$ such that*

(1) *X is Kelley compactly generated.*
(2) *U is not Kelley compactly generated.*

PROOF. Let X be the topological space from Example E.2.4.10. Then X is Kelley compactly generated as noted in the proof of Lemma E.2.4.11. On the other hand, upon choosing $U := Y$ as in the Example, Y is not compactly generated (also seen in the proof of Lemma E.2.4.11).[15] □

Compact generation is inherited by open subspaces by Lemma E.1.2.25, whereas by Lemma E.2.4.12 Kelley compact generation is not.

E.3. Small compactly generated weakly Hausdorff spaces

E.3.1. Preparations on small index categories and small compact Hausdorff spaces. We begin, as in [**SP**, Tags 046U, 000H], by introducing the notion of "size" of a compactly generated weakly Hausdorff topological space.

DEFINITION E.3.1.1. We define a function on cardinals κ:

$$\mathrm{Bound}(\kappa) := \max\{\kappa^{\aleph_0}, 2^{2^\kappa}, \kappa^\kappa\}. \tag{E.3.1.1}$$

Moreover, given a compactly generated weakly Hausdorff topological space X, we define:

$$\mathrm{size}(X) := \max\{\aleph_0, \lambda_1, \lambda_2\}$$

where λ_1 and λ_2 are cardinal numbers defined as follows:

(1) we denote by Σ_X the set of all compact Hausdorff subspaces $K \subset X$, and we define:

$$\lambda_1 := |\Sigma_X|.$$

(2) We define:

$$\lambda_2 := \sup\{|\mathrm{C}(K, \mathbf{C})|, K \in \Sigma_X\}$$

the supremum of the cardinalities of all the C^*-algebras of complex-valued continuous functions on all compact Hausdorff subspaces of X.

NOTATION E.3.1.1. We write, for a topological space X, $\mathrm{C}(X)$ to denote the algebra of complex-valued continuous functions on X.

We recall the construction of the Von Neumann universes, following [**Kunen**, Ch. 3, §2] and [**Jech**, Part I, p. 64]. We also refer the reader to [**SP**, Tag 05N1,000B], where the constructions that follow are used to define categories of schemes, with the aim of defining sites (in the sense of [**SP**, Tag 00VG]) and stacks over sites on categories of schemes. Our conventions concerning sites are aligned, throughout this work, to those in [**SP**, Tags 00UZ, 00VG] and, in particular, we shall not be allowing large categories as the categories underlying the sites to be used.

[15] Must show Y is not Kelley compactly generated.

The discussion in the present section is aimed at constructing a small subcategory of **CGWH** stable under enough categorical operations to serve the purpose of underlying a site, over which we shall be considering stacks in groupoids. We will be, in the process, arranging for the closure of such a category under enough many small limits and colimits to ensure our concrete constructions of interest will involve spaces always contained within it.

For any ordinal α, we construct the set V_α as follows, proceeding recursively. We define:

$$V_0 := \varnothing, \quad V_{\alpha+1} := 2^{V_\alpha},$$

with, recall, 2^{V_α} denoting the power-set of V_α, and for a limit ordinal α we further set:

$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta,$$

determining V_α by transfinite recursion. We refer the reader to [**Kunen**, Ch. III, Def. 2.1] and the discussion in loc.cit. following it, for further background on the sets V_α. We further refer the reader to the discussion before [**Jech**, Lemma 6.3]. We refer the reader to [**SP**, Tag 000B] and the subsequent section as well, where the V_α's are referred to as *partial universes*. We refer to the sets V_α as the Von Neumann's universe of level α. They satisfy the following properties, seen by transfinite induction (see [**Kunen**, Ch. III, Lemma 2.3]):

(1) each V_α is a transitive set,
(2) if $\alpha \leq \alpha'$ then $V_\alpha \subset V_{\alpha'}$.
(3) $\alpha \subset V_\alpha$.

The union **WF** of all the V_α as α varies across all limit ordinals is a proper class, called the class of *well-founded sets*. By [**Kunen**, Ch. III, Lemma 2.14], ultimately resting on the Axiom Schema of Regularity and the Axiom of Choice, every set in the class of all sets, and likewise every topological space in the class of all topological spaces, is an element of **WF**.

Before moving forward, we establish two preparatory results. The following lemma assumes the Axiom of Choice.

LEMMA E.3.1.2. *Let κ be a cardinal, and $\mathbf{Sets}_{\leq\kappa}$ the class of all sets J whose cardinality satisfies $|J| \leq \kappa$. Then there exists a sub-class $\mathrm{Sets}_{\leq\kappa} \subset \mathbf{Sets}_{\leq\kappa}$ such that:*

(1) $\mathrm{Sets}_{\leq\kappa}$ *is a set.*
(2) *For any $J \in \mathbf{Sets}_{\leq\kappa}$ there exists $J' \in \mathrm{Sets}_{\leq\kappa}$ and a bijective set-function $f : J' \to J$.*

PROOF. Let λ be a cardinal satisfying $\lambda \leq \kappa$. By the Axiom of Choice, there exists an ordinal α such that $|\alpha| = \lambda$. We denote $\alpha_\lambda := \inf\{\alpha \mid |\alpha| = \lambda\}$, the initial ordinal associated to λ. As $\alpha_\lambda \subset V_{\alpha_\lambda}$ and, for all cardinals λ, λ' satisfying $\lambda \leq \lambda' \leq \kappa$, we have[16] $\alpha_\lambda \leq \alpha_{\lambda'} \leq \alpha_\kappa$, we deduce the containments $V_{\alpha_\lambda} \subset V_{\alpha_{\lambda'}} \subset V_{\alpha_\kappa}$ by [**Kunen**, Ch. III, Lemma 2.3]. We set $\mathrm{Sets}_{\leq\kappa} := V_{\alpha_\kappa+1}$. For any element J of $\mathbf{Sets}_{\leq\kappa}$, there exists a least ordinal $\alpha_{|J|}$ with $|J| = |\alpha_{|J|}|$, and we have $\alpha_{|J|} \subset V_{\alpha_{|J|}} \subset V_{\alpha_\kappa}$, implying $\alpha_{|J|} \in V_{\alpha_\kappa+1} = \mathrm{Sets}_{\leq\kappa}$. In other words, there exists an element $J' \in \mathrm{Sets}_{\leq\kappa}$, itself a set, and a bijection $f : J' \to J$. □

DEFINITION E.3.1.3. For a cardinal ν, a *ν-small category* is any small category I whose underlying set of objects satisfies $|I| \leq \nu$ and for every pair of objects $i, i' \in I$ we have $|\mathrm{Map}_I(i, i')| \leq \nu$.

Building on Lemma E.3.1.2 we now prove an analog for small index categories[17]. Namely, we prove that there exists a *set* of small categories whose sets of objects and morphisms have cardinality bounded above by a fixed arbitrary cardinal ν, and such that any such small category is isomorphic to a category that is an element of such a set.

[16] Why? Give reference or explanation.

[17] The terminology "index category" is often used but never defined/described. Is it needed? Why not simply "category" and "small category"? Should clarify the meaning and rationale for using it.

LEMMA E.3.1.4. *Let ν be an infinite cardinal. There exists a set $\mathrm{Cat}_{\leq \nu}$ whose elements are ν-small index categories, and such that for every ν-small index category there exist an element $J' \in \mathrm{Cat}_{\leq \nu}$ and an isomorphism of categories $f : J' \to J$.*

We note that the lemma provides an *isomorphism* between ν-small categories, and not merely an equivalence of categories.

PROOF. We let $\mathrm{Sets}_{\leq \nu}$ be the set provided by Lemma E.3.1.2. For sets J_1, J_2 and maps $s, t : J_1 \rightrightarrows J_2$, $c : J_1 \times_{s,J_2,t} J_1$, and $e : J_2 \to J_1$, we define the formulas $P_i(s,t,c,e)$ for $i = 1,2,3,4$ as

$$P_1(s,t,c,e) : s \circ e = t \circ e = \mathrm{id} : J_2 \to J_2, \tag{E.3.1.2}$$

$$P_2(s,t,c,e) : c(s^{-1}(y) \cap t^{-1}(x), s^{-1}(z) \cap t^{-1}(y)) \subset s^{-1}(z) \cap t^{-1}(x), \text{ for all } x,y,z \in J_2 \tag{E.3.1.3}$$

(this encodes the two commutative diagrams in (E.4.1.1)),

$$P_3(s,t,c,e) : c(c(\cdot,\cdot),\cdot) = c(\cdot, c(\cdot,\cdot)) : J_1 \times_{s,J_2,t} J_1 \times_{s,J_2,t} J_1 \to J_1, \tag{E.3.1.4}$$

and

$$P_4(s,t,c,e) : c((e \circ t)(\cdot),\cdot) = c(\cdot,(e \circ s)(\cdot)) = \mathrm{id} : J_1 \to J_1. \tag{E.3.1.5}$$

Let $\mathbf{C}_{\leq \nu}$ be the class of 6-tuples (J_1, J_2, s, t, c, e) where J_1, J_2 are ν-small sets, and (s,t,c,e,i) are elements of the set:

$$(s,t,c,e,i) \in \mathrm{Map}(J_1, J_2) \times \mathrm{Map}(J_1 \times_{s,J_2,t} J_1, J_1) \times \mathrm{Map}(J_2, J_1).$$

We define the class $\mathrm{Cat}'_{\leq \nu}$:

$$\{(J_1, J_2, s, t, c, e),\ J_1, J_2 \in \mathrm{Sets}_{\leq \nu}, (s,t,c,e) \in (\mathrm{Map}(J_1, J_2), \mathrm{Map}(J_1 \times_{s,J_2,t} J_1, J_1), \mathrm{Map}(J_2, J_1)), \wedge_{i=1}^{4} P_i(s,t,c,e)\}.$$

The class $\mathrm{Cat}'_{\leq \nu}$ is a set, as it is contained in the set:

$$\mathcal{C}_{\leq \nu} := \coprod_{J_1, J_2 \in \mathrm{Set}_{\leq \nu}} \{(J_1, J_2)\} \times \mathrm{Map}(J_1, J_2) \times \mathrm{Map}(J_1 \times_{s,J_2,t} J_1, J_1) \times \mathrm{Map}(J_2, J_1)$$

via the injective class function:

$$\phi \subset \mathrm{Cat}'_{\leq \nu} \times \coprod_{J_1, J_2 \in \mathrm{Set}_{\leq \nu}} \{(J_1, J_2)\} \times \mathrm{Map}(J_1, J_2) \times \mathrm{Map}(J_1 \times_{s,J_2,t} J_1, J_1) \times \mathrm{Map}(J_2, J_1)$$

defined as

$$\phi := \{((J_1, J_2, s, t, c, e), (I_1, I_2, s', t', c', e')) \in \mathrm{Cat}'_{\leq \nu} \times \mathcal{C}_{\leq \nu} \mid (J_1, J_2, s, t, c, e) = (I_1, I_2, s', t', c', e')\},$$

the equality $(J_1, J_2, s, t, c, e) = (I_1, I_2, s', t', c', e')$ being defined within $\mathbf{C}_{\leq \nu}$. As $\mathcal{C}_{\leq \nu}$ is a set, so is $\mathrm{Cat}'_{\leq \nu}$ by the Axiom Schema of Replacement.

Let J be a ν-small category. By Lemma E.3.1.2, there exists $J_2 \in \mathrm{Sets}_{\leq \nu}$ together with a bijection $f : J_2 \to J$. We now let $J' := \coprod_{(i,j) \in J \times J} \mathrm{Map}_J(i,j)$. We define $s', t' := J' \rightrightarrows J$ on each $\mathrm{Map}_J(i,j)$ as $s'(a) := i$, $t'(a) := j$, alongside $e' : J \to J'$ defined as $e(i) := \mathrm{id}_i \in J'$, and $c' : J' \times_{s',J,t'} J' \to J'$ defined as $c'(a,b) := b \circ a$. As $|J'| \leq \max\{|J \times J|, \sup_{(i,j) \in J \times J} |\mathrm{Map}_J(i,j)|\}$, and J is a ν-small category with ν infinite, we have $|J'| \leq \nu$. By Lemma E.3.1.2, there exists $J_1 \in \mathrm{Sets}_{\leq \nu}$ and a bijection $g : J_1 \to J'$. We now define $s := f^{-1} \circ s' \circ g$, $t := f^{-1} \circ t' \circ g$, $c := g^{-1} \circ c' \circ (g,g)$ and $e := g^{-1} \circ e' \circ f$.

We have $P_1(s,t,c,e)$ from (E.3.1.2) because we have $P_1(s',t',c',e')$, and likewise $P_2(s,t,c,e)$ from (E.3.1.3), $P_3(s,t,c,e)$ from (E.3.1.4) and $P_4(s,t,c,e)$ from (E.3.1.5) because we have $P_i(s',t',c',e')$ for $i = 2,3,4$. It follows that (J_1, J_2, s, t, c, e) is an element of $\mathrm{Cat}'_{\leq \nu}$. Moreover, we define the category $\mathrm{Cat}(J_1, J_2, s, t, c, e)$

to be the category whose objects are the elements of J_2 and such that for any $(i,j) \in J_2 \times J_2$ we have $\mathrm{Map}(i,j) := s^{-1}(i) \cap t^{-1}(j)$. There is a functor $u : \mathrm{Cat}(J_1, J_2, s, t, c, e) \to J$ defined as $u(i) := f(i)$, and $u(\phi : i \to j) := g(\phi)$. The functor u is a bijection on sets of objects, and it is fully faithful by design of g. It is therefore an isomorphism of categories. We finally define the class function:

$$\psi \subset \mathrm{Cat}'_{\leq \nu} \times \mathbf{Cat}_{\leq \nu}$$

where $\mathbf{Cat}_{\leq \nu}$ is the class of ν-small categories, assigned as follows:

$$\psi := \{((J_1, J_2, s, t, c, e), J) \in \mathrm{Cat}'_{\leq \nu} \times \mathbf{Cat}_{\leq \nu} \mid \mathrm{Cat}(J_1, J_2, s, t, c, e) = J\}$$

where the equality $\mathrm{Cat}(J_1, J_2, s, t, c, e) = J$ is defined within $\mathbf{Cat}_{\leq \nu}$. As $\mathrm{Cat}'_{\leq \nu}$ is a set, by the Axiom Schema of Replacement so is the image of ψ. We define such image to be $\mathrm{Cat}_{\leq \nu}$. The set $\mathrm{Cat}_{\leq \nu}$ satisfies the requirements in the statement of the Lemma. □

We further recall the following well-known structural result on compact Hausdorff spaces:

LEMMA E.3.1.5. *Let K be a compact Hausdorff space, and κ a cardinal satisfying* $\mathrm{size}(K) \leq \kappa$ (*see Definition* E.3.1.1). *There exists a closed immersion*

$$\phi : K \to \prod_{\kappa} [0,1],$$

the product being endowed with the product topology.

PROOF. We call $i : [0,1] \to \mathbf{C}$ the closed immersion of the unit interval in $\mathbf{R}$. Then $i_* : \mathrm{C}(K, [0,1]) \to \mathrm{C}(K)$, defined as $i_*(f) := i \circ f$ for any $f \in \mathrm{C}(K, [0,1])$, is injective, and hence $|\mathrm{C}(K, [0,1])| \leq \kappa$. We will now construct a closed immersion:

$$\phi : K \to [0,1]^{\kappa},$$

where we define $[0,1]^{\kappa}$ to be $\prod_{\alpha_\kappa} [0,1]$ endowed with the product topology (whose α_κ is the least ordinal with cardinality κ). We define

$$\phi_K : K \to \prod_{a \in \mathrm{C}(K,[0,1])} [0,1]$$

by $\phi_K(x) := (a(x))_{a \in \mathrm{C}(K,[0,1])}$. The set-function ϕ_K is continuous. As $|\mathrm{C}(K, [0,1])| \leq \kappa$, we may call λ the cardinality of $\mathrm{C}(K, [0,1])$ with $\lambda \leq \kappa$, and select (by the Axiom of Choice) a bijection $j : \mathrm{C}(K, [0,1]) \to \alpha_\lambda$. We may therefore define $J := \{x \in \alpha_\kappa \mid x > y, \text{ for all } y \in \alpha_\lambda\}$, and we have $\alpha_\kappa = \alpha_\lambda \coprod J$.

We define

$$i_K : \prod_{a \in \mathrm{C}(K,[0,1])} [0,1] \to [0,1]^{\kappa}$$

by $i_K((x_a)_{a \in \mathrm{C}(K,[0,1])}) := (x_c)_{c \in \alpha_\kappa}$ where $x_c = x_a$ in $[0,1]$ for c of the form $c = j(a)$ (where such a is unique by injectivity of j), and $x_c = 0$ otherwise. The map i_K is continuous, and we may therefore define $\phi : K \to [0,1]^{\kappa}$ as $\phi := i_K \circ \phi_K$. The set-function ϕ is therefore continuous. It is also injective by Urysohn's Lemma, as K is compact Hausdorff (hence normal). Since K is compact and $[0,1]^{\kappa}$ is Hausdorff, ϕ is a closed immersion. □

We now construct, for any fixed cardinal κ, a set of representatives of all compact Hausdorff spaces of size at most κ up to homeomorphism.

LEMMA E.3.1.6. *Let κ be a cardinal, and $\mathcal{K}_{\leq \kappa}$ the class of all compact Hausdorff spaces K such that* $\mathrm{size}(K) \leq \kappa$. *Then there exists a sub-class* $\mathrm{CH}_{\leq \kappa} \subset \mathcal{K}_{\leq \kappa}$ *such that:*

(1) $\mathrm{CH}_{\leq \kappa}$ *is a set.*
(2) *For any $K \in \mathcal{K}_{\leq \kappa}$ there exists $K' \in \mathrm{CH}_{\leq \kappa}$ and a homeomorphism $f : K' \to K$.*

PROOF. We note that, by design of $\mathrm{size}(\cdot)$ (see Definition E.3.1.1), for any fixed compact Hausdorff space K with $\mathrm{size}(K) \leq \kappa$, the set of all compact Hausdorff subspaces of K has cardinality bounded above by κ,

and the algebra of continuous complex-valued functions on each compact Hausdorff subspace of K also has cardinality bounded above by κ. In particular,

$$|\mathrm{C}(K)| \leq \kappa.$$

We define $\mathcal{C}_\kappa$ to be the set of all closed subsets of $[0,1]^\kappa$, so all elements of $\mathcal{C}_\kappa$ are compact Hausdorff. We may therefore define $\mathrm{CH}_{\leq\kappa} := \mathcal{C}_\kappa \cap \mathcal{K}_{\leq\kappa}$, the intersection being formed within the class of all compact Hausdorff spaces. For any compact Hausdorff space K such that $\mathrm{size}(K) \leq \kappa$, by Lemma E.3.1.5 there exists a closed immersion $\phi : K \to [0,1]^\kappa$, and hence an element K' in $\mathrm{CH}_{\leq\kappa}$ and a homeomorphism $f : K' \to K$, defined as the inverse of the homeomorphism induced by ϕ from K onto the image $K' := \phi(K)$. □

We can now use Lemma E.3.1.2 and Lemma E.3.1.6 alongside Proposition E.2.1.13 to construct, for each fixed cardinal κ, a set of compactly generated and weakly Hausdorff spaces such that every object of **CGWH** of size at most κ is homeomorphic to an object of such a set. This construction will be used as a crutch to build a small subcategory of **CGWH** stable under enough categorical operations to enable us to make some intrinsic constructions of interest, and that will serve as underlying category for our sought-for "topological site". The following Lemma is an analog of [**SP**, Tag 000I].

Lemma E.3.1.7. *For every cardinal κ, there exists a set S of compactly generated weakly Hausdorff topological spaces, such that for every compactly generated weakly Hausdorff topological space X with* $\mathrm{size}(X) \leq \kappa$*, there exists an element $Y \in S$ and an isomorphism $Y \to X$ in* **CGWH**.

Proof. We consider the set S' defined as follows. For any $J \in \mathrm{Sets}_{\leq\kappa}$, where $\mathrm{Sets}_{\leq\kappa}$ is the set from Lemma E.3.1.2, we let f be a set-function $f : J \to \mathrm{CH}_{\leq\kappa}$, where $\mathrm{CH}_{\leq\kappa}$ is the set from Lemma E.3.1.6 above.

We define $S'_{(J,f)}$ to be the set of all closed equivalence relations in the topological space $(\coprod_J f(j)) \times (\coprod_J f(j))$ (product in **CGWH**, agreeing with the one in **Spaces**), and we further define:

$$S' := \coprod_{J \in \mathrm{Sets}_{\leq\kappa}} \coprod_{f \in \mathrm{Map}(J, \mathrm{CH}_{\leq\kappa})} S'_{(J,f)}$$

itself a set. For any $s \in S'$, there exists $J \in \mathrm{Sets}_{\leq\kappa}$ and $f \in \mathrm{Map}(J, \mathrm{CH}_{\leq\kappa})$, and a closed equivalence relation $R \subset (\coprod_J f(j)) \times (\coprod_J f(j))$, with $s = R$. We define $q(s) := (\coprod_J f(j))/R$, a compactly generated and weakly Hausdorff space by Lemma E.2.1.11.

We now define the class function

$$F := \{(s, X) \mid s \in S', X \in \mathrm{Ob}(\mathbf{CGWH}), q(s) = X\}$$

where the formula $q(s) = X$ holds in the class $\mathrm{Ob}(\mathbf{CGWH})$. We call S the image of F. By the Axiom Schema of Replacement, S is a set. By Proposition E.2.1.13, any compactly generated and weakly Hausdorff topological space X with $\mathrm{size}(X) \leq \kappa$ is a quotient of a coproduct of all compact Hausdorff subspaces of X indexed by a set J of cardinality at most κ (the set of all compact Hausdorff subspaces of X). By Lemma E.3.1.6, such subspaces are each homeomorphic to a compact Hausdorff space which is an element of $\mathrm{CH}_{\leq\kappa}$, and the index set is bijective to a set in $\mathrm{Sets}_{\leq\kappa}$. The closed equivalence relation on the coproduct of all compact Hausdorff subspaces of X is therefore homeomorphic to an element of S', and via the class function F there exists a compactly generated weakly Hausdorff space Y which is an element of S and a homeomorphism $Y \to X$. □

E.3.2. α-small compactly generated and weakly Hausdorff spaces. We assign the following:

Definition E.3.2.1. Let α be an ordinal, and ν be a cardinal.

(1) The category $\mathbf{CGWH}_\alpha$ has as its objects the compactly generated weakly Hausdorff topological spaces whose underlying sets are elements of V_α, and its morphisms are continuous maps of topological spaces. We call α-*small* the compactly generated and weakly Hausdorff spaces that are the objects of $\mathbf{CGWH}_\alpha$.

(2) For $\mathcal{C}$ a category, I a ν-small category, and $P : I \to \mathcal{C}$ a diagram, we call P a *ν-small diagram* and denote by $\lim_I P$ respectively $\mathrm{colim}_I P$ its ν-small limit respectively ν-small colimit if representable in $\mathcal{C}$.
(3) We say that a category $\mathcal{C}$ *contains all ν-small limits* respectively *contains all ν-small colimits* (or that ν-small limits, respectively ν-small colimits, are *representable in* $\mathcal{C}$) if and only if for any ν-small diagram $P : I \to \mathcal{C}$ the object $\lim_I P$ respectively $\mathrm{colim}_I P$ exists in $\mathcal{C}$.

Remark E.3.2.2. On Definition E.3.2.1, we make the following observation. Every topological space is homeomorphic to one whose underlying set is in the class **WF**. This is [**Kunen**, Lemma 2.14], and justifies the approach taken throughout to bound the size of the topological spaces in our sought-for small subcategory of the category **CGWH** of all compactly generated weakly Hausdorff topological spaces. We refer the reader to [**Jech**, Lemma 6.3] that establishes that every set is in some V_α, as a consequence of the Axiom of Regularity in ZF^-. In particular, this makes no use of the Axiom of Choice.

The following result is an analog of [**SP**, Tag 000J]. We keep the notation similar so as to enable the reader familiar with loc.cit. to closely follow the analogous steps within the arguments.

Lemma E.3.2.3. *Let S_0 be a set of compactly generated weakly Hausdorff topological spaces and ν an arbitrary infinite cardinal. Then there exists a limit ordinal α, depending on S_0 and ν, such that:*

(1) *S_0 is contained in V_α.*
(2) *For any α-small space X and any compactly generated weakly Hausdorff topological space Y with*

$$\mathrm{size}(Y) \leq \mathrm{Bound}(\mathrm{size}(X))$$

(see (E.3.1.1)*) there exists an α-small space X' and a homeomorphism $X' \to Y$.*
(3) *ν-small limits and colimits are representable in $\mathbf{CGWH}_\alpha$ and the forgetful functor to* **CGWH** *preserves them.*

The following argument is an adaptation to the context of this work of the argument of [**SP**, Tag 000J], with minor adjustments and complements.

Proof. We define a function f on ordinals as follows. We set $f(0) = 0$. Suppose $f(\alpha)$ is already defined. We define $f(\alpha + 1)$ to be the smallest ordinal β such that:

(1) $\max\{\alpha + 1, f(\alpha)\} \leq \beta$.
(2) For any object T of $\mathbf{CGWH}_{f(\alpha)}$ and any compactly generated weakly Hausdorff X with $\mathrm{size}(X) \leq \mathrm{Bound}(\mathrm{size}(T))$, there exists an object $T'$of $\mathbf{CGWH}_\beta$ and a homeomorphism $T' \to X$.
(3) For any ν-small index category I and any functor $F : I \to \mathbf{CGWH}_{f(\alpha)}$, $\lim_I F$ and $\mathrm{colim}_I F$ are isomorphic to objects of $\mathbf{CGWH}_\beta$.

We proceed in a very similar manner as in [**SP**, Tag 000J] to prove that β indeed exists.

We set $\kappa := \sup_T \mathrm{Bound}(\mathrm{size}(T))$ where T varies across all objects of $\mathbf{CGWH}_{f(\alpha)}$, which is a small category by design. The cardinal κ therefore exists. We apply Lemma E.3.1.7 to κ to obtain the set S of that Lemma. We now repeat the argument at this stage of [**SP**, Tag 000J]. Up to isomorphism of small index categories, we may require that I in (3) is an element of the set of ν-small categories $\mathrm{Cat}_{\leq \nu}$ provided by Lemma E.3.1.4, so the *pair* (I, F) in part (3) therefore varies over a *set* of pairs (I, F) where I is an element of $\mathrm{Cat}_{\leq \nu}$, and F is a functor $I \to \mathbf{CGWH}_{f(\alpha)}$. We may therefore find a set S' such that for any pair (I, F) as in (3) above, the limit and colimit are isomorphic to elements of S'. Upon choosing any β such that $\beta \geq \max\{\alpha + 1, f(\alpha)\}$ and $S \cup S' \subset V_\beta$, all the required axioms hold for β. Upon minimizing over such ordinals β, we see that $f(\alpha + 1)$ is well-defined by transfinite recursion. One then extends f to limit ordinals by setting:

$$f(\alpha) := \sup_{\beta < \alpha} f(\beta).$$

We now choose an ordinal γ such that $S_0 \subset V_\gamma$. As $f(\delta) \geq \delta$ for all ordinals δ. and $f(\delta') \geq f(\delta)$ whenever $\delta' \geq \delta$ by design of f, we see that

$$S_0 \subset V_{f(\gamma)} \subseteq V_{f(\beta)}$$

for all $\beta \geq \gamma$. We choose a limit ordinal $\gamma' > \gamma$ with cofinality greater than ν and set $\alpha := f(\gamma')$. The verification that α is the sought-for ordinal is now identical to that in [**SP**, Tag 000J]. We repeat it here merely because we wish to record an argument for compactly generated weakly Hausdorff topological spaces without referring to an argument for schemes, even though formally identical.

Property (1) is satisfied because we chose $\gamma' > \gamma$. We have, as γ' is a limit ordinal:

$$f(\gamma') = \sup_{\beta<\gamma'} f(\beta).$$

The set $\{f(\beta) \mid \beta < \gamma'\} \subset f(\gamma')$ is a cofinal subset, and we get:

$$V_\alpha = V_{f(\gamma')} = \bigcup_{\beta<\gamma'} V_{f(\beta)}.$$

For any object T in $\mathbf{CGWH}_\alpha$, we call $\beta(T)$ the ordinal such that T is an object of $\mathbf{CGWH}_{f(\beta)}$, minimal with respect to this property. It follows $\beta(T) < \gamma'$. Since the objects of $\mathbf{CGWH}_{\beta(T)+1}$ are contained in $\mathbf{CGWH}_{f(\beta+1)}$ with $\beta + 1 < \gamma'$, so they are objects of $\mathbf{CGWH}_\alpha$, the second property is also seen to be satisfied.

The last part of the argument is identical to the last paragraph in [**SP**, Tag 000J], with only notational changes; we repeat it here for the sake of completeness. Suppose that for a set J with $|J| \leq \nu$, we have a J-indexed collection $\{S_j, j \in J\}$ of objects of $\mathbf{CGWH}_\alpha$. The function $\nu \to \gamma'$ given by $j \mapsto \beta(S_j)$ is such that its image is not a cofinal subset, since the cofinality of γ' is greater than ν. There therefore exists $\beta < \gamma'$ such that $\{S_j, j \in J\} \subset \mathbf{CGWH}_{f(\beta)}$, and so every functor $F : I \to \mathbf{CGWH}_\alpha$ as in part (3) of the Lemma factors through a subcategory $\mathbf{CGWH}_{f(\beta)}$ for some appropriate $\beta < \gamma'$ depending on I and F. The limit, respectively colimit, of the diagram $F : I \to \mathbf{CGWH}_\alpha$ is, by design of f, therefore isomorphic to an object of $\mathbf{CGWH}_{f(\beta+1)}$, which is in turn a subcategory of $\mathbf{CGWH}_\alpha$. □

E.3.3. Size-bound criteria. We have the following Lemma, in analogy with [**SP**, Tag 000P].

LEMMA E.3.3.1. *Let X be a compact Hausdorff topological space. Then:*

$$\mathrm{size}(X) \leq \max\{\aleph_0, 2^{|\mathrm{C}(X)|}\}.$$

See Definition E.3.1.1 for the definition of size$(\cdot)$ on $\mathbf{CGWH}$.

PROOF. The number of compact Hausdorff subspaces of X is at most $\max\{\aleph_0, 2^{|\mathrm{C}(X)|}\}$ because if $K \subset X$ is compact Hausdorff then by Urysohn's Lemma K is the zero locus of an ideal $I \subset \mathrm{C}(X)$ (namely, I is the ideal of f vanishing on X), and the number of such ideals is at most $2^{|\mathrm{C}(X)|}$. □

LEMMA E.3.3.2. *Let X be a compactly generated weakly Hausdorff topological space, and $\{K_i, i \in I\}$ a collection of compact Hausdorff subspaces of X such that:*

$$\bigcup_{i\in I} K_i = X.$$

Then we have:

$$\mathrm{size}(X) \leq \max\{\aleph_0, (\sup_{i\in I} \mathrm{size}(K_i))^{|I|}\}.$$

PROOF. Let K be a compact Hausdorff subspace of X, and call $H_i := K \cap K_i$ for all $i \in I$. As K is compact Hausdorff and X is weakly Hausdorff, K is closed in X. In particular, H_i is closed in K_i, hence

compact Hausdorff and closed in X. Since $\bigcup_{i\in I} K_i = X$, we have $\bigcup_{i\in I} H_i = K$. As the map

$$q : \coprod_{i\in I} H_i \to K$$

is surjective, the set-function

$$\mathrm{C}(K) \to \prod_{i\in I} \mathrm{C}(H_i)$$

is injective. We deduce the bound

$$|\mathrm{C}(K)| \leq |\prod_{i\in I} \mathrm{C}(H_i)|.$$

As K_i is normal and $H_i \subset K_i$ is closed, Tietze's Extension Theorem gives that $\mathrm{C}(K_i) \to \mathrm{C}(H_i)$ is surjective for every $i \in I$. We deduce the bound:

$$|\prod_{i\in I} \mathrm{C}(H_i)| \leq |\prod_{i\in I} \mathrm{C}(K_i)|.$$

As $\sup_{i\in I} |\mathrm{C}(K_i)| \leq \sup_{i\in I} \mathrm{size}(K_i) := \lambda$, there exists an initial ordinal ω_λ for λ and we further obtain the cardinal bound $|\prod_{i\in I} \mathrm{C}(K_i)| \leq |\prod_{i\in I} \omega_\lambda| = \lambda^{|I|}$. We deduce the cardinal bound:

$$|\mathrm{C}(K)| \leq \lambda^{|I|}.$$

The set of compact Hausdorff subspaces of X has cardinality at most the cardinality of the set

$$\prod_{i\in I} \Sigma_i$$

where Σ_i is the set of compact Hausdorff subspaces of K_i. Indeed, upon writing Σ_X for the set of compact Hausdorff subspaces of X, the set-function $\phi : \Sigma_X \to \prod_{i\in I} \Sigma_i$ assigned by $\phi(K) := (K \cap K_i)_{i\in I}$ is injective, and hence $|\Sigma_X| \leq |\prod_{i\in I} \Sigma_i|$. Since each K_i is itself compact Hausdorff, so $|\Sigma_i| \leq \mathrm{size}(K_i)$ for every $i \in I$,

$$|\Sigma_X| \leq (\sup_{i\in I} \mathrm{size}(K_i))^{|I|}.$$

We may therefore assemble the bounds obtained to deduce that we have

$$\mathrm{size}(X) = \max\{\aleph_0, |\Sigma_X|, \sup\{|\mathrm{C}(K)|, K \in \Sigma_X\}\} \leq \max\{\aleph_0, \lambda^{|I|}\}.$$

□

Lemma E.3.3.3. *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of compactly generated weakly Hausdorff topological spaces. Then:*

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\mathrm{Bound}(\mathrm{size}(X)), \mathrm{Bound}(\mathrm{size}(Z))\}.$$

Proof. Let $\{K_i, i \in I_Y\}$ be a collection of compact Hausdorff topological subspaces of Y without repetition such that $\coprod_{i\in I_Y} K_i \to Y$ is submersive. This can be found in light of Proposition E.2.1.13. We have $|I_Y| \leq \mathrm{size}(Y)$. The closed preimages $X_i := f^{-1}(K_i)$ and $Z_i := g^{-1}(K_i)$ in X respectively Z are compactly generated by Lemma E.1.2.25 and weakly Hausdorff. For each $i \in I_Y$, by Proposition E.2.1.13, there exist index sets I_{X_i} and I_{Z_i} with $|I_{X_i}| \leq |\Sigma_{X_i}|$ and $|I_{Z_i}| \leq |\Sigma_{Z_i}|$, with Σ_{X_i} and Σ_{Z_i} the sets of compact Hausdorff subspaces of X_i, Z_i, respectively, and compact Hausdorff subspaces $H_{ij} \subset X_i$ and $F_{im} \subset Z_i$ for every $j \in I_{X_i}$, $m \in I_{Z_i}$, such that we have submersions

$$\coprod_{j\in I_{X_i}} H_{ij} \to X_i, \quad \coprod_{m\in I_{Z_i}} F_{im} \to Z_i.$$

It follows that, upon denoting $J := \coprod_{i\in I_Y} I_{X_i} \times I_{Z_i}$, we have, upon denoting (j,m) the elements of J, with $j \in I_{X_i}, m \in I_{Z_i}$ for a unique $i \in I_Y$:

$$\bigcup_{(j,m)\in J} (H_{ij} \times_{f,K_i,g} F_{im}) = X \times_{f,Y,g} Z.$$

As H_{ij} and F_{im} are compact Hausdorff for every $i \in I_Y$, $j \in I_{X_i}$ and $m \in I_{Z_i}$, we have that $H_{ij} \times^0_{f,K_i,g} F_{im}$ is closed in $H_{ij} \times_0 F_{im}$, which is compact Hausdorff. It follows that $H_{ij} \times^0_{f,K_i,g} F_{im}$ is compact Hausdorff and hence compactly generated and weakly Hausdorff by Lemma E.2.3.1. We therefore have that the topological fibre product $H_{ij} \times^0_{f,K_i,g} F_{im}$ computes the fibre product $H_{ij} \times_{f,K_i,g} F_{im}$ in **CGWH** by Lemma E.2.2.5(1) and Lemma E.1.2.2(1). By Lemma E.3.3.2, we have

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, \sup_{(j,m)\in J} (\mathrm{size}(H_{ij} \times^0_{f,K_i,g} F_{im}))^{|J|}\}$$

and by Lemma E.3.3.1 we further have

$$\mathrm{size}(H_{ij} \times^0_{f,K_i,g} F_{im}) \leq \max\{\aleph_0, 2^{|\mathrm{C}(H_{ij} \times^0_{f,K_i,g} F_{im})|}\},$$

so

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, (\sup_{(j,m)\in J} \max\{\aleph_0, 2^{|\mathrm{C}(H_{ij} \times^0_{f,K_i,g} F_{im})|}\}^{|J|}\}$$

and hence

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, \aleph_0^{|J|}, (\sup_{(j,m)\in J} 2^{|\mathrm{C}(H_{ij} \times^0_{f,K_i,g} F_{im})|})^{|J|}\}.$$

We now check that

$$|\mathrm{C}(H_{ij} \times^0_{f,K_i,g} F_{im})| = |\mathrm{C}(H_{ij}) \otimes_{\mathrm{C}(K_i)} \mathrm{C}(F_{im})| \leq \max\{\aleph_0, |\mathrm{C}(H_{ij})|, |\mathrm{C}(F_{im})|\},$$

where the tensor product is an algebraic tensor product. The elements of the Banach algebra $\mathrm{C}(H_{ij} \times^0_{f,K_i,g} F_{im})$ extend to $\mathrm{C}(H_{ij} \times F_{im})$ by the Tietze Extension Theorem, and in the latter algebra everything is the limit of a sequence from $\mathrm{C}(H_{ij}) \otimes_{\mathrm{C}(K_i)} \mathrm{C}(F_{im})$ (density by Stone-Weierstrass), so

$$|\mathrm{C}(H_{ij}) \otimes_{\mathrm{C}(K_i)} \mathrm{C}(F_{im})| \leq |\mathrm{C}(H_{ij})| \times |\mathrm{C}(F_{im})| \times \aleph_0.$$

By [**Kunen**, Ch. I, Cor. 10.13] the right side is, in turn, $\max\{\aleph_0, |\mathrm{C}(H_{ij})|, |\mathrm{C}(F_{im})|\}$. As

$$(\sup_{(j,m)\in J} \max\{\aleph_0, |\mathrm{C}(H_{ij})|, |\mathrm{C}(F_{im})|\})^{|J|} = \max\{\aleph_0^{|J|}, (\sup_{(j,m)\in J} |\mathrm{C}(H_{ij})|)^{|J|}, (\sup_{(j,m)\in J} |\mathrm{C}(F_{im})|)^{|J|}\}$$

and

$$(\sup_{(j,m)\in J} |\mathrm{C}(H_{ij})|)^{|J|} \leq (\mathrm{size}(X))^{|J|}, \quad (\sup_{(j,m)\in J} |\mathrm{C}(F_{im})|)^{|J|} \leq (\mathrm{size}(Z))^{|J|},$$

upon assembling all the bounds so far obtained we have

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, \aleph_0^{|J|}, (2^{\mathrm{size}(X)})^{|J|}, (2^{\mathrm{size}(Z)})^{|J|}\}. \tag{E.3.3.1}$$

We further have:

$$|J| = \sup_{i\in I_Y} |I_{X_i}| \times |I_{Z_i}| \leq \sup_{i\in I_Y} |\Sigma_{X_i}| \times |\Sigma_{Z_i}|.$$

By [**Kunen**, Ch. I, Cor. 10.13], the right side is $\max\{\sup_{i\in I_Y} |\Sigma_{X_i}|, \sup_{i\in I_Y} |\Sigma_{Z_i}|\}$, which is bounded above by $\max\{\mathrm{size}(X), \mathrm{size}(Z)\}$, as X_i is closed in X and Z_i is closed in Z, and hence the maps $\Sigma_{X_i} \to \Sigma_X$ and $\Sigma_{Z_i} \to \Sigma_Z$ sending a compact Hausdorff subspace K of X_i, respectively Z_i, to itself as a compact Hausdorff subspace of X, respectively Z, are injective. We deduce from (E.3.3.1) the cardinal bound:

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, 2^{\max\{\mathrm{size}(X),\mathrm{size}(Z)\}}, (\mathrm{size}(X))^{\max\{\mathrm{size}(X),\mathrm{size}(Z)\}}, (\mathrm{size}(Z))^{\max\{\mathrm{size}(X),\mathrm{size}(Z)\}}\}$$

which in turn yields:

$$\mathrm{size}(X \times_{f,Y,g} Z) \leq \max\{\aleph_0, 2^{\max\{\mathrm{size}(X),\mathrm{size}(Z)\}}, \max\{(\mathrm{size}(X))^{\mathrm{size}(X)}, (\mathrm{size}(Z))^{\mathrm{size}(Z)}\}\}.$$

Finally, the right side is bounded above by $\max\{\mathrm{Bound}(\mathrm{size}(X)), \mathrm{Bound}(\mathrm{size}(Z)\}$ and the Lemma is proved. □

LEMMA E.3.3.4. *For S_0 and V as in Lemma* E.3.2.3*, let α be as constructed there. Let Y be a compactly generated weakly Hausdorff topological space and $\{K_i, i \in I\}$ a collection of compact Hausdorff subspaces of Y such that $\coprod_{i\in I} K_i \to Y$ us submersive. Assume that there exists an object X in $\mathbf{CGWH}_\alpha$ such that*

(1) $\mathrm{size}(K_i) \leq \mathrm{size}(X)$ *for all* $i \in I$,
(2) $|I| \leq \mathrm{size}(X)$.

Then Y is homeomorphic to an object of $\mathbf{CGWH}_\alpha$.

PROOF. This holds as the size of Y is, by Lemma E.3.3.2, at most:

$$\max\{\aleph_0, (\sup_{i\in I} \mathrm{size}(K_i))^{|I|}\} \leq \mathrm{Bound}(\mathrm{size}(X)).$$

By Lemma E.3.2.3(2) there exists an object X' of $\mathbf{CGWH}_\alpha$ and a homeomorphism $X' \to Y$. □

E.3.4. Closure properties of $\mathbf{CGWH}_\alpha$. We collect here the properties that the category $\mathbf{CGWH}_\alpha$ enjoys.

PROPOSITION E.3.4.1. *For S_0 a set of compactly generated weakly Hausdorff topological spaces, ν an infinite cardinal, and α a limit ordinal as provided by Lemma E.3.2.3, the category $\mathbf{CGWH}_\alpha$ satisfies the following properties:*

(1) *ν-small limits and colimits are representable in $\mathbf{CGWH}_\alpha$ and the forgetful functor to $\mathbf{CGWH}$ commutes with them.*
(2) *Fibre products exist in $\mathbf{CGWH}_\alpha$ and the forgetful functor to $\mathbf{CGWH}$ commutes with them.*
(3) *Countable coproducts exist in $\mathbf{CGWH}_\alpha$ and the forgetful functor to $\mathbf{CGWH}$ commutes with them.*
(4) *For any object X in $\mathbf{CGWH}_\alpha$ and any immersion $Z \to X$ with Z compactly generated weakly Hausdorff, there exists an object Z' in $\mathbf{CGWH}_\alpha$ and a homeomorphism $Z' \to Z$.*
(5) *For any object X in $\mathbf{CGWH}_\alpha$ and any open, respectively closed, respectively locally closed, immersion $Z \to X$ (so Z is compactly generated weakly Hausdorff by Lemma* E.1.2.25*), there exists an object Z' in $\mathbf{CGWH}_\alpha$ and a homeomorphism $Z' \to Z$.*

PROOF. By Lemma E.3.2.3(3), part (1) holds. Parts (2) and (3) are a consequence of part (1) since ν is infinite. We prove parts (4) and (5). In both cases, the cardinality of the set I of compact Hausdorff subspaces of Z is bounded above by $\mathrm{size}(X)$. Likewise, for any compact Hausdorff subspace K of Z, we have $\mathrm{size}(K) \leq \mathrm{size}(X)$, as $\Sigma_K \hookrightarrow \Sigma_X$ because the image of K under the immersion $Z \to X$ is a compact Hausdorff subspace of X as well. We now observe that the set I defined above is such that $\bigcup_{K\in I} K$ is Z by Proposition E.2.1.13. By Lemma E.3.3.4, we deduce that Z is homeomorphic to an object of $\mathbf{CGWH}_\alpha$. This proves (4) and (5). □

REMARK E.3.4.2. We do not address whether the function space $\mathscr{C}ont(X, Y)$, for α-small compactly generated topological spaces X and Y, is homeomorphic to an object of $\mathbf{CGWH}_\alpha$, as this would be of no use for the purposes of this appendix. Such objects would not endow $\mathbf{CGWH}_\alpha$ with the structure of a cartesian closed symmetric monoidal category. Indeed, the obstruction to $\mathbf{CGWH}_\alpha$ being a cartesian closed symmetric monoidal category *such that the forgetful functor to* $\mathbf{CGWH}$ *is a strict closed functor*; i.e., it preserves internal Hom-bifunctors, is that the category $\mathbf{CGWH}_\alpha$ is not replete. This is because it is a small category by design, whereas $\mathbf{CGWH}$ is large. If $\mathbf{CGWH}_\alpha$ was cartesian closed, and if the forgetful functor (which, in light of Proposition E.3.4.1(1) is product-preserving and hence automatically a "strong

closed functor" between symmetric monoidal categories) were a *strict* closed functor then $\mathbf{CGWH}_\alpha$ would have to contain all function spaces of all pairs of its objects. This cannot be, as Example E.3.4.3 below shows. Proposition E.3.4.1 shows, however, that despite not being cartesian closed, the category $\mathbf{CGWH}_\alpha$ does still enjoy important consequences of cartesian-closedness of $\mathbf{CGWH}$ that we have collected in the previous sections. Most importantly, formation of products in $\mathbf{CGWH}_\alpha$ commutes with *representable* colimits. Proposition E.3.4.1 also ensures that ν-small colimits (and ν-small limits) are representable.

Example E.3.4.3. We let $X_0 := \varnothing$, equipped with the empty topology. We note that X is compactly generated and weakly Hausdorff (by definition, or, for instance, by Remark E.2.1.2). We further let $Y := \{0,1\}$ endowed with the discrete topology, so Y is also compactly generated and weakly Hausdorff. For any ordinal β, we call:

$$X_{\beta+1} := \mathscr{C}ont(X_\beta, Y)$$

therefore assigning X_β by transfinite recursion. All the X_β are compactly generated and weakly Hausdorff by Proposition E.2.2.10. However, the size of the underlying set of $X_{\beta+1}$ is $\beth_\beta$ (see [**Kunen**, Ch. I, §10, Def. 10.44]) which, for large enough β, exceeds the size of the von Neumann universe V_α (see [**Kunen**, Ch. III, §3, Lemma 2.13]) that contains all of the objects of $\mathbf{CGWH}_\alpha$. We let $\gamma := \inf\{\beta \mid X_\beta \notin V_\alpha\}$, so X_γ is an example of an object of $\mathbf{CGWH}_\alpha$ such that $\mathscr{C}ont(X_\gamma, Y)$ is not an object of $\mathbf{CGWH}_\alpha$ but only is homeomorphic to such an object.

We denote by:

$$u_\alpha : \mathbf{CGWH}_\alpha \to \mathbf{CGWH} \tag{E.3.4.1}$$

the forgetful functor. The functor u_α is fully faithful by design, and it commutes with ν-small limits and ν-small colimits by Proposition E.3.4.1. In particular, it commutes with finite limits and finite colimits.

We conclude this subsection with a description of important classes of morphisms within $\mathbf{CGWH}_\alpha$, in the spirit of Proposition E.2.3.15.

Proposition E.3.4.4. *We have:*

(1) *The monomorphisms in* $\mathbf{CGWH}_\alpha$ *are the injective continuous maps.*
(2) *The epimorphisms in* $\mathbf{CGWH}_\alpha$ *are the continuous maps with dense image.*
(3) *The regular monomorphisms in* $\mathbf{CGWH}_\alpha$ *are the closed immersions.*
(4) *The regular epimorphisms in* $\mathbf{CGWH}_\alpha$ *are the submersive maps.*
(5) *The effective monomorphisms in* $\mathbf{CGWH}_\alpha$ *are the closed immersions.*
(6) *The effective epimorphisms in* $\mathbf{CGWH}_\alpha$ *are the submersive maps.*
(7) *The universally effective epimorphisms in* $\mathbf{CGWH}_\alpha$ *are the submersive maps.*
(8) *The isomorphisms in* $\mathbf{CGWH}_\alpha$ *are the homeomorphisms.*
(9) *Monomorphisms in* $\mathbf{CGWH}_\alpha$ *are stable under products, coproducts and compositions.*
(10) *Regular monomorphisms in* $\mathbf{CGWH}_\alpha$ *are stable under products, coproducts and compositions.*
(11) *Epimorphisms in* $\mathbf{CGWH}_\alpha$ *are stable under finite products, coproducts and compositions.*
(12) *Regular epimorphisms in* $\mathbf{CGWH}_\alpha$ *are stable under finite products, coproducts and compositions.*
(13) *Every morphism in* $\mathbf{CGWH}_\alpha$ *is universally the composition of a regular epimorphism followed by a monomorphism.*
(14) $\mathbf{CGWH}_\alpha$ *is a regular category.*

We recall the notion of regular category (see [**vO**, Def.4.1]): a category is regular when it contains finite limits, the kernel pair of any morphism has a co-equalizer, and regular epimorphisms are stable under arbitrary base change.

Proof. As the forgetful functor u_α is faithful, it reflects monomorphisms, proving (1), and epimorphisms, proving (2). As u_α is fully faithful, it is conservative, i.e. it reflects isomorphisms. We now prove (3) and

(5). By Lemma E.3.4.1(1), $\mathbf{CGWH}_\alpha$ contains equalizers and co-kernel pairs, and the forgetful functor u_α to $\mathbf{CGWH}$ commutes with them. It follows that u_α reflects effective monomorphisms, and that a monomorphism in $\mathbf{CGWH}_\alpha$ is regular if and only if it is effective. Suppose $f : X \to Y$ is a closed immersion of objects of $\mathbf{CGWH}_\alpha$. Then $u_\alpha(f)$ is a closed immersion in $\mathbf{CGWH}$, hence an effective monomorphism in $\mathbf{CGWH}$ by Proposition E.2.3.15. It follows f that is an effective monomorphism in $\mathbf{CGWH}_\alpha$. Conversely, if f is an effective monomorphism in $\mathbf{CGWH}_\alpha$, then $u_\alpha(f)$ is an effective monomorphism in $\mathbf{CGWH}$, hence a closed immersion by Proposition E.2.3.15(5) and therefore f is a closed immersion.

We now prove (4) and (6). By part (1) of Lemma E.3.4.1, the category $\mathbf{CGWH}_\alpha$ contains co-equalizers and kernel pairs. It follows that epimorphisms are regular if and only if they are effective. Moreover, as u_α is fully faithful and commutes with finite limits and colimits by part (1) of Lemma E.3.4.1, u_α reflects effective epimorphisms. Suppose $f : X \to Y$ is a submersive map of objects in $\mathbf{CGWH}_\alpha$. Then $u_\alpha(f)$ is a submersive map in $\mathbf{CGWH}$, hence an effective epimorphism in $\mathbf{CGWH}$ by part (4) of Proposition E.2.3.15. As u_α reflects effective epimorphisms, it follows that f is an effective epimorphism in $\mathbf{CGWH}_\alpha$, equivalently a regular epimorphism. Conversely, assume f is an effective epimorphism. As u_α commutes with co-equalizers by part (1) of Lemma E.3.4.1, $u_\alpha(f)$ is a regular epimorphism, hence a submersive map, implying that f is a submersive map. This proves (4) and (6).

We now prove part (7). Since $\mathbf{CGWH}_\alpha$ contains fibre products, and u_α commutes with them, submersive maps between objects of $\mathbf{CGWH}_\alpha$ remain submersive under base change along an arbitrary morphism in $\mathbf{CGWH}_\alpha$, since the same is true in $\mathbf{CGWH}$ by part (4) of Lemma E.2.3.6. By part (6), submersive maps are universally effective epimorphisms. Conversely, since universally effective epimorphisms are, in particular, effective epimorphisms, they are submersive maps by (6). Part (8) follows, for instance, from parts (2) and (3). For parts (9) and (11), we observe that since u_α reflects monomorphisms and epimorphisms, then part (9) and part (11) follow from part (9) and Proposition E.2.3.15(11). As u_α reflects effective monomorphisms and effective epimorphisms, then by (3) and (4), parts (10) and (12) follow from Proposition E.2.3.15(10, 12). Part (13) is a formal consequence of the fact that $\mathbf{CGWH}_\alpha$ is closed under finite limits and finite colimits by Lemma E.3.4.1(1), and regular epimorphisms are stable under arbitrary base change by (4) and (7). These three conditions say that $\mathbf{CGWH}_\alpha$ is a *regular category* (see [**vO**, Def. 4.1]) and by [**vO**, Prop. 4.2], which proves (14). □

The above Proposition E.3.4.4 is saying that even though $\mathbf{CGWH}_\alpha$ is not cartesian closed as a cartesian symmetric monoidal subcategory of $\mathbf{CGWH}$ (i.e. a cartesian symmetric monoidal category such that the forgetful functor u_α is a strict closed functor), it inherits the good properties of $\mathbf{CGWH}$ collected in this appendix, many of which are a consequence of cartesian closure of $\mathbf{CGWH}$.

E.3.5. The site of small compactly generated weakly Hausdorff spaces. We let:

(1) S_0 be an arbitrary set of compactly generated weakly Hausdorff spaces.
(2) ν an arbitrary infinite cardinal.

We shall refer to the datum (S_0, ν) as *basic setup*. By Lemma E.3.2.3 there exists a limit ordinal α and a *small* category $\mathbf{CGWH}_\alpha$ with the following properties:

(1) S_0 is contained in the objects of $\mathbf{CGWH}_\alpha$.
(2) I-indexed limits and colimits exist in $\mathbf{CGWH}_\alpha$ and are preserved by the forgetful functor to $\mathbf{CGWH}$, as long as the cardinality of I is at most ν.

When paired with Proposition E.3.4.1 on closure properties of $\mathbf{CGWH}_\alpha$, we obtained Proposition E.3.4.4 that gives a topological description of key classes of morphisms in $\mathbf{CGWH}_\alpha$. We are allowed to choose a basic setup (S_0, ν) that fits our requirements.

We are now going to use Proposition E.3.4.4 to construct a site. The topology of our choosing will be the canonical topology itself. Proposition E.3.4.4 enables a concrete description for it (see Lemma E.3.5.7).

The methods in use within this subsection follow [SP, Tag 00VH], and [KS, Ch. 16-17], and particularly [KS, Ch. 16, §16.1] and [KS, Ch. 17, §17.1-17.2].

We begin with the following construction:

Construction E.3.5.1. We let S_0' be the smallest set of compactly generated weakly Hausdorff topological spaces, satisfying the following properties:

(1) S_0' contains the following topological groups:
   (a) $\mathbf{Q}$, $\mathbf{Q}^\times$ with the discrete topology.
   (b) $\mathbf{Z}$ and all its finite abelian quotients, all with the discrete topology.
   (c) $\mathbf{Z}_p^\times$, $\mathbf{Z}_p$ with the p-adic topology.
   (d) $\mathbf{R}$, $\mathbf{R}^\times$, $\mathbf{C}$, $\mathbf{C}^\times$, all with the Euclidean topology.
   (e) $\mathbf{A}_{\mathbf{Q}}$ with the adelic topology.
   (f) $\mathbf{A}_{\mathbf{Q}}^\times$ with the idelic topology.
   (g) The closed subgroups of all of the above.
(2) S_0' contains $\mathbf{CP}^1$.
(3) S_0' is closed under countable products and coproducts in $\mathbf{CGWH}$.
(4) For any $X \in S_0'$, all closed equivalence relations $R \subset X \times X$ are contained in S_0, and the quotients X/R are contained in S_0'.

We let $\nu := \mathrm{Bound}(\sup_{X \in S_0'}\{\mathrm{size}(X)\})$ (see (E.3.1.1) from Definition E.3.1.1 for the definition of the "size" of a compactly generated weakly Hausdorff space, alongside the Bound$(\cdot)$ function on cardinals) and call S_0'' the set $\mathrm{CH}_{\leq \nu}$ provided by Lemma E.3.1.6. For $S_0 := S_0' \cup S_0''$. the pair (S_0, ν) is our choice of basic setup. We let α be the limit ordinal provided by Lemma E.3.2.3 associated to this choice of S_0 and ν.

We refer the reader to §E.3.1 for the construction of $\mathrm{CH}_{\leq \nu}$, and likewise for the construction of $\mathrm{Sets}_{\leq \nu}$ and $\mathrm{Cat}_{\leq \nu}$. These are sets, the first a set of representatives for the homeomorphism classes of all compact Hausdorff spaces of size bounded above by ν, the second a set of representatives for all sets of cardinality bounded above by ν, the third a set of representatives for the isomorphism classes of small categories, whose sets of objects and morphisms has cardinality bounded above by ν. Constructing the three sets was carried out in Lemma E.3.1.2, Lemma E.3.1.4, and Lemma E.3.1.6.

We now construct three classes of coverings on $\mathbf{CGWH}_\alpha$, for α the limit ordinal constructed in Construction E.3.5.1, depending on the basic setup (S_0, ν) chosen in Construction E.3.5.1.

Definition E.3.5.2. We define:

(1) The class $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ whose elements are families of closed immersions $\{f_i : T_i \to T\}_{i \in I}$ for $I \in \mathrm{Sets}_{\leq \nu}$ and $T \in S_0$, for (S_0, ν) the basic setup constructed in Construction E.3.5.1, and T_i a compact Hausdorff subspace of T and f_i the resulting closed (since T is weakly Hausdorff) immersion for all $i \in I$.
(2) The class $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)$ whose elements are families of continuous maps $\{f_i : T_i \to T\}_{i \in I}$ of α-small compactly generated weakly Hausdorff spaces such that the induced continuous map
$$\coprod_{i \in I} f_i : \coprod_{i \in I} T_i \to T$$
is a submersive map.
(3) The class $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)$ whose elements are families of open immersions $\{f_i : T_i \to T\}_{i \in I}$ of α-small compactly generated weakly Hausdorff spaces such that the induced continuous map
$$\coprod_{i \in I} f_i : \coprod_{i \in I} T_i \to T$$
is surjective.

We call the elements of the class $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ *fundamental coverings*, the elements of the class $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)$ *submersive coverings*, and the elements of the class $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)$ *open coverings*.

We need the following:

LEMMA E.3.5.3. *The class* $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ *is a set.*

PROOF. The class $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ is a sub-class of:

$$\coprod_{(T,I)\in S_0\times \mathrm{Sets}_{\leq \nu}} \mathrm{Map}(I, \mathrm{CH}(T))$$

where $\mathrm{CH}(T)$ is the set of all compact Hausdorff subspaces of T, and $\mathrm{Map}(I, \mathrm{CH}(T))$ is the set of set-functions $I \to \mathrm{CH}(T)$. Indeed, given an element $\mathcal{U} := \{f_i : T_i \to T\}_{i\in I}$ of $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$, we may define $q(\mathcal{U}) := f \in \mathrm{Map}(I, \mathrm{CH}(T))$, where $f : I \to \mathrm{CH}(T)$ is defined as $f(i) := T_i$. We see that q is an injective class function:

$$q : \mathrm{Cov}_0(\mathbf{CGWH}_\alpha) \to \coprod_{(T,I)\in S_0\times \mathrm{Sets}_{\leq \nu}} \mathrm{Map}(I, \mathrm{CH}(T))$$

and by the Axiom of Separation $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ is a set since $\mathrm{Map}(I, \mathrm{CH}(T))$ is, and the co-product is indexed by a product of two sets, so it is therefore itself a set. □

We proceed as in [SP, Tag 000W] to obtain a site in the sense of [SP, Tag 00VH], and [KS, Ch. 16-17].

CONSTRUCTION E.3.5.4. By [SP, Tag 000W], we pick out a *set* of coverings in $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)$ and $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)$, as follows. There exists a cardinal κ and a limit ordinal β, such that:

$$\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}, \text{ for } i = 1, 2 \tag{E.3.5.1}$$

is a set of submersive coverings, for $i = 1$, respectively open coverings, for $i = 2$, contained within the classes $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)$, respectively $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)$, such that:

(1) $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ contains $\mathrm{Cov}_0(\mathbf{CGWH}_\alpha)$ for $i = 1, 2$.
(2) $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ is contained in the Von Neumann universe V_β for $i = 1, 2$.
(3) For every $\{f_i : T_i \to T\}_{i\in I} \in \mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$, for $i = 1, 2$, we have $|I| \leq \kappa$.
(4) For every $\{f_i : T_i \to T\}_{i\in I} \in \mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$, for $i = 1, 2$, the defining axioms for coverings (see [SP, Tag 00VH]) hold.
(5) every element in $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)$, respectively $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)$, is combinatorially equivalent to (informally: has a refinement coming from) some element of $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$, for $i = 1, 2$ respectively.

That large enough κ and β as in Construction E.3.5.4 exist is the content of [SP, Tag 000W].

DEFINITION E.3.5.5. We define:

(1) J_1 the topology on $\mathbf{CGWH}_\alpha$ generated by $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$. We refer to J_1 as the *submersive-covering topology* on $\mathbf{CGWH}_\alpha$.
(2) J_2 the topology on $\mathbf{CGWH}_\alpha$ generated by $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$. We refer to J_2 as the *open-covering topology* on $\mathbf{CGWH}_\alpha$.

REMARK E.3.5.6. On Definition E.3.5.5, we make the following observations.

(1) The pairs $(\mathbf{CGWH}_\alpha, \mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta})$ are sites in the sense of [SP, Tag 00VH], and in the sense of [KS, Ch. 17, Def. 17.2.1]. We further refer the reader to [KS, Ch. 16, Def. 16.1.2], [KS, Ch. 16, Def. 16.1.5], and [SP, Tag 00YW] on how to relate the two approaches to the construction of categories of sheaves of sets on sites whose underlying categories are small, through the concept of topology generated by a set of coverings (see [KS, Ch. 16, Def. 16.1.5] and the discussion from [KS, Ch. 16, p. 392] following the Definition, and [SP, Tags 00YZ, 00Z1]). In particular, the pairs $(\mathbf{CGWH}_\alpha, J_1)$ and $(\mathbf{CGWH}_\alpha, J_2)$ are sites in the sense of [KS, Ch. 17, Def. 17.2.1].

(2) We recall that the topology (in the sense of [SP, Tag 00Z4], which is also [KS, Ch. 16, Def. 16.1.2]) on the small category (pre-site, in the terminology of [KS, Ch. 17, Def. 17.1.1]) $\mathbf{CGWH}_\alpha$ *generated* by $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ for $i = 1, 2$ respectively is the assignment $T \rightsquigarrow J_i(T)$, for every object $T \in \mathrm{Ob}(\mathbf{CGWH}_\alpha)$, where $J_i(T)$ is the set of sieves $S \subset h_T$ of the form $S_{\mathcal{U}} = S(\{f_j : T_j \to T\}_{j\in I})$, for some element $\mathcal{U} = \{f_j : T_j \to T\}_{j\in I} \in \mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\alpha}$, where $S_{\mathcal{U}}$ is the unique smallest sieve S such that each f_j belongs to $S(T_j)$. Such $S_{\mathcal{U}}$ indeed exists by part (4) of [SP, Tag 00YZ] (see also the discussion on [KS, Ch. 16, §1, p. 392] after [KS, Ch. 16, Def. 16.1.5]).

(3) We note that the topology J_1 is finer than the topology J_2 (see [SP, Tag 00Z6]), which is to say that for every object T of $\mathbf{CGWH}_\alpha$, $J_2(T) \subset J_1(T)$. Indeed, for any open covering $\{f_i : T_i \to T\}_{i\in I}$, the map:

$$\coprod_{i\in I} f_i : \coprod_{i\in I} T_i \to T$$

is open and surjective, and hence submersive by [SP, Tag 02YB]. The element $\{f_i : T_i \to T\}_{i\in I}$ of $\mathrm{Cov}_2(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ is therefore an element of $\mathrm{Cov}_1(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ and the sieve it generates is a covering sieve for J_1, i.e. it is contained in $J_1(T)$. This shows the containment $J_2(T) \subset J_1(T)$. In particular, to show that J_2 is sub-canonical, it is enough to show that J_1 is.

(4) We note that the ordinal β and the cardinal κ in the definition of $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$ provided by [SP, Tag 000X] for $i = 1, 2$ respectively are a priori distinct, and should be denoted β_1, κ_1 and β_2, κ_2. Upon inspection of the proof of [SP, Tag 000X], one can replace each β_i, respectively κ_i, with some $\beta \geq \max\{\beta_1, \beta_2\}$, respectively $\kappa \geq \max\{\kappa_1, \kappa_2\}$, and arrange for the same β and the same κ to ensure the existence of $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$, for $i = 1, 2$, satisfying the desired properties.

(5) We note that the dependency of the topologies J_1, J_2 on $\mathbf{CGWH}_\alpha$ on β and κ, while a priori stringent, is mitigated by [SP, Tag 00VY], from which we deduce that we may replace $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)_{\kappa,\beta}$, for $i = 1, 2$, respectively, with any other set of coverings of the same class $\mathrm{Cov}_i(\mathbf{CGWH}_\alpha)$ respectively (see parts (2) and (3) of Definition E.3.5.2) such that every element $\mathcal{U} \in \mathrm{Cov}_i(\mathbf{CGWH}_\alpha)$ is combinatorially equivalent to an element of the set, without changing the topos of sheaves of sets with respect to the resulting topology. In other words, the dependency of the topoi $\mathrm{Shv}(\mathbf{CGWH}_\alpha, J_i)$ for $i = 1, 2$ on β and κ is not too stringent.

We conclude this section with one final consideration on the topologies J_1 and J_2. It is that J_1 is sub-canonical (and hence so is J_2 by part (3) of Remark E.3.5.6). In fact, we shall show that J_1 is the canonical topology itself on $\mathbf{CGWH}_\alpha$ (which is, recall, the coarsest topology with respect to which all representable pre-sheaves are sheaves; see [SP, Tag 00ZA], and also [SP, Tags 00Z8,00Z9] applied to the collection of all pre-sheaves of sets).

LEMMA E.3.5.7. *The topology J_1 on $\mathbf{CGWH}_\alpha$ is the canonical topology.*

PROOF. This is a direct consequence of Proposition E.3.4.4(7), which implies that the covering sieves of J_1 are exactly those generated by jointly surjective families of universal effective epimorphisms. By [SGA3, Exp .IV, Prop. 2.3], J_1 is the canonical topology on $\mathbf{CGWH}_\alpha$. □

REMARK E.3.5.8. We point out to the reader [Les, Prop. 4.7], where a characterization of the canonical topology on **CGWH** is provided. This can be used to deduce Lemma E.3.5.7. We caution the reader that we have endeavoured to only define topologies on "pre-sites" in the sense of [KS, Ch. 17, Def. 17.1.1], hence without allowing large categories. One can, however, define the concept of a Grothendieck topology on any category, as in [SGA4, Exp. II, §1], and hence define the canonical topology as in [SGA4, Exp. II, §2, Def. 2.5]. We further point out that, in this sense, one can define the canonical topology on the large category **Spaces**. It turns out that it is defined by those covering sieves generated by families of maps that are jointly surjective and jointly universally submersive, where a continuous map is universally submersive if and only if it remains

submersive under arbitrary topological base change. It is a feature of **CGWH** that submersive maps are stable under arbitrary base change internally in **CGWH** (see Lemma E.2.3.6(6)), and one that does not hold not only in **Spaces** but also in **CG**, the category of compactly generated spaces (see Example E.1.2.16). This is one of the reasons for adopting **CGWH** as ambient category for the relevant site.

We conclude by introducing some terminology, to be used throughout the remainder of this appendix.

Definition E.3.5.9. For the choice of limit ordinal α made in Construction E.3.5.1, and for the choices of cardinal κ and limit ordinal β made in Construction E.3.5.4, we denote by:

(1) $\mathbf{CGWH}_{\alpha,\tau}$ the site $(\mathbf{CGWH}_\alpha, J_1)$.
(2) $\mathbf{CGWH}_{\alpha,\sigma}$ the site $(\mathbf{CGWH}_\alpha, J_2)$.

We refer to $\mathbf{CGWH}_{\alpha,\tau}$ as the *topological τ-site*, and to $\mathbf{CGWH}_{\alpha,\sigma}$ as the *topological σ-site*.

We will be mostly using the topological τ-site throughout the remainder of this appendix, with only occasional reference to the σ-site. In other words, the "open-covering" topology will play a very limited role.

E.4. Topological groupoids

The stacks of interest for the purposes of this appendix will always be fibered in groupoids. In fact they will always be quotient stacks associated to explicit topological groupoids. The aim of this section is to collect here what is needed on the subject of topological groupoids for our later use with 2-categorical constructions in §E.5-§E.6.

E.4.1. Groupoids in sets and internal groupoids. There exist several sources within the literature covering some of the material of this section. We will be occasionally following [**Hig**], [**CdSW**, Part VI] and [**N**, §3.2]. In terms of notation, we follow the notational choices of [**SP**, Tag 022L] closely, in analogy with [**SP**, Tag 0437] for "groupoids in schemes" and "groupoids in spaces", so as to facilitate the reader who wishes to skip this section and the next on a first reading and rely on their own familiarity with the algebraic counterparts of the topics discussed within them.

We first recall the following:

Definition E.4.1.1. A *groupoid in sets* is the datum (X, R, s, t, c, e, i) of:

(i) a set denoted X which we shall refer as the *set of objects*,
(ii) a set denoted R which we shall refer as the *set of morphisms*,
(iii) set-functions $s, t : R \rightrightarrows X$ respectively called *source* and *target* morphisms,
(iv) a set-function
$$c : R \times_{s,X,t} R \to R$$
called the *composition law*,
(v) a set-function $e : X \to R$ called the *unit section*,
(vi) a set-function $i : R \to R$ called the *inversion*,

satisfying the following axioms:

(1) Associativity:
(a) the diagrams

(E.4.1.1)
$$\begin{array}{ccc} R \times_{s,X,t} R & \xrightarrow{c} & R \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow[t]{} & X \end{array} \qquad \begin{array}{ccc} R \times_{s,X,t} R & \xrightarrow{c} & R \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle s} \\ R & \xrightarrow[s]{} & X \end{array}$$

commute,

(b) we have the equality of set-functions

$$c(\mathrm{id}_R, c) = c \circ (c, \mathrm{id}_R) : R \times_{s,X,t} R \times_{s,X,t} R \rightrightarrows R \tag{E.4.1.2}$$

(2) Identity:

(a) we have the equality of set-functions

$$s \circ e = t \circ e = \mathrm{id} : X \rightrightarrows X \tag{E.4.1.3}$$

(b) we have the equalities of pairs of set-functions

$$c \circ (\mathrm{id}, (e \circ s)) = c \circ ((e \circ t), \mathrm{id}) = \mathrm{id} : R \rightrightarrows R \tag{E.4.1.4}$$

(3) Inverse:

(a) As set-functions $R \to X$, we have the equality:

$$s \circ i = t, t \circ i = s : R \to X \tag{E.4.1.5}$$

(b) As set-functions $R \to R$, we have the equality:

$$c \circ (\mathrm{id}, i) = e \circ t, c \circ (i, \mathrm{id}) = e \circ s : R \to R \tag{E.4.1.6}$$

We call the set-functions s, t, c, e, i the *structure morphisms* of the groupoid.

Morphisms of groupoids are pairs of set-functions

$$f = (f_1 : X \to X', f_2 : R \to R') \tag{E.4.1.7}$$

compatible with the structure morphisms of (X, R, s, t, c, e, i) and $(X', R', s', t', c', e', i')$; i.e., the following diagrams commute:

(1) (Compatibility with source and target)

(E.4.1.8)
$$\begin{array}{ccc} R & \longrightarrow & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ X & \longrightarrow & X' \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ X & \longrightarrow & X' \end{array}$$

(2) (Compatibility with the composition laws)

(E.4.1.9)
$$\begin{array}{ccc} R \times_{s,X,t} R & \xrightarrow{(f_2,f_2)} & R' \times_{s',X',t'} R' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ R & \xrightarrow[f_2]{} & R' \end{array}$$

(3) (Compatibility with units)

(E.4.1.10)
$$\begin{array}{ccc} X & \xrightarrow{f_1} & X' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ R & \xrightarrow[f_2]{} & R' \end{array}$$

(4) (Compatibility with inversions)

(E.4.1.11)
$$\begin{array}{ccc} R & \xrightarrow{f_2} & R' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ R & \xrightarrow[f_2]{} & R' \end{array}$$

We define the category **Groupoids** whose objects are the class of all groupoids in sets and morphisms are morphisms of groupoids.

NOTATION E.4.1.1. We abbreviate the notation (X, R, s, t, c, e, i) to $R \rightrightarrows X$, or (X, R, s, t, c), when no confusion is expected to arise. Moreover, for a morphism between groupoids assigned by a pair of set-functions (f_1, f_2) as in (E.4.1.7), we often write f for both f_1 and f_2, so as to simplify the notation. Source and target of each f are expected to leave no ambiguity as to whether an f is a set-function on objects or on morphisms.

REMARK E.4.1.2. On Definition E.4.1.1, we make the following observations:

(1) We note that the datum of a groupoid in sets is equivalent to the datum of an (effective) small category (see Definition E.5.1.15) all of whose morphisms are isomorphisms. Indeed, given a groupoid in sets (X, R, s, t, c, e, i), we may construct the category $\mathcal{C}_R$ whose set of objects is defined to be $\mathrm{Ob}(\mathcal{C}_R) := X$, and whose morphisms are defined as follows. For any pair of objects $x, y \in \mathrm{Ob}(\mathcal{C}_R)$, $\mathrm{Hom}_{\mathcal{C}_R}(x, y) := s^{-1}(x) \cap t^{-1}(y)$. The axioms from Definition E.4.1.1 ensure that the above assignment indeed is a (small) category. We will occasionally use this perspective, especially throughout the development and use of the desired theory of stacks in groupoids on a suitable site on topological spaces. For $g \in \mathcal{R}$ with $s(g) = x$ and $t(g) = y$, we may write $g : x \to y$ to view g as a morphism $x \to y$ in $\mathcal{C}_R$.
(2) In (E.4.1.8) for the definition of morphisms of groupoids, concerning compatibility with the composition laws we note that by (E.4.1.9) on compatibility with source and target morphisms the set-function $(f_2, f_2) : R \times R \to R' \times R'$ restricted to $R \times_{s,X,t} R \subset R \times R$ factors through a set-function $R \times_{s,X,t} R \to R'_{s',X',t'} R'$; it is *this* set-function that is being referred to in Axiom (2) on compatibility with composition laws. We denote this set-function by (f_2, f_2) again, as it is uniquely determined by $(f_2, f_2) : R \times R \to R' \times R'$, expecting no confusion to arise.

We now prove a first lemma that reduces the task of verifying the defining compatibilities of a morphism of groupoids:

$$(f_1, f_2) : (X, R, s, t, c, e, i) \to (X', R', s', t', c', e', i')$$

to the task of verifying compatibility of f_1 and f_2 with (s, t, c) and (s', t', c') only.

LEMMA E.4.1.3. *Let (X, R, s, t, c, e, i) and $(X', R', s', t', c', e', i')$ be two groupoids in sets. Let $f_1 : X \to X'$ and $f_2 : R \to R'$ be set-functions. The following conditions are equivalent:*

(1) *(f_1, f_2) is a morphism of groupoids.*
(2) *f_1 and f_2 satisfy the following compatibilities:*
   (a) *$s' \circ f_2 = f_1 \circ s$.*
   (b) *$t' \circ f_2 = f_1 \circ t$.*
   (c) *$f_2 \circ c(\cdot, \cdot) = c'(f_2(\cdot), f_2(\cdot))$.*

PROOF. In terms of the associated small categories $\mathcal{C}_R$ and $\mathcal{C}_{R'}$, condition (2) says (f_1, f_2) defines a covariant functor $F : \mathcal{C}_R \to \mathcal{C}_{R'}$. Since any covariant functor respects identity maps and inverses of isomorphisms, we are done. □

The following result is also a direct consequence of the axioms, and will be used later.

LEMMA E.4.1.4. *Let (X, R, s, t, c, e, i) be a groupoid in sets, and $x, y, z, w \in R$ such that:*

(1) *(x, y), (z, w) are elements of $R \times_{s,X,t} R$.*
(2) *$c(x, y) = c(z, w)$.*

Then:

$$c(c(i(y), i(x)), c(z, w)) = (e \circ s)(y) = (e \circ s)(w).$$

PROOF. In terms of the associated small category $\mathcal{C}_R$, we're given a commutative diagram of isomorphisms

$$\begin{array}{ccc} a & \xrightarrow{y} & b \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle x} \\ b' & \xrightarrow[z]{} & c \end{array}$$

and the assertion is $y^{-1}x^{-1}(zw) = \mathrm{id}_a$, which is clear. □

E.4.2. Groupoids internal to CGWH. We now assign the notion of groupoid internal to any category. While it will be useful to have the general concept within this appendix, we shall not need it beyond this appendix apart from the notion of groupoids internal to the category **CGWH**, the large category of topological spaces constructed in Definition E.2.2.1.

DEFINITION E.4.2.1. Let $\mathscr{A}$ be a category, and assume fibre products are representable in $\mathscr{A}$. An *$\mathscr{A}$-groupoid* is the datum $(X, \mathcal{R}, s, t, c, e, i)$ with $X, \mathcal{R}$ objects of $\mathscr{A}$, and $s, t : \mathcal{R} \rightrightarrows X$, $c : \mathcal{R} \times_{s,X,t} \mathcal{R} \to \mathcal{R}$, $i : \mathcal{R} \to \mathcal{R}$, $e : X \to \mathcal{R}$ morphisms in $\mathscr{A}$ such that the diagrams as in Definition E.4.1.1 commute. We denote the category of $\mathscr{A}$-groupoids by **$\mathscr{A}$-Grpd**.

A *map* of groupoids $(X, \mathcal{R}) \to (X', \mathcal{R}')$ is a pair of morphisms $f_1 : X \to X'$ and $f_2 : \mathcal{R} \to \mathcal{R}'$ such that the analogues of (E.4.1.8)–(E.4.1.11) commute.

REMARK E.4.2.2. We define $\mathrm{PShv}(\mathscr{A})$ the functor category $\mathbf{Functors}(\mathscr{A}^{\mathrm{opp}}, \mathbf{Sets})$ and refer to it as the category of *pre-sheaves of sets on* $\mathscr{A}$, following [**SGA4**, Exp. I, §1, Def. 1.3] and [**SP**, Tag 00V1]. We denote by:

$$h : \mathscr{A} \to \mathrm{PShv}(\mathscr{A})$$

the functor-of-points functor (see [**SP**, Tag 001O]), defined by the assignment, for every object X of $\mathscr{A}$:

$$T \rightsquigarrow h_X(T) := \mathrm{Hom}_{\mathscr{A}}(T, X),$$

and we write $h(f)$ for the effect of h on morphisms f in $\mathscr{A}$, as opposed to h_f. In particular, we shall denote by $h(f)(T)$ the value at the object T of $\mathscr{A}$ of the morphism of pre-sheaves of sets $h(f) : h_X \to h_Y$, for a morphism $f : X \to Y$ in $\mathscr{A}$. We may then assign the notion of *groupoid-object* in $\mathscr{A}$ by means of analogy with Definition E.4.1.1: a groupoid-object in $\mathscr{A}$ is the datum of objects $X, \mathcal{R}$ of $\mathscr{A}$, and morphisms:

$$s, t : \mathcal{R} \rightrightarrows X, \quad c : \mathcal{R} \times_{s,X,t} \mathcal{R} \to \mathcal{R}, \quad e : X \to \mathcal{R}, \quad i : \mathcal{R} \to \mathcal{R}$$

such that for every object T in $\mathscr{A}$, the datum:

$$(h_X(T), h_{\mathcal{R}}(T), h(s)(T), h(t)(T), h(c)(T), h(e)(T), h(i)(T))$$

is a groupoid in sets in the sense of Definition E.4.1.1. Morphisms of internal $\mathscr{A}$-groupoids are likewise pairs of morphisms $(f_1, f_2) : (X, \mathcal{R}) \to (X', \mathcal{R}')$ in $\mathscr{A}$ such that $(h(f_1)(T), h(f_2)(T))$ is a morphism of groupoids in sets in the sense of Definition E.4.1.1 for every object T of $\mathscr{A}$.

DEFINITION E.4.2.3. We define the category of *topological groupoids* to be the category **CGWH-Grpd** of Definition E.4.2.1. We shall be referring to the objects of **CGWH-Grpd** as *topological groupoids*.

The above definition means that we are assuming that both sets of objects and morphisms of the groupoid are compactly generated weakly Hausdorff topological spaces, and that all the structure morphisms are continuous maps. Note that **CGWH** does contain all fibre products, by Lemma E.2.2.5(1).

REMARK E.4.2.4. We observe that for an arbitrary subspace $T \subset X$, the subspaces $s^{-1}(T)$ and $t^{-1}(T)$ of $\mathcal{R}$ may not be compactly generated. When $T = \{x\}$ is a point, however, they both are, since s, t are continuous maps between compactly generated weakly Hausdorff spaces, and weakly Hausdorff spaces are T_1-spaces (every

point is the continuous image of $\{*\}$ and thus is closed), hence T is closed, and so are $s^{-1}(T)$ and $t^{-1}(T)$. By Lemma E.1.2.25, these subspaces are compactly generated, and they are weakly Hausdorff (as are all subspaces). This is, more generally, the case when T is open, or locally closed, again by Lemma E.1.2.25.

We have the following:

LEMMA E.4.2.5. *Let $(X, \mathcal{R}, s, t, c, e, i)$ be a topological groupoid. Then:*

(1) *the unit map e is a closed immersion.*
(2) *The source and target maps $s, t : \mathcal{R} \rightrightarrows X$ are submersive.*

PROOF. The assertions follow from Lemma E.2.3.4, as e is a continuous section to both s, t and all topological spaces are objects of **CGWH**. □

LEMMA E.4.2.6. *Let $(X, \mathcal{R}, s, t, c, e, i)$ be a topological groupoid. Then the fibre product $\mathcal{R} \times^0_{s,X,t} \mathcal{R}$ is closed as a subspace of $\mathcal{R} \times \mathcal{R}$ and so belongs to* **CGWH** *coinciding with $\mathcal{R} \times_{s,X,t} \mathcal{R}$.*

PROOF. This is a special case of Lemma E.2.2.5(1), but we review the proof in this case for the reader's convenience. Indeed, we have, as subspaces of $\mathcal{R} \times_0 \mathcal{R}$:

$$\mathcal{R} \times^0_{s,X,t} \mathcal{R} = (s, t)^{-1}(\Delta_X(X)).$$

Since X is compactly generated weakly Hausdorff, by Lemma E.2.1.6 the diagonal subspace $\Delta_X(X) \subset X \times_0 X$ is k-closed. As $(s, t) : \mathcal{R} \times_0 \mathcal{R} \to X \times_0 X$ is continuous, the subspace $\mathcal{R} \times^0_{s,X,t} \mathcal{R} \subset \mathcal{R} \times_0 \mathcal{R}$ is also k-closed. It follows that $\mathcal{R} \times^0_{s,X,t} \mathcal{R}$ is a closed subspace of $\mathcal{R} \times \mathcal{R}$, and as such it is compactly generated by Lemma E.1.2.25(2), and weakly Hausdorff (as are all subspaces of $\mathcal{R} \times \mathcal{R}$). In particular, $\mathcal{R} \times^0_{s,X,t} \mathcal{R} = \mathcal{R} \times_{s,X,t} \mathcal{R}$, as desired. □

LEMMA E.4.2.7. *Let $(X, \mathcal{R}, s, t, c, e, i)$ be a topological groupoid. For any $x \in X$, the subspace of $\mathcal{R}$:*

$$\mathcal{R}_x := s^{-1}(x) \cap t^{-1}(x) \tag{E.4.2.1}$$

is a compactly generated weakly Hausdorff topological group, closed as a subspace of $\mathcal{R}$.

PROOF. As compactly generated weakly Hausdorff spaces are T_1-spaces, $\mathcal{R}_x$ is T_1 as a subspace of $\mathcal{R}$, and as such it is compactly generated by Lemma E.1.2.25 and weakly Hausdorff (as are all subspaces of $\mathcal{R}$). Since the group law, and the inversion homomorphisms are induced by c and i on $\mathcal{R}$, they are continuous, making $\mathcal{R}_x$ into a topological group. □

There's an evident formal analogy between the datum of a groupoid in sets and that of an equivalence relation on a set. We make this analogy precise:

REMARK E.4.2.8. We make the following observations:

(1) Let (X, R, s, t, c, e, i) be a groupoid in sets. Then the image of the set-function:

$$(t, s) : R \to X \times X$$

is an equivalence relation on X. Note the order of source and target morphisms being exchanged in the definition of (t, s). Conversely, given an equivalence relation $R \subset X \times X$, there exists, up to isomorphism of groupoids in sets, a unique groupoid:

$$(X, R', s, t, c, e, i)$$

such that:
(1) $j : R' \to X \times X$ is injective.
(2) $R = j(R') \subset X \times X$.

Indeed, we may set $R' := R$, and s, t the compositions of the inclusion $R \subset X \times X$ with the two projections $p_2, p_1 : X \times X \to X$, respectively, where p_1 is the left projection, and p_2 is the right projection.

(2) If $(X, \mathcal{R}, s, t, c, e, i)$ is a topological groupoid, then the image of (t, s) endowed with the subspace topology is a topological equivalence relation on $X \times X$, and the above construction provides a method to construct a topological groupoid from a topological equivalence relation. We caution the reader that such relation may not be closed, as is typically the case when the groupoid is an action groupoid associated to the continuous, non-proper action of a (compactly generated weakly Hausdorff) topological group.

Beware that the topology on the image of (t, s) is, by convention, the quotient topology and this may not be the subspace topology inherited from $X \times X$; i.e., (t, s) may not be a strict map (see part (3) of Terminology E.0.0.2). Topological groupoids therefore generalize the assignment of topological equivalence relations, allowing more flexibility beyond the case of immersed equivalence relations in $X \times X$. The resulting quotients therefore necessitate a more robust geometric theory than that of topological quotients, which leads us to the necessity of an appropriate theory of stacks, to supply the otherwise missing geometric objects serving as the quotients.

E.4.2.1. *Constant groupoids.* The following construction and the subsequent Lemma explain how one can view a space X as a "constant" topological groupoid.

Construction E.4.2.9. Let X be an object in **CGWH**. We denote by $\mathbf{Const}_X$ the datum:

$$(X, \Delta(X), s = \mathrm{id}, t = \mathrm{id}, e = \mathrm{id}, c = \mathrm{id}, i = \mathrm{id}) \tag{E.4.2.2}$$

where $\Delta(X)$ is the diagonal subspace within $X \times X$. Then $\mathbf{Const}_X$ is a topological groupoid.

Definition E.4.2.10. For X a compactly generated weakly Hausdorff space, we refer to $\mathbf{Const}_X$ as the *constant groupoid* on X.

Lemma E.4.2.11. *The assignment* Const *defined as* $X \rightsquigarrow \mathbf{Const}_X$ *on objects of* **CGWH** *and as* $(f, f) : \mathbf{Const}_X \to \mathbf{Const}_Y$ *on morphisms* $f : X \to Y$ *in* **CGWH**, *is a fully faithful functor:*

$$\mathrm{Const} : \mathbf{CGWH} \to \mathbf{CGWH\text{-}Grpd}. \tag{E.4.2.3}$$

Proof. The set $\mathrm{Hom}_{\mathbf{CGWH}}(X, Y)$ is sent under Const onto the subset of

$$\mathrm{Hom}_{\mathbf{CGWH\text{-}Grpd}}(\mathbf{Const}_X, \mathbf{Const}_Y)$$

given by the pairs of maps $f = (f_1 : X \to Y, f_2 : \Delta(X) \to \Delta(Y))$ where $f_1 = f_2$. The spaces of morphisms of $\mathbf{Const}_X, \mathbf{Const}_Y$ contain the identities only, and so this subset is the entire Hom-set in **CGWH-Grpd**, since the only morphisms $(f, f') : \mathbf{Const}_X \to \mathbf{Const}_Y$ of topological groupoids must, for every $x \in X$, preserve the identities; i.e., $f'(e(x)) = e'(f(x))$. This forces $f = f'$, which is what we needed to show. □

E.4.2.2. *Action groupoids.* We show how to assign a topological groupoid to the datum of an action of a compactly generated weakly Hausdorff topological group G on a compactly generated weakly Hausdorff space X. For such G and X, we assume that G acts on X continuously via a morphism in **CGWH**:

$$a : G \times X \to X. \tag{E.4.2.4}$$

Construction E.4.2.12. We denote by $G \times X \rightrightarrows X$ the topological groupoid assigned by the datum $(X, \mathcal{R}, s, t, c, e, i)$, where

$$\mathcal{R} := G \times X \tag{E.4.2.5}$$

and

$$s, t : \mathcal{R} \rightrightarrows X, \quad (g, x) \rightsquigarrow s(g, x) := x, \quad (g, x) \mapsto t(g, x) := a(g, x), \tag{E.4.2.6}$$

with $a : G \times X \to X$ the action map from (E.4.2.4). We further define:

$$c : \mathcal{R} \times_{s,X,t} \mathcal{R} \to \mathcal{R}, \quad ((g,x),(g',x')) \mapsto c((x,g),(x',g')) := (gg',x') \tag{E.4.2.7}$$

and:

$$e : X \to \mathcal{R}, \quad x \mapsto e(x) := (1,x) \tag{E.4.2.8}$$

$$i : \mathcal{R} \to \mathcal{R} \quad (g,x) \mapsto i(g,x) := (g^{-1}, a(g,x)). \tag{E.4.2.9}$$

The datum $G \times X \rightrightarrows X$ indeed assigns a topological groupoid in the sense of Definition E.4.2.3.

E.5. 2-categorical preparations

In this subsection we collect background material on 2-categories, up to and including the concept of 2-fibre products. We will then include some examples of interest, and explain how the category of $\mathscr{A}$-groupoids is a 2-category in a natural manner.

2-categories were first introduced, in their "strict" form by J. Bénabou in [**Be**, Example (5)] and J.-M. Maranda in [**Mar**, Def. 1], building in some measure on Ehresmann's work on double categories and structured categories ([**Ehr1**] and [**Ehr2**]). Our main references will be [**Bor1**, Ch. 7], [**GR**, Ch. 2], M. Hakim's Ph.D. thesis [**Ha**, Ch. 1], alongside [**SP**, Tag 003G] and [**SP**, Tag 003O]. Our aim is to give a self-contained treatment of a special case of the theory of 2-categories, encompassing only basic terminology and constructions, and including a concrete discussion of 2-fibre products in (strict) $(2,1)$-categories. These will suffice for all intended applications, insofar the category of stacks in groupoids on the topological site $\mathbf{CGWH}_{\alpha,\tau}$ that was constructed in §E.3.5 will be one example of such a (strict) $(2,1)$-category, and all constructions of interest to this appendix will be performed intrinsically within it.

The concept of a 2-category serves the purpose of keeping track of morphisms between morphisms, a notion that often arises when studying moduli spaces of structures equipped with non-trivial symmetries.

We begin by describing the structures that are present within the category of all small categories, for later axiomatizing them into the concept of a 2-category. This example will serve as guidance and will justify the terminology that we shall be introducing accordingly afterwards. It will also enable us to construct more examples and, most importantly, once we formally endow the category of all small categories with the structure of a 2-category it will contain, in the appropriate 2-categorical sense, all examples of 2-categories of main focus within this appendix. The various operations and constructions that will be discussed explicitly for it will thereby encompass all cases of interest. In this section, functors are understood to be *covariant*.

E.5.1. The category of all small categories. Main sources for the material discussed in this section are [**SP**, Tag 003D] and [**Bor1**, Ch. 7, §7.1].

E.5.1.1. *The categories* **Cat** *and* $\mathbf{Cat}^*$. We begin with a Definition:

Definition E.5.1.1. We call **Cat** the category whose objects are small categories, and whose morphisms are functors between small categories.

The category **Cat** is endowed with more structure than that of an ordinary category. Namely, for any pair of small categories A, B, and any two functors $F, F' : A \rightrightarrows B$, we also have all natural transformations of functors, denoted $F \Rightarrow F'$, and it is the purpose of this section to recall how to record such additional structure within **Cat** in a robust manner.

Before axiomatizing it into the notion of a 2-category, of which **Cat** will turn out to be a first example, we describe some operations on objects, morphisms and natural transformations of functors within **Cat**.

E.5.1.2. *Structures on the category of all small categories.* For a triple of small categories A, B, C, and functors $F : A \to B$ and $G : B \to C$, we may compose functors to obtain the functor:

$$G \circ F : A \to C. \tag{E.5.1.1}$$

For a triple of functors between a fixed pair of small categories $F, F', F'' : A \rightrightarrows B$, we may consider natural transformations $t : F \Rightarrow F'$ and $t' : F' \Rightarrow F''$, and compose them to obtain a natural transformation which we denote:

$$t' \odot t : F \Rightarrow F''. \tag{E.5.1.2}$$

We call $\odot$ from (E.5.1.2) the *vertical composition* of natural transformations of functors.

NOTATION E.5.1.1. We let A, B be small categories and $F : A \to B$ a functor.

(1) We denote by $\mathbf{1}_F$ the identity natural transformation $\mathbf{1}_F : F \Rightarrow F$.
(2) We denote, for $A = B$, by id_A the identity functor $F : A \to A$, sending every object and morphism of A to itself.

In particular, for any small category A, $\mathbf{1}_{\mathrm{id}_A}$ denotes the identity natural transformation of the identity functor on A.

LEMMA E.5.1.2. *Let A, B be small categories, $F, F' : A \rightrightarrows B$ functors and $t : F \Rightarrow F'$ a natural transformation of functors. Then we have:*

$$\mathbf{1}_F \odot t = t \odot \mathbf{1}_F = t \tag{E.5.1.3}$$

making $\mathbf{1}_F$ a unit for the vertical composition $\odot$ of natural transformations. Moreover, $\odot$ is associative.

PROOF. For an arbitrary object a of A, we have:

$$(\mathbf{1}_F \odot t)_a = \mathrm{id}_{F(a)} \circ t_a = t_a = t_a \circ \mathrm{id}_{F(a)} = (t \odot \mathbf{1}_F)_a$$

where the innermost compositions are as morphisms in B. Associativity of $\odot$ is a direct consequence of associativity of composition of morphisms in any category. □

As a direct consequence of the definition in (E.5.1.2) and Lemma E.5.1.2, for small categories A, B we may define:

$$\mathbf{Hom}_{\mathbf{Cat}}(A, B) \tag{E.5.1.4}$$

to be the category of functors $A \to B$, whose morphisms are natural transformations of functors, and whose composition law is $\odot$.

SITUATION E.5.1.3. We fix a triple of small categories A, B, C and two pairs of functors $F, F' : A \rightrightarrows B$ and $G, G' : B \rightrightarrows C$, we consider natural transformations $t : F \Rightarrow F'$ and $s : G \Rightarrow G'$.

We can compose t and s from Situation E.5.1.3 as well, to obtain a natural transformation:

$$s \star t : G \circ F \Rightarrow G' \circ F'. \tag{E.5.1.5}$$

We shall call it *horizontal composition* of natural transformations of functors. Constructing $s \star t$ (mainly that two possible definitions agree) and proving its properties is one of the main purposes of this subsection.

REMARK E.5.1.4. We recall for the reader, in the hope of motivating this endeavor, that all examples of 2-categories of interest throughout this work will be encompassed by **Cat** in the appropriate sense. Vertical and horizontal compositions of natural transformations of functors will, in particular, be taking place, or at least be computable, within **Cat**. The constructions contained within this subsection will therefore cover all cases of interest.

We start with the case $G = G'$ and $s = \mathbf{1}_G$.

CONSTRUCTION E.5.1.5. Let A, B, C be a triple of small categories, and $F, F' : A \rightrightarrows B$ a pair of functors. Let $G : B \to C$ be a functor, and $t : F \Rightarrow F'$ a natural transformation as in Situation E.5.1.3. We call $\mathbf{1}_G \star t$ the natural transformation:

$$\mathbf{1}_G \star t : G \circ F \Rightarrow G \circ F' \tag{E.5.1.6}$$

defined as follows. For any object $a \in A$, we assign:

$$(\mathbf{1}_G \star t)_a := G(t_a) : (G \circ F)(a) \to (G \circ F')(a) \tag{E.5.1.7}$$

where t_a is the morphism $F(a) \to F'(a)$ in B induced by the natural transformation $t : F \Rightarrow F'$ evaluated at the object $a \in \mathrm{Ob}(A)$. One checks that $\mathbf{1}_G \star t$ indeed is a natural transformation since for any morphism $f : a \to a'$ in A, the following diagram in B commutes

$$\begin{array}{ccc} F(a) & \xrightarrow{t_a} & F'(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(a') & \xrightarrow[t_{a'}]{} & F'(a') \end{array} \tag{E.5.1.8}$$

and upon evaluating G on the (E.5.1.8), we obtain a morphism

$$(G \circ F)(f) : (G \circ F)(a) \to (G \circ F)(a') \tag{E.5.1.9}$$

exhibiting naturality of $G(t_a)$ as a consequence of naturality of t_a with respect to morphisms in A. Compatibility with compositions of morphisms in A is a consequence of both naturality of t and functoriality of F and G.

The following is straightforward.

LEMMA E.5.1.6. *Let A, B, C be a triple of small categories, and F, F', F' a triple of functors from A to B. Let $G : B \to C$ be a functor, and $t : F \Rightarrow F'$, $t' : F' \Rightarrow F''$ natural transformations. We have:*

$$\mathbf{1}_G \star \mathbf{1}_F = \mathbf{1}_{G \circ F}, \quad (\mathbf{1}_G \star t') \odot (\mathbf{1}_G \star t) = \mathbf{1}_G \star (t' \odot t) \tag{E.5.1.10}$$

and:

$$\mathbf{1}_{\mathrm{id}_A} \star t = t \tag{E.5.1.11}$$

In particular, the assignment:

$$G_* := G \circ (\cdot) : \mathbf{Hom}_{\mathbf{Cat}}(A, B) \to \mathbf{Hom}_{\mathbf{Cat}}(A, C)$$

is a functor.

Within Situation E.5.1.3, we now construct horizontal compositions of natural transformations in the case $F = F'$, $t = \mathbf{1}_F$ and s is an arbitrary natural transformation $s : G \Rightarrow G'$.

CONSTRUCTION E.5.1.7. Let A, B, C be a triple of small categories, and $F : A \to B$ a functor. Let $G, G' : B \rightrightarrows C$ be a pair of functors, and $s : G \Rightarrow G'$ a natural transformation. We call $s \star \mathbf{1}_F$ the natural transformation:

$$s \star \mathbf{1}_F : G \circ F \Rightarrow G' \circ F \tag{E.5.1.12}$$

defined as follows. For any object $a \in A$, we assign:

$$(s \star \mathbf{1}_F)_a := s_{F(a)} : (G \circ F)(a) \to (G \circ F')(a) \tag{E.5.1.13}$$

where $s_{F(a)}$ is the morphism $(G \circ F)(a) \to (G' \circ F)(a)$ in C induced by the natural transformation $s : G \Rightarrow G'$. To check that $s \star \mathbf{1}_F$ indeed is a natural transformation, we observe that $s_{F(a)}$ is natural with respect to morphisms on a by functoriality of F and naturality of s.

LEMMA E.5.1.8. *Let A, B, C be a triple of small categories, and $F : A \to B$ a functor. Let G, G', G'' a triple of functors from B to C, and $s : G \Rightarrow G'$, $s' : G' \Rightarrow G''$ natural transformations. We have:*

$$(s' \star \mathbf{1}_F) \odot (s \star \mathbf{1}_F) = (s' \odot s) \star \mathbf{1}_F \tag{E.5.1.14}$$

and:

$$s \star \mathbf{1}_{\mathrm{id}_B} = s \tag{E.5.1.15}$$

In particular, the assignment:

$$F^* := (\cdot) \circ F : \mathbf{Hom_{Cat}}(B, C) \to \mathbf{Hom_{Cat}}(A, C)$$

is a functor.

PROOF. To check all the identities, it is enough to evaluate them on an arbitrary object a of A, respectively an arbitrary object b of B, as the reader may carry out. □

We further note that these constructions are compatible with one another in the following sense:

LEMMA E.5.1.9. *We have:*

$$\mathbf{1}_{G_1} \star (\mathbf{1}_{G_2} \star t) = \mathbf{1}_{G_1 \circ G_2} \star t \tag{E.5.1.16}$$

for any natural transformation of functors $t : F \Rightarrow F' : A \to B$, and functors $G_2 : B \to C$ and $G_1 : C \to D$. We further have:

$$(s \star \mathbf{1}_{F_1}) \star \mathbf{1}_{F_2} = s \star \mathbf{1}_{F_1 \circ F_2} \tag{E.5.1.17}$$

for any natural transformation of functors $s : G \Rightarrow G' : C \to D$ and functors $F_2 : A \to B$ and $F_1 : B \to C$. Moreover, for any natural transformation of functors $s : G \Rightarrow G' : B \to C$, and any functors $F : A \to B$ and $H : C \to D$, we have:

$$\mathbf{1}_H \star (s \star \mathbf{1}_F) = (\mathbf{1}_H \star s) \star \mathbf{1}_F. \tag{E.5.1.18}$$

PROOF. The result is obtained analogously to Lemma E.5.1.6 and Lemma E.5.1.8, by evaluating all natural transformations on objects and deducing the desired identities from the definitions alone, using functoriality of the component functors and naturality of the component natural transformations. □

We have the following:

LEMMA E.5.1.10. *We let $F, F' : A \rightrightarrows B$ and $G, G' : B \rightrightarrows C$ be functors between small categories, and let $t : F \Rightarrow F'$ and $s : G \Rightarrow G'$ be natural transformations of functors. Then the following identity of natural transformations of functors $A \to C$ holds:*

$$(\mathbf{1}_{G'} \star t) \odot (s \star \mathbf{1}_F) = (s \star \mathbf{1}_{F'}) \odot (\mathbf{1}_G \star t). \tag{E.5.1.19}$$

This result is obtained analogously to Lemma E.5.1.6 and Lemma E.5.1.8, by evaluating all natural transformations on objects and deducing the desired identities from the definitions alone, using functoriality of the component functors and naturality of the component natural transformations. We further refer the reader to [**SP**, Tag 003D].

DEFINITION E.5.1.11. For $t : F \Rightarrow F'$ and $s : G \Rightarrow G'$ two natural transformations of the functors $F, F' : A \rightrightarrows B$ and $G, G' : B \rightrightarrows C$, define

$$s \star t := (\mathbf{1}_{G'} \star t) \odot (s \star \mathbf{1}_F) : G \circ F \Rightarrow G' \circ F'. \tag{E.5.1.20}$$

In light of Lemma E.5.1.10, the construction of $s \star t$ is well-posed. This is the alternate notion of composition for natural transformations of functors promised in (E.5.1.5), which we shall refer to as *horizontal composition*.

We now establish properties of the two composition laws $\star$ and $\odot$ on natural transformations.

LEMMA E.5.1.12. *The composition $\star$ is associative, and for every small category A, the identity natural transformation $\mathbf{1}_{\mathrm{id}_A}$ of the identity functor is a unit for both $\odot$ and $\star$.*

PROOF. We let A, B, C, D be four small categories, and $F, F' : A \rightrightarrows B$, $G, G' : B \rightrightarrows C$, $H, H' : C \rightrightarrows D$ three pairs of functors. We let $t : F \Rightarrow F'$, $s : G \Rightarrow G'$ and $r : H \Rightarrow H'$ be three natural transformations of functors. For an arbitrary object a of A, we have:

$$\begin{aligned}((r \star s) \star t)_a &= ((\mathbf{1}_{H' \circ G'} \star t) \odot ((r \star s) \star \mathbf{1}_F))_a \\ &= ((\mathbf{1}_{H' \circ G'} \star t) \odot (((\mathbf{1}_{H'} \star s) \odot (r \star \mathbf{1}_G)) \star \mathbf{1}_F))_a \\ &= (\mathbf{1}_{H' \circ G'} \star t)_a \circ ((\mathbf{1}_{H'} \star s) \odot (r \star \mathbf{1}_G))_{F(a)} \\ &= H'(G'(t_a)) \circ H'(s_{F(a)}) \circ r_{G(F(a))}.\end{aligned}$$

We also have:

$$\begin{aligned}(r \star (s \star t))_a &= ((\mathbf{1}_{H'} \star (s \star t)) \odot (r \star \mathbf{1}_{G \circ F}))_a \\ &= H'((s \star t)_a) \circ r_{G(F(a))} \\ &= H'((\mathbf{1}_{G'} \star t)_a \circ (s \star \mathbf{1}_F)_a) \circ r_{G(F(a))} \\ &= H'(G'(t_a)) \circ H'(s_{F(a)}) \circ r_{G(F(a))}.\end{aligned}$$

We deduce that $(r \star s) \star t = r \star (s \star t)$. For every small category A, the identity natural transformation of the identity functor $\mathbf{1}_{\mathrm{id}_A}$ is a unit for both $\odot$ by Lemma E.5.1.2 applied to $F = \mathbf{1}_{\mathrm{id}_A}$, and for $\star$ by (E.5.1.11) from Lemma E.5.1.6 and (E.5.1.15) from Lemma E.5.1.8. □

LEMMA E.5.1.13. *Let A, B, C be three small categories, and consider triples of functors F, F', F'' from A to B and G, G', G'' from B to C equipped with natural transformations $t : F \Rightarrow F'$, $t' : F' \Rightarrow F''$, $s : G \Rightarrow G'$, and $s' : G' \Rightarrow G''$. Then we have:*

$$(s' \odot s) \star (t' \odot t) = (s' \star t') \odot (s \star t). \tag{E.5.1.21}$$

PROOF. For any object a of A, we have:

$$\begin{aligned}((\mathbf{1}_{G''} \star (t' \odot t)) \odot ((s' \odot s) \star \mathbf{1}_F))_a &= (\mathbf{1}_{G''} \star (t' \odot t))_a \circ ((s' \odot s) \star \mathbf{1}_F)_a \\ &= G''(t'_a \circ t_a) \circ (s'_{F(a)} \circ s_{F(a)}) \\ &= G''(t'_a) \circ G''(t_a) \circ s'_{F(a)} \circ s_{F(a)} \\ &= ((\mathbf{1}_{G''} \star t') \odot (\mathbf{1}_{G''} \star t) \odot (s' \star \mathbf{1}_F) \odot (s \star \mathbf{1}_F))_a \\ &= ((\mathbf{1}_{G''} \star t') \odot ((s' \star \mathbf{1}_{F'}) \odot (\mathbf{1}_{G'} \star t)) \odot (s \star \mathbf{1}_F))_a\end{aligned}$$

and by Lemma E.5.1.2 on associativity of $\odot$, we deduce:

$$\begin{aligned}(\mathbf{1}_{G''} \star t') \odot ((s' \star \mathbf{1}_{F'}) \odot (\mathbf{1}_{G'} \star t)) \odot (s \star \mathbf{1}_F) &= ((\mathbf{1}_{G''} \star t')) \odot (s' \star \mathbf{1}_{F'}) \odot ((\mathbf{1}_{G'} \star t) \odot (s \star \mathbf{1}_F)) \\ &= (s' \star t') \odot (s \star t).\end{aligned}$$

□

LEMMA E.5.1.14. *Let A, B, C be small categories. Then the assignment:*

$$c_{A,B,C}(\cdot,\cdot) := (\circ, \star) : \mathbf{Hom_{Cat}}(A, B) \times \mathbf{Hom_{Cat}}(B, C) \to \mathbf{Hom_{Cat}}(A, C) \tag{E.5.1.22}$$

defined as the composition of functors from (E.5.1.1) *on objects, and the horizontal composition of natural transformations from* (E.5.1.20) *on morphisms, is a functor. Moreover, we have the following identities:*

(1) For small categories A, B, functors $F, F' : A \rightrightarrows B$, and a natural transformation $t : F \Rightarrow F'$, we have:
  (a) $c_{A,A,B}(\mathrm{id}_A, F) = c_{A,B,B}(F, \mathrm{id}_B) = F$.
  (b) $c_{A,A,B}(\mathbf{1}_{\mathrm{id}_A}, t) = c_{A,B,B}(t, \mathbf{1}_{\mathrm{id}_B}) = t$.

(2) For small categories A, B, C, D, functors $F, F' : A \rightrightarrows B$, $G, G' : B \rightrightarrows C$, $H, H' : C \rightrightarrows D$, and natural transformations $t : F \Rightarrow F'$, $s : G \Rightarrow G'$, $r : H \Rightarrow H'$, we have:
  (a) $c_{A,B,D}(c_{A,B,C}(F, G), H) = c_{A,B,D}(F, c_{B,C,D}(G, H))$.
  (b) $c_{A,B,C}(c_{A,B,C}(t, s), r) = c_{A,B,D}(t, c_{B,C,D}(s, r))$.

We recall that the product $\mathbf{Hom_{Cat}}(A, B) \times \mathbf{Hom_{Cat}}(B, C)$ is the product of two small categories (see [**SP**, Tag 001K]). We further refer the reader to [**SP**, Tag 003F].

Proof. This is a direct consequence of Lemma E.5.1.12, which can be applied by Lemma E.5.1.13. Explicitly, we let (F, G), (F', G') and (F'', G'') be three objects in the product category:

$$\mathbf{Hom_{Cat}}(A, B) \times \mathbf{Hom_{Cat}}(B, C)$$

and $t : F \Rightarrow F'$, $s : G \Rightarrow G'$, $t' : F' \Rightarrow F''$, $s' : G' \Rightarrow G''$ natural transformations of functors defining morphisms (t, s) and (t', s') in the product category. Then

$$(\circ, \star)(F, G) = G \circ F, \quad (\circ, \star)(F', G') = G' \circ F', \quad (\circ, \star)(F'', G'') = G'' \circ F''$$

on objects, and

$$(\circ, \star)(t, s) = s \star t, \quad (\circ, \star)(t', s') = s' \star t'$$

on morphisms.

We are to check, upon denoting by $\square$ the composition of morphisms in the product category:

$$(\circ, \star)((t', s') \square (t, s)) = ((\circ, \star)(t', s')) \star ((\circ, \star)(t, s)).$$

Since we have $(t', s') \square (t, s) = (t' \odot t, s' \odot s)$, we deduce the identity of morphisms in $\mathbf{Hom_{Cat}}(A, C)$:

$$((\circ, \star)(t', s')) \star ((\circ, \star)(t, s)) = (s' \star t') \odot (s \star t)$$

and we have $(s' \odot s) \star (t' \odot t) = (s' \star t') \odot (s \star t)$ from Lemma E.5.1.13, which is what we needed to show. The desired identities are a reformulation of (E.5.1.11) from Lemma E.5.1.6 and of (E.5.1.15) from Lemma E.5.1.8, for part (b) of (1), and of Lemma E.5.1.12 for part (b) of (2). Parts (a) of (1) and of (2) is a consequence of the fact that on objects, $c_{\bullet,\bullet,\bullet}$ is defined via composition of functors between small categories. □

E.5.1.3. *Effective small categories.* We now revisit the category **Cat** under a different perspective, which will be of use throughout and will give us the opportunity to make examples. We begin with a definition:

Definition E.5.1.15. Call $\mathbf{Cat}^*$ the category whose objects are 6-tuples (X, A, s, t, c, e) as for groupoids in sets in Definition E.4.1.1 but without the inversion axioms (this is why we write "A" instead of "R"), and likewise for morphisms.[18] A *functor* between objects in $\mathbf{Cat}^*$ is a map between such 6-tuples as in (E.4.1.7) but without the inversion aspect.

Objects in $\mathbf{Cat}^*$ are called *effective small categories*. There is an evident analogy between $\mathbf{Cat}^*$ and the category **Cat** from §E.5.1.1. We define the analog of natural transformations of functors:

Definition E.5.1.16. Let (X, A, s, t, c, e) and (X', A', s', t', c', e') be in $\mathbf{Cat}^*$ and

$$F := (f, g), F' := (f', g') : (X, A, s, t, c, e) \to (X', A', s', t', c', e')$$

[18] Previously had not included compatibility with e's, which was surely an oversight.

two functors.

We define a *natural transformation* $t : F \Rightarrow F'$ to be a set-function:

$$T : X \to A' \tag{E.5.1.23}$$

such that for all $x \in X$ we have $T(x) : f(x) \to f(x')$ and for every $a : x \to y$ the diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{g(a)} & f(y) \\ {\scriptstyle T(x)}\downarrow & & \downarrow{\scriptstyle T(y)} \\ f'(x) & \xrightarrow[g'(a)]{} & f'(y) \end{array} \tag{E.5.1.24}$$

commutes.

As for the category **Cat**, we highlight here additional structures that the category $\mathbf{Cat}^*$ is equipped with.

Construction E.5.1.17. Let (X, A, s, t, c, e), (X', A', s', t', c', e') and $(X'', A'', s'', t'', c'', e'')$ be three effective small categories.

(1) We write

$$\mathrm{Hom}_{\mathbf{Cat}^*}((X, A, s, t, c, e), (X', A', s', t', c', e')) \tag{E.5.1.25}$$

for the set of functors $(X, A, s, t, c, e) \to (X', A', s', t', c', e')$.

(2) For functors $F = (f_1, f_2), F' = (f_1', f_2') : (X, A, s, t, c, e) \rightrightarrows (X', A', s', t', c', e')$, we write $\text{2-Morph}_{F,F'}$ for the sets of natural transformations $F \Rightarrow F'$. We write:

$$\text{2-Morph}((X, A, s, t, c, e), (X', A', s', t', c', e')) := \coprod_{(F,F')} \text{2-Morph}_{F,F'} \tag{E.5.1.26}$$

where the co-product is over all ordered pairs

$$(F, F') \in \mathrm{Hom}_{\mathbf{Cat}^*}((X, A, s, t, c, e), (X', A', s', t', c', e')) \times \mathrm{Hom}_{\mathbf{Cat}^*}((X, A, s, t, c, e), (X', A', s', t', c', e'))$$

and we define

(a) For a triple of functors F, F', F'' from (X, A, s, t, c, e) to (X', A', s', t', c', e') and natural transformations:

$$T : F \Rightarrow F', \quad T' : F' \Rightarrow F''$$

we define

$$\odot : \text{2-Morph}_{F,F'} \times \text{2-Morph}_{F',F''} \to \text{2-Morph}_{F,F''} \tag{E.5.1.27}$$

by

$$(T' \odot T)(x) := c'(T(x), T'(x)) \tag{E.5.1.28}$$

for all $x \in X$ and refer to $\odot$ as the *vertical composition* of natural transformations.

(b) For functors

$$F = (f_1, f_2), F' = (f_1', f_2') : (X, A, s, t, c, e) \rightrightarrows (X', A', s', t', c', e')$$

and

$$G = (g_1, g_2), G' = (g_1', g_2') : (X', A', s', t', c', e') \rightrightarrows (X'', A'', s'', t'', c'', e'')$$

we define, for natural transformations $T : F \Rightarrow F'$ and $S : G \Rightarrow G'$:

$$\star : \text{2-Morph}_{F,F'} \times \text{2-Morph}_{G,G'} \to \text{2-Morph}_{G\circ F, G'\circ F'} \tag{E.5.1.29}$$

by

$$(S \star T)(x) := c''(S(f_1(x)), g_2'(T(x))) \tag{E.5.1.30}$$

for all $x \in X$ and refer to it as the *horizontal composition* of natural transformations.

(3) For a functor $F = (f_1, f_2) : (X, A, s, t, c, e) \to (X', A', s', t', c', e')$, we denote by $\mathbf{1}_F$ the set-function

$$\mathbf{1}_F(x) := (f_2 \circ e)(x) = (e' \circ f_1)(x).$$

We now briefly state and prove analogs for the category $\mathbf{Cat}^*$ of the results established on $\odot$ and $\star$ in the foregoing, concerning the category $\mathbf{Cat}$.

LEMMA E.5.1.18. *Let (X, A, s, t, c, e), (X', A', s', t', c', e'), $(X'', A'', s'', t'', c'', e'')$, $(X''', A''', s''', t''', c''', e''')$ be effective small categories. Let $F = (f_1, f_2)$, $F' = (f_1', f_2')$, and $F'' = (f_1'', f_2'')$ be triples of functors*

$$(X, A, s, t, c, e) \to (X', A', s', t', c', e')$$

and $G = (g_1, g_2)$, $G' = (g_1', g_2')$, and $G'' = (g_1'', g_2'')$ be triples of functors

$$(X', A', s', t', c', e') \to (X'', A'', s'', t'', c'', e''),$$

and $H = (h_1, h_2) : (X'', A'', s'', t'', c'', e'') \to (X''', A''', s''', t''', c''', e''')$ another such functor.

Let $T : F \Rightarrow F'$, $T' : F' \Rightarrow F''$, $S : G \Rightarrow G'$, $S' : G' \Rightarrow G''$ be natural transformations. We have the identities:

$$\mathbf{1}_{F'} \odot T = T \odot \mathbf{1}_F \tag{E.5.1.31}$$

$$\mathbf{1}_G \star \mathbf{1}_F = \mathbf{1}_{G \circ F}, \quad (\mathbf{1}_G \star T') \odot (\mathbf{1}_G \star T) = \mathbf{1}_G \star (T' \odot T), \quad \mathbf{1}_{\mathrm{id}_{(X,A)}} \star T = T \tag{E.5.1.32}$$

$$(S' \star \mathbf{1}_F) \odot (S \star \mathbf{1}_F) = (S' \odot S) \star \mathbf{1}_F, \quad S \star \mathbf{1}_{\mathrm{id}_{(Y,B)}} = S \tag{E.5.1.33}$$

$$\mathbf{1}_H \star (S \star \mathbf{1}_F) = (\mathbf{1}_H \star S) \star \mathbf{1}_F \tag{E.5.1.34}$$

$$(\mathbf{1}_{G'} \star T) \odot (S \star \mathbf{1}_F) = (S \star \mathbf{1}_{F'}) \odot (\mathbf{1}_G \star T) \tag{E.5.1.35}$$

$$(S' \odot S) \star (T' \odot T) = (S' \star T') \odot (S \star T). \tag{E.5.1.36}$$

PROOF. For (E.5.1.31), we have:

$$\begin{aligned}(\mathbf{1}_{F'} \odot T)(x) &= c'(T(x), (f_2' \circ e)(x)) \\ &= c'((f_2 \circ e)(x), T(x)) \\ &= (T \odot \mathbf{1}_F)(x)\end{aligned}$$

by (E.5.1.24). For (E.5.1.32), we have:

$$\begin{aligned}(\mathbf{1}_G \star \mathbf{1}_F)(x) &= c''(\mathbf{1}_G(f_1(x)), g_2(\mathbf{1}_F(x))) \\ &= c''((g_2 \circ e' \circ f_1)(x), (g_2 \circ f_2 \circ e)(x)) \\ &= c''((g_2 \circ f_2)(e(x)), (g_2 \circ f_2)(e(x))) \\ &= \mathbf{1}_{G \circ F}(x).\end{aligned}$$

and the second identity of (E.5.1.32) is proved in the same way as the first in (E.5.1.33) that we now check:

$$\begin{aligned}((S' \star \mathbf{1}_F) \odot (S \star \mathbf{1}_F))(x) &= c''((S \star \mathbf{1}_F)(x), (S' \star \mathbf{1}_F)(x)) \\ &= c''(c''(S(f_1(x)), g_2'(f_2(e(x)))), c''(S'(f_1(x)), g_2''(f_2(e(x))))) \\ &= c''(c''(c''(S(f_1(x)), g_2'(f_2(e(x)))), S'(f_1(x))), g_2''(f_2(e(x)))) \\ &= c''(c''(S(f_1(x)), c''(g_2'(f_2(e(x))), S'(f_1(x)))), g_2''(f_2(e(x)))) \\ &= c''(c''(S(f_1(x)), c''(e''(g_1'(f_1(x))), S'(f_1(x))), g_2''(f_2(e(x))))) \\ &= c''(c''(S(f_1(x)), c''((e'' \circ t'')(S'(f_1(x))), S'(f_1(x))), g_2''(f_2(e(x))))) \\ &= c''(c''(S(f_1(x)), S'(f_1(x))), g_2''(f_2(e(x)))) \\ &= c''((S' \odot S)(f_1(x)), g_2''(f_2(e(x)))) \\ &= ((S' \odot S) \star \mathbf{1}_F)(x)\end{aligned}$$

The third identity in (E.5.1.32) is proved in the same way as the second in (E.5.1.33) that we now check:

$$(\mathbf{1}_{\mathrm{id}_A} \star T)(x) = c'(e(f_1(x)), \mathrm{id}(T(x))) = T(x)$$

where the last identity follows from $(t \circ T)(x) = f_1(x)$ and (**??**). For (E.5.1.34), we have:

$$\begin{aligned}(\mathbf{1}_H \star (S \star \mathbf{1}_F))(x) &= c'''(\mathbf{1}_H(g_1(f_1(x))), h_2((S \star \mathbf{1}_F)(x))) \\ &= c'''(h_2(e''(g_1(f_1(x)))), h_2(c''(S(f_1(x)), g_2'(f_2(e(x)))))) \\ &= c'''(h_2(e''(g_1(f_1(x)))), c'''(h_2(S(f_1(x))), h_2(g_2'(f_2(e(x))))))) \\ &= c'''(c'''(h_2(e''(g_1(f_1(x)))), h_2(S(f_1(x)))), h_2(g_2'(f_2(e(x)))))) \\ &= c'''((\mathbf{1}_H \star S)(f_1(x)), h_2(g_2'(f_2(e(x)))))) \\ &= ((\mathbf{1}_H \star S) \star \mathbf{1}_F)(x).\end{aligned}$$

For (E.5.1.35), we have:

$$\begin{aligned}((\mathbf{1}_{G'} \star T) \odot (S \star \mathbf{1}_F))(x) &= c''((S \star \mathbf{1}_F)(x), (\mathbf{1}_{G'} \star T)(x)) \\ &= c''(c''(S(f_1(x)), g_2'(f_2(e(x)))), c''(g_2'(e'(f_1(x))), g_2'(T(x)))) \\ &= c''(c''(S(f_1(x)), e''(g_1'(f_1(x)))), g_2'(c'(e'(f_1(x)), T(x)))) \\ &= c''(S(f_1(x)), g_2'(T(x))) \\ &= c''(g_2(T(x)), S(f_1'(x)) \\ &= c''(c''(g_2(e'(f_1(x))), g_2(T(x))), c''(S(f_1'(x)), g_2'(f_2'(e(x))))) \\ &= c''((\mathbf{1}_G \star T)(x), (S \star \mathbf{1}_{F'})(x))\end{aligned}$$

where we used naturality of S (from (E.5.1.24)) to obtain the identity $c''(S(f_1(x)), g_2'(T(x))) = c''(g_2(T(x)), S(f_1'(x))$. For (E.5.1.36), the proof is formally identical to the proof of Lemma E.5.1.13. □

We have the following analog of Lemma E.5.1.14:

LEMMA E.5.1.19. *Let:*

$$(X, A, s, t, c, e, i), (Y, B, p, q, d, f), (Z, C, \sigma, \tau, \gamma, \varepsilon), (W, D, \phi, \psi, \theta, \eta)$$

be effective small categories, and:

$$F = (f_i), F' = (f_i') : (X, A) \rightrightarrows (Y, B), G = (g_i), G' = (g_i') : (Y, B) \rightrightarrows (Z, C)$$
$$H = (h_i), H' = (h_i') : (Z, C) \rightrightarrows (W, D)$$

be functors for $i = 1, 2$*, let:*

$$T : F \Rightarrow F', S : G \Rightarrow G', R : H \Rightarrow H'$$

be natural transformations. Then the assignment:

$$c_{A,B,C}(\cdot,\cdot) := (\circ, \star) : \mathbf{Hom}_{\mathbf{Cat}^*}((X,A),(Y,B)) \times \mathbf{Hom}_{\mathbf{Cat}^*}((Y,B),(Z,C)) \to \mathbf{Hom}_{\mathbf{Cat}^*}((X,A),(Z,C))$$

defined as the composition of functors on objects, and the horizontal composition of natural transformations from (E.5.1.30) *on morphisms, is a functor. Moreover, we have the following identities:*

(1) We have:
  (a) $c_{A,A,B}(\mathrm{id}_{(X,A)}, F) = c_{A,B,B}(F, \mathrm{id}_{(Y,B)}) = F$.
  (b) $c_{A,A,B}(\mathbf{1}_{\mathrm{id}_{(X,A)}}, T) = c_{A,B,B}(T, \mathbf{1}_{\mathrm{id}_{(Y,B)}}) = T$.

(2) We have:
  (a) $c_{A,B,D}(c_{A,B,C}(F,G),H) = c_{A,B,D}(F, c_{B,C,D}(G,H))$.
  (b) $c_{A,B,C}(c_{A,B,C}(T,S),R) = c_{A,B,D}(T, c_{B,C,D}(S,R))$.

Proof. Given Lemma E.5.1.18, which provides analogs to Lemma E.5.1.2, Lemma E.5.1.6, Lemma E.5.1.8, Lemma E.5.1.9, Lemma E.5.1.10, Lemma E.5.1.12 and Lemma E.5.1.13, the proof of the Lemma proceeds in a formally identical manner as for Lemma E.5.1.14. □

Remark E.5.1.20. On the categories **Cat** and $\mathbf{Cat}^*$, we make the following remarks.

(1) Both **Cat** and $\mathbf{Cat}^*$ have a final object. It is given, for **Cat**, by the small category with only one object, which we denote $\bullet$, whose only morphism is the identity, which we denote $e_\bullet$, and for $\mathbf{Cat}^*$ by the datum (X, A, s, t, c, e) with $X = \{1\}$, $A = X$, $s = t = \mathrm{id}$, $c : A \times_{s,X,t} A = A$ the unique morphism, $e : X \to A$ the identity. In both cases, we denote the final object by $\mathbf{1}$.

(2) Throughout the next section we shall use the two examples of **Cat** and $\mathbf{Cat}^*$ as the archetypal example of a 2-category. We shall be proving that, according to the appropriate notion of equivalence of 2-categories, they are equivalent (in fact, isomorphic) as 2-categories, and they may therefore be thought of as the same, for all practical purposes. This will provide us with a rigorous dictionary to turn results on groupoids and topological groupoids into assertions about small categories, functors and natural transformations, while at the same time enabling us to construct more examples.

   Most importantly, all 2-categories of interest throughout this appendix will be contained as sub 2-categories of **Cat**, whether directly in its category of small categories incarnation, or as sub 2-categories of $\mathbf{Cat}^*$. In particular, the reader may think of the operations occurring denoted $\circ, \odot, \star$ occurring throughout this appendix as being exactly those constructed explicitly in this subsection.

E.5.2. 2-categories. We discuss here the basic definitions. We refer the reader to [**Bor1**, Ch.7], [**GR**, Ch.2], [**Ha**, Ch.I,§1], and [**SP**, Tag 003H], for the corresponding subjects within the sources. We will closely follow the former two, while for the most part adopting the notation in use in the latter two. The main difference between this subsection and the referred sources is that we allow the objects of our 2-categories to be elements of proper classes. We reserve the terminology *small 2-category* to refer to those 2-categories whose objects form a *set*. Our envisioned examples are: groupoids internal to the large category **CGWH**, the 2-category **Groupoids** of all groupoids in sets introduced in Definition E.4.1.1 (see part (3) of [**SP**, Tag 003J]) and the 2-category of stacks in groupoids on a site. Large categories will have a limited role throughout, but we shall need the language to speak of 2-functors and 2-fibre products within some.

E.5.2.1. *Basic notions on 2-categories.*

Definition E.5.2.1. A *2-category* $\mathcal{C}$ is the datum of:

(1) A class of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair of objects $(A, B) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$, a small category $\mathbf{Hom}_{\mathcal{C}}(A, B)$.

(a) We call 1-*morphisms* the objects of $\mathbf{Hom}_{\mathcal{C}}(A,B)$, denoted $f : A \to B$.
(b) We call 2-*morphisms* the morphisms of $\mathbf{Hom}_{\mathcal{C}}(A,B)$, denoted $\alpha : f \Rightarrow g$, for two objects $f, g : A \rightrightarrows B$.
(c) For two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, we write $\beta \odot \alpha$ for their composition. We call *vertical composition* the operation $\odot$.

(3) For each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a *composition bifunctor*:

$$c_{A,B,C} : \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,C) \to \mathbf{Hom}_{\mathcal{C}}(A,C).$$

Given 1-morphisms $f : A \to B$ and $g : B \to C$, and 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, we write:
(a) $g \circ f$ for $c_{A,B,C}(f,g) : A \to C$.
(b) $\beta \star \alpha$ for $c_{A,B,C}(\alpha,\beta) : f \Rightarrow h$.

(4) For every object $A \in \mathrm{Ob}(\mathcal{C})$, a *unit functor*:[19]

$$u_A : \mathbf{1} \to \mathbf{Hom}_{\mathcal{C}}(A,A)$$

where $\mathbf{1}$ denotes the final object of $\mathbf{Cat}$ (see Remark E.5.1.20).

The bifunctors $c_{A,B,C}$ and the functor u_A are assumed to satisfy the following axioms:

(1) (Unit Axiom) The following diagram is commutative:

(E.5.2.1)
$$\begin{array}{ccccc}
\mathbf{1} \times \mathbf{Hom}_{\mathcal{C}}(A,B) & \longleftarrow & \mathbf{Hom}_{\mathcal{C}}(A,B) & \longrightarrow & \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{1} \\
\downarrow{\scriptstyle (u_A,\mathbf{id})} & & \| & & \downarrow{\scriptstyle (\mathbf{id},u_A)} \\
\mathbf{Hom}_{\mathcal{C}}(A,A) \times \mathbf{Hom}_{\mathcal{C}}(A,B) & \xrightarrow[c_{A,A,B}]{} & \mathbf{Hom}_{\mathcal{C}}(A,B) & \xleftarrow[c_{A,B,B}]{} & \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,B)
\end{array}$$

where the top horizontal arrows are equivalences of categories, and where $\mathbf{id}$ denotes, within (E.5.2.1), the identity functor.

(2) (Associativity of Compositions Axiom) The following diagram is commutative:

(E.5.2.2)
$$\begin{array}{ccc}
\mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,C) \times \mathbf{Hom}_{\mathcal{C}}(C,D) & \xrightarrow{(\mathbf{id},c_{B,C,D})} & \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,D) \\
\downarrow{\scriptstyle (c_{A,B,C},\mathbf{id})} & & \downarrow{\scriptstyle c_{A,B,D}} \\
\mathbf{Hom}_{\mathcal{C}}(A,C) \times \mathbf{Hom}_{\mathcal{C}}(C,D) & \xrightarrow[c_{A,C,D}]{} & \mathbf{Hom}_{\mathcal{C}}(A,D)
\end{array}$$

where $\mathbf{id}$ denotes, within (E.5.2.2), the identity functor. We say a 2-category is *small* if the class $\mathrm{Ob}(\mathcal{C})$ is a set.

Remark E.5.2.2. We make the following observations:

(1) The functors $\mathbf{Hom}_{\mathcal{C}}(A,B) \to \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{1}$ on the top row of (E.5.2.1) are defined by sending an object $f \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A,B))$ to $(f, \bullet) \in \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{1}$, and likewise sending a morphism $\alpha : f \Rightarrow g$ to $(\alpha, e_\bullet)$.
(2) The functor u_A is equivalent to the datum of a distinguished object $\bullet_A \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A,A))$, together with its identity, which we denote $e_A : \bullet_A \to \bullet_A$. See also [**GR**, Ch. 2, §2.1, p. 71].
(3) In light of (1) above, we may rephrase the Unit Axiom in the above Definition E.5.2.1 as follows. For each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, we have, for any 1-morphisms $f, f' \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A,B))$ and every 2-morphism $\alpha : f \Rightarrow f'$:

$$c_{A,A,B}(\bullet_A, f) = c_{A,B,B}(f, \bullet_B) = f$$

[19] Do we need to axiomatize that $u_A(\cdot) : A \to A$ is an isomorphism, or is this automatic?

and:

$$c_{A,A,B}(e_A, \alpha) = c_{A,B,B}(\alpha, e_B) = \alpha.$$

We note that we are not assuming $\bullet_A$ and $\bullet_B$ to be the identity functors of A and B respectively, although this will be the case in all situations of interest throughout this work (since this is indeed the case for **Cat** and $\mathbf{Cat}^*$).

(4) In light of (1) above, we may rephrase the Associativity of Compositions Axiom in the above Definition E.5.2.1 as follows. For every quadruple of objects $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$, and every $f, f' \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A, B))$, $g, g' \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(B, C))$ and $h, h' \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(C, D))$, we have:

$$c_{A,B,D}(c_{A,B,C}(f, g), h) = c_{A,B,D}(f, c_{B,C,D}(g, h))$$

and for every 2-morphisms $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow$ and $\gamma : h \Rightarrow h'$, we have:

$$c_{A,B,D}(c_{A,B,C}(\alpha, \beta), \gamma) = c_{A,B,D}(\alpha, c_{B,C,D}(\beta, \gamma)).$$

(5) Definition E.5.2.1 above is [**Bor1**, Ch. 7,Def. 7.1.1]. It is also the Definition introduced in [**GR**, Ch. 2, §2.1], with one difference. We are allowing $\mathrm{Ob}(\mathcal{C})$ to be a proper class, as opposed to a set. None of the arguments in loc.cit. for the results that we require are affected by this additional amount of generality. We shall in any event recall the required arguments here, exhibiting explicitly that this is indeed the case. We shall follow [**GR**, Ch.2§2.1] to treat pseudo-functors, and we shall then specialize to the case of strict $(2, 1)$-categories only, with the aim of constructing two types of 2-fibre products in strict $(2, 1)$-categories.

(6) In light of parts (1), (2), and (3) of this Remark, small 2-categories in the sense of Definition E.5.2.1 agree with 2-categories as defined in [**Ha**, Ch. I, §1].

(7) For a pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we think of the objects of $\mathbf{Hom}_{\mathcal{C}}(A, B)$ as arrows $f : A \to B$, and are occasionally also referred to as 1-*cells* in the literature, such as in [**Bor1**, Ch.7] and [**GR**, Ch.2]. For two 1-morphisms $f, g \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A, B))$, morphisms $f \Rightarrow g$ are also referred to as 2-*cells* in the literature, as done in [**Bor1**, Ch. 7] and [**GR**, Ch. 2]. Our preferred choice of terminology follows [**Ha**, Ch. I, §1, (1.3)], where it is referred back to Bénabou's paper [**Be**], and it is also the terminology from [**SP**, Tags 003G, 003O, 02XG] and [**SP**, Tags 02XP, 02XQ].

E.5.2.2. *Basic notions on* 1*-morphisms.* Throughout this subsection, we fix a 2-category $\mathcal{C}$.

We now define when two objects of a 2-category are "equivalent", respectively "isomorphic", and what it means for 1-morphisms and 2-morphisms to be "invertible". The definition below is [**Bor1**, Def.7.1.2], and it is directly analogous to the concept defined right after [**Ha**, Ch.I,(1.3.3)], [**SP**, Tag 003L], and [**GR**, Ch.2, Def.2.1.3].

Definition E.5.2.3. Let $\mathcal{C}$ be a 2-category, and $A, B \in \mathrm{Ob}(\mathcal{C})$ be objects, and $f : A \to B$ a 1-morphism.

(1) We say that f is an *equivalence*, or *essentially invertible*, if and only if there exist two 1-morphisms $g, h : B \rightrightarrows A$, and two 2-isomorphisms:

$$\alpha : \bullet_A \Rightarrow g \circ f, \quad \beta : f \circ h \Rightarrow \bullet_B.$$

(2) We say that f is an *isomorphism* if and only if there exist two 1-morphisms $g, h : B \rightrightarrows A$ such that the elements $\bullet_A$ and $g \circ f$ of the set $\mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A, A))$ are the same element, and the elements $\bullet_B$ and $f \circ h$ of the set $\mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(B, B))$ are the same element. In this case, we define α and β to be the identities.

(3) We say that a 1-morphism $g : B \to A$ is right-adjoint to $f : A \to B$ if and only if there exists 2-morphisms $\epsilon : f \circ g \Rightarrow \bullet_B$ and $\eta : \bullet_A \Rightarrow g \circ f$ such that the following *triangular identities* hold:

$$(\mathbf{1}_g \star \epsilon) \odot (\eta \star \mathbf{1}_g) = \mathbf{1}_g, \quad (\epsilon \star \mathbf{1}_f) \odot (\mathbf{1}_f \star \eta) = \mathbf{1}_f.$$

We also say that f is *left-adjoint to* g, and call ϵ the *unit* and η the *co-unit* of the adjunction (f, g).

(4) For a 1-morphism $g : A \to B$, a 2-morphism $\alpha : f \Rightarrow g$ is *invertible* if and only if it is an isomorphism in the category $\mathbf{Hom}_{\mathcal{C}}(A, B)$. We call an invertible 2-morphism a 2-*isomorphism*.

We now introduce terminology to define what it means for a diagram to commute in $\mathcal{C}$, or commute up to 2-isomorphisms.

DEFINITION E.5.2.4. Let $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ be objects, and $f : A \to B$, $g : B \to C$, $h : A \to D$, $k : D \to C$, 1-morphisms. We consider the following (possibly non-commutative) diagram:

(E.5.2.3)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[k]{} & C \end{array}$$

(1) For objects $A, B \in \mathrm{Ob}(\mathcal{C})$, 1-morphisms $f, f' : A \rightrightarrows B$, and a 2-morphism $\alpha : f \Rightarrow f'$, we write:

$$A \overset{f}{\underset{f'}{\Downarrow \alpha}} B$$

to denote the cumulative datum of $((A, B), (f, f'), \alpha)$.

(2) We say that (E.5.2.3) is *oriented* if and only if there exists a 2-morphism $\alpha : g \circ f \Rightarrow k \circ h$. In this case, we say that the diagram (E.5.2.3) is *oriented by* α. We also call α the *orientation* of the diagram.

(3) We say that (E.5.2.3) is *commutative* if and only if $g \circ f = k \circ h$ as objects of $\mathbf{Hom}_{\mathcal{C}}(A, C)$. As a consequence, commutative diagrams are oriented (by the identity on $g \circ f = k \circ h$).

(4) We say (E.5.2.3) is 2-*commutative* if and only if it is oriented by a 2-morphism $\alpha : g \circ f \Rightarrow k \circ h$, and α is an isomorphism. We write

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle g} \\ D & \xrightarrow[k]{} & C \end{array}$$

Thus, 2-commutative diagrams are oriented diagrams whose orientation is a 2-isomorphism.

REMARK E.5.2.5. We note that [**GR**, Ch. 2, Rem. 2.1.4] applies to our Definition E.5.2.3 as well, despite our assumption that the objects of $\mathcal{C}$ be allowed to be a class, as opposed to a set. In particular, we have:

(1) If a 1-morphism $f : A \to B$ in $\mathcal{C}$ is an equivalence in the sense of part (1) of Definition E.5.2.3, then we can choose $g = h$ in part (1) of Definition E.5.2.3. The argument to see this is identical to [**GR**, Ch.2,Rem.2.1.4,(i)], because the objects A, B are fixed, and $\mathbf{Hom}_{\mathcal{C}}(A, B)$ is a small category.

(2) For $f : A \to B$ and $g : B \to C$ equivalences in $\mathcal{C}$, the composition $g \circ f : A \to C$ is also an equivalence. The argument to see this is identical to [**GR**, Ch.2,Rem.2.1.4(ii)].

(3) Let $f, g : A \rightrightarrows B$ be 1-morphisms, and $\alpha : f \Rightarrow g$ a 2-isomorphism. Then f is an equivalence if and only if g is. In other words, the notion of *equivalence* introduced on 1-morphisms of $\mathcal{C}$ in part (3) of Definition E.5.2.3 propagates along 2-isomorphisms between 1-morphisms. The argument to see this is identical to [**GR**, Ch.2,Rem.2.1.4,(iii)].

(4) Suppose (f, g) is an adjunction, with $f : A \to B$ and $g : B \to A$ two 1-morphisms. Let $g' : B \to A$ be another right-adjoint for f. Then g and g' are 2-isomorphic, i.e. there exists a 2-isomorphism $\alpha : g \Rightarrow g'$. The argument to see this is identical to the argument in [**GR**, Ch.2,Rem.2.1.4(v)].

(5) Suppose (f, g) is an adjunction, with $f : A \to B$ and $g : B \to A$ two 1-morphisms. Let $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ be 2-isomorphisms. Then (f', g') is also an adjunction. The argument to see this is identical to that in [**GR**, Ch. 2, Rem. 2.1.4(iv)].

E.5.2.3. *2-categories and ordinary categories.* We first describe a construction to associate an ordinary category to a 2-category:

Construction E.5.2.6. We let $\mathcal{C}$ be a 2-category. We define $\mathrm{Cat}(\mathcal{C})$ to be the locally small category whose objects are the objects of $\mathcal{C}$, $\mathrm{Ob}(\mathcal{C})$, and whose morphisms are the 1-morphisms of $\mathcal{C}$. Composition of morphisms in $\mathrm{Cat}(\mathcal{C})$ is defined to be the composition of 1-morphisms of $\mathcal{C}$ via the functor $c_{\bullet,\bullet,\bullet}$ of part (3) of Definition E.5.2.1.

We now discuss a construction to associate a 2-category to an ordinary category. This will enable us to "view" ordinary categories as a special instance of 2-categories, and will give us access to the concept of pre-sheaves on an ordinary category, valued in a 2-category.

Construction E.5.2.7. We let $\mathcal{C}$ be a locally small category. We define $\text{2-}\mathrm{Cat}(\mathcal{C})$ to be the 2-category assigned as follows.

(1) We define:

$$\mathrm{Ob}(\text{2-}\mathrm{Cat}(\mathcal{C})) := \mathrm{Ob}(\mathcal{C}) \tag{E.5.2.4}$$

which is to say that the objects of $\text{2-}\mathrm{Cat}(\mathcal{C})$ are the objects of $\mathcal{C}$.

(2) For any pair of objects $A, B \in \mathrm{Ob}(\text{2-}\mathrm{Cat}(\mathcal{C}))$, we define the category $\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B)$ as follows:
   (a) we define:

$$\mathrm{Ob}(\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B)) := \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{E.5.2.5}$$

   which is to say that the objects of $\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B)$ are the elements of the Hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of $\mathcal{C}$.
   (b) We define, for any $f, g \in \mathrm{Ob}(\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B))$ the set $\mathrm{Hom}_{\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}}(f, g)$ to be the singleton set if $f = g$ as elements of $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and the empty set otherwise. We denote by $\mathbf{1}_f$ the only element of $\mathrm{Hom}_{\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}}(f, f)$.

   This is to say that the objects of $\text{2-}\mathrm{Cat}(\mathcal{C})$ are defined to be exactly the objects of $\mathcal{C}$, the 1-morphisms of $\text{2-}\mathrm{Cat}(\mathcal{C})$ are defined to be the morphisms of $\mathcal{C}$, and the 2-morphisms of $\text{2-}\mathrm{Cat}(\mathcal{C})$ are defined to be all and only the identities on the 1-morphisms.

(3) We define, for any triple of objects $A, B, C \in \mathrm{Ob}(\text{2-}\mathrm{Cat}(\mathcal{C}))$:

$$c_{A,B,C} : \mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B) \times \mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(B, C) \to \mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, C) \tag{E.5.2.6}$$

   as follows:
   (a) for any $f \in \mathrm{Ob}(\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, B))$ and $g \in \mathrm{Ob}(\mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(B, C))$, we define:

$$c_{A,B,C}(f, g) := g \circ f \tag{E.5.2.7}$$

   where $g \circ f$ is the composition of f and g as morphisms of $\mathcal{C}$.
   (b) for any 1-morphisms $f, f' : A \rightrightarrows B$ and $g, g' : B \rightrightarrows C$, and for any $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$, we define $c_{A,B,C}(\alpha, \beta)$ to be $c_{A,B,C}(\alpha, \beta) := \mathbf{1}_{g \circ f}$ if $f = f'$ and $g = g'$.

   To see that the assignment in part (b) of (3) is well-defined, we observe that for $f \neq f'$ or $g \neq g'$, $c_{A,B,C}$ is not defined since either there is no 2-morphism $f \Rightarrow f'$ or there is no 2-morphism $g \Rightarrow g'$.

(4) For every object $A \in \mathrm{Ob}(\text{2-}\mathrm{Cat}(\mathcal{C}))$, we set:

$$u_A : \mathbf{1} \to \mathbf{Hom}_{\text{2-}\mathrm{Cat}(\mathcal{C})}(A, A)$$

by defining, having $\mathrm{Ob}(\mathbf{1}) = \{\bullet\}$ and $\mathrm{Hom}_{\mathbf{1}}(\bullet,\bullet) = \{e_\bullet\}$ (see part (1) of Remark E.5.1.20 and part (2) of Remark E.5.2.2), $u_A(\bullet) := \mathrm{id}_A$, the identity map on the object A of $\mathcal{C}$, and $u_A(e_\bullet) := \mathbf{1}_{\mathrm{id}_A} : \mathrm{id}_A \to \mathrm{id}_A$, the identity on id_A.

We now check that the above assignments indeed satisfy the axioms from Definition E.5.2.1. For the Unit Axiom (Axiom (1) from Definition E.5.2.1), we have, for every triple of objects $A, B, C \in \mathrm{Ob}(\text{2-Cat}(\mathcal{C}))$ and $f \in \mathrm{Ob}(\mathbf{Hom}_{\text{2-Cat}(\mathcal{C})}(A,B))$:

$$c_{A,A,B}(\mathbf{1}_{\mathrm{id}_A}, f) = f = c_{A,B,B}(f, \mathbf{1}_{\mathrm{id}_B})$$

and the same axiom on 2-morphisms is indeed satisfied as the relevant Hom-sets are either empty, in which case the assertion is vacuously true, or contain only the identity endomorphism $\mathbf{1}_{\mathrm{id}_A}$. Finally, the Associativity Axiom (Axiom (2) from Definition E.5.2.1) is satisfied as composition of morphisms in $\mathcal{C}$ is associative, as $\mathcal{C}$ is a category. We deduce that 2-Cat($\mathcal{C}$) is a 2-category in the sense of Definition E.5.2.1.

E.5.3. Pseudo-functors. The purpose of this subsection is that of providing the language that is necessary to define sub 2-categories and their properties relative to an "ambient" 2-category. All the examples of 2-categories throughout this appendix will be constructed as sub 2-categories of the 2-category **Cat** of all small categories (see §E.5.1). We will be returning to the examples of **Cat** and its variant $\mathbf{Cat}^*$ from Definition E.5.1.15 after we collect the basic concepts around functors between 2-categories.

E.5.3.1. *Basic notions on pseudo-functors.* The definition below is a special case of [**Bor1**, Ch.7,Def.7.5.1], and a direct analog of [**GR**, Ch.2,Def.2.1.14]. We will follow to some large extent the notation of [**Bor1**, Ch.7] and [**GR**, Ch.2], to facilitate the reader in consulting the referred source materials.

DEFINITION E.5.3.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two 2-categories. We call a *pseudo-functor* $\mathcal{C} \to \mathcal{C}'$ the datum of:

(1) A class function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$.
(2) For all pairs of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a functor:

$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A,B) \to \mathbf{Hom}_{\mathcal{C}}(F(A),F(B)).$$

(3) For all triples of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a 2-isomorphism $\gamma_{A,B,C}$ in **Cat** making the following diagram 2-commutative:

(E.5.3.1)
$$\begin{array}{ccc} \mathbf{Hom}_{\mathcal{C}}(A,B) \times \mathbf{Hom}_{\mathcal{C}}(B,C) & \xrightarrow{c_{A,B,C}} & \mathbf{Hom}_{\mathcal{C}}(A,C) \\ \downarrow{\scriptstyle (F_{A,B},F_{B,C})} & \overset{\gamma_{A,B,C}}{\nearrow} & \downarrow{\scriptstyle F_{A,C}} \\ \mathbf{Hom}_{\mathcal{C}}(F(A),F(B)) \times \mathbf{Hom}_{\mathcal{C}}(F(B),F(C)) & \xrightarrow[c_{F(A),F(B),F(C)}]{} & \mathbf{Hom}_{\mathcal{C}}(F(A),F(C)) \end{array}$$

(4) For every object $A \in \mathrm{Ob}(\mathcal{C})$, a natural isomorphism between the two functors defined as the compositions in the diagram (E.5.3.3) below from the upper leftmost vertex to the lower rightmost vertex, between the categories $\mathbf{1} \rightrightarrows \mathbf{Hom}_{\mathcal{C}}(F(A),F(A))$:

(E.5.3.2)
$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{u_A} & \mathbf{Hom}_{\mathcal{C}}(A,A) \\ \| & \overset{\delta_A}{\nearrow} & \downarrow{\scriptstyle F_{A,A}} \\ \mathbf{1} & \xrightarrow[u_{F(A)}]{} & \mathbf{Hom}_{\mathcal{C}}(F(A),F(A)) \end{array}$$

The datum $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$ is assumed to satisfy the following properties:

(1) (Unit Axiom) For every 1-morphism $f : A \to B$, the following diagrams commute:

$$\begin{array}{ccc} F_{A,B}(f) \circ \mathbf{1}_{F(A)} & \xRightarrow{\mathbf{1}_{F_{A,B}(f)} \star (\delta_A)_*} & F_{A,B}(f) \circ F_{A,A}(\mathbf{1}_A) \\ {\scriptstyle \mathbf{1}_{F_{A,B}(f)}}\Downarrow & & \Downarrow {\scriptstyle \gamma_{A,A,B}(\mathbf{1}_A, f)} \\ F_{A,B}(f) & \xRightarrow[\mathbf{1}_{F_{A,B}(f)}]{} & F_{A,B}(f \circ \mathbf{1}_A) \end{array} \tag{E.5.3.3}$$

where $(\delta_A)_*$ denotes the 2-morphism $\mathbf{1}_A \Rightarrow F_{A,A}(\mathbf{1}_A)$ induced by δ_A, and

$$\begin{array}{ccc} \mathbf{1}_{F(B)} \circ F_{A,B}(f) & \xRightarrow{(\delta_B)_* \star F_{A,A}(f)} & F_{B,B}(\mathbf{1}_B) \circ F_{A,B}(f) \\ {\scriptstyle \mathbf{1}_{F_{A,B}(f)}}\Downarrow & & \Downarrow {\scriptstyle \gamma_{A,B,B}(f, \mathbf{1}_B)} \\ F_{A,B}(f) & \xRightarrow[\mathbf{1}_{F_{A,B}(f)}]{} & F_{A,B}(\mathbf{1}_B \circ f) \end{array} \tag{E.5.3.4}$$

where $(\delta_B)_*$ denotes the 2-morphism $\mathbf{1}_B \Rightarrow F_{B,B}(\mathbf{1}_B)$ induced by δ_B.

(2) (Associativity of Compositions) For all triples of 1-morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F_{C,D}(h) \circ F_{B,C}(g) \circ F_{A,B}(f) & \xRightarrow{\mathbf{1}_{F_{C,D}(h)} \star \gamma_{A,B,C}(f,g)} & F_{C,D}(h) \circ F_{A,C}(g \circ f) \\ {\scriptstyle \gamma_{B,C,D}(g,h) \star \mathbf{1}_{F_{A,B}(f)}}\Downarrow & & \Downarrow {\scriptstyle \gamma_{A,C,D}(g \circ f, h)} \\ F_{B,D}(h \circ g) \circ F_{A,B}(f) & \xRightarrow[\gamma_{A,B,D}(f, h \circ g)]{} & F_{A,D}(h \circ g \circ f) \end{array} \tag{E.5.3.5}$$

We denote by $F : \mathcal{C} \to \mathcal{C}'$ the datum $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$, and call $(\gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$ the *coherence constraint* of F.

We illustrate the meaning of the two axioms from Definition E.5.3.1 above. Let $f : A \to B$ and $g : B \to C$ be two 1-morphisms. Then we may evaluate $F_{A,B}$ and $F_{B,C}$ on each, and then compose. We obtain:

$$c_{F(A),F(B),F(C)}(F_{A,B}(f), F_{B,C}(g)) = F_{B,C}(g) \circ F_{A,B}(f)$$

according to the notation introduced in Terminology E.5.2.4. We may also first compose to obtain $c_{A,B,C}(f,g)$, and then evaluate $F_{A,C}$ on the result, to get $F_{A,C}(g \circ f)$. $\gamma_{A,B,C}(f,g)$ is a 2-isomorphism between the two 1-morphisms $F_{B,C}(g) \circ F_{A,B}(f)$ and $F_{A,C}(g \circ f)$. Specifically, we have the 2-isomorphism:

$$\gamma_{A,B,C}(f,g) : F_{B,C}(g) \circ F_{A,B}(f) \Rightarrow F_{A,C}(g \circ f)$$

naturally in f, g.

Example E.5.3.2. In the case of Construction E.5.2.6, to a pseudo-functor between 2-categories $F : \mathcal{C} \to \mathcal{C}'$, we associate the functor $\mathrm{Cat}(F) : \mathrm{Cat}(\mathcal{C}) \to \mathrm{Cat}(\mathcal{C}')$ given by the class function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ induced by F on objects, and by the effect of F on 1-morphisms prescribed by part (2) of Definition E.5.3.1, on morphisms of $\mathrm{Cat}(\mathcal{C})$ and $\mathrm{Cat}(\mathcal{C}')$, respectively.

Example E.5.3.3. In the special case of Construction E.5.2.7, we have an ordinary (locally small) category $\mathcal{C}$, and its associated 2-category 2-Cat($\mathcal{C}$). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary (locally small) categories. We may associate to it the datum $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$ of a pseudo-functor 2-Cat($\mathcal{C}$) $\to$ 2-Cat($\mathcal{D}$) as per Definition E.5.3.1, by defining the class function F on objects to be the same class function from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$, and for any pair of objects $A, B \in \mathrm{Ob}(\text{2-Cat}(\mathcal{C}))$:

$$F_{A,B} : \mathbf{Hom}_{\text{2-Cat}(\mathcal{C})}(A,B) \to \mathbf{Hom}_{\text{2-Cat}(\mathcal{D})}(F(A), F(B))$$

is defined on objects to be the effect of F on morphisms of $\mathcal{C}$, and on morphisms to be the unique assignment sending the identities to the identities. In this case, $\gamma_{A,B,C}$ and δ_A are the identity for all A, B, C objects of 2-Cat($\mathcal{C}$).

The Unit Axiom from Definition E.5.3.1 encodes the correct behavior of the datum $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$ with respect to passage from $\mathbf{1}_{F(A)}$ to $F_{A,A}(\mathbf{1}_A)$. Namely, we are *not* assuming that pseudo-functors preserve the identities, i.e. $\mathbf{1}_{F(A)}$ and $F_{A,A}(\mathbf{1}_A)$ may not be equal as elements of the set of objects of the category $\mathbf{Hom}_{\mathcal{C}}(F(A), F(A))$, but they are isomorphic, and the axiom defines a notion of naturality for this isomorphism, codifying the compatibility of a pseudo-functor with it.

We first note that the composition of the leftmost vertical arrow followed by the bottom horizontal arrow in (E.5.3.3) is always a 2-isomorphism. This is due to the fact that in light of the Unit Axiom in the very definition of a 2-category, i.e. Axiom (1) from Definition E.5.2.1.[20]

Specifically, for any 1-morphism $f : A \to B$, the composition $F_{A,B}(f) \circ \mathbf{1}_{F(A)}$ is $F_{A,B}(f)$, which in turn can be written as $F_{A,B}(f \circ \mathbf{1}_A)$. "Writing $F_{A,B}(f)$ as $F_{A,B}(f \circ \mathbf{1}_A)$", and likewise "writing $\mathbf{1}_B \circ F_{A,B}(f)$ as $F_{A,B}(f)$", involves passage from one to the other via the composition bifunctor, which by Axiom (2) from Definition E.5.2.1 satisfies $c_{F(A),F(A),F(B)}(\mathbf{1}_{F(A)}, F_{A,B}(f)) = F_{A,B}(f)$, and hence $F_{A,B}(c_{A,A,B}(\mathbf{1}_A, f)) = F_{A,B}(f)$. This is the reason why we write $\mathbf{1}_{F_{A,B}(f)}$ for the canonical 2-(iso)morphisms placed as the leftmost vertical arrow and the bottom horizontal arrow in (E.5.3.3) and (E.5.3.4).

Axiom (1) in Definition E.5.3.1 compares the canonical 2-isomorphism $F_{A,B} \circ \mathbf{1}_{F(A)} \Rightarrow F_{A,B}(f \circ \mathbf{1}_A)$ obtained by composition of the two $\mathbf{1}_{F_{A,B}(f)}$ as discussed above, to the 2-morphism obtained instead by using $(\delta_A)_* : \mathbf{1}_{F(A)} \Rightarrow F_{A,A}(\mathbf{1}_A)$ to replace $\mathbf{1}_{F(A)}$ in $F_{A,B}(f) \circ \mathbf{1}_{F(A)}$ with $F_{A,A}(\mathbf{1}_A)$, for then passing to $F_{A,B}(f \circ \mathbf{1}_A)$ using $\gamma_{A,A,B}(\mathbf{1}_A, f)$. The first operation is a 2-isomorphism $F_{A,B}(f) \circ \mathbf{1}_{F(A)} \Rightarrow F_{A,B}(f) \circ F_{A,A}(\mathbf{1}_A)$, given by $\mathbf{1}_{F_{A,B}(f)} \star (\delta_A)_*$. The content of the axiom is that the two compositions, namely $F_{A,B}(f) \circ \mathbf{1}_{F(A)}$ and $F_{A,B}(f \circ \mathbf{1}_A)$, are 2-isomorphic.

As for the Axiom on Associativity of Compositions, it similarly encodes that the two possible results of grouping compositions together in a triple composition are 2-isomorphic.

NOTATION E.5.3.1. We introduce the following notation:

(1) We denote, for any $(f, g) \in \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(A, B)) \times \mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(B, C))$, the "$(f, g)$-section" $\gamma_{A,B,C}(f, g)$ of the natural transformation $\gamma_{A,B,C}$ evaluated at the object (f, g) of $\mathbf{Hom}_{\mathcal{C}}(A, B) \times \mathbf{Hom}_{\mathcal{C}}(B, C)$, by $\gamma_{f,g}$, omitting A, B, C from the notation when no confusion is expected to arise.

(2) We denote, for any 1-morphism $f : A \to B$, by $F(f)$ the 1-morphism $F_{A,B}(f) : F(A) \to F(B)$.

REMARK E.5.3.4. We make the following observations on Definition E.5.3.1. In the special case when δ_A is the identity for every object $A \in \mathrm{Ob}(\mathcal{C})$, the axioms satisfied by the datum of a pseudo-functor $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet)$ as per Definition E.5.3.1 can be rephrased as follows, and when $\mathcal{C}$ is a small 2-category, they recover the axioms from [**Ha**, Ch. I, §1, (1.5)]:

(1) (Unit Axiom) For all objects $A \in \mathrm{Ob}(\mathcal{C})$, we have $F_{A,A}(\mathbf{1}_A) = \mathbf{1}_{F(A)}$. For all pairs of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we have $\gamma_{A,B,B}(\cdot, \mathbf{1}_B) = \mathbf{1}_{F_{A,B}}$, and $\gamma_{A,A,B}(\mathbf{1}_A, \cdot) = \mathbf{1}_{F_{A,B}}$. This is [**Ha**, Ch. I, §1, (1.5), Axioms (i)′,(ii)′].

(2) (Associativity of Compositions) For all quadruples of objects $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$, one has the identity of 2-isomorphisms in **Cat**:

$$\gamma_{A,B,D}(\cdot,\cdot) \odot (\gamma_{B,C,D}(\cdot,\cdot) \star \mathbf{1}_{F_{A,B}}) = \gamma_{A,C,D}(\cdot,\cdot) \odot (\mathbf{1}_{F_{C,D}} \star \gamma_{A,B,C}(\cdot,\cdot)).$$

This is [**Ha**, Ch. I, §1, (1.5), Axiom (iii)′], upon observing that our conventions are consistent with the ones in loc.cit., as they are with the ones in [**GR**, Ch. 2], and that the identity in [**Ha**, Ch. I, §1, (1.5), Axiom (iii)′] is equivalent to the one above upon evaluating the functor μ' of loc.cit. on it.

[20] This is not a sentence; what is it meant to be saying? Rewrite as an actual sentence.

REMARK E.5.3.5. We let $\mathcal{I}$ be a locally small category, and now describe the pseudo-functors from 2-Cat($\mathcal{I}$), where 2-Cat($\mathcal{I}$) is the 2-category associated to $\mathcal{I}$ constructed in Construction E.5.2.7, to a fixed 2-category $\mathcal{C}$. In the special case when $\mathcal{C}$ is small, the description we shall obtain will match the definition of *weak 2-functor* from part (2) of [**SP**, Tag 003N], up to a discrepancy concerning the coherence constraint, and specifically the transformation $\gamma_{\bullet,\bullet,\bullet}$. In our conventions, which follow closely those in [**GR**, Ch. 2], $\gamma_{\bullet,\bullet,\bullet}$ is the inverse of the analogous transformation α from [**SP**, Tag 003N].

A pseudo-functor 2-Cat($\mathcal{I}$) $\to \mathcal{C}$ in the sense of Definition E.5.3.1 is the datum of:

(1) a class function $F : \mathrm{Ob}(\mathcal{I}) \to \mathrm{Ob}(\mathcal{C})$.
(2) For each morphism $f : x \to y$ in $\mathcal{I}$, a 1-morphism $F(f) : F(x) \to F(y)$ in $\mathcal{C}$.
(3) For every object $x \in \mathcal{I}$, a 2-morphism $\delta_x : F(\mathbf{1}_x) \Rightarrow \mathbf{1}_{F(x)}$ in $\mathcal{C}$.
(4) For every pair of morphisms $f : x \to y$, $g : y \to z$, a 2-morphism $\gamma_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ in $\mathcal{C}$.

The above data is then required to satisfy the Unit Axiom and the Associativity of Compositions Axiom as per Definition E.5.3.1, which in this special case take the following form:

(1) $\gamma_{f,g}$ and δ_x are 2-isomorphisms for all $x \in \mathrm{Ob}(\mathcal{I})$ and all morphisms f, g in $\mathcal{I}$.
(2) (Unit Axiom) For any morphism $f : x \to y$ in $\mathcal{I}$, we have:
$$\mathbf{1}_{F(f)} \star \delta_x = \gamma_{f,\mathbf{1}_x}$$
and:
$$\delta_y \star \mathbf{1}_{F(f)} = \gamma_{\mathbf{1}_y,f}.$$
(3) (Associativity of Compositions Axiom) For any triple of morphisms $f : x \to y$, $g : y \to z$, $h : z \to w$ in $\mathcal{I}$, we have:
$$\gamma_{f,h\circ g} \odot (\gamma_{g,h} \star \mathbf{1}_{F(f)}) = \gamma_{g\circ f,h} \odot (\mathbf{1}_{F(h)} \star \gamma_{f,g}).$$

If we rename $\gamma^{-1}_{\bullet,\bullet}$ as $\alpha_{\bullet,\bullet}$ and $\odot$ as $\circ$ (and we therefore reverse the order of all compositions), we recover the definition of *weak functor* from part (2) of [**SP**, Tag 003N].

E.5.3.2. *Degrees of faithfulness.* We introduce the following:

DEFINITION E.5.3.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be 2-categories, and $F : \mathcal{C} \to \mathcal{C}'$ a pseudo-functor. We say:

(1) F is *weakly faithful* if and only if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor:
$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}}(F(A), F(B))$$
is faithful.
(2) F is *faithful* if and only if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor:
$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}}(F(A), F(B))$$
is fully faithful.
(3) F is *strongly faithful* if and only if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor
$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}}(F(A), F(B))$$
is fully faithful and the set-function induced on objects is injective.
(4) F is *fully faithful* if and only if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor:
$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}'}(F(A), F(B))$$
is an equivalence of categories.
(5) F is *strongly fully faithful* if and only if for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor:
$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}'}(F(A), F(B))$$
is an isomorphism of categories.

(6) F is a 2-*equivalence* if and only if it is fully faithful and for every object $B \in \mathrm{Ob}(\mathcal{C}')$ there exists an object $A \in \mathrm{Ob}(\mathcal{C})$ and an equivalence $F(A) \to B$ in $\mathcal{C}$.
(7) F is a *strong* 2-*equivalence* if and only if it is fully faithful and for every object $B \in \mathrm{Ob}(\mathcal{C}')$ there exists an object $A \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $F(A) \to B$ in $\mathcal{C}$.
(8) F is an *isomorphism* if and only if it is strongly fully faithful and a strong 2-equivalence.
(9) F is *strict* if and only if the structure natural transformations $\gamma_{\bullet,\bullet,\bullet}$ and $\delta_\bullet$ in the coherence constraint $(\gamma^F_{\bullet,\bullet,\bullet}, \delta^F_\bullet)$ for F are the identity.

We can now define the concept of a sub 2-category of a 2-category:

Definition E.5.3.7. Let $\mathcal{C}$ be a 2-category. We say:

(1) $\mathcal{C}'$ is a *sub* 2-*category* of $\mathcal{C}$ if and only if:
  (a) $\mathcal{C}'$ is a 2-category.
  (b) The class of the objects of $\mathcal{C}'$ is a sub-class of the class of the objects of $\mathcal{C}$.
  (c) The inclusion class function $F : \mathrm{Ob}(\mathcal{C}') \to \mathrm{Ob}(\mathcal{C})$ is part of the datum of a pseudo-functor $(F, F_{\bullet,\bullet}, \gamma_{\bullet,\bullet,\bullet}, \delta_\bullet) : \mathcal{C}' \to \mathcal{C}$. We call F the *inclusion functor*.
  (d) The inclusion functor F is weakly faithful.
(2) $\mathcal{C}'$ is a *strict sub* 2-*category* of $\mathcal{C}$ if and only if is is a sub 2-category and the inclusion functor is strict.
(3) $\mathcal{C}'$ is a *full sub* 2-*category* of $\mathcal{C}$ if and only if $\mathcal{C}'$ is a sub 2-category and the inclusion functor is fully faithful.
(4) $\mathcal{C}'$ is a *strong sub* 2-*category* of $\mathcal{C}$ if and only if $\mathcal{C}'$ is a sub 2-category and the inclusion functor is strongly faithful.

E.5.4. Comparing Cat* and Cat. We consider the following assignment.

Construction E.5.4.1. For any effective small category (X, A, s, t, c, e) (see Definition E.5.1.15), we associate the category $\mathcal{C}_{(X,A)}$ such that:

(1) $\mathrm{Ob}(\mathcal{C}_{(X,A)}) := X$.
(2) For any two objects $x, y \in \mathrm{Ob}(\mathcal{C}_{(X,A)})$, we define $\mathrm{Hom}_{\mathcal{C}_{(X,A)}}(x, y) := s^{-1}(x) \cap t^{-1}(y)$.
(3) For any two morphisms $f \in \mathrm{Hom}_{\mathcal{C}_{(X,A)}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{C}_{(X,A)}}(y, z)$, we define $g \circ f := c(f, g)$.

For any functor $F = (f_1, f_2) : (X, A, s, t, c, e) \to (X', A', s', t', c', e')$:

(1) for any object $x \in \mathcal{C}_{(X,A)}$, we define $\phi_F(x) := f_1(x)$.
(2) For any morphism $f \in \mathrm{Hom}_{\mathcal{C}_{(X,A)}}(x, y)$, we define $\phi_F(f) := f_2(f)$.

We call $\phi_F : \mathcal{C}_{(X,A)} \to \mathcal{C}_{(X',A')}$ the assignment defined above on objects and morphisms.

We have the following consistency checks:

Lemma E.5.4.2. *For any effective small category (X, A, s, t, c, e), the datum $\mathcal{C}_{(X,A)}$ is a small category. The assignment:*

$$(X, A, s, t, c, e) \rightsquigarrow \mathcal{C}_{(X,A)} \tag{E.5.4.1}$$

is functorial, the functoriality being defined by ϕ constructed in Construction E.5.4.1.

Proof. The very definition of ϕ ensures compatibility with composition of morphisms on source and target, since, for any functor $F = (f_1, f_2) : (X, A, s, t, c, e) \to (X', A', s', t', c', e')$, ϕ_F is defined to be f_2 on morphisms, and (f_1, f_2) is a morphism of effective small categories. In particular, f_2 is compatible with the composition laws c, c', which translates into ϕ_F being compatible with composition of morphisms, as desired. □

We have the following:

LEMMA E.5.4.3. *The assignment:*

$$\phi : \mathbf{Cat}^* \to \mathbf{Cat} \tag{E.5.4.2}$$

defined in Construction E.5.4.1 is an isomorphism of locally small categories.

PROOF. Indeed, for any two effective small categories (X, A, s, t, c, e) and (X', A', s', t', c', e'), the functor ϕ induces an injective set-function:

$$\phi : \mathrm{Hom}_{\mathbf{Cat}^*}((X, A, s, t, c, e), (X', A', s', t', c', e')) \to \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}_{(X,A)}, \mathcal{C}_{(X',A')}).$$

To see this, we observe that by design of ϕ, for any functor $F = (f_1, f_2) : (X, A, s, t, c, e) \to (X', A', s', t', c', e')$, f_2 is uniquely determined by ϕ_F. As for f_1, one recovers it by computing the effect of ϕ_F on objects of $\mathcal{C}_{(X,A)}$, which are, by definition, the set X. We now prove surjectivity of the map that ϕ induces on Hom-sets. Let $F : \mathcal{C}_{(X,A)} \to \mathcal{C}_{(X',A')}$ be a functor. We may define $f_1 : X \to X$ to be $f_1(x) := F(x)$, as $X = \mathrm{Ob}(\mathcal{C}_{(X,A)})$ by definition of $\mathcal{C}_{(X,A)}$. We can reconstruct A as:

$$A = \coprod_{(x,y) \in \mathrm{Ob}(\mathcal{C}_{(X,A)}) \times \mathrm{Ob}(\mathcal{C}_{(X,A)})} \mathrm{Hom}_{\mathcal{C}_{(X,A)}}(x, y)$$

and so $f_2(a) := F(a)$ defines a pair of set-functions $(f_1, f_2) : (X, A) \to (X', A')$.

Compatibility with s, t, c, e follows from the definition of $\mathcal{C}_{(X,A)}$ and the composition of morphisms in $\mathcal{C}_{(X,A)}$. This proves that ϕ is fully faithful. For $\mathcal{C}$ a small category, we call $X := \mathrm{Ob}(\mathcal{C})$, $A := \coprod_{(x,y) \in X \times X} \mathrm{Hom}_{\mathcal{C}}(x, y)$, and define $s, t : A \rightrightarrows X$ as follows. For any $a \in A$, there exists a unique pair $(x, y) \in X \times X$ such that $a \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. We define $s(a) := x$ and $t(a) := t$. For $a, a' \in A$ such that $s(a) = t(a')$, we have $a \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ and $a' \in \mathrm{Hom}_{\mathcal{C}}(z, x)$, and so the composition $a \circ a'$ makes sense. We denote it by $c(a', a)$, and this provides a set-function $A \times_{s,X,t} A \to A$. Finally, we define $e : X \to A$ by $e(x) := \mathrm{id}_x$. We therefore obtain a 6-tuple (X, A, s, t, c, e) from $\mathcal{C}$, and the axioms (**??**), (**??**), (**??**) and (**??**) from Definition E.5.1.15 are seen to be satisfied by (s, t, c, e) upon inspection of the defining axioms of a category. This shows that $\mathcal{C} = \mathcal{C}_{(X,A)}$, i.e. $\mathcal{C}$ is isomorphic as a category to a category of the form $\mathcal{C}_{(X,A)}$. This shows that $\phi : \mathbf{Cat}^* \to \mathbf{Cat}$ from (E.5.4.2) is an isomorphism of categories. □

We can promote ϕ to a pseudo-functor as follows:

CONSTRUCTION E.5.4.4. For any natural transformation $T : F \Rightarrow F'$ between functors $F = (f_1, f_2), F' = (f_1', f_2') : (X, A, s, t, c, e) \rightrightarrows (X', A', s', t', c', e')$, we define $\phi(T)$ to be the natural transformation of functors $\phi_F \Rightarrow \phi_{F'}$ characterized by the following property: for any object x in $\mathcal{C}_{(X,A)}$, the datum of $\phi(T)_x$ is the morphism $f_1(x) \to f_1'(x)$ in $\mathcal{C}_{(X',A')}$ given by $T(x) \in (s')^{-1}(f_1(x)) \cap (t')^{-1}(f_1'(x)) = \mathrm{Hom}_{\mathcal{C}_{(X',A')}}(f_1(x), f_1'(x))$. Naturality of $\phi(T)_x$ with respect to x is a consequence of (E.5.1.24) from Definition E.5.1.16. See also Remark **??**.

LEMMA E.5.4.5. *The locally small categories* $\mathbf{Cat}$ *and* $\mathbf{Cat}^*$ *are, in a canonical way,* 2*-categories, whose composition bifunctors are the bifunctors* $c_{\bullet,\bullet,\bullet}$ *constructed in Lemma E.5.1.14 and Lemma E.5.1.19. The functor* $\phi : \mathbf{Cat}^* \to \mathbf{Cat}$ *is a strict isomorphism of* 2*-categories.*

PROOF. Indeed, since the composition of morphisms in $\mathcal{C}_{(X,A)}$ for (X, A, s, t, c, e) an effective small category, is defined by means of the composition law c, and ϕ is defined in terms of the component set-functions of functors between effective small categories (see Definition E.5.1.15 and Construction E.5.4.4), ϕ is compatible with composition of 1-morphisms and 2-morphisms on the nose, forcing $\gamma_{\bullet,\bullet,\bullet}$ and $\delta_\bullet$ to be the identity, so ϕ is a strict pseudo-functor. By Lemma E.5.4.3 it induces an essentially surjective functor on Hom-categories. Such functor is also fully faithful because, given a natural transformation of functors $T : F \Rightarrow F'$ between effective small categories $F = (f_1, f_2), F' = (f_1', f_2') : (X, A) \rightrightarrows (X', A')$, T is a set-function $T : X \to A'$,

and knowledge of the associated natural transformation is knowledge of a coherent family of set-functions $f_1(x) \to f_1'(x)$ for every $x \in X$.

To reconstruct T, one defines $T(x)$ to be the element $a' \in A'$ corresponding to the morphism $f_1(x) \to f_1'(x)$ under ϕ, and seeing that indeed this construction recovers T is a simple consequence of the definitions. More generally, any natural transformation $T : F \Rightarrow F'$ between functors of small categories $\mathcal{C} \to \mathcal{C}'$ produces, by means of the above construction, a set-function $T : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Arrows}(\mathcal{C}')$ satisfying the defining axioms (see Definition E.5.1.16), and hence the functor induced by ϕ on Hom-categories is an isomorphism of categories, so ϕ is strongly fully faithful. Upon combining strong full faithfulness with Lemma E.5.4.3 and with the strictness of ϕ proved in the foregoing, we deduce ϕ is a strict isomorphism of 2-categories, as desired. □

REMARK E.5.4.6. In light of Lemma E.5.4.5, we can now give meaning to the sentence "we view the datum of (X, A, s, t, c, e) as a small category", which is commonly done when discussing, for instance, groupoids and stacks in groupoids on a site. Conversely, we may "view a small category as the datum of (X, A, s, t, c, e)", for appropriate structure morphisms (s, t, c, e), and this sentence now has an un-ambiguous meaning provided by Lemma E.5.4.5, insofar it refers to the functor $\phi : \mathbf{Cat}^* \to \mathbf{Cat}$ being a strict isomorphism of 2-categories.

We have the following simple but useful Lemma:

LEMMA E.5.4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be small categories. If all the morphisms of $\mathcal{A}'$ are isomorphisms, then for any pair of functors $F, F' : \mathcal{A} \rightrightarrows \mathcal{B}$, any natural transformation $t : F \Rightarrow F'$ is an isomorphism of functors.*

PROOF. For every object a of $\mathcal{A}$, $t_a : F(a) \to F'(a)$ is a morphism in $\mathcal{A}'$. By assumption, every morphism in $\mathcal{A}'$ is an isomorphism, and hence for every object a of $\mathcal{A}$, there exists a necessarily unique inverse t_a^{-1} to t_a in $\mathcal{A}'$. This gives a natural transformation $t^{-1} : F' \Rightarrow F$. That t^{-1} is the inverse of t follows from the definition of t_a^{-1} for every object a of $\mathcal{A}$. □

We now consider the inclusion of the category **Groupoids** into the category $\mathbf{Cat}^*$, assigned by sending the datum of a groupoid in sets (X, R, s, t, c, e, i) (see Definition E.4.1.1) to the effective small category (X, R, s, t, c, e) ("forget i"). We call:

$$u_{\mathrm{gpd}} : \mathbf{Groupoids} \to \mathbf{Cat}^* \tag{E.5.4.3}$$

the fully faithful inclusion functor of ordinary categories.

LEMMA E.5.4.8. *The category* **Groupoids** *is a 2-category upon endowing it with the composition bifunctor constructed on effective small categories in Lemma E.5.1.14. The functor $u_{\mathrm{gpd}} : \mathbf{Groupoids} \to \mathbf{Cat}^*$ from* (E.5.4.3) *extends to a strongly fully faithful strict pseudo-functor.*

PROOF. All we need to check is that for any two groupoids (X, R, s, t, c, e, i) and $(X', R', s', t', c', e', i')$ and any functor $F = (f_1, f_2) : (X, R) \to (X', R')$, any natural transformation $T : X \to R'$ is invertible. This is indeed the case, as we may define T^{-1} to be $i' \circ T$, and verify it is indeed inverse to T upon proceeding similarly to Lemma E.5.4.7. □

REMARK E.5.4.9. The pseudo-functor $\phi : \mathbf{Groupoids} \to \mathbf{Cat}$ induces a strict isomorphism of 2-categories between the 2-category **Groupoids** and the sub 2-category of all small categories whose morphisms are all isomorphisms. We may therefore view any such category in its effective-small-category incarnation, and we shall do so without mentioning ϕ, knowing that interchanging between the two perspectives amounts to appealing to ϕ being a strict isomorphism of 2-categories. We will call **Groupoids** both the category defined in Definition E.4.1.1, and the image of ϕ, and we shall be identifying their 2-category structures.

We have the following example of a pseudo-functor that is not strongly fully faithful (nor fully faithful) but is strongly faithful.

LEMMA E.5.4.10. *The category of topological groupoids* **CGWH-Grpd** *is a* 2*-category. Moreover, the forgetful functor* $u : \mathbf{CGWH\text{-}Grpd} \to \mathbf{Groupoids}$ *is a strict faithful pseudo-functor.*

PROOF. The forgetful functor induces injective set-functions on objects of the Hom-categories by faithfulness of the forgetful functor $\mathbf{CGWH} \to \mathbf{Sets}$. For any two objects of **CGWH-Grpd**, the induced functor between Hom-categories is faithful for the same reason. Strictness of u follows from the fact that morphisms between topological groupoids are defined analogously to morphisms of groupoids in sets (see Definition E.4.1.1) and hence the structure maps of the coherence constraint of u are the identity. □

E.5.5. 2-category of 2-commutative diagrams. In this section we shall describe in detail the 2-category of 2-commutative diagrams in a 2-category. The concept of 2-commutativity for diagrams in a 2-category was introduced in Definition E.5.2.4. This construction will enable us to discuss 2-categorical fibre products in a concrete manner in §E.5.7 below without a general discussion on 2-limits.

E.5.5.1. *Construction of the* 2*-category of* 2*-commutative diagrams.* We fix a 2-category $\mathcal{C}$ and objects A, B, C, with 1-morphisms $f : B \to A$, $g : C \to A$.

CONSTRUCTION E.5.5.1. We define the class Diagrams(f, g), whose elements are 4-tuples (D, a, b, α) where $D \in \mathrm{Ob}(\mathcal{C})$, $a : D \to B$ and $b : D \to C$ are 1-morphisms, and $\phi : f \circ a \Rightarrow g \circ b$ is a 2-isomorphism. We represent the elements of Diagrams(f, g) pictorially as (not necessarily commutative) diagrams:

(E.5.5.1)
$$\begin{array}{ccc} D & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & \swarrow^{\phi} & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & A \end{array}$$

We note that Diagrams(f, g) is a class, since it is contained in the union of the sets:

$$\mathrm{Ob}(\mathbf{Hom}_{\mathcal{C}}(D, B) \times \mathbf{Hom}_{\mathcal{C}}(D, C)) \times \mathrm{Arrows}(\mathbf{Hom}_{\mathcal{C}}(D, A))$$

indexed by the elements D of the class $\mathrm{Ob}(\mathcal{C})$. In particular, Diagrams(f, g) is a set if $\mathcal{C}$ is a *small* 2-category.

NOTATION E.5.5.1. We note that in Definition E.5.2.4, we denoted the 2-isomorphism rendering a diagram 2-commutative in the 2-category $\mathcal{C}$ as follows:

$$\begin{array}{ccc} D & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & \swarrow^{\phi} & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & A \end{array}$$

It is our preference to favor a more symmetric notation, in light of the definition of 1-morphisms between 2-commutative diagrams, to be discussed below, in order to avoid conveying the incorrect impression of a choice being made of a preferred order between the morphisms $f \circ a$ and $g \circ b$.

CONSTRUCTION E.5.5.2. For any two 2-commutative diagrams (D, a, b, ϕ) and (D', c, d, ψ), we define the category $\mathbf{Hom}((D, a, b, \phi), (D', c, d, \psi))$ whose:

(1) objects are triples (k, α, β) where:
   (a) $k : D' \to D$ is a 1-morphism in $\mathcal{C}$.
   (b) $\alpha : c \Rightarrow a \circ k$, $\beta : d \Rightarrow b \circ k$ are 2-isomorphisms in $\mathcal{C}$, such that the following identity of 2-morphisms $f \circ c \Rightarrow g \circ b \circ k$ in $\mathcal{C}$ holds:

(E.5.5.2)
$$(\mathbf{1}_g \star \beta) \odot \psi = (\phi \star \mathbf{1}_k) \odot (\mathbf{1}_f \star \alpha).$$

(2) Morphisms $(k, \alpha, \beta) \Rightarrow (k', \alpha', \beta')$ are 2-morphisms $\gamma : k \Rightarrow k'$ such that the following identity as 2-morphisms $c \Rightarrow a \circ k'$ in $\mathcal{C}$ holds:

$$\alpha' = (\mathbf{1}_a \star \gamma) \odot \alpha$$

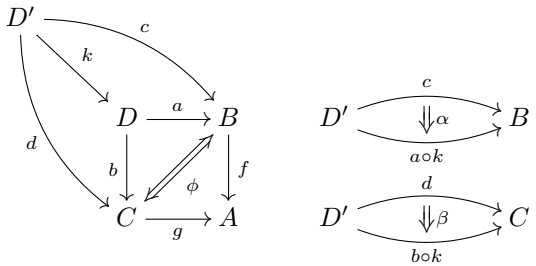
and the following identity as 2-morphisms $d \Rightarrow b \circ k'$ in $\mathcal{C}$ holds:

$$\beta' = (\mathbf{1}_b \star \gamma) \odot \beta.$$

We compose morphisms of $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ by composing the 2-morphisms γ using $\odot$ within 2-morphisms in $\mathcal{C}$, and hence $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ is indeed a (small) category.

NOTATION E.5.5.2. We depict the objects of $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ pictorially as follows:

(E.5.5.3)



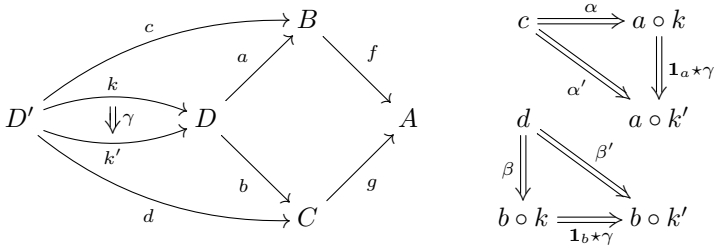
where condition (E.5.5.2) encodes commutativity of the following diagram of 2-(iso)morphisms in $\mathcal{C}$:

(E.5.5.4)

$$\begin{array}{ccc} f \circ c & \xRightarrow{\mathbf{1}_f \star \alpha} & f \circ a \circ k \\ {\scriptstyle\psi}\Downarrow & & \Downarrow{\scriptstyle \phi \star \mathbf{1}_k} \\ g \circ d & \xRightarrow[\mathbf{1}_g \star \beta]{} & g \circ b \circ k \end{array}$$

We depict the morphisms of $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ as follows:

(E.5.5.5)



Succintly, we write:

$$(D', c, d, \psi) \underset{(k', \alpha', \beta')}{\overset{(k, \alpha, \beta)}{\Downarrow \gamma}} (D, a, b, \phi)$$

LEMMA E.5.5.3. *Let (D, a, b, ϕ), (D', c, d, ψ) be two 2-commutative diagrams, and k, k', k'' be three morphisms $(D', c, d, \psi) \to (D, a, b, \phi)$ of 2-commutative diagrams. Given $\gamma : k \Rightarrow k'$ and $\gamma' : k' \Rightarrow k''$ be two morphisms in $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$, the assignment:*

$$(\gamma, \gamma') \rightsquigarrow \gamma' \odot \gamma$$

endows $\mathbf{Hom}(((D', c, d, \psi), (D, a, b, \phi))$ with the structure of a small category.

PROOF. Indeed, since $\odot$ is associative on 2-morphisms of $\mathcal{C}$ by Axiom (2) of Definition E.5.2.1, the assignment $(\gamma, \gamma') \rightsquigarrow \gamma' \odot \gamma$ defines an associative composition law

$$\mathrm{Hom}(k, k') \times \mathrm{Hom}(k', k'') \to \mathrm{Hom}(k, k'').$$

As the objects of $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ are contained in the product of the set of 1-morphisms $D' \to D$, the set of 2-morphisms $c \Rightarrow k \circ a$ in $\mathcal{C}$, and the set of 2-morphisms $d \Rightarrow k \circ b$ in $\mathcal{C}$, the class of the objects of $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ is a set. Moreover, morphisms between two such objects (k, α, β) and (k', α', β') are contained in the set of 2-morphisms $k \Rightarrow k'$, and hence are themselves a set. It follows that $\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$ is a small category. □

For a triple (D, a, b, ϕ), (D', c, d, ψ), (D'', r, s, λ) of elements of Diagrams(f, g), we define a functor

$$c(\bullet, \bullet) : \mathbf{Hom}((D'', r, s, \lambda), (D', c, d, \psi)) \times \mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi)) \to \mathbf{Hom}((D'', r, s, \lambda), (D, a, b, \phi))$$

by setting, for objects $(k, \alpha, \beta) : (D', c, d, \psi) \to (D, a, b, \phi)$ and $(k', \alpha', \beta') : (D'', r, s, \lambda) \to (D', c, d, \psi)$:

$$c((k', \alpha', \beta'), (k, \alpha, \beta)) := (k \circ k', (\alpha' \star \mathbf{1}_k) \odot \alpha, (\beta' \star \mathbf{1}_k) \odot \beta) \tag{E.5.5.6}$$

and given morphisms

$$(D'', r, s, \lambda) \underset{(k_2, \alpha_2, \beta_2)}{\overset{(k', \alpha', \beta')}{\Downarrow \gamma_2}} (D', c, d, \psi) \underset{(k_1, \alpha_1, \beta_1)}{\overset{(k, \alpha, \beta)}{\Downarrow \gamma_1}} (D, a, b, \phi) \tag{E.5.5.7}$$

we define

$$c(\gamma_2, \gamma_1) := \gamma_1 \star \gamma_2 \tag{E.5.5.8}$$

where $\gamma_1 \star \gamma_2$ is the morphism making the following joined diagram 2-commute

$$(D, a, b, \phi) \underset{c((k_1, \alpha_1, \beta_1), (k_2, \alpha_2, \beta_2))}{\overset{c((k, \alpha, \beta), (k', \alpha', \beta'))}{\Downarrow \gamma_2 \star \gamma_1}} (D'', r, s, \lambda) \tag{E.5.5.9}$$

LEMMA E.5.5.4. *Let $\mathcal{C}$ be a 2-category, and A, B, C objects of $\mathcal{C}$, with 1-morphisms $f : B \to A$, $g : C \to A$. The datum of:*

(1) The class Diagrams(f, g) *constructed in Construction* E.5.5.1.

(2) For all pairs of 2-commutative diagrams (D, a, b, ϕ) and (D', c, d, ψ), the category:

$$\mathbf{Hom}((D', c, d, \psi), (D, a, b, \phi))$$

constructed in Construction E.5.5.2.

(3) The bifunctor $c(\bullet, \bullet)$ constructed in (E.5.5.6) *and* (E.5.5.8).

is a 2-category, denoted $\mathbf{Diagrams}(f, g)$.

PROOF. We have to check that indeed $c((k', \alpha', \beta'), (k, \alpha, \beta))$ is an object of $\mathbf{Hom}((D'', r, s, \lambda), (D, a, b, \phi))$. To see this, we are to check that the following diagram of 2-morphisms in $\mathcal{C}$:

$$\begin{array}{ccccc} f \circ r & \xRightarrow{\mathbf{1}_f \star \alpha'} & f \circ c \circ k' & \xRightarrow{\mathbf{1}_f \star \alpha \star \mathbf{1}_{k'}} & f \circ a \circ (k \circ k') \\ \lambda \Downarrow & & \psi \star \mathbf{1}_{k'} \Downarrow & & \Downarrow \phi \star \mathbf{1}_{k \circ k'} = \phi \star \mathbf{1}_k \star \mathbf{1}_{k'} \\ g \circ s & \xRightarrow[\mathbf{1}_g \star \beta']{} & g \circ d \circ k' & \xRightarrow[\mathbf{1}_g \star \beta \star \mathbf{1}_{k'}]{} & g \circ b \circ (k \circ k') \end{array} \tag{E.5.5.10}$$

is commutative. This is in turn checked using Axiom (2) from Definition E.5.2.1 on associativity of horizontal compositions, to deduce commutativity of the rightmost square, alongside the identity $\mathbf{1}_{k \circ k'} = \mathbf{1}_k \star \mathbf{1}_{k'}$, and,

upon calling $\alpha'' := (\alpha' \star \mathbf{1}_k) \odot \alpha$ and $\beta'' := (\beta' \star \mathbf{1}_k) \odot \beta$, the identity:

$$(\mathbf{1}_f \star \alpha \star \mathbf{1}_{k'}) \circ (\mathbf{1}_f \star \alpha') = (\mathbf{1}_f \circ \mathbf{1}_f) \star ((\alpha' \star \mathbf{1}_{k'}) \odot \alpha) = \mathbf{1}_f \star \alpha''$$

and:

$$(\mathbf{1}_g \star \beta \star \mathbf{1}_{k'}) \odot (\mathbf{1}_g \star \beta') = (\mathbf{1}_g \circ \mathbf{1}_g) \star ((\beta' \star \mathbf{1}_{k'}) \odot \beta) = \mathbf{1}_g \star \beta''.$$

Both of the above identities follow from Axioms (1) and (2) of Definition E.5.2.1. We deduce that the definition of $c(\bullet, \bullet)$ is well-posed. That it satisfies the axioms from Definition E.5.2.1 is a consequence of $\mathcal{C}$ being a 2-category. □

E.5.6. 2-final objects in Diagrams(f, g). We will now define the concept of 2-fibre products, and we shall then discuss a series of examples of 2-categories of interest that contain 2-fibre products. First, we introduce the concept of *final objects* in a 2-category. One way to assign conditions on objects of a 2-category $\mathcal{C}$ is to assign such conditions on the ordinary category $\mathrm{Cat}(\mathcal{C})$ associated to $\mathcal{C}$ (see Construction E.5.2.6). We recall that an object x in a locally small category $\mathcal{D}$ is *weakly terminal* if and only if for every object $y \in \mathrm{Ob}(\mathcal{D})$, there exists a morphism $y \to x$ in $\mathcal{D}$ (see [**Ka**, §2], especially the discussion after [**Ka**, Thm. 2.1]).

Definition E.5.6.1. Let $\mathcal{C}$ be a 2-category. An object A of $\mathcal{C}$ is *final in* $\mathcal{C}$ if and only if:

(1) As an object of $\mathrm{Cat}(\mathcal{C})$, A is weakly terminal.
(2) For any object B of $\mathcal{C}$ and any two 1-morphisms $f, g : B \rightrightarrows A$, there exists a unique 2-isomorphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$.

Several ambiguities arise around the concept of final objects in a 2-category, and one has to start distinguishing among varying degrees of strength of the notions of final objects. We refer the reader to [**GR**, Ch. 2, §2.5] for a discussion of 2-limits and 2-colimits and related (non-)uniqueness aspects, and especially the discussion between [**GR**, Ch. 2, §2, Def. 2.5.14] and [**GR**, Ch. 2, §2, Prop. 2.5.15], for a useful special situation when final objects can be defined.

The definition of final objects provided in Definition E.5.6.1 is sufficient for all our purposes. In particular, by introducing a suitable diagram 2-category **Diagrams**(f, g) as provided by Lemma E.5.5.4, we can define the notion of 2-fibre products that will be used to assign conditions on 1-morphisms of stacks in groupoids. Such notion can be introduced in wider generality, and we refer the interested reader to [**GR**, Ch. 2, §2, Rem. 2.5.2,(v)].

We can now assign the following:

Definition E.5.6.2. Let $\mathcal{C}$ be a 2-category, $A, B, C \in \mathrm{Ob}(\mathcal{C})$, $f : B \to A$ and $g : C \to A$ two 1-morphisms. We define a 2-*fibre product* for (f, g) in $\mathcal{C}$ to be any final object of the 2-category **Diagrams**(f, g).

The above Definition E.5.6.2 in particular means that a 2-fibre product of f, g is the datum of an object P of $\mathcal{C}$, equipped with 1-morphisms $a : P \to B$, $b : P \to C$ and a 2-isomorphism $\phi : f \circ a \Rightarrow g \circ b$ making the following diagram 2-commute:

$$\begin{array}{ccc} P & \xrightarrow{b} & C \\ {\scriptstyle a}\downarrow & \swarrow\!\!\!\swarrow \phi & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \end{array}$$

such that every other object of **Diagrams**(f, g) has a 1-morphism to (P, a, b, ϕ), and for any two 1-morphisms $p = (k, \alpha, \beta) : (D, a', b', \phi') \to (P, a, b, \phi)$ and $q = (k', \alpha', \beta') : (D'', a'', b'', \phi'') \to (P, a, b, \phi)$, there exists a unique 2-isomorphism $\gamma : p \Rightarrow q$ in **Diagrams**(f, g), i.e. a unique 2-isomorphism:

$$\gamma : k \Rightarrow k'$$

in $\mathcal{C}$ such that:

$$(\mathbf{1}_a \star \gamma) \odot \alpha = \alpha', \quad (\mathbf{1}_b \star \gamma) \odot \beta = \beta'.$$

NOTATION E.5.6.1. We denote by $(B \times^2_{f,A,g} C, p_1, p_2, \phi)$ any 2-fibre product (P, a, b, ϕ) in $\mathbf{Diagrams}(f, g)$, if it exists.

E.5.7. 2-fibre products and 1-fibre products. We have collected basic examples of 2-categories and of pseudo-functors. We will now briefly focus on those 2-categories all of whose 2-morphisms are 2-isomorphisms. These will be the only 2-categories of interest throughout the development of the sought-for theory of stacks in groupoids on a suitable site of topological spaces.

In this section, we shall recall results concerning the existence of 2-fibre products in 2-categories of interest. Our main source within the literature will be [SP, Tags 003O, 02XG, 02XJ, 003S], to which we shall refer the reader for the relevant arguments, with occasional exceptions. This will conclude the 2-categorical preparations needed later in this appendix.

We begin with a definition:

DEFINITION E.5.7.1. Let $\mathcal{C}$ be a 2-category. We say that $\mathcal{C}$ is a *strict* $(2,1)$*-category* if and only if every 2-morphism in $\mathcal{C}$ is a 2-isomorphism.

The notion of strict $(2,1)$-category is also known, in the literature, under the terminology of *groupoid-enriched category* (see [FM, §1]). For every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$ of a strict $(2,1)$-category $\mathcal{C}$, the category $\mathbf{Hom}_{\mathcal{C}}(A, B)$ is a groupoid (see [SP, Tag 003I]) whose set of objects is the set of 1-morphisms $f : A \to B$, and whose set of morphisms is the set of 2-morphisms $f \Rightarrow f'$, all isomorphisms.

E.5.7.1. *Strict* $(2,1)$*-categories of interest.* We will now discuss four examples of strict $(2,1)$-categories, ultimately focusing on the example of stacks in groupoids over a site, for which we will discuss existence of 2-fibre products.

EXAMPLE E.5.7.2. We recall that upon fixing a small category $\mathcal{D}$, we may consider the following datum:

(1) the class of all pairs $(\mathcal{S}, p)$ where $\mathcal{S}$ is a small category and $p : \mathcal{S} \to \mathcal{D}$ is a functor.
(2) For any two pairs $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$, the category $\mathbf{Hom}((\mathcal{S}, p), (\mathcal{S}', p'))$ whose:
   (a) objects are functors $F : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ F = p$.
   (b) Morphisms $t : F \Rightarrow F'$ for $F, F' : (\mathcal{S}, p) \to (\mathcal{S}', p')$ are natural transformations of functors such that $p'(t_A) = \mathbf{1}_{p(A)}$ for every object $A \in \mathrm{Ob}(\mathcal{S})$.

We note that for any two pairs $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$, there is a faithful forgetful functor:

$$\mathbf{Hom}((\mathcal{S}, p), (\mathcal{S}', p')) \to \mathbf{Hom}_{\mathbf{Cat}}(\mathcal{S}, \mathcal{S}')$$

given by sending a functor $F : \mathcal{S} \to \mathcal{S}'$ to itself forgetting its compatibility with p and p', and sending a natural transformation $t : F \Rightarrow F'$ to itself. The bifunctor $(\circ, \star)$ constructed in §E.5.1 (see Definition E.5.1.11) restricts to a bifunctor:

$$c_{(\mathcal{S},p),(\mathcal{S}',p'),(\mathcal{S}'',p'')} : \mathbf{Hom}((\mathcal{S}, p), (\mathcal{S}', p')) \times \mathbf{Hom}((\mathcal{S}', p'), (\mathcal{S}'', p'')) \to \mathbf{Hom}((\mathcal{S}, p), (\mathcal{S}'', p''))$$

which satisfies the Axioms from Definition E.5.2.1 by Lemma E.5.1.14. We denote the resulting 2-category by $\mathbf{Cat}/\mathcal{D}$. We refer to it as the 2-*category of categories over* $\mathcal{D}$. The compositions within $\mathbf{Cat}/\mathcal{D}$ are inherited from those on $\mathbf{Cat}$, and hence the worked example of $\mathbf{Cat}$ (see Definition E.5.1.20 and Lemma E.5.1.14) can be used to compute compositions within $\mathbf{Cat}/\mathcal{D}$.

We note that the 2-category described in Example E.5.7.2 is not a strict $(2,1)$-category. However, we can associate to any 2-category a strict $(2,1)$-category by re-defining the sets of 2-morphisms to be comprised only of those 2-morphisms that are isomorphisms (see [SP, Tag 02X8], and specifically [SP, Tag 003K]).

Within the strict $(2,1)$-category $(\mathbf{Cat}/\mathcal{D})^{\mathrm{strict}}$ associated to $\mathbf{Cat}/\mathcal{D}$ from Example E.5.7.2, we make the following Construction.

CONSTRUCTION E.5.7.3. We let $F : (\mathcal{X}, p_{\mathcal{X}}) \to (\mathcal{S}, p_{\mathcal{S}})$ and $G : (\mathcal{Y}, p_{\mathcal{Y}}) \to (\mathcal{S}, p_{\mathcal{S}})$ be morphisms of categories over $\mathcal{D}$. We define $\mathcal{X} \times_{F,\mathcal{S},G} \mathcal{Y}$ to be the small category whose:

(1) objects are 4-tuples (A, x, y, ϕ) with $A \in \mathrm{Ob}(\mathcal{D})$, $x \in \mathrm{Ob}(\mathcal{X}_A)$, $y \in \mathrm{Ob}(\mathcal{Y}_A)$, $\phi : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}$.
(2) Morphisms $(A, x, y, \phi) \to (A', x', y', \phi')$ are given by pairs (a, b) with $a : x \to x'$ a morphism in $\mathcal{X}$, $b : y \to y'$ a morphism in $\mathcal{Y}$, satisfying the following properties:
   (a) $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(b)$ as morphisms of $\mathcal{D}$.
   (b) As morphisms of $\mathcal{S}$, we have the identity:

$$\phi' \circ F(a) = G(b) \circ \phi.$$

We further define the functors $p_1 : \mathcal{X} \times^2_{F,\mathcal{S},G} \mathcal{Y} \to \mathcal{X}$ and $p_2 : \mathcal{X} \times^2_{F,\mathcal{S},G} \mathcal{Y} \to \mathcal{Y}$ to be the forgetful functors. We note that $\mathcal{X} \times^2_{F,\mathcal{S},G} \mathcal{Y}$ can be made into an object of $(\mathbf{Cat}/\mathcal{D})^{\mathrm{strict}}$ by equipping it with the projection to $\mathcal{D}$ given by the composition $\pi := p_{\mathcal{X}} \circ p_1 = p_{\mathcal{Y}} \circ p_2$. We call $\psi : F \circ p_{\mathcal{X}} \to G \circ p_{\mathcal{Y}}$ the natural transformation of functors assigned on the arbitrary object $\xi := (A, x, y, \phi)$ by:

$$\psi_\xi := \phi : F(p_{\mathcal{X}}(\xi)) = F(x) \to G(y) = G(p_{\mathcal{Y}}(\xi)).$$

We denote by:

$$\mathcal{X} \times^2_{F,\mathcal{S},G} \mathcal{Y} \tag{E.5.7.1}$$

the datum $(\mathcal{X} \times^2_{F,\mathcal{S},G} \mathcal{Y}, p_1, p_2, \psi)$.

For the strict $(2,1)$-category associated to $\mathbf{Cat}/\mathcal{D}$, we have the following:

LEMMA E.5.7.4. *Let $\mathcal{D}$ be a small category. Let $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of categories over $\mathcal{D}$. A 2-fibre product in the strict $(2,1)$-category $(\mathbf{Cat}/\mathcal{D})^{\mathrm{strict}}$ of all small categories over $\mathcal{D}$ for (F, G) exists, and it is equivalent to $\mathcal{X} \times^2_{\mathcal{S}} \mathcal{Y}$ from Construction E.5.7.3.*

PROOF. This is [SP, Tag 0040]. We further refer to the reader to [Ac, Ch. I, Lemma 1.2.1]. □

We recall here for the reader's convenience the following definition (see [SP, Tag 02XK]):

DEFINITION E.5.7.5. Let $\mathcal{D}$ be a small category, $p : \mathcal{S} \to \mathcal{D}$ a small category over $\mathcal{D}$. For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{S})$, a *strongly $\mathcal{D}$-cartesian morphism* $f : A \to B$ is a morphism in $\mathcal{S}$ such that for every object $C \in \mathrm{Ob}(\mathcal{C})$ the map:

$$\mathrm{Hom}_{\mathcal{S}}(C, A) \to \mathrm{Hom}_{\mathcal{S}}(C, B) \times_{\mathrm{Hom}_{\mathcal{D}}(p(C), p(B))} \mathrm{Hom}_{\mathcal{D}}(p(C), p(A))$$

assigned by sending $g : C \to B$ to $(f \circ g, p(g))$, is a bijection.

We observe that for any object A of $\mathcal{S}$ with $p(A) = D$ in $\mathcal{D}$, and any morphism $\phi : D' \to D$ in $\mathcal{D}$, if a strongly cartesian morphism $f : A' \to A$ exists such that $p(f) = \phi$, then the pair (A', f) is unique up to unique isomorphism in $\mathcal{S}$ by Yoneda's Lemma. We may therefore denote refer to the pair (A', f) via the notation $\phi^* A \to A$. Strongly cartesian morphisms satisfy the following basic permanence properties:

LEMMA E.5.7.6. *Let $\mathcal{D}$ be a small category, $p : \mathcal{S} \to \mathcal{D}$ a small category over $\mathcal{D}$.*

(1) Strongly cartesian morphisms in $\mathcal{S}$ are stable under finite compositions.
(2) For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{S})$, if $f : A \to B$ is an isomorphism, then f is strongly cartesian.
(3) Let $f : A \to B$ be a strongly cartesian morphism between objects of $\mathcal{S}$. If $p(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism.

PROOF. We refer the reader to [SP, Tag 02XL] and [Ac, Ch. I, Lemma 1.3.1]. □

Following [SP, Tag 02XM], we recall the following:

DEFINITION E.5.7.7. Let $\mathcal{D}$ be a small category and $p : \mathcal{S} \to \mathcal{D}$ a small category over $\mathcal{D}$. We say $p : \mathcal{S} \to \mathcal{D}$ is a *fibered category* if and only if for every object $A \in \mathrm{Ob}(\mathcal{S})$, and any morphism $\phi : D \to p(A)$ in $\mathcal{D}$, there exists a strongly cartesian morphism $f : \phi^* A \to A$ with $p(f) = \phi$.

EXAMPLE E.5.7.8. (The 2-category of fibered categories over a fixed category) We let $\mathcal{D}$ be a small category. We define $\mathbf{Fib}/\mathcal{D}$ to be the sub 2-category of the 2-category $\mathbf{Cat}/\mathcal{D}$ assigned as follows:

(1) The objects of $\mathbf{Fib}/\mathcal{D}$ are small fibered categories $p : \mathcal{S} \to \mathcal{D}$.
(2) The 1-morphisms $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ are 1-morphisms in $\mathbf{Cat}/\mathcal{D}$ that, in addition, satisfy the property of preserving strongly cartesian morphisms.
(3) The 2-morphisms of $\mathbf{Fib}/\mathcal{D}$ are those natural transformations of functors $t : F \Rightarrow F'$, for 1-morphisms $F, F' : (\mathcal{S}, p) \to (\mathcal{S}', p')$, such that t is a 2-morphism in $\mathbf{Cat}/\mathcal{D}$.

We define the composition bifunctor:

$$c(\bullet, \bullet) : \mathbf{Hom}_{\mathbf{Fib}/\mathcal{D}}((\mathcal{S}, p), (\mathcal{S}', p')) \times \mathbf{Hom}_{\mathbf{Fib}/\mathcal{D}}((\mathcal{S}', p'), (\mathcal{S}'', p'')) \to \mathbf{Hom}_{\mathbf{Fib}/\mathcal{D}}((\mathcal{S}, p), (\mathcal{S}'', p''))$$

to be the restriction of the composition bifunctor on $\mathbf{Cat}/\mathcal{D}$, which is in turn the restriction of the composition bifunctor $(\circ, \star)$ from §E.5.1. The above datum is a 2-category, denoted $\mathbf{Fib}/\mathcal{D}$. We shall refer to it as the 2-*category of categories fibered over* $\mathcal{D}$.

The above example is [**SP**, Tag 02XP], and as noted in loc.cit., $\mathbf{Fib}/\mathcal{D}$ is a genuine 2-category. In order to state the next Lemma, we therefore pass to its associated strict $(2,1)$-category, $(\mathbf{Fib}/\mathcal{D})^{\mathrm{strict}}$. The following Lemma is [**SP**, Tag 02XQ]:

LEMMA E.5.7.9. *Let $\mathcal{D}$ be a small category. Then the strict $(2,1)$-category associated to $\mathbf{Fib}/\mathcal{D}$ contains 2-fibre products, and they are computed by the 2-fibre products in $(\mathbf{Cat}/\mathcal{D})^{\mathrm{strict}}$ from Lemma E.5.7.4. Moreover, for any object T of $\mathcal{D}$, and any 1-morphisms $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ of categories fibered over $\mathcal{D}$, we have a canonical isomorphism of fibre categories:*

$$(\mathcal{X} \times^2_{f,\mathcal{S},g} \mathcal{Y})_T = \mathcal{X}_T \times^2_{f_T, \mathcal{S}_U, g_T} \mathcal{Y}_T$$

where the 2-fibre product $\mathcal{X}_U \times^2_{f_U, \mathcal{S}_U, g_U} \mathcal{Y}_U$ is formed in $\mathbf{Cat}$.

PROOF. This is [**SP**, Tag 02XQ] and [**SP**, Tag 02XI]. We further refer the reader to [**Ac**, Ch.I,Lemma 1.3.6] and to [**Ac**, Ch.I,Lemma 1.2.2]. □

EXAMPLE E.5.7.10. We may now consider the full sub 2-category $\mathbf{Groupoids}/\mathcal{D}$ of $\mathbf{Fib}/\mathcal{D}$ of those fibered categories $p : \mathcal{S} \to \mathcal{D}$ whose fibre categories over $\mathcal{D}$ are all groupoids. Then $\mathbf{Groupoids}/\mathcal{D}$ is a strict $(2,1)$-category with 2-fibre products.

LEMMA E.5.7.11. *Let $\mathcal{D}$ be a small category. Then the strict $(2,1)$-category $\mathbf{Groupoids}/\mathcal{D}$ of small categories fibered in groupoids over $\mathcal{D}$ has 2-fibre products, and they are computed by the 2-fibre products in $(\mathbf{Cat}/\mathcal{D})^{\mathrm{strict}}$.*

PROOF. This is [**SP**, Tag 0041]. We further refer the reader to [**Ac**, Ch.I,Lemma 1.5.3]. □

REMARK E.5.7.12. In Lemma E.5.7.11, we may choose $\mathcal{D} = \mathbf{1}$, the final object in $\mathbf{Cat}$ (see part (1) of Remark E.5.1.20). In this case, $\mathbf{Groupoids}/\mathcal{D} = \mathbf{Groupoids}$ as strict $(2,1)$-categories, and the Lemma implies the existence of 2-fibre products of groupoids in sets, described by Construction E.5.7.3.

Finally, the last example is the one of our interest.

LEMMA E.5.7.13. *Let $\mathcal{D}$ be a small category equipped with the structure of a site (in the sense of* [**SP**, Tag 00VH]*). We define $\mathbf{Stacks}/\mathcal{D}$ to be the full sub 2-category of stacks in groupoids over $\mathcal{D}$. This is a strict $(2,1)$-category and has 2-fibre products, and they are computed as the stackification of the 2-fibre products in* $\mathbf{Groupoids}/\mathcal{D}$.

PROOF. This is [SP, Tag 02ZL]. □

E.5.7.2. *1-fibre products.* We want a notion of fibre products for stacks in groupoids on a site that is rarely used within the literature. It will enable us to compute the stack-theoretic intersection of certain "graph" and "diagonal" sub-stacks of a topological stack of key interest to our work. We review the construction here concerning such an alternate version of fibre products for categories over a fixed small category $\mathcal{D}$, and we will recall criteria for its existence which, contrary to 2-fibre products and as illustrated in Lemma E.5.7.4 , does not always hold within the strict $(2,1)$-category of stacks in groupoids on a site. Our main source within the literature for the material within this subsection is [Ac, Ch.I-II], with particular emphasis on [Ac, Ch. I, Lemma 1.2.3, Prop. 1.3.1,Thm. 1,Thm. 2] and [Ac, Ch. II, Thm. 4].

DEFINITION E.5.7.14. Let $\mathcal{C}$ be a 2-category, $A, B, C \in \mathrm{Ob}(\mathcal{C})$, $f : B \to A$, $g : C \to A$ two 1-morphisms in $\mathcal{C}$. We define the 1-*fibre product* for (f, g) in $\mathcal{C}$, denoted $B \times^1_{f,A,g} C$, to be the fibre product of (f, g) in the ordinary locally small category $\mathrm{Cat}(\mathcal{C})$ of Construction E.5.2.6.

LEMMA E.5.7.15. *The* 2*-category* **Cat** *contains* 1*-fibre products.*

PROOF. Let $A, B, C \in \mathrm{Ob}(\mathcal{C})$, $f : B \to A$, $g : C \to A$ two 1-morphisms in **Cat**. We define $B \times^1_{f,A,g} C$ to be the category whose class of objects is:

$$\mathrm{Ob}(B \times^1_{f,A,g} C) := \mathrm{Ob}(B) \times_{f,\mathrm{Ob}(A),g} \mathrm{Ob}(C)$$

and for any two objects $(x, y), (x', y')$, we define the set of morphisms to be:

$$\mathrm{Hom}((x, y), (x', y')) := \mathrm{Hom}_B(x, x') \times_{f,\mathrm{Hom}_A(z,z'),g} \mathrm{Hom}_C(y, y')$$

where $z := f(x) = g(y)$ and $z' := f(x') = g(y')$. Composition of morphisms in $B \times^1_{f,A,g} C$ is defined component-wise. The forgetful functors $p_1 : B \times^1_{f,A,g} C \to B$ and $p_2 : B \times^1_{f,A,g} C \to C$ satisfy $f \circ p_1 = g \circ p_2$ as 1-morphisms in **Cat**. For the proof that $(B \times^1_{f,A,g} C, p_1, p_2)$ verifies the universal property of fibre products in $\mathrm{Cat}(\mathbf{Cat})$, we refer the reader to [Ac, Ch.I,Lemma 1.2.3], applied to $\mathcal{C} = \mathbf{1}$, the final object in **Cat** (see part (1) of Remark E.5.1.20). □

In the 2-category of all small categories over a fixed small category $\mathcal{D}$, 1-fibre products exist. This is the content of the following:

LEMMA E.5.7.16. *Let* $\mathcal{D}$ *be a small category. The* 2*-category* $\mathbf{Cat}/\mathcal{D}$ *contains* 1*-fibre products.*

PROOF. To see this, it is enough to check that 1-fibre products in **Cat** are contained in the sub 2-category $\mathbf{Cat}/\mathcal{D}$. Equivalently, that given objects $(A, p_A : A \to \mathcal{D})$,$(B, p_B : B \to \mathcal{D})$, $(C, p_C : C \to \mathcal{D})$ of $\mathbf{Cat}/\mathcal{D}$, and 1-morphisms $f : B \to A$ and $g : C \to A$ in $\mathbf{Cat}/\mathcal{D}$, the 1-fibre product $B \times^1_{f,A,g} C$ formed in **Cat** is itself a category over $\mathcal{D}$. This is seen by defining $p : B \times^1_{f,A,g} C \to \mathcal{D}$ as the composition $p_A \circ f \circ p_1 = p_A \circ g \circ p_2$, and then checking that the resulting object of $\mathbf{Cat}/\mathcal{D}$ indeed satisfies the universal property of fibre products in $\mathrm{Cat}(\mathbf{Cat}/\mathcal{D})$. We refer the reader to the proof of [Ac, Ch.I,Lemma 1.2.3] for this verification. □

It is not always the case that 1-fibre products exist in $\mathbf{Fib}/\mathcal{D}$, the 2-category of all small categories fibered over the fixed small category $\mathcal{D}$. For a discussion on 1-fibre products in fibered categories we refer the reader to [Ac, Ch.I,p.31-33], with emphasis on [Ac, Ch. I, Examples 4, 5]. The obstruction to the existence of 1-fibre products of fibered categories is the fact that it is not generally true that, given 1-morphisms $f : B \to A$, $g : C \to A$, for all objects U of $\mathcal{D}$ we have $(d^*x, d^*y) \in \mathrm{Ob}(B \times^1_{f,A,g} C)$ for all $(x, y) \in \mathrm{Ob}(B \times^1_{f,A,g} C)$ and $d : U' \to U$ a morphism in $\mathcal{D}$. This is also the only obstruction. We have the following definition, which is [Ac, Ch. I, Def. 1.3.5]:

DEFINITION E.5.7.17. Let $(A, p_A), (B, p_B), (C, p_C)$ be three small categories fibered over $\mathcal{D}$, and $f : B \to A, g : C \to A$ be two 1-morphisms of small categories fibered over $\mathcal{D}$. The 1-fibre product in $\mathbf{Cat}/\mathcal{D}$,

$B \times^1_{f,A,g} C$, has *enough component-wise pullbacks* if and only if for all $(x, y) \in \mathrm{Ob}(B \times^1_{f,A,g} C)$ and $d : U' \to U$ any morphism in $\mathcal{D}$, we have:

(1) (d^*x, d^*y) is an object of $B \times^1_{f,A,g} C$.
(2) $(d^*x \to x, d^*y \to y)$ is a morphism of $B \times^1_{f,A,g} C$.

TERMINOLOGY E.5.7.1. We note that the terminology being adopted here of "1-fibre products" corresponds to the terminology "fibre products" of [**Ac**, Ch.I], and that the condition of having "enough component-wise pullbacks" corresponds to having "componentwise pullbacks", as defined in [**Ac**, Ch. I, Def. 1.3.5].

LEMMA E.5.7.18. *Let $\mathcal{D}$ be a small category, and $(A, p_A), (B, p_B), (C, p_C)$ be three small categories fibered over $\mathcal{D}$. Let $f : B \to A, g : C \to A$ be two 1-morphisms of small categories fibered over $\mathcal{D}$, such that the 1-fibre product in $\mathbf{Cat}/\mathcal{D}$, $B \times^1_{f,A,g} C$, has* enough component-wise pullbacks. *Then a 1-fibre product for (f, g) exists in the 2-category $\mathbf{Fib}/\mathcal{D}$, and it is computed by the 1-fibre product $B \times^1_{f,A,g} C$ for (f, g) in $\mathbf{Cat}/\mathcal{D}$.*

PROOF. We refer the reader to [**Ac**, Ch.I,Thm.1] for the proof of the first assertion in the Lemma. To verify that $B \times^1_{f,A,g} C$ satisfies the universal property, it is enough to observe: as soon as $B \times^1_{f,A,g} C$ is an object of $\mathbf{Fib}/\mathcal{D}$ it induces an object in $\mathrm{Cat}(\mathbf{Fib}/\mathcal{D})$, the forgetful functor $\mathrm{Cat}(\mathbf{Fib}/\mathcal{D}) \to \mathrm{Cat}(\mathbf{Cat}/\mathcal{D})$ induced by the forgetful pseudo-functor $\mathbf{Fib}/\mathcal{D} \to \mathbf{Cat}/\mathcal{D}$ is conservative, and conservative functors reflect representable limits. □

REMARK E.5.7.19. In Lemma E.5.7.18, we may choose $\mathcal{D} = \mathbf{1}$, the final object in **Cat** (see part (1) of Remark E.5.1.20). In this case, $\mathbf{Fib}/\mathcal{D} = \mathbf{Cat}$ as 2-categories. The Lemma implies the existence of 1-fibre products of small categories, defined in Definition E.5.7.14.

Within $\mathbf{Cat}/\mathcal{D}$, 1-fibre products and 2-fibre products are related according to the following:

LEMMA E.5.7.20. *Let $(A, p_A), (B, p_B), (C, p_C)$ be three categories over $\mathcal{D}$, and $f : B \to A, g : C \to A$ two 1-morphisms of categories over $\mathcal{D}$. There exists a canonical 1-morphism of categories over $\mathcal{D}$:*

$$p : B \times^1_{f,A,g} C \to B \times^2_{f,A,g} C \tag{E.5.7.2}$$

such that:

(1) p is a fully faithful functor of small categories.
(2) If $(A, p_A), (B, p_B), (C, p_C)$ are fibered categories over $\mathcal{D}$, and $B \times^1_{f,A,g} C$ in $\mathbf{Cat}/\mathcal{D}$ has enough component-wise pullbacks, then p is a 1-morphism of fibered categories over $\mathcal{D}$.

PROOF. We refer the reader to [**Ac**, Ch. I, Prop. 1.3.1] for the proof of the Lemma. □

We note that the functor p from (E.5.7.2) in Lemma E.5.7.20 above may not be essentially surjective (see [**Ac**, Ch. I, Example 7] for an explicit example).

LEMMA E.5.7.21. *Let $\mathcal{D}$ be a small category. Let $(A, p_A), (B, p_B), (C, p_C)$ be three small categories fibered in groupoids over $\mathcal{D}$, and $f : B \to A, g : C \to A$ two 1-morphisms of categories fibered in groupoids over $\mathcal{D}$. Then a 1-fibre product for (f, g) exists in the 2-category $\mathbf{Groupoids}/\mathcal{D}$ of all small categories fibered in groupoids over $\mathcal{D}$ if and only if a 1-fibre product for (f, g) exists in the 2-category $\mathbf{Fib}/\mathcal{D}$ of all small categories fibered over $\mathcal{D}$.*

PROOF. The inclusion functor $\mathbf{Groupoids}/\mathcal{D} \to \mathbf{Fib}/\mathcal{D}$ is strict and strongly fully faithful, and hence the induced ordinary functor $\mathrm{Cat}(\mathbf{Groupoids}/\mathcal{D}) \to \mathrm{Cat}(\mathbf{Fib}/\mathcal{D})$ is fully faithful. As fully faithful functors reflect representable limits, a 1-fibre product for (f, g) exists in $\mathbf{Groupoids}/\mathcal{D}$ if and only if $B \times^1_{f,A,g} C$, formed in $\mathbf{Cat}/\mathcal{D}$, has enough component-wise pullbacks. Indeed, if so then $B \times^1_{f,A,g} C$ is fibered over $\mathcal{D}$ by Lemma E.5.7.18, and fibered in groupoids, hence an object of $\mathbf{Groupoids}/\mathcal{D}$. As the forgetful inclusion functor $\mathrm{Cat}(\mathbf{Groupoids}/\mathcal{D}) \to \mathrm{Cat}(\mathbf{Fib}/\mathcal{D})$ is fully faithful and hence reflects representable limits, $B \times^1_{f,A,g} C$

is a fibre product in $\mathrm{Cat}(\mathbf{Groupoids}/\mathcal{D})$ if and only if it is a fibre product in $\mathrm{Cat}(\mathbf{Fib}/\mathcal{D})$. Conversely, if a 1-fibre product for (f,g) exists in $\mathbf{Groupoids}/\mathcal{D}$, then its image under the forgetful functor to $\mathbf{Fib}/\mathcal{D}$ is also a 1-fibre product for (f,g), in the 2-category $\mathbf{Fib}/\mathcal{D}$. This is so because $\mathrm{Cat}(\mathbf{Groupoids}/\mathcal{D}) \to \mathrm{Cat}(\mathbf{Fib}/\mathcal{D})$ preserves representable fibre products. It follows $B \times^1_{f,A,g} C$ has enough component-wise pullbacks. □

Remark E.5.7.22. In Lemma E.5.7.21, we may choose $\mathcal{D} = \mathbf{1}$, the final object in **Cat** (see part (1) of Remark E.5.1.20). In this case, $\mathbf{Groupoids}/\mathcal{D} = \mathbf{Groupoids}$ as strict $(2,1)$-categories. The Lemma implies the existence of 1-fibre products of groupoids in sets, since they exist in $\mathbf{Fib}/\mathcal{D} = \mathbf{Cat}$ by Remark E.5.7.19.

We will need a criterion for the existence of 1-fibre products in $\mathbf{Fib}/\mathcal{D}$. First, we recall a key construction:

Construction E.5.7.23. We let $\mathcal{D}$ be a small category, and $F : \mathcal{D}^{\mathrm{opp}} \to \mathbf{Groupoids}$ a functor. We define the category $\mathcal{S}_F$ whose objects are defined to be the set:

$$\mathrm{Ob}(\mathcal{S}_F) := \{(U,x) \mid U \in \mathrm{Ob}(\mathcal{D}), x \in \mathrm{Ob}(F(U))\} \tag{E.5.7.3}$$

and, for any pair of objects (U',x') and (U,x) of $\mathcal{S}_F$, we define the set of morphisms to be:

$$\mathrm{Hom}_{\mathcal{S}_F}((U',x'),(U,x)) := \coprod_{f \in \mathrm{Hom}_{\mathcal{D}}(U',U)} \mathrm{Hom}_{F(U')}(x', F(f)(x)). \tag{E.5.7.4}$$

Composition of morphisms is defined as follows. For $(f,\phi) \in \mathrm{Hom}_{\mathcal{S}_F}((U',x'),(U,x))$, with $f \in \mathrm{Hom}_{\mathcal{D}}(U',U)$ and $\phi \in \mathrm{Hom}_{F(U')}(x', F(f)(x))$, and for $(U'',x'') \in \mathrm{Ob}(\mathcal{S}_F)$ and $(g,\psi) \in \mathrm{Hom}_{\mathcal{S}_F}((U'',x''),(U',x'))$, we define:

$$(g,\psi) \circ (f,\phi) := (g \circ f, F(g)(\phi) \circ \psi). \tag{E.5.7.5}$$

We further define the forgetful functor

$$p_F : \mathcal{S}_F \to \mathcal{D} \tag{E.5.7.6}$$

by $p_F(U,x) := U$, and $p_F(f,\phi) := f$.

Such fibered categories (E.5.7.6) are called *split*. We'll be interested in the case $\mathcal{D} = \mathbf{CGWH}_\alpha$ and F the "functor of points" oof a fixed groupoid in $\mathbf{CGWH}_\alpha$. The category over $\mathcal{D}$ assigned by $(\mathcal{S}_F, p_F)$ is a fibered category by [**SP**, Tag 02XV]. Since $F(U)$ is a groupoid for every object U in $\mathcal{D}$, $(\mathcal{S}_F, p_F)$ is fibered in groupoids by [**SP**, Tag 003V]. We further note that formation of $\mathcal{S}_F$ is natural in F, in the sense that the assignment $F \rightsquigarrow \mathcal{S}_F$ is a strict pseudo-functor $\mathbf{Functors}(\mathcal{D}^{\mathrm{opp}}, \mathbf{Groupoids}) \to \mathbf{Fib}/\mathcal{D}$, where the source is an ordinary category regarded as a 2-category with all its 2-morphisms being the identities.

Lemma E.5.7.24. *Let $\mathcal{D}$ be a small category, and F, G, H three functors $\mathcal{D}^{\mathrm{opp}} \to \mathbf{Groupoids}$. Let $t : F \Rightarrow H, s : G \Rightarrow H$ two natural transformations. Then there exists a canonical isomorphism of categories over $\mathcal{D}$:*

$$\mathcal{S}_F \times^1_{t,\mathcal{S}_H,s} \mathcal{S}_G \to \mathcal{S}_{F \times_{t,H,s} G} \tag{E.5.7.7}$$

In particular, $\mathcal{S}_F \times^1_{t,\mathcal{S}_H,s} \mathcal{S}_G$ has enough component-wise pullbacks, and is the 1-fibre product of t, s in $\mathbf{Groupoids}/\mathcal{D}$.

Proof. By [**Ac**, Ch. I, Thm. 2], if F, G, H are valued in **Cat** then $\mathcal{S}_F \times^1_{t,\mathcal{S}_H,s} \mathcal{S}_G$ as a small category over $\mathcal{D}$ contains enough component-wise pull-backs, and it is a 1-fibre product in $\mathbf{Fib}/\mathcal{D}$, which implies by Lemma E.5.7.21 that it is a 1-fibre product in $\mathbf{Groupoids}/\mathcal{D}$. □

We now assume the fixed small category $\mathcal{D}$ is endowed with the structure of a site, so stackification of categories fibered in groupoids over $\mathcal{D}$ (denoted $(\cdot)^{\mathrm{a}}$) makes sense. We refer the reader to [**SP**, Tag 02ZM] for background on stacks associated to categories fibered over a site. The following Lemma asserts that if 1-fibre products exist in $\mathbf{Groupoids}/\mathcal{D}$ then they do in the strict $(2,1)$-category of stacks in groupoids over $\mathcal{D}$ and are computed by the stack associated to the 1-fibre product of categories fibered in groupoids over $\mathcal{D}$.

LEMMA E.5.7.25. *Let $\mathcal{D}$ be a small category endowed with the structure of a site. Let $(A, p_A), (B, p_B), (C, p_C)$ be three small categories fibered in groupoids over $\mathcal{D}$, and $f : B \to A, g : C \to A$ two 1-morphisms of categories fibered in groupoids over $\mathcal{D}$. If a 1-fibre product exists for (f, g) in* **Groupoids**$/\mathcal{D}$ *then a 1-fibre product of the 1-morphisms $(f^{\mathrm{a}}, g^{\mathrm{a}})$ of stacks in groupoids over $\mathcal{D}$ exists, and it is computed by the stack associated to the category $B \times^1_{f,A,g} C$ fibered in groupoids over $\mathcal{D}$, i.e., upon calling $\mathcal{X} := B^{\mathrm{a}}$, $\mathcal{Y} := C^{\mathrm{a}}$, $\mathcal{S} := A^{\mathrm{a}}$, then:*

$$\mathcal{X} \times^1_{f^{\mathrm{a}},\mathcal{S},g^{\mathrm{a}}} \mathcal{Y} = (B \times^1_{f,A,g} C)^{\mathrm{a}}. \tag{E.5.7.8}$$

PROOF. We refer the reader to [**Ac**, Ch.II,Thm.4] for the proof that the stackification pseudo-functor $(\cdot)^{\mathrm{a}} : \mathbf{Groupoids}/\mathcal{D} \to \mathbf{Stacks}/\mathcal{D}$ preserves 1-fibre products, if they exist. □

We will be applying Lemma E.5.7.24 and Lemma E.5.7.25 to show[21] that the stackification of the split fibered category $\mathcal{S}_F$ for

$$F := \mathrm{Hom}(\cdot, (X_1, \mathcal{R}_1) \times_{(Y,\mathcal{R})} (X_2, \mathcal{R}_2))$$

(given $(X_i, \mathcal{R}_i) \to (Y, \mathcal{R})$ in $\mathbf{CGWH}_\alpha$) is a 1-fibre product $[X_1/\mathcal{R}_1] \times^1_{[Y/\mathcal{R}]} [X_2/\mathcal{R}_2]$ of (quotient) stacks in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$. The analogous result holds for the stackification of the split category fibered in groupoids induced by the 2-fibre product of α-small topological groupoids, which computes the 2-fibre product of (quotient) stacks in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$.

We conclude this section by discussing an example of a situation where 1-fibre products of stacks in groupoids over a site indeed exist not only as a category over the base site, but as a stack in groupoids.

LEMMA E.5.7.26. *Let A, B, C be three categories fibered in groupoids over a small category $\mathcal{D}$, and let $f : B \to A$ and $g : C \to A$ two 1-morphisms. If A is fibered in sets then the 1-fibre product $B \times^1_{f,A,g} C$ as a category over $\mathcal{D}$ is fibered in groupoids and isomorphic to the 2-fibre product $B \times^2_{f,A,g} C$.*

PROOF. For any object U of $\mathcal{D}$, the objects of $(B \times^2_{f,A,g} C)_U = B_U \times^2_{f_U,A_U,g_U} C_U$ are triples (x, y, ϕ) with $x \in \mathrm{Ob}(B_U)$, $y \in \mathrm{Ob}(C_U)$, and α an isomorphism $\alpha : f_U(x) \to g_U(y)$. Since A is fibered in sets, A_U is a discrete category, forcing $f_U(x) = g_U(x)$. It follows that the very construction of 2-fibre products agrees, in this case, with that of 1-fibre products. In particular, the functor p of Lemma E.5.7.20, which is fully faithful by the Lemma, is surjective on objects, and hence an isomorphism of categories over $\mathcal{D}$. In particular, $B \times^1_{f,A,g} C$ contains enough component-wise pullbacks, and hence it is a fibered category over $\mathcal{D}$ by Lemma E.5.7.18. □

E.6. Stacks on $\mathbf{CGWH}_{\alpha,\tau}$

We are now ready to introduce stacks over the site $\mathbf{CGWH}_{\alpha,\tau}$. We follow the definition of [**SP**, Tag 0026F] closely throughout.

E.6.1. Basic concepts. We now give a 2-categorical generalization of Construction E.5.7.23 of split fibered categories in sets and in groupoids.

CONSTRUCTION E.6.1.1. We let $F : \text{2-Cat}(\mathbf{CGWH}^{\mathrm{opp}}_{\alpha,\tau}) \to \mathbf{Cat}$ be a (necessarily strict) pseudo-functor between 2-categories (see Example E.5.2.7 for the construction 2-Cat$(\cdot)$, and Definition E.5.3.1 for the concept of a pseudo-functor between 2-categories). We define the datum $\mathcal{S}_F$ of a class $\mathrm{Ob}(\mathcal{S}_F)$ defined as the class of all pairs (U, x) with U an object of $\mathbf{CGWH}_{\alpha,\tau}$, and x an object of the groupoid $F(U)$. Since both the category $F(U)$ and the category $\mathbf{CGWH}_\alpha$ are small, the class $\mathrm{Ob}(\mathcal{S}_F)$ is, in fact, a set.

Given two pairs $(U, x), (U', x')$, we define the set:

$$\mathrm{Hom}_{\mathcal{S}_F}((U', x'), (U, x)) := \{(\phi, f) \mid \phi \in \mathrm{Hom}_{\mathbf{CGWH}_\alpha}(U', U), f \in \mathrm{Hom}_{F(U')}(x', F(\phi)(x))\}. \tag{E.6.1.1}$$

[21] where?

Composition of morphisms:

$$\mathrm{Hom}_{\mathcal{S}_F}((U'',x''),(U',x')) \times \mathrm{Hom}_{\mathcal{S}_F}((U',x'),(U,x)) \to \mathrm{Hom}_{\mathcal{S}_F}((U'',x''),(U,x)) \tag{E.6.1.2}$$

is defined as:

$$((\phi,f),(\psi,g)) \rightsquigarrow (\psi,g)\circ(\phi,f) := (\psi\circ\phi, F(f)(\psi)\circ\phi). \tag{E.6.1.3}$$

We further define the assignment:

$$p_F : \mathcal{S}_F \to \mathbf{CGWH}_{\alpha,\tau}, \quad (U,x) \rightsquigarrow p_F(U,x) := U,\ (\phi,f) \rightsquigarrow p_F(\phi,f) := \phi \tag{E.6.1.4}$$

Lemma E.6.1.2. *Let:*

$$F : \text{2-Cat}(\mathbf{CGWH}_{\alpha,\tau}^{\mathrm{opp}}) \to \mathbf{Cat}$$

be a strict pseudo-functor between 2-categories. The datum $\mathcal{S}_F$ is a small category, the assignment $p_F : \mathcal{S}_F \to \mathbf{CGWH}_{\alpha,\tau}$ is a functor, and the pair $(\mathcal{S}_F, p_F)$ is a fibered category. Moreover, if:

1. *F factors as a functor through the inclusion pseudo-functor $\mathbf{Groupoids} \to \mathbf{Cat}$, then $(\mathcal{S}_F, p_F)$ is fibered in groupoids.*
2. *F factors as a functor through the inclusion pseudo-functor $\mathbf{Sets} \to \mathbf{Cat}$ (sending a set to the discrete category over it, then $(\mathcal{S}_F, p_F)$ is fibered in sets).*

Proof. A verification of the assertion on $\mathcal{S}_F$ being a category can be found in [**Ac**, Ch.I,§1.4,p.40,41]. Since $\mathrm{Ob}(\mathcal{S}_F)$ constructed in Construction E.6.1.1 is a set by design, $\mathcal{S}_F$ is a small category. A verification of the assertion that $p_F : \mathcal{S}_F \to \mathbf{CGWH}_{\alpha,\tau}$ is a functor is a direct consequence of the definition of the composition of morphisms of $\mathcal{S}_F$ in (E.6.1.3) of Construction E.6.1.1. A verification of the assertion on $p_F : \mathcal{S}_F \to \mathbf{CGWH}_{\alpha,\tau}$ being a fibered category can be found in [**Ac**, Ch.I,§1.4,p.41,42,43]. That p_F is a fibration in discrete categories is seen by forming, for any object U of $\mathbf{CGWH}_{\alpha,\tau}$, the fibre category $(\mathcal{S}_F)_U$ and observing that

$$\mathrm{Ob}((\mathcal{S}_F)_U) = F(U)$$

and for any two objects $x, x' \in (\mathcal{S}_F)_U$,

$$\mathrm{Hom}_{(\mathcal{S}_F)_U}(x',x) = \mathrm{Hom}_{F(U)}(x',x)$$

and hence $(\mathcal{S}_F)_U = F(U)$. The Lemma therefore follows from [**SP**, Tag 003V] and [**SP**, Tag 02Y2]. □

Remark E.6.1.3. We make the following observations on the preceding lemma.

(i) In part (1), the pseudo-functor F "factoring through" the inclusion pseudo-functor $\mathbf{Groupoids} \to \mathbf{Cat}$ means that F is the composition of a pseudo-functor $F' : \text{2-Cat}(\mathbf{CGWH}_{\alpha,\tau}^{\mathrm{opp}}) \to \mathbf{Groupoids}$ followed by the inclusion pseudo-functor $\mathbf{Groupoids} \to \mathbf{Cat}$. Likewise, in part (2), the pseudo-functor F "factoring through" the inclusion pseudo-functor $\mathbf{Sets} \to \mathbf{Cat}$ means that F is the composition of a pseudo-functor $F' : \text{2-Cat}(\mathbf{CGWH}_{\alpha,\tau}^{\mathrm{opp}}) \to \mathbf{Sets}$ followed by the inclusion pseudo-functor $\mathbf{Sets} \to \mathbf{Cat}$.

(ii) The fibered category $(\mathcal{S}_F, p_F)$ constructed in Construction E.6.1.1 and Lemma E.6.1.2 is an example of a *split* fibered category. The general notion of split fibered category is rendered invariant under isomorphism of fibered categories by declaring (see [**SP**, Tag 02XW]) that split fibered categories are precisely those fibered categories over a fixed small category that are isomorphic in the 2-category $\mathbf{Fib}/\mathcal{D}$ (see part (2) of Definition E.5.2.3) to a fibered category of the form $(\mathcal{S}_F, p_F)$. To assign this concept using equivalences in $\mathbf{Fib}/\mathcal{D}$ as opposed to isomorphisms would render it unimportant, insofar every fibered category is equivalent to a split one (see [**SP**, Tag 004A]).

A reference for background around the following definition is [**GR**, Def.3.4.10(i)].

Definition E.6.1.4. For a small category $(\mathcal{X}, p)$ fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$, and for any α-small compactly generated weakly Hausdorff space T, a covering sieve $F \in J_1(T)$ is called of *effective descent* if and

only if the functor:

$$\mathbf{Hom}(\mathcal{S}_T, \mathcal{X}) \to \mathbf{Hom}(\mathcal{S}_F, \mathcal{X})$$

between the Hom-categories within the 2-category of small categories fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$, is an equivalence of categories.

REMARK E.6.1.5. As in [GR, Rem.3.4.11], if F is the covering sieve generated by the set of morphisms $\{T_i \to T, i \in I\}$, we have a diagram of functors:

$$\mathbf{Hom}(\mathcal{S}_F, \mathcal{X}) \to \prod_{i\in I} \mathbf{Hom}(\mathcal{S}_{T_i}, \mathcal{X}) \rightrightarrows \prod_{(i,j)\in I\times I} \mathbf{Hom}(\mathcal{S}_{T_{ij}}, \mathcal{X}) \tag{E.6.1.5}$$

with $T_{ij} := T_i \times_T T_j$, and with the first functor $e : \mathbf{Hom}(\mathcal{S}_F, \mathcal{X}) \to \prod_{i\in I} \mathbf{Hom}(\mathcal{S}_{T_i}, \mathcal{X})$ being induced by the morphisms of pre-sheaves of sets $h_{T_i} \to F$ and the two rightmost functors being induced by the morphisms of pre-sheaves of sets $h_{T_{ij}} \to h_{T_i}$ induced by the two projections $T_{ij} \to T_i$ and $T_{ij} \to T_j$. The above diagram is a 2-equalizer in the 2-category of all small categories (see [GR, Lemma 3.4.6]). The condition in Definition E.6.1.4 ensures that $\mathbf{Hom}(\mathcal{S}_F, \mathcal{X})$ is equivalent to $\mathbf{Hom}(\mathcal{S}_T, \mathcal{X})$, and it avoids any mention of 2-equalizers by replacing a covering with the sieve it generates, and repackaging the 2-equalizer property of (E.6.1.5) into that of Definition E.6.1.4.

We are now ready to assign the notion of stacks in groupoids on topological spaces:

DEFINITION E.6.1.6. A *stack in groupoids* on $\mathbf{CGWH}_{\alpha,\tau}$ is the datum of a small category $p : \mathcal{X} \to \mathbf{CGWH}_{\alpha,\tau}$ fibered in groupoids which satisfies the following conditions:

(1) For any object T of $\mathbf{CGWH}_{\alpha,\tau}$ and any two objects $x, y \in \mathcal{X}_T$, the pre-sheaf $\mathrm{Isom}(x, y)$ (see [SP, Tag 02ZB]) on the site $\mathbf{CGWH}_{\alpha,\tau}/T$ is a sheaf.
(2) For any object T of $\mathbf{CGWH}_{\alpha,\tau}$, any covering sieve $F \in J_1(T)$ is of effective descent.

REMARK E.6.1.7. The above Definition is formally the same as in [SP, Tag 026F]. The definition also agrees with [GR, Def.5.1.1] for $\varphi = p$ and $i = 2$ under the additional assumption that p be a fibration in groupoids.

E.6.2. Quotient stacks associated to α-small topological groupoids. We first assign a definition:

DEFINITION E.6.2.1. Let $(X, \mathcal{R}, s, t, c, e, i)$ be a topological groupoid (see Definition E.4.2.3). We say that $(X, \mathcal{R}, s, t, c, e, i)$ is *α-small* if and only if the compactly generated weakly Hausdorff spaces X and $\mathcal{R}$ are α-small (see Definition E.3.2.1(2))

We make the following:

CONSTRUCTION E.6.2.2. We define the following functor:

$$F_{(X,\mathcal{R})} : \mathbf{CGWH}_\alpha^{\mathrm{opp}} \to \mathbf{Groupoids} \tag{E.6.2.1}$$

by the assignment, for T varying across arbitrary objects in $\mathbf{CGWH}_\alpha$:

$$T \rightsquigarrow F_{(X,\mathcal{R})}(T) := (h_X(T), h_{\mathcal{R}}(T), h_s(T), h_t(T), h_c(T), h_e(T), h_i(T)) \tag{E.6.2.2}$$

where for a morphism $f : X \to Y$ in $\mathbf{CGWH}_\alpha$, we write h_f for the natural transformation of functors $h_X \to h_Y$ induced by f. We further define the natural transformation of functors $F_{(X,X)} \to F_{(X,\mathcal{R})}$ induced under [SP, Tag 001P] by the morphism of α-small topological groupoids:

$$p : \mathbf{Const}_X \to (X, \mathcal{R}, s, t, c, e, i) \tag{E.6.2.3}$$

where $\mathbf{Const}_X = (X, X, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$ is the constant groupoid on X from (E.4.2.2) and p is the morphism defined by (id_X, e).

REMARK E.6.2.3. We observe that for any α-small topological groupoid $(X, \mathcal{R}, s, t, c, e, i)$, $F_{(X,\mathcal{R})}$ from (E.6.2.1) assigned in (E.6.2.2) is a sheaf of groupoids in sets on $\mathbf{CGWH}_{\alpha,\tau}$ since both h_X and $h_{\mathcal{R}}$ are sheaves of sets.

DEFINITION E.6.2.4. Let $(X, \mathcal{R}, s, t, c, e, i)$ be an α-small topological groupoid. We define:

(1) the split category fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$:

$$p_{(X,\mathcal{R})} : \mathcal{S}_{(X,\mathcal{R})} \to \mathbf{CGWH}_{\alpha,\tau}. \tag{E.6.2.4}$$

to be the split fibered category $(\mathcal{S}_F, p_F)$ from (E.6.1.4), for $F := F_{(X,\mathcal{R})}$ from (E.6.2.1) defined in (E.6.2.2).

(2) We denote the stack associated to $p_{(X,\mathcal{R})}$ by $[X/\mathcal{R}]$, and write:

$$p_{[X/\mathcal{R}]} : [X/\mathcal{R}] \to \mathbf{CGWH}_{\alpha,\tau}. \tag{E.6.2.5}$$

for the canonical 1-morphism induced by $p_{(X,\mathcal{R})}$ under stackification.

(3) Let G be an α-small compactly generated weakly Hausdorff topological group, X an object of $\mathbf{CGWH}_{\alpha}$, $a : G \times X \to X$ a continuous action of G on X. We define the split category fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$:

$$p_{(X,G)} : \mathcal{S}_{(X,G)} \to \mathbf{CGWH}_{\alpha,\tau}. \tag{E.6.2.6}$$

to be the split fibered category $(\mathcal{S}_F, p_F)$ from (E.6.1.4), for $F := F_{(X,\mathcal{R})}$ from (E.6.2.2), where $(X, \mathcal{R}, s, t, c, e, i)$ is the action groupoid associated to the action a of G on X (see Construction E.4.2.12).

(4) Let X be an α-small compactly generated weakly Hausdorff space. We define the split category fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$:

$$p_X : \mathcal{S}_X \to \mathbf{CGWH}_{\alpha,\tau}. \tag{E.6.2.7}$$

to be the split fibered category $(\mathcal{S}_F, p_F)$ from (E.6.1.4), for $F := F_{(X,\mathcal{R})}$ from (E.6.2.2), where $(X, \mathcal{R}, s, t, c, e, i)$ is the constant groupoid $\mathbf{Const}_X$ on X (see (E.4.2.2)).

(5) Let X be an α-small compactly generated weakly Hausdorff space. We define the split category fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$:

$$p_{\underline{X}} : \underline{X} \to \mathbf{CGWH}_{\alpha,\tau}. \tag{E.6.2.8}$$

to be the split fibered category $(\mathcal{S}_F, p_F)$ from (E.6.1.4), for $F := h_X$ from (E.6.2.2).

REMARK E.6.2.5. For any object X in $\mathbf{CGWH}_{\alpha}$, we have two ways to associate a stack in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$ to X.

(1) through the assignment $X \rightsquigarrow \mathcal{S}_X$ from (E.6.2.7).
(2) Through the assignment $X \rightsquigarrow \underline{X}$ from (E.6.2.8)

The two split fibered categories $\mathcal{S}_X$ and $\underline{X}$ are isomorphic as categories fibered in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$ (and hence as categories fibered in setoids by [SP, Tag 02Y1], and fibered in sets by [SP, Tag 04S8]). For any object T of $\mathbf{CGWH}_{\alpha}$, the isomorphism is induced on T-fibre categories by the canonical isomorphism in $\mathbf{CGWH}_{\alpha}$:

$$\mathrm{can} : \mathrm{coeq}(\mathrm{id}_X, \mathrm{id}_X : X \rightrightarrows X) \to X \tag{E.6.2.9}$$

We further observe that since the topology on $\mathbf{CGWH}_{\alpha,\tau}$ is sub-canonical, both $\mathcal{S}_X$ and $\underline{X}$ are stacks in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$. They are therefore isomorphic as stacks in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$ (i.e. not merely equivalent) and the isomorphism is canonical.

We further observe that by [SP, Tag 04SF], there exists a functor between small categories:

$$\mathbf{Hom}((\mathcal{S}_X, p_X), (\mathcal{S}_Y, p_Y)) \to \mathrm{Hom}(h_X, h_Y) \tag{E.6.2.10}$$

where $\mathbf{Hom}((\mathcal{S}_X, p_X), (\mathcal{S}_Y, p_Y))$ is the Hom-category in the 2-category of stacks in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$, which is also the Hom-category of the 2-category of small categories fibered in groupoids over $\mathbf{CGWH}_\alpha$, and where $\mathrm{Hom}(X, Y)$ is the discrete category of morphisms of sheaves of sets (equivalent to the discrete category of continuous maps $\mathrm{Hom}_{\mathbf{CGWH}_\alpha}(X, Y)$). The functor in (E.6.2.10) sends a 1-morphism of categories fibered in groupoids to its 2-isomorphism class, and 2-morphisms to identities. It is surjective on objects and a bijection on isomorphism classes of objects.

The dependency of $\mathcal{S}_X$ and $\underline{X}$, both split fibered categories, on X is as follows: for any object X of $\mathbf{CGWH}_\alpha$, and for the homeomorphism can of (E.6.2.9), there exists a canonical 1-morphism:

$$\underline{X} \to \mathcal{S}_X \tag{E.6.2.11}$$

determined by the natural transformation of sheaves of groupoids can:

$$\mathrm{can} : (\mathrm{coeq}(\mathrm{id}, \mathrm{id} : h_X \rightrightarrows h_X), \mathrm{coeq}(\mathrm{id}, \mathrm{id} : h_X \rightrightarrows h_X), \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}) \to (h_X, h_X, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}) \tag{E.6.2.12}$$

and for any other 1-morphism as in (E.6.2.11) in the same 2-isomorphism class, there exists a unique 2-isomorphism relating the two; see [**SP**, Tag 02Y2]). In particular, the two available ways to assign a stack in groupoids over $\mathbf{CGWH}_{\alpha,\tau}$ to any object of $\mathbf{CGWH}_{\alpha,\tau}$ are un-ambiguous, up to unique 1-isomorphism, itself unique up to unique 2-isomorphism. We shall be using the notation $\underline{X}$ and $\mathcal{S}_X$ interchangeably.

Definition E.6.2.6. We define:

$$\pi : \underline{X} \to [X/\mathcal{R}] \tag{E.6.2.13}$$

to be the 1-morphism of stacks in groupoids over $\mathbf{CGWH}_\alpha$ induced by the morphism p of α-small topological groupoids from (E.6.2.3).

E.6.3. Inertia stack and coarse space. Inertia stacks are designed to keep track of automorphisms of objects of a stack that lift the identity on the base in a suitable sense, depending on the situation.

The general construction for fibered categories is in [**SP**, Tag 04Z2], and we refer to [**SP**, Tag 034I,(2)] for the definition of the absolute inertia fibered category. We now recall the definition in [**SP**, Tag 034I,(2)], spelled out within the intended framework of this appendix.

Definition E.6.3.1. Let $p : \mathcal{X} \to \mathbf{CGWH}_\alpha$ be a stack in topological groupoids over $\mathbf{CGWH}_{\alpha,\tau}$. The *inertia stack* $\mathcal{I}_\mathcal{X}$ of $\mathcal{X}$ is the category $\mathcal{I}_\mathcal{X}$ fibered over $\mathbf{CGWH}_\alpha$ whose:

(1) objects are pairs (x, f) with x an object of $\mathcal{X}$ and $f : x \to x$ an automorphism with $p(f) = \mathrm{id}$.
(2) Morphisms $(x, f) \to (y, g)$ are given by morphisms $\phi : x \to y$ in $\mathcal{X}$ making the following diagram commute:

$$\begin{array}{ccc} x & \xrightarrow{\phi} & y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow[\phi]{} & y \end{array}$$

The category $\mathcal{I}_\mathcal{X}$ is equipped with a functor $q : \mathcal{I}_\mathcal{X} \to \mathbf{CGWH}_\alpha$ defined on objects as $(x, f) \mapsto p(x)$.

Before discussing inertia groups, their meaning, and how they can be viewed as being classified by $\mathcal{I}_\mathcal{X}$, we recall the following basic result which is [**SP**, Tag 034H].

Lemma E.6.3.2. *Let $\mathcal{X}$ be a stack in topological groupoids over $\mathbf{CGWH}_{\alpha,\tau}$. Then there is an equivalence in the $(2,1)$-category of categories over $\mathbf{CGWH}_\alpha$:*

$$\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_{\Delta,(\mathcal{X}\times_{\mathbf{CGWH}_\alpha}\mathcal{X}),\Delta} \times\mathcal{X}$$

and $q : \mathcal{I}_\mathcal{X} \to \mathbf{CGWH}_\alpha$ makes $\mathcal{I}_\mathcal{X}$ into a fibered category.

REMARK E.6.3.3. We further recall (see [SP, Tag 042H] and [SP, Tag 04Z3]) that for a stack in topological groupoids $\mathcal{X}$, there is always a canonical 1-morphism:

$$\iota : \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$$

defined on objects as $(x, f) \mapsto x$ and on morphisms as $(\phi : (x, f) \to (y, g)) \mapsto \phi$, with the last $\phi : x \to y$ regarded as a morphism in $\mathcal{X}$. There moreover exists a section

$$\varepsilon : \mathcal{X} \to \mathcal{I}_{\mathcal{X}}$$

defined on objects as $x \mapsto (x, \mathrm{id}_x)$ and on morphisms by sending $\phi : x \to y$ to $\phi : (x, \mathrm{id}_x) \to (y, \mathrm{id}_y)$ viewed as a morphism in $\mathcal{I}_{\mathcal{X}}$.

TERMINOLOGY E.6.3.1. For a stack $\mathcal{X}$ in topological groupoids, we say the inertia stack is *trivial* if the canonical 1-morphism:

$$\iota : \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$$

is an equivalence. We shall see how this means that all the inertia groups of $\mathcal{X}$ at all points of $\mathcal{X}$ are trivial.

We will only ever be interested in the inertia stacks and groups of quotient stacks in topological groupoids. We therefore describe the inertia stack of such a stack in topological groupoids here below. First, a definition.

DEFINITION E.6.3.4. Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid. Its *inertia group* $I_{[\mathcal{G} \rightrightarrows X]}$ is defined as the fibre product in $\mathbf{CGWH}_\alpha$:

$$I_{[\mathcal{G} \rightrightarrows X]} := X \times_{\Delta, X \times X, (s,t)} \mathcal{G}.$$

This is made into a relative topological group over X by the continuous multiplication law over X induced by the composition law c on $[\mathcal{G} \rightrightarrows X]$.

LEMMA E.6.3.5. *Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid. Then $I_{[\mathcal{G} \rightrightarrows X]}$ is closed in $\mathcal{G}$ with respect to the topology induced by the second projection:*

$$I_{[\mathcal{G} \rightrightarrows X]} = X \times_{\Delta, X \times X, (s,t)} \mathcal{G} \to \mathcal{G}.$$

Moreover, if X is Hausdorff then $I_{[\mathcal{G} \rightrightarrows X]}$ can be computed in **Spaces**.

PROOF. Set-theoretically, $I_{[\mathcal{G} \rightrightarrows X]}$ is the union $\bigcup_{x \in X} I_x$, and so $I_{[\mathcal{G} \rightrightarrows X]}$ is the set-theoretic equalizer of s, t. It is also the topological equalizer, since the forgetful functor $\mathbf{CGWH}_\alpha \to \mathbf{Sets}$ is faithful and commutes with small limits. Thus $I_{[\mathcal{G} \rightrightarrows X]}$ is closed in $\mathcal{G}$ by Lemma E.2.3.9 as long as $\Delta : X \to X \times X$ is a closed immersion in **Spaces**, which it is by Lemma E.2.1.6 since all objects in $\mathbf{CGWH}_\alpha$ are weakly Hausdorff. If X is Hausdorff then by Lemma E.2.3.1 $I_{[\mathcal{G} \rightrightarrows X]}$ can be computed in **Spaces** because $X \times X$ can, and $X \times_{\Delta, X \times X, (s,t)} \mathcal{G}$ is a fibre product in **Spaces** by Lemma E.2.3.9. □

We have the following description of the inertia stack of the quotient stack $[X/\mathcal{G}]$ in terms of the chart $\mathcal{G} \rightrightarrows X$ for $[X/\mathcal{G}]$:

LEMMA E.6.3.6. *Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid. We set $\mathcal{G}' := \mathcal{G} \times_{s,X} I_{[\mathcal{G} \rightrightarrows X]}$, and further define:*

(1) *$s' : \mathcal{G}' \to I_{[\mathcal{G} \rightrightarrows X]}$, by $s'(r, g) := g$.*
(2) *$t' : \mathcal{G}' \to I_{[\mathcal{G} \rightrightarrows X]}$, by $t'(r, g) := c(r, c(g, i(r)))$.*
(3) *$c' : \mathcal{G}' \times_{s', I_{[\mathcal{G} \rightrightarrows X]}, t'} \mathcal{G}' \to \mathcal{G}'$, by $c'((r, g), (r', g')) := (c(r, r'), g)$.*

Then $(I_{[\mathcal{G} \rightrightarrows X]}, \mathcal{G}', s', t', c')$ together with the uniquely determined unit section e' and inverse map i', is a topological groupoid. Moreover, we have a canonical equivalence of stacks in topological groupoids:

$$\mathcal{I}_{[X/\mathcal{G}]} = [I_{[\mathcal{G} \rightrightarrows X]}/\mathcal{G}'].$$

PROOF. The proof proceeds in a manner that is identical to the case of groupoids in schemes, and we record it here to provide more details and to remind the reader of the construction of the morphism:

$$\mathcal{I}_{[X/\mathcal{G}]_{\text{prestack}}} \to [I_{[\mathcal{G}\rightrightarrows X]}/\mathcal{G}']_{\text{prestack}}.$$

We refer the reader to [SP, Tag 06PB] for the direct analog for groupoids in schemes. By [SP, Tag 06NS], it is enough to prove this on the level of pre-stacks.

We need to show that there is an equivalence of fibered categories:

$$\mathcal{I}_{[X/\mathcal{G}]_{\text{prestack}}} \to [I_{[\mathcal{G}\rightrightarrows X]}/\mathcal{G}']_{\text{prestack}}.$$

For T in $\mathbf{CGWH}_\alpha$, a T-object in the inertia fibred category $\mathcal{I}_{[X/\mathcal{G}]_{\text{prestack}}}$ is the datum of an object x in the *fibre* category over T and an automorphism g of x. We may interpret x as a morphism $x : T \to X$, and g as a morphism $g : T \to \mathcal{G}$, such that $s \circ g = t \circ g = x$. This means that g factors through $I_{[\mathcal{G}\rightrightarrows X]}$. A morphism in the inertia fibred category $(x, g) \to (y, h)$ over the space T is the datum of a morphism $r : T \to \mathcal{G}$ such that $s \circ r = x$ and $t \circ r = y$, and the diagram:

$$\begin{array}{ccc} x & \xrightarrow{r} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow[r]{} & y \end{array}$$

commutes. This is by [SP, Tag 034H], and means the identity:

$$c(r, g) = c(h, r)$$

using the composition law c. Incorporating the effect of the inverse map i, we can also write:

$$c(c(r, g), i \circ r) = c(r, c(g, i \circ r))$$

which means "$r \circ (g \circ r^{-1}) = h$" and "$(r \circ g) \circ r^{-1} = h$", respectively. Once again this gives a unique element of $\mathcal{G}'$ acting as a morphism between objects, i.e. elements in $I_{[\mathcal{G}\rightrightarrows X]}$. Not only this constructs a morphism of fibred categories $\mathcal{I}_{[X/\mathcal{G}]_{\text{prestack}}} \to [I_{[\mathcal{G}\rightrightarrows X]}/\mathcal{G}']_{\text{prestack}}$, but it proves it is both fully faithful and essentially surjective. □

We have the following simple criterion for the triviality of the inertia stack of the quotient stack associated to a topological groupoid:

LEMMA E.6.3.7. *Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid. Assume $(s, t) : \mathcal{G} \to X \times X$ is injective. Then the canonical 1-morphism:*

$$\iota : \mathcal{I}_{[X/\mathcal{G}]} \to [X/\mathcal{G}]$$

is an equivalence.

PROOF. The section ε to ι discussed in Remark E.6.3.3 shows that the functor

$$\mathcal{I}_{[X/\mathcal{G}]} \to [X/\mathcal{G}]$$

is essentially surjective. If $(s, t) : \mathcal{G} \to X \times X$ is injective then the morphism $I_{[\mathcal{G}\rightrightarrows X]} \to X$ given by the first projection in the construction of $I_{[\mathcal{G}\rightrightarrows X]}$[22] is also injective, hence surjective by the existence of a section to ι. This forces ι to be faithful. Moreover, it also forces $\mathcal{G}' \to \mathcal{G}$ to be an isomorphism, and therefore ι is fully faithful, hence an equivalence. □

We further have the following representability criterion:[23]

[22] Clarify here with a reference

[23] There seems to be no actual discussion of coarse spaces anywhere here, complementing this result, yet coarse spaces are the *entire purpose* of all of this effort. Must address it.

PROPOSITION E.6.3.8. *Let $[\mathcal{G} \rightrightarrows X]$ be a topological groupoid. Assume:*

(1) *$(s,t) : \mathcal{G} \to X \times X$ is a closed immersion,*

(2) *the continuous map $X \to X/\mathcal{G}$ is a quotient map.*

Then $[X/\mathcal{G}]$ is representable, and the representing object of $\mathbf{CGWH}_\alpha$ is the quotient topological space $X/\mathcal{G}$. Equivalently, there is a canonical equivalence:

$$[X/\mathcal{G}] \simeq \underline{X/\mathcal{G}}.$$

The condition on the map $X \to X/\mathcal{G}$ being a quotient map is typically easy to arrange. The usual obstruction to representability is the discrepancy between the intrinsic topology on $\mathcal{G}$ and the induced topology on $\mathcal{G}$ from $X \times X$, with the latter product being in **Spaces**. If X is Hausdorff this is indeed the case. More generally, this can fail and then one can get a quotient stack in topological groupoids which is not representable despite having trivial inertia stack.

PROOF OF PROPOSITION E.6.3.8. Under the assumptions of the Proposition, $X/\mathcal{G}$ is compactly generated and weakly Hausdorff. By Lemma E.6.3.7, the inertia stack of $[X/\mathcal{G}]$ is trivial. This implies that not only $X/\mathcal{G}$ is an object of $\mathbf{CGWH}_\alpha$ (it is α-small since X is), but that the morphism $[X/\mathcal{G}] \to X/\mathcal{G}$ is representable, and so $[X/\mathcal{G}]$ is itself representable. Let T be the representing object. By Yoneda's Lemma, the morphism $[X/\mathcal{G}] \to X/\mathcal{G}$ is therefore induced by a unique continuous map $T \to X/\mathcal{G}$. Moreover, the morphism $X \to [X/\mathcal{G}]$ is always representable by quotient maps, which forces $X \to T$ to be itself a quotient map. One checks, by assumption (1) and Lemma E.6.3.7, that the induced equivalence relation $R := X \times_T X$ on X yields a trivial inertia stack $\mathcal{I}_{[R \rightrightarrows X]}$, which forces the map $R \to \mathcal{G}$ to be surjective. By the final assertion in Lemma E.2.3.5, the map $R \to \mathcal{G}$ is therefore a closed immersion and hence a homeomorphism, and T receives a homeomorphic quotient map from $X/\mathcal{G}$ which is a section to the map $T \to X/\mathcal{G}$, so the Proposition is proved. □

We now recall foundational material about inertia groups, highlighting how to view them as fibers of the morphism $\iota : \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$.

LEMMA E.6.3.9. *Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid. For a point $x \in X$, let $I_x = \{y \in \mathcal{G} : s(y) = x = t(y)\}$. Then I_x equipped with the topology induced by $\mathcal{G}$ is a topological group.*

PROOF. A quick way to see this is by observing that the first projection $I_{[\mathcal{G} \rightrightarrows X]} \to X$ makes $I_{[\mathcal{G} \rightrightarrows X]}$ into a relative topological group over X, whose multiplication X-morphism is induced by the composition law on $\mathcal{G}$ of $[\mathcal{G} \rightrightarrows X]$. The fiber of $I_{[\mathcal{G} \rightrightarrows X]} \to X$ over x is exactly I_x with the induced topology from the topology on $\mathcal{G}$. □

LEMMA E.6.3.10. *Let (X, G, s, t, c, e, i) be a locally compact Hausdorff topological group. Then all the topological groups I_x are locally compact Hausdorff.*

PROOF. Closed subspaces of locally compact Hausdorff topological spaces are locally compact Hausdorff, and I_x is closed in $\mathcal{G}$ since $x \in X$ is closed (as X is weakly Hausdorff). □

For $x : T \to \mathcal{X}$, define a sheaf $\mathcal{I}_{x:\underline{T} \to \mathcal{X}}$ of groups on $\mathbf{CGWH}_{\alpha,\tau}$ whose sections interpolate the topological groups I_y. For any object T' of $\mathbf{CGWH}_\alpha$, let

$$\mathcal{I}_{x:\underline{T} \to \mathcal{X}}(T')$$

denote the set of pairs (f, ϕ) with $f : T' \to T$ a morphism and ϕ an automorphism of $f^*(x)$ as an object of $\mathcal{X}(T')$. The map

$$\mathcal{I}_{x:\underline{T} \to \mathcal{X}} \to h_T$$

sending a T'-section (f, ϕ) to $f \in h_T(T')$ makes $\mathcal{I}_{x:\underline{T} \to \mathcal{X}}$ into a sheaf of groups over T.

For the canonical map $x : \underline{X} \to \mathcal{X} := [X/\mathcal{G}]$, $\mathcal{I}_{x:\underline{X} \to \mathcal{X}}$ is nothing but $I_{[\mathcal{G} \rightrightarrows X]}$ from Definition E.6.3.4, with its own map to X.

NOTATION E.6.3.1. We often abbreviate $\mathcal{I}_{x:\underline{T}\to\mathcal{X}}$ as $\mathcal{I}_x$ when no confusion is expected to arise.

LEMMA E.6.3.11. *Let $(X, \mathcal{G}, s, t, c, e, i)$ be a topological groupoid and $\mathcal{X} = [X/\mathcal{G}]$ its quotient stack. Let T be an object of $\mathbf{CGWH}_\alpha$. Then for any T-point $x : \underline{T} \to \mathcal{X}$ of $\mathcal{X}$, we have a cartesian diagram:*

$$\begin{array}{ccc} \mathcal{I}_{x:\underline{T}\to\mathcal{X}} & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \iota \\ \underline{T} & \xrightarrow[x]{} & \mathcal{X} \end{array}$$

PROOF. We let $T' \to T$ be a morphism of spaces, and consider the induced diagram of maps of topological groupoids

$$\begin{array}{ccc} \mathcal{I}_{x:\underline{T}\to\mathcal{X}}(T') & \longrightarrow & \mathcal{I}_{\mathcal{X}}(T') \\ \downarrow & & \downarrow \iota \\ h_T(T') & \xrightarrow[x]{} & \mathcal{X}(T') \end{array} \tag{E.6.3.1}$$

where $\mathcal{X}(T')$ is short-hand notation for the topological groupoid of morphisms $\underline{T}' \to \mathcal{X}$ and the maps in (E.6.3.1) are defined as follows. The bottom horizontal map is the map sending $f \in h_T(T')$ to the composition $f^*x = x \circ f : \underline{T'} \to \underline{T} \to \mathcal{X}$, an object of the groupoid $\mathcal{X}(T')$. The left vertical map is $(f, \phi) \mapsto f$, and the top horizontal map is

$$(f, \phi) \mapsto ((f^*(x), f^*(x)), \phi) \in (\mathcal{X} \times_{\Delta, \mathcal{X}\times\mathcal{X}, \Delta} \mathcal{X})(T').$$

The right vertical map ι sends an element (a, ϕ) of $\mathcal{I}_{\mathcal{X}}(T')$ (with $a \in X(T')$ and $\phi \in \mathcal{G}(T')$ an automorphism of a such that $\Delta_X(a) = (s, t)(\phi)$ in $X \times X$) to $a \in X(T')$ viewed as an object of $\mathcal{X}(T')$.

We have:

$$\mathcal{X}(T') = [\mathcal{G}(T') \rightrightarrows X(T')],\ \mathcal{I}_{\mathcal{X}}(T') = [\mathcal{G}' \rightrightarrows I_{[\mathcal{G}(T')\rightrightarrows X(T')]}]$$

where $\mathcal{G}'$ refers to the notation from Lemma E.6.3.6. The groupoid-theoretic fibre product thereby recovers the description of $\mathcal{I}_{x:\overline{T}\to\mathcal{X}}(T')$ as in the construction of $\mathcal{I}_{x:\overline{T}\to\mathcal{X}}$. □

REMARK E.6.3.12. In the special case $T = X$ and x is the quotient $\underline{X} \to [X/\mathcal{G}]$, $\mathcal{I}_{x:\underline{T}\to\mathcal{X}} = I_{[\mathcal{G}\rightrightarrows X]}$ and for any point $y \in X$ the induced morphism $y : \underline{\{y\}} \to \underline{X} \to \mathcal{X}$ fits into the assertion of the Lemma with $T = \{y\}$ and gives:

$$\mathcal{I}_{y:\underline{\{y\}}\to\mathcal{X}} = \underline{\{y\}} \times_{y,\mathcal{X},\iota} \mathcal{I}_{\mathcal{X}} = \{y\} \times_{y,X} I_{[\mathcal{G}\rightrightarrows X]} = I_y.$$

In other words, the inertia stack interpolates the inertia groups of all points of X.

E.6.4. Equivalences. In this subsection we focus on stacks in topological groupoids that are equipped with a chart, introduce related some concepts and establish some consistency checks.

We have associated a stack in topological groupoids to a topological groupoid in a functorial manner. We will now compare the two notions to study how far this functorial assignment is from being fully faithful.

LEMMA E.6.4.1. *Let $(X, \mathcal{G}, s, t, c, e, i)$ and $(X', \mathcal{G}', s', t', c', e', i')$ be two topological groupoids, and:*

$$f : [\mathcal{G} \rightrightarrows X] \to [\mathcal{G}' \rightrightarrows X']$$

a morphism of topological groupoids. We write $f = (f_1, f_2)$ to denote the map $f_1 : X \to X'$ on Ob-sets and the map $f_2 : \mathcal{G} \to \mathcal{G}'$ on Mor-sets. Let f also denote the induced morphism:

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$$

of quotient stacks. The following are equivalent:

(1) *f is fully faithful.*

(2) *The natural map:*

$$\mathcal{G} \to (X \times X) \times_{X' \times X'} \mathcal{G}'$$

is an isomorphism.

PROOF. The argument is identical to that in the proof of [SP, Tag 046S], noting that one has to use the topology on $\mathbf{CGWH}_{\alpha,\tau}$ to work locally and pick local sections, and that the analog of [SP, Tag 044V] still holds for topological groupoids and their quotient stacks, with an identical proof. □

LEMMA E.6.4.2. *Let $(X, \mathcal{G}, s, t, c, e, i)$ and $(X', \mathcal{G}', s', t', c', e', i')$ be two topological groupoids, and:*

$$f : [\mathcal{G} \rightrightarrows X] \to [\mathcal{G}' \rightrightarrows X']$$

a morphism of topological groupoids. We write $f = (f_1, f_2)$ to denote the map $f_1 : X \to X'$ on Ob-sets and the map $f_2 : \mathcal{G} \to \mathcal{G}'$ on Mor-sets. Let f also denote the induced morphism:

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$$

of quotient stacks. The following are equivalent:

(1) *f is an equivalence.*
(2) *f satisfies the following two conditions:*
   (a) *the natural map*

$$\mathcal{G} \to (X \times X) \times_{X' \times X'} \mathcal{G}'$$

   is an isomorphism,
   (b) *the composition*

$$X \times_{f_1, X', t'} \mathcal{G} \xrightarrow{\mathrm{pr}_2} \mathcal{G}' \xrightarrow{s'} X'$$

   is a submersion.

PROOF. Just as for the analog of the result for groupoids on algebraic spaces (see [SP, Tag 046T]), one uses [SP, Tag 046N] which characterizes equivalences between stacks on a site. The argument to check that the assumptions of [SP, Tag 046N] are satisfied, using full faithfulness of f which is a consequence of Lemma E.6.4.1, is identical to the argument in [SP, Tag 046T] upon replacing all objects in the site with objects in $\mathbf{CGWH}_\alpha$ and replacing the fppf topology on algebraic spaces with the topology on $\mathbf{CGWH}_{\alpha,\tau}$. □

DEFINITION E.6.4.3. Let $(X, \mathcal{G}, s, t, c, e, i)$ and $(X', \mathcal{G}', s', t', c', e', i')$ be two topological groupoids, and:

$$f : [\mathcal{G} \rightrightarrows X] \to [\mathcal{G}' \rightrightarrows X']$$

a morphism of topological groupoids. We write $f = (f_1, f_2)$ to denote the map $f_1 : X \to X'$ on Ob-sets and the map $f_2 : \mathcal{G} \to \mathcal{G}'$ on Mor-sets. Let f also denote the induced morphism:

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$$

of quotient stacks. We say f is a *standard Morita equivalence* if and only if:

(1) $f_1 : X \to X'$ is a submersion,
(2) the induced map $\mathcal{G}' \to (X \times X) \times_{(f_1, f_1), X' \times X', (s', t')} \mathcal{G}'$ is an isomorphism.

We have the following:

LEMMA E.6.4.4. *Suppose $(X, \mathcal{G}, s, t, c, e, i)$ and $(X', \mathcal{G}', s', t', c', e', i')$ are two topological groupoids, and:*

$$f : [\mathcal{G} \rightrightarrows X] \to [\mathcal{G}' \rightrightarrows X']$$

is a morphism of topological groupoids. If f is a standard Morita equivalence then the induced morphism of quotient stacks:

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$$

is an equivalence.

PROOF. Indeed, submersiveness of f_1 implies submersiveness of $X \times_{f_1, X', t'} \mathcal{G}' \to \mathcal{G}'$ (by Lemma E.2.3.6(6)), and since $\mathcal{G}' \to X'$ is submersive (as it has a continuous section) we get the result by Lemma E.6.4.2. □

A partial converse holds:

LEMMA E.6.4.5. *Suppose* $(X, \mathcal{G}, s, t, c, e, i)$ *and* $(X', \mathcal{G}', s', t', c', e', i')$ *are two topological groupoids, and*

$$f : [\mathcal{G} \rightrightarrows X] \to [\mathcal{G}' \rightrightarrows X']$$

is a morphism of topological groupoids. Suppose that the induced morphism between quotient stacks

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$$

is an equivalence. Then there exists a topological groupoid $(X'', \mathcal{G}'', s'', t'', c'', e'', i'')$ *and morphisms of topological groupoids*

$$g : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G} \rightrightarrows X], h : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G}' \rightrightarrows X']$$

such that g *and* h *are standard Morita equivalences, and* $f \circ g = h$.

Not all equivalences of quotient stacks are induced by a morphism of topological groupoids. However, they are up to standard Morita equivalence.

LEMMA E.6.4.6. *Suppose* $(X, \mathcal{G}, s, t, c, e, i)$ *and* $(X', \mathcal{G}', s', t', c', e', i')$ *are two topological groupoids, and we are given a morphism between quotient stacks*

$$f : [X/\mathcal{G}] \to [X'/\mathcal{G}'].$$

Then there exists a topological groupoid $(X'', \mathcal{G}'', s'', t'', c'', e'', i'')$*, a standard Morita equivalence*

$$g : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G} \rightrightarrows X]$$

and a morphism of topological groupoids

$$h : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G}' \rightrightarrows X']$$

such that $f \circ g = h$.

PROOF. The idea is to form the 2-fiber product:

$$\begin{array}{ccc} \mathcal{X} := [X/\mathcal{G}] \times_{[X'/\mathcal{G}']} \underline{X}' & \longrightarrow & \underline{X}' \\ \downarrow & & \downarrow \\ [X/\mathcal{G}] & \longrightarrow & [X'/\mathcal{G}'] \end{array}$$

and observe that there exists an induced epimorphism:

$$\underline{X} \times_{[X/\mathcal{G}]} \underline{X}' \to \mathcal{X}$$

with $\underline{X} \times_{[X/\mathcal{G}]} \underline{X}'$ representable. Let X'' be the representing object, so there are evident maps $X'' \rightrightarrows X, X'$. Then one checks that upon calling $\mathcal{G}'' := \underline{X}'' \times_{[X/\mathcal{G}]} \underline{X}''$, and upon defining the structure maps to make a topological groupoid $[\mathcal{G}'' \rightrightarrows X'']$ accordingly, the resulting morphism of topological groupoids $g : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G} \rightrightarrows X]$ (using $\mathcal{G}'' \to X \times_{[X/\mathcal{G}]} X = \mathcal{G}$) is a standard Morita equivalence. One then uses $\mathcal{G}' \to X' \times_{[X'/\mathcal{G}']} X' = \mathcal{G}'$ (resting on $f : [X/\mathcal{G}] \to [X'/\mathcal{G}']$) to define

$$h : [\mathcal{G}'' \rightrightarrows X''] \to [\mathcal{G}' \rightrightarrows X'],$$

and this works. □

REMARK E.6.4.7. Standard Morita equivalences in $\mathbf{CGWH}_{\alpha,\tau}$-**Grpd** satisfy the right-Ore conditions. The localization of the category of topological groupoids at the class of standard Morita equivalences is equivalent, as a 2-category, to the 2-category of stacks in topological groupoids $\mathcal{X} \to \mathbf{CGWH}_{\alpha,\tau}$ equipped with a representable epimorphism $\underline{X} \to \mathcal{X}$ from an object of $\mathbf{CGWH}_{\alpha}$.

EXAMPLE E.6.4.8. For X in $\mathbf{CGWH}_{\alpha}$ equipped with a continuous action by a locally compact Hausdorff topological group G and a closed subgroup $H \subset G$ whose action on X is free,[24] the topological quotient X/H belongs[25] to $\mathbf{CGWH}_{\alpha}$ and is equipped with an evident continuous action by the topological group G/H. We then have[26]

$$[X/G] = [(X/H)/(G/H)].$$

For a number field F, a finite set S of places of F containing the Archimedean places, and a finite place $v_0 \in S$, we apply this to $X = \{0\} \times \prod_{v \in S - \{v_0\}} F_v^{\times}$, $G = (\prod_{v \in S} F_v^{\times})^1$, and $H = (\prod_{v \in S - \{v_0\}} F_v^{\times})^1$. Since $X/H \simeq \mathbf{R}_{>0}$ (via $|\cdot|_{F,S}$) and $G/H = \mathcal{O}_{F_{v_0}}^{\times}$ (with trivial action on X/H), we get $[X/G] = [\mathbf{R}_{>0}/\mathcal{O}_{F_{v_0}}^{\times}]$.[27]

[24] do we need more precision than set-theoretic freeness?

[25] give reference

[26] explain, including with references

[27] This should be expanded, including the groups without norm-1 condition to make the contact with $\mathbf{R}_{>0}/q_{v_0}^{\mathbf{Z}}$ as the coarse space for a specific quotient stack. It also has to be explained how exactly this relates to the preceding discussion, or put it elsewhere.

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